

DIRICHLET POLYHEDRA FOR SIMPLEX GROUPS OF SPHERICAL, EUCLIDEAN OR HYPERBOLIC SPACES

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Abstract. Komori and Umemoto detected combinatorial shapes of Dirichlet polyhedra for simplex groups acting on hyperbolic space. We generalize their result to simplex groups acting on Euclidean or spherical spaces.

1. Introduction

Several methods are known to decompose manifolds into simpler pieces. For manifolds obtained as the quotient space of a finitely compact metric space by the action of a discrete group of isometries, Dirichlet polyhedra are one such decomposition. One of the reasons why Dirichlet polyhedra are so important for studying such manifolds is that they provide combinatorial representations of discrete groups.

In general, it is not obvious to determine Dirichlet polyhedra explicitly. For constructing a Dirichlet polyhedron, we need to choose not only a discrete group but also a point in the metric space, which will be called the *center* of the Dirichlet polyhedron. Take the set of points which are the images of the center under the action of the group. The *Dirichlet polyhedron* is then defined by the Voronoi tessellation with respect to this set: it is the Voronoi cell containing the center. Since the Voronoi tessellation is determined by infinitely many points, it is not obvious in general to determine the explicit geometric shape of Dirichlet polyhedron, even in the case of finitely sided polyhedron.

For a given discrete group, we focus on possible combinatorial shapes of Dirichlet polyhedra. Though each Dirichlet polyhedron is determined by the group together with the center, possible combinatorial shapes depend only on the group. As we have mentioned before, it is difficult to determine Dirichlet polyhedra for a general discrete group. There are, however, specialized cases where the combinatorial shapes of Dirichlet polyhedra are known. A discrete group of orientation-preserving isometries acting on a two-dimensional hyperbolic space is called a *Fuchsian* group. Näätänen determined in [Nää85] combinatorial shapes of Dirichlet polyhedra (or more precisely Dirichlet *polygon* in this case) for a Fuchsian group which yields a particularly symmetric genus two closed orientable surface as its quotient space. Other examples were provided by Umemoto in [Ume11] for Fuchsian groups that are index two subgroups of triangle groups. Komori and Umemoto announced in [KU11] that Umemoto's results are able to generalize to higher-dimensional hyperbolic space.

Based on Komori and Umemoto's previous studies, what we will prove here is that their results also hold for Euclidean or spherical reflection groups.

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2. Preliminaries

For a positive integer n , we use the symbol \mathbf{X}^n to denote either the n -dimensional spherical, Euclidean or hyperbolic spaces. Let ρ and μ be the distance function and the volume function on \mathbf{X}^n respectively. Let I_n be the set $\{1, 2, \dots, n\}$ consisting of n integers. For $k \in I_n$, a k -dimensional *plane* in \mathbf{X}^n is a subset of \mathbf{X}^n that is isometric to \mathbf{X}^k . In particular, $(n - 1)$ -dimensional planes are called *hyperplanes*. For a non-empty subset $S \subset \mathbf{X}^n$, we denote by $\langle S \rangle$ the plane in \mathbf{X}^n which contains S and which is minimal with respect to dimension. We also denote by ∂S and $\text{Int}(S)$ the *boundary* and the *interior* of S in $\langle S \rangle$ with respect to its subspace topology, respectively.

For $k \in I_n$, points $v_1, v_2, \dots, v_{k+1} \in \mathbf{X}^n$ are said to be in a *general position* if they are affinely independent, that is, no $(k - 1)$ -dimensional plane in \mathbf{X}^n contains them. A single point is always regarded in a general position. For points $v_1, v_2, \dots, v_{k+1} \in \mathbf{X}^n$ in a general position, we denote by $v_1 v_2 \cdots v_{k+1}$ the k -dimensional *simplex*, or simply k -*simplex*, in \mathbf{X}^n . In other words, $v_1 v_2 \cdots v_{k+1}$ is the convex hull for v_1, v_2, \dots, v_{k+1} in \mathbf{X}^n . The points v_1, v_2, \dots, v_{k+1} , which are also regarded as 0-dimensional simplexes, are called the *vertices* of $v_1 v_2 \cdots v_{k+1}$. The *join* of two simplexes is denoted by the binary operator $*$.

Let $\text{Isom}^+(\mathbf{X}^n)$ be the group of all orientation-preserving isometries on \mathbf{X}^n . We denote by $\iota \in \text{Isom}^+(\mathbf{X}^n)$ the identity mapping. For any $f \in \text{Isom}^+(\mathbf{X}^n)$, let $\text{Fix}(f)$ be the set of all fixed points by the action of f on \mathbf{X}^n :

$$\text{Fix}(f) := \{p \in \mathbf{X}^n \mid f(p) = p\}.$$

For $f \in \text{Isom}^+(\mathbf{X}^n)$ and for $p \in \mathbf{X}^n$, let $H_p(f)$ be a closed half-space defined as follows:

$$H_p(f) := \{q \in \mathbf{X}^n \mid \rho(q, p) \leq \rho(q, f(p))\}.$$

Let G be a discrete subgroup of $\text{Isom}^+(\mathbf{X}^n)$. As in [Rat06, §6.7], we define the *Dirichlet polyhedron* $D_p(G)$ of G with center $p \in \mathbf{X}^n - \bigcup_{f \in G - \{\iota\}} \text{Fix}(f)$ as

$$D_p(G) := \bigcap_{f \in G - \{\iota\}} H_p(f).$$

It is a locally finite convex fundamental polyhedron for G .

For an n -simplex $\Delta := v_1 v_2 \dots v_{n+1}$ in \mathbf{X}^n , let

$$\Delta_i := \Delta \cap \langle \{v_1, v_2, \dots, v_{n+1}\} - \{v_i\} \rangle$$

be the $(n - 1)$ -dimensional simplex which is opposite in Δ to the vertex v_i , and let R_i be the reflection of \mathbf{X}^n along the hyperplane $\langle \Delta_i \rangle$. From now on, Δ is assumed to yield a *simplex group* Γ_Δ defined in [KU11] and in [Rat06, §7.2]. That is, Γ_Δ is a discrete subgroup of $\text{Isom}^+(\mathbf{X}^n)$ generated by all transformations of the form $R_i \circ R_j$ for any $i, j \in I_{n+1}$.

We study combinatorial shapes of Dirichlet polyhedra for Γ_Δ with respect to centers. For our purpose, points in $\text{Int}(\Delta) \sqcup \bigsqcup_{i \in I_{n+1}} \text{Int}(\Delta_i)$ are enough to be taken as centers, since the images of the closure of $\text{Int}(\Delta) \sqcup \bigsqcup_{i \in I_{n+1}} \text{Int}(\Delta_i)$ by the action of

Γ_Δ gives a tessellation of \mathbf{X}^n . Furthermore, the set of points in Δ that are not fixed by any non-trivial elements of Γ_Δ is disjointly decomposed as follows:

$$\Delta - \bigcup_{f \in \Gamma_\Delta - \{1\}} \text{Fix}(f) = \text{Int}(\Delta) \sqcup \bigsqcup_{i \in I_{n+1}} \text{Int}(\Delta_i),$$

since each generator $R_i \circ R_j \in \Gamma_\Delta$ is a rotation along the plane $\langle \Delta_i \cap \Delta_j \rangle$ of codimension two, and since $\partial \Delta_i \subset \bigcup_{j \in I_{n+1} - \{i\}} \langle \Delta_i \cap \Delta_j \rangle$. Based on these observations, the following theorem is enough to detect combinatorial shapes of Dirichlet polyhedra for Γ_Δ :

Theorem. *Let Γ_Δ be a discrete simplex group defined by an n -simplex Δ . The Dirichlet polyhedron $D_p(\Gamma_\Delta)$ of Γ_Δ centered at a point p is given as follows:*

$$(1) \quad D_p(\Gamma_\Delta) = \begin{cases} \bigcap_{i, j \in I_{n+1}, i \neq j} H_p(R_i \circ R_j), & \text{for } p \in \text{Int}(\Delta), \\ \bigcap_{j \in I_{n+1} - \{i\}} (H_p(R_i \circ R_j) \cap H_p(R_j \circ R_i)), & \text{for } p \in \text{Int}(\Delta_i), \end{cases}$$

where Δ_i is the $(n-1)$ -dimensional simplex which is opposite in Δ to the vertex v_i , R_i is the reflection of \mathbf{X}^n along the hyperplane $\langle \Delta_i \rangle$ containing Δ_i , and $H_p(R_i \circ R_j)$ is the closed half-space defined as $H_p(R_i \circ R_j) := \{q \in \mathbf{X}^n \mid \rho(q, p) \leq \rho(q, R_i \circ R_j(p))\}$.

Remark. When \mathbf{X}^2 is the hyperbolic plane, the same result has already been obtained by [Ume11]. In [KU11], Komori and Umemoto announced that the same result holds when \mathbf{X}^n is the n -dimensional hyperbolic space. Furthermore, they also mentioned that the same result holds when vertices of Δ are ideal.

3. Proof of Theorem

We start by introducing the notion for dihedral angles of Δ , which will be used to state the following lemma of a higher-dimensional generalization of Ceva's theorem. For a subset S in \mathbf{X}^n bounded by two transversely intersecting hyperplanes, we denote by $\theta(S)$ its dihedral angle. In particular, for $e \in v_i v_j$, we also denote by $\theta(v_j e)$ the dihedral angle in Δ which is opposite to the 1-simplex ev_j . That is, the dihedral angle between the hyperplanes $\langle \{v_1, v_2, \dots, v_{n+1}, e\} - \{v_i, v_j\} \rangle$ and $\langle \{v_1, v_2, \dots, v_{n+1}, e\} - \{v_i, e\} \rangle$. The composition $R_i \circ R_j$ for any mutually distinct $i, j \in I_{n+1}$ is a rotation from $\langle \Delta_j \rangle$ toward $\langle \Delta_i \rangle$ along $\langle \Delta_i \cap \Delta_j \rangle$, which is opposite to $v_i v_j$ in Δ . So we have

$$\theta(H_j \cap (R_i \circ R_j)(R_j(H_j))) = \theta(H_j \cap R_i(H_j)) = 2\theta(v_i v_j),$$

where $H_j := H_{v_j}(R_j)$, meaning that H_j is the closed halfspace supported by $\langle \Delta_j \rangle$ and containing Δ .

Lemma. [Ush19, Theorem 3] *For an n -simplex Δ and points $e_{i,j} = e_{j,i} \in v_i v_j$ for any $i, j \in I_{n+1}$, the intersection of $(n-1)$ -simplexes $v_{i_1} \dots v_{i_{n-1}} e_{k,m}$, where indexes are taken all possible choices satisfying $\{i_1, \dots, i_{n-1}, k, m\} = I_{n+1}$, consists of a point in Δ if and only if the equation*

$$\frac{\sin \theta(v_i e_{i,j}) \sin \theta(v_j e_{j,k}) \sin \theta(v_k e_{k,i})}{\sin \theta(v_i e_{i,k}) \sin \theta(v_k e_{k,j}) \sin \theta(v_j e_{j,i})} = 1$$

holds for any mutually distinct $i, j, k \in I_{n+1}$.

The key fact used in the proof of [Ume11, Theorem 24] is this lemma in the case that \mathbf{X}^n is the hyperbolic plane, which is also known as Ceva's theorem for the hyperbolic plane. If we want to prove a generalization of Umemoto's result for discrete simplex groups in $\text{Isom}^+(\mathbf{X}^n)$ following the strategy of her proof, we need to prepare a corresponding generalization of Ceva's theorem, that is this lemma.

Proof of Theorem. To obtain (1), it is enough to show that the sets on each side of the equation have equal volumes. Note that by definition, the Dirichlet polyhedron $D_p(\Gamma_\Delta)$ is a subset of each set appearing on the right-hand side of (1). Since all sets in question are constructed as the intersection of closed halfspaces, if $D_p(\Gamma_\Delta)$ is a proper subset, then its volume must be strictly less than the volume of the sets appearing on the right-hand side of (1). If the volumes are equal, however, then the sets on each side of (1) must coincide.

The volume $\mu(D_p(\Gamma_\Delta))$ is computed as follows. The fundamental regions $D_p(\Gamma_\Delta)$ and $\Delta \cup R_i(\Delta)$ are *proper* in the sense that the volumes of their boundary are zero. Thus they have the same volume by [Rat06, Theorem 6.7.2], which implies

$$\mu(D_p(\Gamma_\Delta)) = \mu(\Delta \cup R_i(\Delta)) = \mu(\Delta) + \mu(R_i(\Delta)) = 2\mu(\Delta).$$

The aim of our proof thus comes to show that the volumes of the sets on the right-hand side of (1) are $2\mu(\Delta)$.

For $p \in \text{Int}(\Delta_i)$ for any $i \in I_{n+1}$, we actually have

$$\bigcap_{j \in I_{n+1} - \{i\}} (\mathbb{H}_p(R_i \circ R_j) \cap \mathbb{H}_p(R_j \circ R_i)) = \Delta \cup R_i(\Delta),$$

since $\mathbb{H}_p(R_j \circ R_i) = \mathbb{H}_j$, $\mathbb{H}_p(R_i \circ R_j) = R_i(\mathbb{H}_j)$, and since $R_i(\mathbb{H}_j)$ is a halfspace supported by $\langle R_i(\Delta_j) \rangle$ and containing $R_i(\Delta)$.

We will show that $\mu(\bigcap_{i,j \in I_{n+1}, i \neq j} \mathbb{H}_p(R_i \circ R_j)) = 2\mu(\Delta)$ for any $p \in \text{Int}(\Delta)$ from now on. We first study the shape of $R_i(\mathbb{H}_i) \cap \bigcap_{j \in I_{n+1} - \{i\}} \mathbb{H}_p(R_i \circ R_j)$. As we have seen at the beginning of this section, $R_i \circ R_j$ is the rotation along $\langle \Delta_i \cap \Delta_j \rangle$ with rotation angle $2\theta(v_i v_j)$ from $\langle \Delta_j \rangle$ toward $\langle \Delta_i \rangle$. So we have

$$R_i(\mathbb{H}_i) \cap \mathbb{H}_p(R_i \circ R_j) \subset R_i(\mathbb{H}_i) \cap R_i(\mathbb{H}_j) = R_i(\mathbb{H}_i \cap \mathbb{H}_j)$$

for any $j \in I_{n+1} - \{i\}$. This implies that the set in question is an n -dimensional pyramid with base $R_i(\Delta_i) = \Delta_i$ and apex, say, u_i in $R_i(\Delta)$.

Showing the coincidence of the points $R_1(u_1), R_2(u_2), \dots, R_{n+1}(u_{n+1})$ is enough to complete the proof by the following reason. We have already shown that

$$R_i(\mathbb{H}_i) \cap \bigcap_{j \in I_{n+1} - \{i\}} \mathbb{H}_p(R_i \circ R_j) = u_i * \Delta_i.$$

This means

$$\bigcap_{i,j \in I_{n+1}, i \neq j} \mathbb{H}_p(R_i \circ R_j) = \Delta \cup \bigcup_{i \in I_{n+1}} (u_i * \Delta_i).$$

So we have

$$\begin{aligned} \mu\left(\bigcap_{i,j \in I_{n+1}, i \neq j} \mathbb{H}_p(R_i \circ R_j)\right) &= \mu\left(\Delta \cup \bigcup_{i \in I_{n+1}} (u_i * \Delta_i)\right) = \mu(\Delta) + \sum_{i=1}^{n+1} \mu(u_i * \Delta_i) \\ &= \mu(\Delta) + \sum_{i=1}^{n+1} \mu(R_i(u_i) * \Delta_i). \end{aligned}$$

Since each $R_i(u_i)*\Delta_i$ is an n -dimensional pyramid contained in Δ with base Δ_i for any $i \in I_{n+1}$, if the points in question coincide with each other, then $\sum_{i=1}^{n+1} \mu(R_i(u_i)*\Delta_i)$ must be $\mu(\Delta)$ and the proof is complete.

In order to show the coincidence of $R_1(u_1), R_2(u_2), \dots, R_{n+1}(u_{n+1})$, we first see that $\langle R_i(u_i)*(\Delta_i \cap \Delta_j) \rangle$ coincides with $\langle R_j(u_j)*(\Delta_i \cap \Delta_j) \rangle$ for any mutually distinct $i, j \in I_{n+1}$. Since $u_i \in \partial H_p(R_i \circ R_j)$, we have $R_i(u_i) \in \partial R_i(H_p(R_i \circ R_j))$. We also have $\Delta_i \cap \Delta_j \subset \partial R_i(H_p(R_i \circ R_j))$. So we have $\langle R_i(u_i)*(\Delta_i \cap \Delta_j) \rangle = \partial R_i(H_p(R_i \circ R_j))$. Let $e_{i,j} = e_{j,i}$ be the point defined as the intersection between $v_i v_j$ and the hyperplane $\langle \{v_1, v_2, \dots, v_{n+1}, p\} - \{v_i, v_j\} \rangle$. Then we have

$$\theta(H_i \cap R_i(H_p(R_i \circ R_j))) = \theta(R_i(H_i) \cap H_p(R_i \circ R_j)) = \theta(e_{i,j}v_i),$$

and similarly

$$\begin{aligned} \langle R_j(u_j)*(\Delta_i \cap \Delta_j) \rangle &= \partial R_j(H_p(R_i \circ R_j)), \\ \theta(H_j \cap R_j(H_p(R_j \circ R_i))) &= \theta(e_{j,i}v_j) \end{aligned}$$

hold. So we have

$$\begin{aligned} \theta(H_i \cap R_i(H_p(R_i \circ R_j))) + \theta(H_j \cap R_j(H_p(R_j \circ R_i))) &= \theta(e_{i,j}v_i) + \theta(e_{j,i}v_j) \\ &= \theta(H_i \cap H_j), \end{aligned}$$

which implies the coincidence of $\partial R_i(H_p(R_i \circ R_j))$ and $\partial R_j(H_p(R_j \circ R_i))$.

Since the vertices in question lie on hyperplanes $\langle R_i(u_i)*(\Delta_i \cap \Delta_j) \rangle$, for showing the coincidence of the vertices, it is enough to show that these hyperplanes intersect at one point. In order to do it, let $d_{i,j} = d_{j,i}$ be the point of the intersection of the 1-simplex $v_i v_j$ and the hyperplane $\langle R_i(u_i)*(\Delta_i \cap \Delta_j) \rangle = \langle R_j(u_j)*(\Delta_i \cap \Delta_j) \rangle$. Then, by the Lemma, it is enough to show the following equation:

$$\frac{\sin \theta(v_i d_{i,j}) \sin \theta(v_j d_{j,k}) \sin \theta(v_k d_{k,i})}{\sin \theta(v_i d_{i,k}) \sin \theta(v_k d_{k,j}) \sin \theta(v_j d_{j,i})} = 1.$$

Since $\theta(v_i d_{i,j}) = \theta(v_j e_{i,j})$, the desired equation holds, again, by the Lemma for the center $p \in \Delta$ of $D_p(\Gamma_\Delta)$. \square

References

- [Bea83] BEARDON, A. F.: The geometry of discrete groups. - Grad. Texts in Math. 91, Springer-Verlag, New York, 1983.
- [KU11] KOMORI, Y., and Y. UMEMOTO: On Dirichlet polyhedra for generalized simplex groups. - In: Proceedings of MSJ Autumn Meeting, Complex Analysis Section, 2011, 20.
- [Nää85] NÄÄTÄNEN, M.: On the stability of identification patterns for Dirichlet regions. - Ann. Acad. Sci. Fenn. Ser. A I Math. 10, 1985, 411–417.
- [Rat06] RATCLIFFE, J. G.: Foundations of hyperbolic manifolds. - Grad. Texts in Math. 149, Springer, New York, second edition, 2006.
- [Ume11] UMEMOTO, Y.: On Dirichlet fundamental domains for Fuchsian groups. - Master's thesis, Osaka City University, Osaka, 2011.
- [Ush19] USHIJIMA, A.: Ceva's and Menelaus' theorems for higher-dimensional simplexes. - J. Geom. 110, 2019, 13.