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ON CARLESON MEASURES INDUCED BY BELTRAMI COEFFICIENTS BEING COMPATIBLE WITH FUCHSIAN GROUPS

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Abstract. Let μ be a Beltrami coefficient on the unit disk, which is compatible with a finitely generated Fuchsian group G of the second kind. In this paper we show that if $\frac{|\mu|^2}{1-|z|^2}\,dx\,dy$ satisfies the Carleson condition on the infinite boundary of the Dirichlet fundamental domain of G, then $\frac{|\mu|^2}{1-|z|^2}\,dx\,dy$ is a Carleson measure on the unit disk.

1. Introduction

A Fuchsian group is a discrete Möbius group G acting on the unit disk Δ. A Fuchsian group is said to be of the first kind if its limit set is the entire circle and of the second kind otherwise. A Fuchsian group G is called cocompact if Δ/G is compact and is called convex cocompact if G is finitely generated without parabolic elements. All cocompact groups are first kind and convex cocompact groups minus cocompact groups are second kind. A Fuchsian group G is of divergence type if

$$\Sigma_{g\in G}(1-|g(0)|)=\infty \quad \text{or} \quad \sum_{g\in G}\exp(-\rho(0,g(0)))=\infty,$$

where $\rho(0,g(0))$ is the hyperbolic distance between 0 and $g(0)$. Otherwise, we say that it is of convergence type. All second kind groups are of convergence type. For more details about Fuchsian groups, see [9].

For g in G, we denote by $\mathcal{D}_z(g)$ the closed hyperbolic half-plane containing z, bounded by the perpendicular bisector of the segment $[z,g(z)]_h$. The Dirichlet fundamental domain $\mathcal{F}_z(G)$ of G centered at z is the intersection of all the sets $\mathcal{D}_z(g)$ with g in $G-\{id\}$. For simplicity, in this paper we use the notation $\mathcal{F}$ for the Dirichlet fundamental domain $\mathcal{F}_z(G)$ of G centered at $z=0$.

A positive measure λ defined in a simply connected domain Ω is called a Carleson measure if there exists some constant C which is independent of r such that, for all $0<r<\operatorname{diameter}(\partial\Omega)$ and $z\in\partial\Omega$,

$$\lambda(\Omega\cap D(z,r))\le Cr.$$

The infimum of all such C is called the Carleson norm of λ, denoted by $\|\lambda\|_*$. Let Δ be the unit disk. In this paper, we mainly focus our attention on the case $\Omega=\Delta$. We denote by $CM(\Delta)$ the set of all Carleson measures on Δ.

We say that a measurable function $\mu(z)$ belongs to $CM^*(\Delta)$ if the measure

$$\frac{|\mu|^2}{1-|z|^2}\,dx\,dy\in CM(\Delta).$$

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The importance of the class $CM^*(\Delta)$ lies in the fact that it plays a crucial role in the theory of BMOA-Teichmüller space, see [1, 5, 8, 14] etc. If G is a Fuchsian group and $\mu(z)$ a bounded measurable function on Δ which satisfies

$$||\mu(z)||_\infty < 1 \quad \text{and} \quad \mu(z) = \mu(g(z))\overline{g'(z)}/g'(z)$$

for every $g \in G$, then we say μ is a G-compatible Beltrami coefficient (or complex dilatation). We denote by $M(G)$ the set of all G-compatible Beltrami coefficients. For a G-compatible Beltrami coefficient μ, if the measure

$$\frac{|\mu|^2}{1-|z|^2}\,dx\,dy$$

is a Carleson measure on Δ, when the Carleson norm is small, then $f_\mu(\partial\Delta)$ is a rectifiable (chord-arc) curve, where f_μ is the quasiconformal mapping of the complex plane $\mathbf{C}$ with i, 1 and $-i$ fixed, whose Beltrami coefficient equals to μ a.e. on the unit disk and equals to zero on the outside of the unit disk. This is essential for the proof of the convergence-type first-kind Fuchsian groups failing to have Bowen's property, see [2]. It is also the method to prove that some convergence-type Fuchsian groups fail to have Ruelle's property, see [12, 11].

It is important to investigate under which condition the G-compatible Beltrami coefficients belong to $CM^*(\Delta)$. We call the intersection of $\overline{\mathcal{F}}$ with the unit circle $\partial\Delta$ the boundary at infinity of $\mathcal{F}$, denoted by $\mathcal{F}(\infty)$. In this paper, we first prove:

Theorem 1.1. *Let G be a convex cocompact Fuchsian group of the second kind and $\mathcal{F}$ the Dirichlet fundamental domain of G centered at 0. Let $\mu \in M(G)$: if there exists a constant C such that, for any $\xi \in \mathcal{F}(\infty)$(i.e. ξ is in the free edges of $\mathcal{F}$) and for any $0 < r < 1$,*

$$\iint_{B(\xi,r)} \frac{|\mu|^2\chi_{\mathcal{F}}}{1-|z|^2}\,dx\,dy \le Cr,$$

then μ is in $CM^*(\Delta)$, where $\chi_{\mathcal{F}}$ is the characteristic function of the Dirichlet fundamental domain $\mathcal{F}$.

Notice that Theorem 1.1 fails for the case of convex cocompact groups of the first kind (i.e. cocompact groups), since Bowen [6] showed that cocompact groups hold a rigidity property, now called Bowen's property, i.e. the image of the unit circle under any quasiconformal map whose Beltrami coefficient compatible with a cocompact group, is either a circle or has Hausdorff dimension bigger than 1. Hence for any μ being compatible with cocompact groups, the measure $\frac{|\mu|^2}{1-|z|^2}\,dx\,dy$ is not a Carleson measure.

Furthermore, Theorem 1.1 can be generalized to the finitely generated Fuchsian group of the second kind with some parabolic elements. We have

Theorem 1.2. *Let G be a finitely generated Fuchsian group of the second kind with some parabolic elements and $\mathcal{F}$ the Dirichlet fundamental domain of G centered at 0. Let $\mu \in M(G)$: if there exists a constant C such that, for any $\xi \in \mathcal{F}(\infty)$ and for any $0 < r < 1$,*

$$\iint_{B(\xi,r)} \frac{|\mu|^2\chi_{\mathcal{F}}}{1-|z|^2}\,dx\,dy \le Cr,$$

then μ is in $CM^*(\Delta)$.

This theorem means that the Carleson property of the measures which are compatible with the finitely generated Fuchsian groups can be checked from the points in the set $\mathcal{F}(\infty)$ i.e., the boundary at infinity of the Dirichlet domain $\mathcal{F}$.

Notation. In this paper χ_A always denotes the characteristic function of the set A.

2. Some lemmas

The following lemma will be used several times in this paper. I give a short proof here.

Lemma 2.1. *Let μ be a essentially bounded measurable function on Δ. If the measure $\frac{|\mu|^2}{1-|z|^2}\,dx\,dy$ is in $CM(\Delta)$, then there exists a constant C such that, for any $\xi \in \overline{\Delta}$ and all $0 < r < 2$,*

$$\iint_{B(\xi,r)\cap\Delta} \frac{|\mu|^2}{1-|z|^2}\,dx\,dy \le Cr,$$

where the constant C depends only on the Carleson norm of the measure $\frac{|\mu|^2}{1-|z|^2}\,dx\,dy$ and the essential norm of μ.

Proof. We first choose $0 < r < 2$ and fix it. For any $\xi \in \overline{\Delta}$, if $\xi \in \partial\Delta$, there is nothing to prove. We suppose $\xi \in \Delta$. If $dist(\xi, \partial\Delta) \ge 2r$ (this case only happens when $0 < r < 0.5$), where $dist(\cdot,\cdot)$ denotes the Euclidean distance. Then we have

$$\iint_{B(\xi,r)} \frac{|\mu|^2}{1-|z|^2}\,dx\,dy \le \frac{||\mu||_\infty \pi r^2}{1-|1-r|^2} = \frac{||\mu||_\infty \pi r}{2-r} \le \pi||\mu||_\infty r.$$

For the case $\mathrm{dist}(\xi,\partial\Delta) \le 2r$, we can choose a point $\eta \in \partial\Delta$ such that $\mathrm{dist}(\eta,\xi) < 2r$. Then we have $B(\xi, r) \subset B(\eta, 4r)$ and

$$\iint_{B(\xi,r)\cap\Delta} \frac{|\mu|^2}{1-|z|^2}\,dx\,dy \le \iint_{B(\eta,4r)\cap\Delta} \frac{|\mu|^2}{1-|z|^2}\,dx\,dy \le 4C^* r, \tag{2.1}$$

where C^* is the Carleson norm of the measure $\dfrac{|\mu|^2}{1-|z|^2}\,dx\,dy$.

Hence we let $C = \max\{\pi||\mu||_\infty, 4C^*\}$ and the lemma follows. □

Remark. By this lemma we see that for any simply connected domain $\Omega \subset \Delta$, If $\frac{|\mu|^2}{1-|z|^2}\,dx\,dy$ is a Carleson measure on Δ, then it is also a Carleson measure on Ω.

In order to prove Theorem 1.1, we will need the following lemma which essentially belongs to Astala and Zinsmeister, see [1], or [2].

Lemma 2.2. *For a convergence-type Fuchsian group G and μ in $M(G)$, if there exists a $0 < t < 1$ such that the support set of $\mu\chi_{\mathcal{F}}$ is contained in the ball $B(0,t)$ with center 0 and radius t. Then μ is in $CM^*(\Delta)$.*

For the readers to see more clearly about the property of μ, we give the detail of proof of this lemma here.

Proof. Recall that a sequence $\{z_j\}$ is called an interpolating sequence of Δ if

$$\text{(i)}\quad \exists\delta > 0,\ \rho(z_j, z_k) \ge \delta \text{ if } j \ne k;$$

$$\text{(ii)}\quad \sum (1-|z_i|^2)\delta_{z_i} \in CM(\Delta),$$

where δ_z stands for the Dirac mass at z.

We first show that the sequence $\{g(0)\}_{g\in G}$ is an interpolating sequence of the unit disk Δ. The sequence $\{g(0)\}_{g\in G}$ satisfies the property (i) of the interpolating sequence immediately from the action of Fuchsian group being discrete. For the property (ii), by a result due to Carleson [7], we know that

$$\sum_{g\in G}(1-|g(0)|^2)\delta_g \in CM(\Delta) \tag{3.1}$$

is equivalent to

$$\inf_{g_i}\prod_{g\in G, g\neq g_i}|\frac{g_i(0)-g(0)}{1-\overline{g_i(0)}g(0)}| \geq \delta > 0. \tag{3.2}$$

In order to show (3.2), it is enough to prove that for any $g_i \neq g_k$,

$$\prod_{g\in G, g\neq g_i}\left|\frac{g_i(0)-g(0)}{1-\overline{g_i(0)}g(0)}\right| \equiv \prod_{g\in G, g\neq g_k}\left|\frac{g_k(0)-g(0)}{1-\overline{g_k(0)}g(0)}\right|. \tag{3.3}$$

Note that

$$\left|\frac{g_i(0)-g(0)}{1-\overline{g_i(0)}g(0)}\right| = \tanh 2\rho(g_i(0), g(0)),$$

where $\rho(g_i(0), g(0))$ denotes the hyperbolic distance between $g_i(0)$ and $g(0)$. Similarly,

$$\left|\frac{g_k(0)-g(0)}{1-\overline{g_k(0)}g(0)}\right| = \tanh 2\rho(g_k(0), g(0)).$$

Let $\gamma = g_k \circ g_i^{-1}$, we have $g_k = \gamma \circ g_i$ and

$$\begin{aligned}
&\prod_{g\in G, g\neq g_k}\left|\frac{g_k(0)-g(0)}{1-\overline{g_k(0)}g(0)}\right| = \prod_{g\in G, g\neq g_k}\tanh(2\rho(g_k(0), g(0)))\\
&= \prod_{g\in G, g\neq g_k}\tanh(2\rho(\gamma\circ g_i(0), g(0))) = \prod_{g\in G, g\neq g_k}\tanh(2\rho(g_i(0), \gamma^{-1}\circ g(0)))\\
&= \prod_{g\in G, g\neq g_i}|\frac{g_i(0)-g(0)}{1-\overline{g_i(0)}g(0)}|.
\end{aligned}$$

Let $g_i = id$ and in this case

$$\prod_{g\in G, g\neq g_i}\left|\frac{g_i(0)-g(0)}{1-\overline{g_i(0)}g(0)}\right| = \prod_{g\neq id}|g(0)| = \exp\left(\sum_{g\neq id}\ln|g(0)|\right) \geq \exp\left(C\sum_{g\neq id}(1-|g(0)|)\right),$$

where C is some universal constant.

Thus by the definition of the convergence-type property it follows that the sequence $\{g(0)\}_{g\in G}$ is an interpolating sequence.

We now prove Lemma 2.2. Suppose the support set of $\mu\chi_{\mathcal{F}}$, denoted by $Supp(\mu_{\mathcal{F}})$ which is contained in the ball $B(0,t)$. For any $\xi \in \partial\Delta$ and $0 < r \leq 2$, we have

$$\begin{aligned}\iint_{\Delta\cap B(\xi,r)} \frac{|\mu(z)|^2}{1-|z|^2}\,dx\,dy &= \sum_{g\in G}\iint_{g(B(0,t))\cap B(\xi,r)} \frac{|\mu(z)|^2}{1-|z|^2}\,dx\,dy \\ &= \sum_{g\in G}\iint_{g(B(0,t))} \frac{|\mu(z)|^2}{1-|z|^2}\chi_{B(\xi,r)}\,dx\,dy \\ &\leq \sum_{g\in G} \parallel \mu \parallel_\infty^2 \iint_{g(B(0,t))} \frac{1}{1-|z|^2}\chi_{B(\xi,r)}\,dx\,dy.\end{aligned}$$

It is easy to see that the hyperbolic radius t_ρ of the Euclidean disk $B(0,t)$ is $\ln\frac{1+t}{1-t}$. Hence for any $g \in G$, the disk $g(B(0,t))$ is a hyperbolic disk with center $g(0)$ and hyperbolic radius t_ρ. By some simple calculation or by [3] we know that the disk $g(B(0,t))$ is contained in the Euclidean disk $B(g(0),R_g)$, where the radius R_g is equal to

$$\frac{(1+|g(0)|)(1-e^{t_\rho})(1-|g(0)|)}{(1+|g(0)|)+e^{t_\rho}(1-|g(0)|)} \leq C(1-|g(0)|),$$

where C is some constant depending only on t.

Combined with the above discussion, we get

$$\begin{aligned}\iint_{\Delta\cap B(\xi,r)} \frac{|\mu(z)|^2}{1-|z|^2}\,dx\,dy &\leq \sum_{g(0)\in B(\xi,r)} \frac{||\mu||_\infty \pi R_g^2}{1-|1-R_g|^2} \\ &\leq C' \sum_{g(0)\in B(\xi,r)} (1-|g(0)|) \leq C^* r,\end{aligned}$$

where the constant C^* depends only on C' and the Carleson norm of the measure $\sum_{g\in G}(1-|g(0)|)\delta_{g(0)}$. Hence the lemma holds. □

Remark. In [5], Bishop used the norm property of Schwarzian derivative of holomorphic function under hyperbolic metric to give another proof of Lemma 2.2 for the case of the Beltrami coefficient μ supported on a compact subset of the surface Δ/G.

A Jordan curve γ is said to be a chord-arc curve if there exists a constant C such that for any two points ξ_1, $\xi_2 \in \gamma$, the length of the arc γ_{ξ_1,ξ_2} satisfies

$$\text{length}(\gamma_{\xi_1,\xi_2}) \leq Cd(\xi_1,\xi_2),$$

where γ_{ξ_1,ξ_2} is the shorter arc of γ with endpoints ξ_1, ξ_2 and $d(\xi_1,\xi_2)$ means the Euclidean distance between ξ_1 and ξ_2.

A result from [15] says that

Lemma 2.3. [15] *Let Ω be a chord-arc domain. Then the following are equivalent:*

(a) *$d\nu$ is a Carleson measure for Ω.*
(b) *For $0 < p < \infty$, and $f \in H^p(\Omega)$,*

$$\iint_\Omega |f|^p\,d\nu \leq C\int_{\partial\Omega} |f|^p\,ds,$$

where $H^p(\Omega) = \{f\colon f \text{is analytic on } \Omega \text{ and } \int_{\partial\Omega}|f|^p\,ds < \infty\}$ and the constant C depends only on the the Carleson norm of $d\nu$.

Remark. Lemma 2.3 was first given by Carleson [[10],Theorem 3.9, P.61] when Ω is the upper half plane. Zinsmeister proved that Carleson's theorem remains true for chord-arc domains, see [15].

After this preparatory work, it is time to give the proof of Theorem 1.1.

3. Proof of Theorem 1.1

Proof. Let G be a second-kind convex cocompact Fuchsian group and $\mathcal{F}$ be the Dirichlet domain of G with center 0. Let μ be an element in $M(G)$. The intersection of the closure of $\mathcal{F}$ with $\partial\Delta$ contains finitely many intervals which are called free edges of $\mathcal{F}$, denoted by $I_1, I_2, \cdots I_n$.

For any $1 \le i \le n$, let $q_{i,1}, q_{i,2}$ be the endpoints of I_i. It is well known that both $q_{i,1}, q_{i,2}$ do not belong to the limit set. Both sides of $q_{i,j}$ ($j = 1$, or 2) are free sides of Dirichlet fundamental domains with different centers.

By the statement of the theorem we know there exists a constant C such that for any $1 \le i \le n$, we can choose a ball B_i such that $B_i \cap \partial\Delta$ contains no limit points of G and $I_i \subset B_i \cap \partial\Delta$ and for any point $\xi \in I_i$ and $0 < r < 2$,

$$\iint_{B(\xi,r)\cap\Delta} \frac{|\mu(z)|^2}{1-|z|^2}\chi_{B_i\cap\Delta}\, dx\, dy \le Cr,$$

furthermore, the set $\overline{\mathcal{F}} - \bigcup_{i=1}^n (B_i \cap \mathcal{F})$ is compact, denoted by $\mathcal{F}_c$.

By Lemma 2.1, we know that the measure

$$\frac{|\mu(z)|^2}{1-|z|^2}\, dx\, dy$$

is a Carleson measure on the domain $B_i \cap \mathcal{F}$. We divide μ into two parts. Let

$$\mu = \sum_{g\in G} \mu\chi_{g(\mathcal{F}_c)} + \sum_{g\in G} \mu\chi_{g(B)},$$

where $B = \bigcup_{i=1}^n (B_i \cap \mathcal{F})$.

By Lemma 2.2, we know that the measure $\sum_{g\in G} \mu\chi_{g(\mathcal{F}_c)}$ is a Carleson measure on Δ. In the following we only need to show that $\sum_{g\in G} \mu\chi_B$ is also a Carleson measure. Without loss of generality, we may assume $\mu = \sum_{g\in G} \mu\chi_B$.

Let ξ be an arbitrary point of $\partial\Delta$ and r a positive real number less than 2. In the following we will find a positive constant C^* which does not depend on ξ and r such that

$$\iint_{B(\xi,r)\cap\Delta} \frac{|\mu|^2}{1-|z|^2}\, dx\, dy \le C^* r. \tag{3.1}$$

We first consider the following special case: there exists $g \in G$ such that $g(B(\xi, r)\cap \Delta) \subset \mathcal{F}$. By Lemma 2.1 we know that $\frac{|\mu|^2}{1-|z|^2}\, dx\, dy$ is a Carleson measure on the domain $g(B(\xi, r) \cap \Delta)$. Then we have

$$\begin{aligned}\iint_{B(\xi,r)\cap\Delta} \frac{|\mu(w)|^2}{1-|w|^2}\, du\, dv &\le \iint_{g(B(\xi,r)\cap\Delta)} \frac{|\mu(g^{-1}(z))|^2}{1-|g^{-1}(z)|^2}|(g^{-1})'(z)|^2\, dx\, dy \\ &= \iint_{g(B(\xi,r)\cap\Delta)} \frac{|\mu(g^{-1}(z))\frac{\overline{(g^{-1})'(z)}}{(g^{-1})'(z)}|^2}{1-|g^{-1}(z)|^2}|(g^{-1})'(z)|^2\, dx\, dy \\ &= \iint_{g(B(\xi,r)\cap\Delta)} \frac{|\mu(z)|^2}{1-|z|^2}|(g^{-1})'(z)|\, dx\, dy.\end{aligned}$$

Since g is a Möbius transformation, the domain $g(B(\xi,r)\cap\Delta)$ is a chord-arc domain. By Lemma 2.3, we have

$$\iint_{g(B(\xi,r)\cap\Delta)} \frac{|\mu(z)|^2}{1-|z|^2}|(g^{-1})'(z)|\,dx\,dy \le C_1 \int_{\partial g(B(\xi,r)\cap\Delta)} |(g^{-1})'(z)|\,ds$$
$$= \int_{\partial(B(\xi,r)\cap\Delta)} ds \le 2\pi C_1 r,$$

where the constant C_1 depends only on the constant C in the statement of the Theorem 1.1. Hence we have

$$\iint_{B(\xi,r)\cap\Delta} \frac{|\mu(w)|^2}{1-|w|^2}\,du\,dv \le 2\pi C_1 r. \tag{3.2}$$

By the above discussion, we easily get that the measure $\frac{|\mu(z)|^2}{1-|z|^2}dxdy$ is a Carleson measure on $B_i\cap\Delta$ for any $1\le i\le n$, since $B_i\cap\partial\Delta$ contains no limit points of G and there are finitely many $g_1,\cdots,g_m$ belonging to G such that

$$(B_i\cap\Delta)\subset\bigcup_1^m g_j(\mathcal{F}).$$

Now we consider the general case. Let G^* be the set of all the elements g in G such that $g(B)\cap B(\xi,r)\neq\emptyset$. If $g\in G^*$ there are at most three possibilities as follows:

(a) there exist $1\le i\le n$, $g(B_i\cap\mathcal{F})\subset B(\xi,r)$;
(b) there exists $1\le i\le n$, $g(B_i)\cap B(\xi,r)\neq\emptyset$ and $g(I_i)\subset B(\xi,r)\cap\partial\Delta$;
(c) there exist $1\le i\le n$, $g(B_i)\cap B(\xi,r)\neq\emptyset$ and $g(I_i)\cap B(\xi,r)\cap\partial\Delta\neq\emptyset$.

In case (a), we have

$$\iint_{g(B_i\cap\mathcal{F})} \frac{|\mu(w)|^2}{1-|w|^2}\,du\,dv \le \iint_{g(B_i\cap\Delta)} \frac{|\mu(w)|^2}{1-|w|^2}\,du\,dv$$
$$= \iint_{B_i\cap\Delta} \frac{|\mu(g(z))\frac{\overline{g'(z)}}{g'(z)}|^2}{1-|g(z)|^2}|g'(z)|^2\,dx\,dy$$
$$= \iint_{B_i\cap\Delta} \frac{|\mu(z)|^2}{1-|z|^2}|g'(z)|\,dx\,dy$$
$$\le C_1\int_{\partial(B_i\cap\Delta)} |g'(z)|ds = C_1\int_{\partial g(B_i\cap\Delta)} ds$$
$$\le C_1\pi\,\text{length}(g(B_i\cap\partial\Delta)),$$

where the second above inequality holds is by Lemma 2.3 and C_1 depends only on the Carleson norm of $\frac{|\mu(z)|^2}{1-|z|^2}|\,dx\,dy$ on $B_i\cap\Delta$.

For case (b) we have

$$\iint_{g(B_i\cap\mathcal{F})\cap B(\xi,r)} \frac{|\mu(w)|^2}{1-|w|^2}|\,du\,dv \le \iint_{g(B_i\cap\Delta)\cap B(\xi,r)} \frac{|\mu(w)|^2}{1-|w|^2}|\,du\,dv$$
$$\le \pi C_1\,\text{length}(B_i\cap\partial\Delta).$$

For case (c), notice that $g(B_i\cap\Delta)\cap B(\xi,r)$ is a triangle with three circle-arc and the angle corresponding to the side $g(B_i\cap\partial\Delta)\cap B(\xi,r)$ is bigger than some constant, we have

$$\mathrm{length}(\partial(g(B_i \cap \Delta) \cap B(\xi, r))) \leq C_2\, \mathrm{length}(g(B_i \cap \partial\Delta) \cap B(\xi, r)).$$

where the constant C_2 depends only on the Carleson norm of $\frac{|\mu(z)|^2}{1-|z|^2}\, dx\, dy$ on $B_i \cap \Delta$ and the angle between ∂B_i and $\partial\Delta$.

By a similar discussion as case(a) we have

$$\iint_{g(B_i \cap \mathcal{F})} \frac{|\mu(w)|^2}{1-|w|^2}\, du\, dv \leq \pi C_2\, \mathrm{length}(g(B_i \cap \partial\Delta) \cap B(\xi, r)).$$

Since for every $1 \leq i \leq n$, the arc $B_i \cap \partial\Delta$ does not contain the limit points of G. Hence for $g_1,\, g_2 \in G^*$ if $g_1(B_i) \cap B(\xi, r) \neq \emptyset$ and $g_2(B_i) \cap B(\xi, r) \neq \emptyset$, the images of $B_i \cap \partial\Delta$ under maps $g_1,\, g_2$, respectively, do not overlap. Hence we have

$$\iint_{B(\xi,r)\cap\Delta} \frac{|\mu(w)|^2}{1-|w|^2}\, du\, dv \leq \pi C^* \sum_{g \in G^*} \mathrm{length}(g(B) \cap B(\xi, r) \cap \partial\Delta)$$
$$\leq \pi C^*\, \mathrm{length}(B(\xi, r) \cap \partial\Delta) \leq 2(\pi)^2 C^* r,$$

where C^* equals to the maximum value of the constants which appeared in the proof of this theorem and $B = \bigcup_i^n (B_i \cap \Delta)$. This completes the proof. □

4. Proof of Theorem 1.2

Proof. Let G be a finitely generated Fuchsian group of second kind with some parabolic elements. Since the generator of G contains finite elements, without loss of generality, we may suppose that the generator of G contains only one parabolic element γ and suppose $\xi \in \mathcal{F}(\infty)$ be the fixed point of the parabolic element γ.

We divide μ into two parts. Let

$$\mu = \sum_{g \in G} \mu \chi_{g(\mathcal{F}^*)} + \sum_{g \in G} \mu \chi_{(g(B \cap \mathcal{F}))},$$

where $\mathcal{F}^* = \mathcal{F} - (B \cap \mathcal{F})$ and B is a disk with center ξ and radius r_0 (that is sufficiently small such that ∂B intersect with the sides of $\mathcal{F}$ which have ξ as a common vertex). Let γ_0 be the arc of ∂B between the sides of $\mathcal{F}$ which have ξ as a common vertex.

By Theorem 1.1, we know that the measure $\sum_{g \in G} \mu \chi_{g(\mathcal{F}^*)}$ is a Carleson measure on Δ. In the following we only need to show that $\sum_{g \in G} \mu \chi_{g(B \cap \mathcal{F})}$ is also a Carleson measure. Without loss of generality, we may assume $\mu = \sum_{g \in G} \mu \chi_{g(B \cap \mathcal{F})}$.

We first show that the hyperbolic area of $B \cap \mathcal{F}$ is finite.

By a conformal mapping $\varphi(z)$ which maps the unit disk Δ onto the upper half plane $\mathbf{H}$, we only need to show that the hyperbolic area of the image $\varphi(B \cap \mathcal{F})$ is finite, since hyperbolic area is unchanged under conformal mapping. Without loss of generality, we may suppose $\varphi(\xi) = 0$. The images of the sides of $\mathcal{F}$ with ξ as a vertex under the mapping φ are contained in two circles, denoted by C_1 and C_2, respectively. Let C_1 be the circle

$$(x - r_1)^2 + y^2 = r_1^2$$

and C_2 the circle

$$(x + r_2)^2 + y^2 = r_2^2.$$

The tangent point of C_1 and C_2 is 0 , see Figure 1. The images of the arc γ_0 is contained in a circle with center 0 and radius r_3.

Then we have

$$\iint_{B\cap\mathcal{F}} \frac{1}{(1-|z|)^2}\,dx\,dy = \iint_{\varphi(B\cap\mathcal{F})} \frac{1}{4v^2}\,du\,dv = \int_0^{r_3} dr \int_{\arccos(\frac{r}{2r_1})}^{\arccos(\frac{-r}{2r_2})} \frac{1}{4r\sin^2\theta}\,d\theta.$$

It is easy to see that the limit

$$\lim_{r\to 0} \int_{\arccos(\frac{r}{2r_1})}^{\arccos(\frac{-r}{2r_2})} \frac{1}{4r\sin^2\theta}\,d\theta = \frac{1}{8}\left(\frac{1}{r_2}+\frac{1}{r_1}\right).$$

Hence by some easy calculation, the hyperbolic area of $B\cap\mathcal{F}$ is finite.

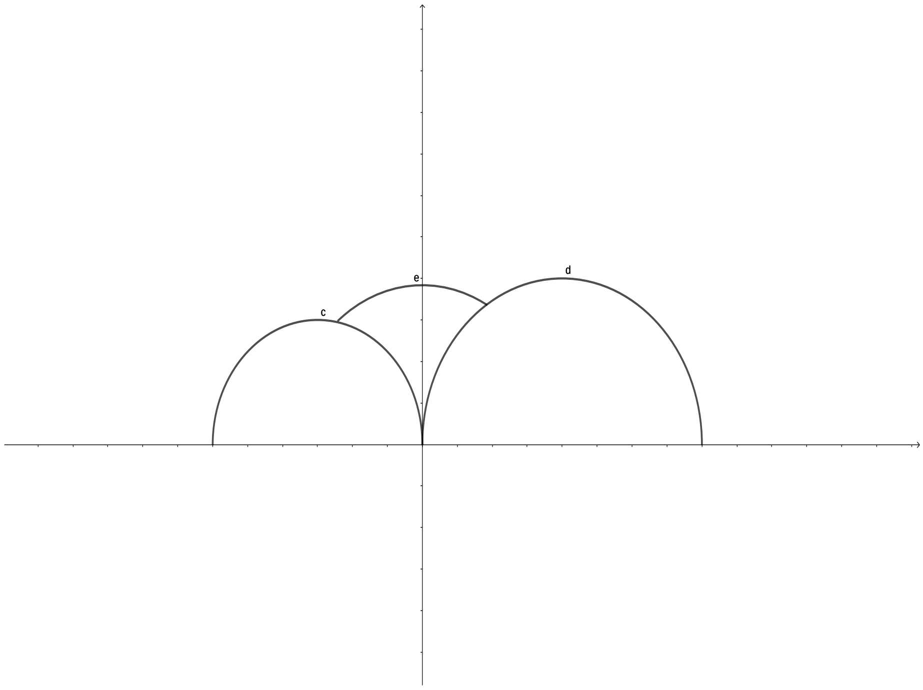

Figure 1.

We continue to prove Theorem 1.2. For any $g\in G$, we have

$$\begin{aligned}\iint_{g(B\cap\mathcal{F})} \frac{|\mu(w)|^2}{1-|w|^2}\,du\,dv &\le \iint_{g(B\cap\mathcal{F})} \frac{1}{1-|w|^2}\,du\,dv\\ &= \iint_{B\cap\mathcal{F}} \frac{|g'(z)|^2}{1-|g(z)|^2}\,dx\,dy\\ &= \iint_{B\cap\mathcal{F}} \frac{1}{1-|z|^2}|g'(z)|\,dx\,dy\\ &= \iint_{B\cap\mathcal{F}} \frac{1-|g(z)|^2}{(1-|z|^2)^2}\,dx\,dy\\ &\le C_1(1-|g(z_0)|)\iint_{B\cap\mathcal{F}} \frac{1}{(1-|z|^2)^2}\,dx\,dy\\ &\le C(1-|g(z_0)|),\end{aligned}$$

where z_0 is any point in γ_0, C_1 depends on z_0 and the hyperbolic length of γ_0, and C depends on C_1 and the hyperbolic area of $B\cap\mathcal{F}$. The fourth above equality holds since

$$\frac{|g'(z)|}{1-|g(z)|^2} = \frac{1}{1-|z|^2} \quad \text{for any } g\in G$$

and the last second above inequality holds since the hyperbolic length of γ_0 is finite.

Let η be any point on the unit circle and $B(\eta, r)$ the disk with center η and radius r, here $0 < r < 1$. By the proof of Lemma 2.2, we know that the sequence $\{g(z_0)\}_{g\in G}$ is an interpolating sequence. Hence the images of $B \cap \mathcal{F}$ under $g \in G$ which are contained in the disk $B(\eta, r)$ satisfy

$$\iint_{B\cap\Delta} \frac{|\mu(w)|^2}{1-|w|^2}\,du\,dv \leq \sum_{g\in G, g(B\cap\mathcal{F})\cap B(\eta,r)\neq\emptyset} \iint_{g(B\cap\mathcal{F})} \frac{|\mu(w)|^2}{1-|w|^2}\,du\,dv \leq Cr,$$

which completes the proof of the theorem. □

In [4], Bishop showed that all divergence type Fuchsian groups have Bowen's property, hence Theorem 1.1 fails for the case of divergence-type groups. By Lemma 2.2 we know that for all convergence-type Fuchsian groups with compact support Beltrami coefficient, the result also holds. We ask the following question: does the result holds for all convergence-type Fuchsian groups?

5. Some applications

In this section we give an application of Theorem 1.1 and 1.2. In [1, p. 617, Theorem 7], Astala and Zinsmeister showed that there exist Fuchsian groups G of the second kind such that the Hausdorff dimension of the quasicircle $f(\partial\Delta)$ is not a real analytic function on Teichmüller space $T(G)$. In [11], the author and Wu Shengjian showed that the result holds for any second kind Fuchsian groups. By Theorem 1.1 and Theorem 1.2, we can give a very short proof of the result for the case of finitely generated Fuchsian groups of second kind. We have

Corollary 5.1. *Let G be any finitely generated Fuchsian groups of the second kind with or without parabolic elements, the Hausdorff dimension of the quasicircle $f(\partial\Delta)$ is not a real analytic function on Teichmüller space $T(G)$.*

Proof. We will construct a $\mu \in CM^*(\Delta) \cap M(G)$ such that the Hausdorff dimension of the quasicircle $f_\mu(\partial\Delta)$, denoted by $\dim(f_\mu(\partial\Delta))$, is bigger than 1. Let $\mu^* \in CM(\Delta)$ such that $\dim(f_{\mu^*}(\partial\Delta)) > 1$, without loss of generality we may suppose $\dim(\partial f_{\mu^*}(\overline{\mathcal{F}} \cap \partial\Delta)) > 1$(the existence of such a μ^* can be found in ([1, p. 624]).

Let $\mu_{\mathcal{F}} = \mu^*(z)$ for $z \in \mathcal{F}$. Now we translate $\mu_{\mathcal{F}}$ by the group G to the whole disk Δ. We get

$$\mu(z) = \begin{cases} \mu^*(z), & z \in \mathcal{F}, \\ \mu^*(g(z)), & g(z) \in \mathcal{F}, g \in G. \end{cases}$$

By Theorem 1.1 or Theorem 1.2, we can see that $\mu \in CM^*(\Delta) \cap M(G)$. Now we consider the mapping $\psi\colon t \to dim(f_{t\mu})$. By [13] we know that there exist a $\varepsilon > 0$ such that, for all $|t| < \varepsilon$, the quasicircle $f_{t\mu}(\partial\Delta)$ is a rectifiable curve. This implies that ψ can not be real analytic on Δ. □

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References

[1] Astala, K., and M. Zinsmeister: Teichmüller spaces and BMOA. - Math. Ann. 289, 1991, 613–625.

[2] Astala, K., and M. Zinsmeister: Rectifiability in Teichmüller theory. - In: Topics in Complex Analysis, Banach Center Publications 31, 1995, 45–52.

[3] Beardon, A. F.: The geometry of discrete group. - Springer-Verlag, 1983.

[4] Bishop, C. J.: Divergence groups have the Bowen property. - Ann. of Math. 154, 2001, 205–217.

[5] Bishop, C. J.: Compact deformations of Fuchsian group. - J. Anal. Math. 87, 2002, 5–36.

[6] Bowen, R.: Hausdorff dimension of quasicircles. - Publ. Math. Inst. Hautes. Études. Sci. 50, 1979, 11–25.

[7] Carleson, L.: An interpolation problem for bounded analytic functions. - Amer. J. Math. 4, 1958, 921–930.

[8] Cui, G.: Integrably asymptotic affine homeomorphisms of the circle and Teichmüller spaces. - Sci. China Ser. A 43, 2000, 267–279.

[9] Dal'Bo, F.: Geodesic and horocyclic trajectories. - Springer, 2011.

[10] Garnett, J. B.: Bounded analytic functions. - Academic Press, New York, 1981.

[11] Huo, S., and S. Wu: The failure of analyticity of Hausdorff dimensions of quasi-circles of Fuchsian groups of the second kind. - Proc. Amer. Math. Soc. 143, 2015, 1101–1108.

[12] Huo, S., and M. Zinsmerster: On Ruelle's property. - Preprint, arXiv:1906.01291v1.

[13] Pommerenke, Ch.: Schlichte Functionen und BMOA. - Comment. Helv. 52, 1977, 591–602.

[14] Shen, Y., and H. Wei: Universal Teichmüller space and BMO. - Adv. Math. 234, 2013, 129–148.

[15] Zinsmeister, M.: Les domaines de Carleson. - Michigan Math. J. 36:2, 1989, 213–220.

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