

QUASIHYPHERBOLIC MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we establish several characterizations for quasihyperbolic mappings in Banach spaces. As an application, we provide a partial solution to a problem left open by Väisälä regarding the local properties of quasihyperbolic mappings.

1. Introduction and main results

The quasihyperbolic metric (briefly, QH metric) was introduced by Gehring and his students Palka and Osgood in the 1970's in the setting of Euclidean spaces \mathbf{R}^n ($n \geq 2$) [5, 6]. Since its first appearance, the quasihyperbolic metric has become an important tool in geometric function theory of Euclidean spaces.

With the aid of the quasihyperbolic metric, from late 1980's onwards, Väisälä developed the theory of (dimension) free quasiconformal mappings in Banach spaces [12, 13, 14, 15, 16]. The main advantage of this approach avoids to make use of volume integrals and conformal modulus, which allows one to study the quasiconformality of mappings in Banach spaces with dimension infinity and metric spaces without volume measures. This research has recently attracted substantial interest in the research community (see e.g., [3, 7, 8, 10, 17] and reference therein).

In this paper, we will investigate a problem raised by Väisälä concerning the quasihyperbolic mappings in the free quasiconformal mapping theory ([16]). We first recall the definition of the quasihyperbolic mappings. In the following, we always assume that E and E' are real Banach spaces with dimension at least 2.

Definition 1.1. Let $D \subsetneq E$ and $D' \subsetneq E'$ be domains (open and connected), and let $f: D \rightarrow D'$ be a homeomorphism. We say that f is M -quasihyperbolic (or briefly M -QH) if for all x and y in D ,

$$\frac{1}{M}k_D(x, y) \leq k_{D'}(f(x), f(y)) \leq Mk_D(x, y),$$

where k_D and $k_{D'}$ are the quasihyperbolic metrics of D and D' , respectively.

In geometric function theory, the class of quasihyperbolic mappings is closely related to the classes of conformal and quasiconformal mappings. It turns out that the quasihyperbolic mappings are useful in the study of quasiconformal extension of \mathbf{R}^n . For instance, Beurling and Ahlfors constructed a quasisymmetric extension from \mathbf{R}^1 to the half plane, which is actually quasihyperbolic [2]. In \mathbf{R}^n , when $n \neq 4$, Tukia and Väisälä demonstrated that two domains of \mathbf{R}^n are quasihyperbolically equivalent if and only if they are quasiconformally equivalent [11].

In [12, 16], Väisälä systematically studied the properties of QH mappings and obtained the following results.

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Theorem A. [12, Theorem 4.7] Suppose that $D \subsetneq E$ and $D' \subsetneq E'$ are domains, and that $f: D \rightarrow D'$ is M -QH. Then for any subdomain $G \subset D$, the restriction of f on G , $f|_G$, is $4M^2$ -QH.

Theorem B. [16, Theorem 5.16] Suppose that $f: D \rightarrow D'$ is a homeomorphism and that each point has a neighborhood $D_1 \subset D$ such that $f|_{D_1}: D_1 \rightarrow f(D_1)$ is M -bilipschitz (or briefly f is locally M -bilipschitz). Then f is M^2 -QH.

Obviously, not every QH mapping is bilipschitz (cf. [12, Section 4] or [16, Section 8]). Thus the following problem posed by Väisälä is natural.

Open Problem 1.1. [16, 13.2.13] Suppose that $f: D \rightarrow D'$ is a homeomorphism and that each point has a neighborhood $D_1 \subset D$ such that $f|_{D_1}: D_1 \rightarrow f(D_1)$ is M -QH. Is f M' -QH with $M' = M'(M)$, where $M'(M)$ means that the constant M' depends only on M ?

The main motivation of this paper is to study Problem 1.1. As the first main result of this paper, we obtain the following equivalent characterizations of the quasi-hyperbolic mappings.

Theorem 1.1. Let $D \subsetneq E$ and $D' \subsetneq E'$ be domains, and let $f: D \rightarrow D'$ be a homeomorphism. Then the following conditions are equivalent with the constants depending only on each other.

- (1) $f: D \rightarrow D'$ is M -QH;
- (2) f and f^{-1} are both (L, ϑ) -locally bilipschitz;
- (3) There are constants $C > 0$ and $\mu \in (0, 1)$ such that for any $x, y \in D$ with $|x - y| < \mu d_D(x)$,

$$(1.2) \quad \frac{1}{C} \frac{|x - y|}{d_D(x)} \leq \frac{|f(x) - f(y)|}{d_{D'}(f(x))} \leq C \frac{|x - y|}{d_D(x)},$$

- (4) f and f^{-1} are both (θ, t_0) -relative with $\theta(t) = M_0 t$, where $0 < t_0 < 1$ and $0 < t < t_0$;
- (5) For all $x \in D$, $f: \mathbf{B}(x, d_D(x)) \rightarrow f(\mathbf{B}(x, d_D(x)))$ is M_1 -QH and $d_{D'}(f(x)) \leq M_2 d_{f(\mathbf{B}(x, d_D(x)))}(f(x))$.

Let us remark that the class of (L, ϑ) -locally bilipschitz mappings in Theorem 1.1(2) (see Definition 2.2 below) was introduced by Bonk, Heinonen and Koskela when they established the theory of uniformizing Gromov hyperbolic spaces [3]. They also proved that a quasisymmetric and (L, ϑ) -locally bilipschitz mapping between uniform metric spaces is quasihyperbolic (see [3, Proposition 4.36]). Based on Theorem 1.1, we shall generalize Proposition 4.36 in [3] to the setting of Banach spaces (see Lemma 3.5 below).

Also, we remark that the condition (1.2) is new and it is useful in the study of the local properties of the quasihyperbolic mappings. Note that the θ -relative mappings stated in Theorem 1.1(4) were first considered by Gehring, and also, the relationship between θ -relative mappings and quasiconformal mappings in \mathbf{R}^n was investigated [4].

With the aid of Theorem 1.1, we obtain the following partial solution to Problem 1.1.

Theorem 1.2. Suppose that $f: D \rightarrow D'$ is a homeomorphism. If there exist a constant $M > 1$ and a homeomorphism $\eta: [0, \infty) \rightarrow [0, \infty)$ such that for each point $x \in D$, $f: \mathbf{B}(x, d_D(x)) \rightarrow f(\mathbf{B}(x, d_D(x)))$ is M -quasihyperbolic and η -quasisymmetric, then f is M_1 -QH with $M_1 = M_1(M, \eta)$.

We remark that the authors in [7] recently considered Problem 1.1 in metric spaces. However, the target spaces in [7] are always assumed to be *proper*. Here a metric space being proper means that every closed ball in this space is compact. Obviously, in all Banach spaces with dimension infinity, this condition is invalid, and also, this condition plays a key role in the discussions in [7]. This shows that the proof methods in [7] cannot be used to prove Theorem 1.2 in the setting of Banach spaces with dimension infinity. We shall establish a different method to overcome this difficulty.

Further, by Theorems 1.1 and 1.2, we get the following result.

Theorem 1.3. *Suppose that $f: D \rightarrow D'$ is a homeomorphism such that for each point $x \in D$, $f|_{\mathbf{B}(x, d_D(x))}$ is quasisimilar with data (η, L, λ) . Then f is M -QH with $M = M(\eta, L, \lambda)$.*

Remark 1.1. The concept of quasisimilarity was introduced in [3] (see Definition 2.4 below). In [3], the authors demonstrated that there is a one-to-one correspondence between the quasi-isometry classes of Gromov hyperbolic spaces and the quasisimilarity classes of uniform metric spaces (see [3, Theorem 1.1]). Note that, in [1, 9], the authors also considered quasisimilar mappings in infinite-dimensional Banach spaces. But these two classes of mappings are not the same.

The rest of this paper is organized as follows. In Section 2, we recall necessary definitions and preliminary results, and the proofs of Theorems 1.1 ~ 1.3 are presented in Section 3.

2. Preliminaries

In this section, we recall some necessary terminology. Throughout this paper, we always assume that E and E' denote real Banach spaces with dimension at least 2, and that $D \subsetneq E$ and $D' \subsetneq E'$ are domains. The norm of a vector z in E is written as $|z|$, and for each pair of points z_1, z_2 in E , the distance between them is denoted by $|z_1 - z_2|$, the closed line segment with endpoints z_1 and z_2 by $[z_1, z_2]$. The distance from $z \in D$ to the boundary ∂D of D is denoted by $d_D(z)$. For a bounded set A in E , $\text{diam}A$ means the diameter of A .

Let

$$\mathbf{B}(x, r) = \{z \in E: |z - x| < r\}, \quad \overline{\mathbf{B}}(x, r) = \{z \in E: |z - x| \leq r\},$$

and

$$\mathbf{S}(x, r) = \{z \in E: |z - x| = r\}.$$

The *quasihyperbolic length* of a rectifiable arc or a curve α with respect to the norm metric in a domain $D \subsetneq E$ is the number (cf. [5, 6]):

$$\ell_k(\alpha) = \int_{\alpha} \frac{|dz|}{d_D(z)}.$$

For each pair of points z_1, z_2 in D , the *quasihyperbolic distance* $k_D(z_1, z_2)$ between z_1 and z_2 is defined in the usual way:

$$k_D(z_1, z_2) = \inf \ell_k(\alpha),$$

where the infimum is taken over all rectifiable curves α joining z_1 to z_2 in D .

For all z_1, z_2 in D , we have the following chain of inequalities (cf. [16])

$$(2.1) \quad k_D(z_1, z_2) \geq \log \left(1 + \frac{|z_1 - z_2|}{\min\{d_D(z_1), d_D(z_2)\}} \right) \geq \left| \log \frac{d_D(z_2)}{d_D(z_1)} \right|.$$

Moreover, we record the following useful result.

Theorem C. [16, Theorem 3.9] *Let $D \subsetneq E$ be a domain, and let $x, y \in D$ be such that $|x - y| \leq \frac{1}{2}d_D(x)$ or $k_D(x, y) \leq 1$. Then*

$$\frac{1}{2} \frac{|x - y|}{d_D(x)} < k_D(x, y) \leq 2 \frac{|x - y|}{d_D(x)}.$$

Next, we recall several concepts from [3, 16].

Definition 2.1. Let $0 < t_0 \leq 1$, and let $\theta: [0, t_0) \rightarrow [0, \infty)$ be an embedding with $\theta(0) = 0$. Suppose that $f: D \rightarrow D'$ is a homeomorphism with $D \neq E$ and $D' \neq E'$. We say that f is (θ, t_0) -relative if

$$\frac{|f(x) - f(y)|}{d_{D'}(f(x))} \leq \theta\left(\frac{|x - y|}{d_D(x)}\right)$$

for all $x, y \in D$ with $|x - y| < t_0 d_D(x)$. If $t_0 = 1$, then we say that f is θ -relative.

Definition 2.2. Let $D \subsetneq E$ and $D' \subsetneq E'$ be domains. A homeomorphism $f: D \rightarrow D'$ is said to be (L, λ) -locally bilipschitz, where $L > 0$ and $\lambda \in (0, 1)$, if for each $x \in D$, there is a constant $c_x > 0$ such that for all $y, z \in B(x, \lambda d_D(x))$,

$$\frac{c_x}{L} |y - z| \leq |f(y) - f(z)| \leq L c_x |y - z|.$$

Definition 2.3. A homeomorphism $f: X \rightarrow X'$ between two metric spaces is said to be η -quasisymmetric if there is a homeomorphism $\eta: [0, \infty) \rightarrow [0, \infty)$ such that

$$d(x, a) \leq t d(x, b) \quad \text{implies} \quad d'(f(x), f(a)) \leq \eta(t) d'(f(x), f(b))$$

for each $t > 0$ and for each triplet x, a, b of points in X .

Definition 2.4. Let $D \subsetneq E$ and $D' \subsetneq E'$ be domains. A homeomorphism $f: D \rightarrow D'$ is said to be *quasisimilar* with data (η, L, λ) if it is both η -quasisymmetric and (L, λ) -locally bilipschitz.

3. Proofs of main results

3.1. Proof of Theorem 1.1. The proof consists of four lemmas: Lemma 3.1 \sim Lemma 3.4. Precisely, the equivalent relationships (1) \iff (2) \iff (3) are proved in Lemma 3.1; The implications (1) \implies (4) \implies (3) are demonstrated in Lemmas 3.2 and 3.3, and the equivalent relationship (1) \iff (5) is shown in Lemma 3.4.

Lemma 3.1. *Let $D \subsetneq E$ and $D' \subsetneq E'$ be domains, and let $f: D \rightarrow D'$ be a homeomorphism. Then the following conditions are equivalent with constants depending only on each other:*

- (1) $f: D \rightarrow D'$ is M -QH;
- (2) f and f^{-1} are both (L, ϑ) -locally bilipschitz;
- (3) there are constants $C > 0$ and $\mu \in (0, 1)$ such that for all $x, y \in D$ with $|x - y| < \mu d_D(x)$,

$$(3.1) \quad \frac{1}{C} \frac{|x - y|}{d_D(x)} \leq \frac{|f(x) - f(y)|}{d_{D'}(f(x))} \leq C \frac{|x - y|}{d_D(x)}.$$

Proof. We prove this lemma by showing the implications (1) \implies (2) \implies (3) \implies (1).

(1) \implies (2): Assume that f is M -QH. Obviously, we only need to show that f is (L, ϑ) -locally bilipschitz for some suitable constants $L \geq 1$ and $\vartheta \in (0, 1)$ both depending only on M .

For any $z \in D$ and for any $x, y \in \mathbf{B}(z, \frac{1}{4M+1}d_D(z))$, we have

$$|x - y| \leq \frac{2}{4M + 1}d_D(z) \quad \text{and} \quad d_D(x) \geq \frac{4M}{4M + 1}d_D(z),$$

which implies

$$|x - y| \leq \frac{1}{2M}d_D(x).$$

Then, by Lemma C, we have

$$k_D(x, y) \leq 2\frac{|x - y|}{d_D(x)} \leq \frac{1}{M}.$$

Moreover, since f is M -QH, we get from (2.1) that

$$(3.2) \quad \log \left(1 + \frac{|f(x) - f(y)|}{d_{D'}(f(x))} \right) \leq k_{D'}(f(x), f(y)) \leq Mk_D(x, y) \\ \leq \min \left\{ 2M\frac{|x - y|}{d_D(x)}, 1 \right\},$$

and thus,

$$|f(x) - f(y)| \leq (e - 1)d_{D'}(f(x)).$$

This yields

$$\frac{|f(x) - f(y)|}{d_{D'}(f(x))} \leq e \log \left(1 + \frac{|f(x) - f(y)|}{d_{D'}(f(x))} \right) \leq 2Me\frac{|x - y|}{d_D(x)}.$$

On the other hand, by (3.2), we see that

$$k_{D'}(f(x), f(y)) \leq 1.$$

Again, by Lemma C, together with (2.1), we know that

$$\log \left(1 + \frac{|x - y|}{d_D(x)} \right) \leq k_D(x, y) \leq Mk_{D'}(f(x), f(y)) \leq \min \left\{ 2M\frac{|f(x) - f(y)|}{d_{D'}(f(x))}, M \right\}.$$

This leads to

$$|x - y| \leq (e^M - 1)d_D(x),$$

and so,

$$\frac{|x - y|}{d_D(x)} \leq e^M \log \left(1 + \frac{|x - y|}{d_D(x)} \right) \leq 2Me^M \frac{|f(x) - f(y)|}{d_{D'}(f(x))}.$$

By letting $L = 2Me^M$ and $\vartheta = \frac{1}{4M+1}$, we know that f is (L, ϑ) -locally bilipschitz.

(2) \Rightarrow (3): Assume that both f and f^{-1} are (L, ϑ) -locally bilipschitz for constants $L \geq 1$ and $\vartheta \in (0, 1)$. Let $\mu = \frac{\vartheta^2}{4L^2}$. We are going to prove that there exists some constant $C \geq 1$ such that for any $x, y \in D$ with $|x - y| < \mu d_D(x)$, (3.1) holds.

We first check the right hand side of the inequalities (3.1). Since f is (L, ϑ) -locally bilipschitz, we see that there is a constant $c_x > 0$ such that for all $u \in \mathbf{B}(x, \vartheta d_D(x))$,

$$(3.3) \quad \frac{c_x}{L}|x - u| \leq |f(x) - f(u)| \leq Lc_x|x - u|.$$

Let $a' \in \partial D'$ be such that

$$|f(x) - a'| \leq 2d_{D'}(f(x)),$$

and let w be such that

$$|x - w| = \frac{1}{2}\vartheta d_D(x) \quad \text{and} \quad w' = f(w) \in [f(x), a'] \cap f(\mathbf{B}(x, \vartheta d_D(x))).$$

Then it follows from (3.3) that

$$\frac{\vartheta c_x}{2L} d_D(x) = \frac{c_x}{L} |w - x| \leq |w' - f(x)| \leq |a' - f(x)| \leq 2d_{D'}(f(x)),$$

and so,

$$(3.4) \quad |f(x) - f(y)| \leq Lc_x |x - y| \leq \frac{4L^2 |x - y|}{\vartheta} d_{D'}(f(x)).$$

Since (3.4) and the choice of μ guarantee that

$$|f(x) - f(y)| \leq \vartheta d_{D'}(f(x)),$$

and since f^{-1} is (L, ϑ) -locally bilipschitz, similarly, we know that the left hand side of the inequalities (3.1) is also true.

(3) \Rightarrow (1): Assume that there are constants $C > 0$ and $\mu \in (0, 1)$ such that for all $x, y \in D$ with $|x - y| < \mu d_D(x)$, (3.1) holds true. Then we infer from (3.1) that

$$L(x, f) := \limsup_{y \rightarrow x} \frac{|f(y) - f(x)|}{|y - x|} \leq \frac{C d_{D'}(f(x))}{d_D(x)}$$

and

$$l(x, f) := \liminf_{y \rightarrow x} \frac{|f(y) - f(x)|}{|y - x|} \geq \frac{d_{D'}(f(x))}{C d_D(x)}.$$

Thus [16, Theorem 5.6] ensures that f is C -QH. \square

Lemma 3.2. *Let $D \subsetneq E$ and $D' \subsetneq E'$ be domains, and let $f: D \rightarrow D'$ be a homeomorphism. If both f and f^{-1} are (θ, t_0) -relative with $\theta(t) = M_0 t$, where $0 < t_0 < 1$ and $0 < t < t_0$, then there are constants $C > 0$ and $\mu \in (0, 1)$ such that for all $x, y \in D$ with $|x - y| < \mu d_D(x)$,*

$$(3.5) \quad \frac{1}{C} \frac{|x - y|}{d_D(x)} \leq \frac{|f(x) - f(y)|}{d_{D'}(f(x))} \leq C \frac{|x - y|}{d_D(x)}.$$

Proof. Let $\mu = \frac{t_0}{M_0}$. Since f is (θ, t_0) -relative, it follows that for all $x, y \in D$ with $|x - y| < \mu d_D(x)$,

$$(3.6) \quad \frac{|f(x) - f(y)|}{d_{D'}(f(x))} \leq M_0 \frac{|x - y|}{d_D(x)},$$

from which the right hand side of the inequalities (3.5) follows.

Because (3.6) and the choice of μ guarantee that

$$|f(x) - f(y)| < t_0 d_{D'}(f(x)),$$

and since f^{-1} is (θ, t_0) -relative, we deduce that

$$\frac{|x - y|}{d_D(x)} \leq M_0 \frac{|f(x) - f(y)|}{d_{D'}(f(x))}.$$

Hence the lemma is proved. \square

Lemma 3.3. *Let $D \subsetneq E$ and $D' \subsetneq E'$ be domains, and let $f: D \rightarrow D'$ be a homeomorphism. If f is M -QH, then both f and f^{-1} are (θ, t_0) -relative with $\theta(t) = M_0 t$, where $0 < t_0 < 1$ and $0 < t < t_0$.*

Proof. For any $x, y \in D$, if $|x - y| < \frac{1}{2M}d_D(x)$, then we see from Lemma C that

$$k_{D'}(f(x), f(y)) \leq Mk_D(x, y) \leq 2M \frac{|x - y|}{d_D(x)} \leq 1,$$

and so,

$$\frac{1}{2} \frac{|f(x) - f(y)|}{d_{D'}(f(x))} \leq k_{D'}(f(x), f(y)) \leq 2M \frac{|x - y|}{d_D(x)}.$$

By letting $t_0 = \frac{1}{2M}$ and $M_0 = 4M$, we know that f is (θ, t_0) -relative with $\theta(t) = M_0t$.

Similarly, we see that f^{-1} is also (θ, t_0) -relative. Hence the lemma is proved. \square

Lemma 3.4. *Let $D \subsetneq E$ and $D' \subsetneq E'$ be domains, and let $f: D \rightarrow D'$ be a homeomorphism. Then the following conditions are equivalent:*

- (1) $f: D \rightarrow D'$ is M -QH;
- (2) For any $x \in D$, the restriction $f: \mathbf{B}(x, d_D(x)) \rightarrow f(\mathbf{B}(x, d_D(x)))$ is M_1 -QH and

$$d_{D'}(f(x)) \leq M_2 d_{f(\mathbf{B}(x, d_D(x)))}(f(x)).$$

Proof. (1) \Rightarrow (2): Assume that $f: D \rightarrow D'$ is M -QH. It follows from Theorem A that for any $x \in D$, the restriction $f: \mathbf{B}(x, d_D(x)) \rightarrow f(\mathbf{B}(x, d_D(x)))$ is M_1 -QH with $M_1 = 4M^2$.

Fix $x \in D$. On the one hand, since $f: D \rightarrow D'$ is M -QH, we deduce from Lemma 3.1 that there are constants $C = C(M) \geq 1$ and $\mu = \mu(M) \in (0, 1)$ such that for all $x, y \in D$ with $|x - y| < \mu d_D(x)$,

$$\frac{1}{C} \frac{|x - y|}{d_D(x)} \leq \frac{|f(x) - f(y)|}{d_{D'}(f(x))} \leq C \frac{|x - y|}{d_D(x)}.$$

On the other hand, since $f: \mathbf{B}(x, d_D(x)) \rightarrow f(\mathbf{B}(x, d_D(x)))$ is M_1 -QH, again, by Lemma 3.1, it follows that there are constants $C_1 = C_1(M) \geq 1$ and $\mu_1 = \mu_1(M) \in (0, 1)$ such that for all $x, y \in D$ with $|x - y| < \mu_1 d_D(x)$,

$$\frac{1}{C_1} \frac{|x - y|}{d_{\mathbf{B}(x, d_D(x))}(x)} \leq \frac{|f(x) - f(y)|}{d_{f(\mathbf{B}(x, d_D(x)))}(f(x))} \leq C_1 \frac{|x - y|}{d_{\mathbf{B}(x, d_D(x))}(x)}.$$

Note that $d_{\mathbf{B}(x, d_D(x))}(x) = d_D(x)$. By taking $M_2 = CC_1$, we know that

$$d_{D'}(f(x)) \leq M_2 d_{f(\mathbf{B}(x, d_D(x)))}(f(x)).$$

(2) \Rightarrow (1): Because $f: \mathbf{B}(x, d_D(x)) \rightarrow f(\mathbf{B}(x, d_D(x)))$ is M_1 -QH, it follows from Lemma 3.1 that there exist constants $C_2 = C_2(M_1) \geq 1$ and $\mu_2 = \mu_2(M_1) \in (0, 1)$ such that for all $x, y \in D$ with $|x - y| < \mu_2 d_{\mathbf{B}(x, d_D(x))}(x)$,

$$\frac{1}{C_2} \frac{|x - y|}{d_{\mathbf{B}(x, d_D(x))}(x)} \leq \frac{|f(x) - f(y)|}{d_{f(\mathbf{B}(x, d_D(x)))}(f(x))} \leq C_2 \frac{|x - y|}{d_{\mathbf{B}(x, d_D(x))}(x)}.$$

Then the facts

$$d_{f(\mathbf{B}(x, d_D(x)))}(f(x)) \leq d_{D'}(f(x)) \leq M_2 d_{f(\mathbf{B}(x, d_D(x)))}(f(x))$$

lead to

$$\frac{1}{C_2 M_2} \frac{|x - y|}{d_D(x)} \leq \frac{|f(x) - f(y)|}{d_{D'}(f(x))} \leq C_2 \frac{|x - y|}{d_D(x)}.$$

Once more, by Lemma 3.1, we know that $f: D \rightarrow D'$ is M -QH with constant $M = M(\mu_2, C_2, M_2) = M(M_1, M_2)$. \square

3.2. Proof of Theorem 1.2. Denote by $B_x = \mathbf{B}(x, d_D(x))$ for $x \in D$. By [15, Theorem 6.12], we may assume that f is η -quasisymmetric on all closed balls $\overline{\mathbf{B}}(x, d_D(x))$ for $x \in D$. To prove that f is M -QH, by Theorem 1.1, we only need to check that for any $x_0 \in D$,

$$(3.7) \quad d_{D'}(f(x_0)) \leq 2\eta(1)d_{f(B_{x_0})}(f(x_0)).$$

Let $\varepsilon \in (0, 1/100)$ be such that

$$\eta(1)\eta(\varepsilon) \leq \frac{1}{2}.$$

First, the similar reasoning as in the proof of [13, Theorem 3.10] guarantees that there is a sequence of points $\{x_i\}_{i \in \mathbf{N}}$ and an end cut γ of D emanating from x_0 such that

- (1) $\gamma_n = \cup_{i=0}^{n-1} [x_i, x_{i+1})$ and $\gamma = \cup_{n=0}^{\infty} \gamma_n$, where $[x_i, x_{i+1})$ is a line segment in \overline{D} ;
- (2) $x_{i+1} \in \mathbf{S}(x_i, d_D(x_i))$ and $d_D(x_{i+1}) \leq \varepsilon_{i+1}d_D(x_i)$, and
- (3) $\lim_{n \rightarrow \infty} x_n = a \in \partial D$.

No loss of generality, we assume that $x_i \in D$ for all $i \in \mathbf{N}$. Then we claim that for all $i \in \mathbf{N}$,

$$|f(x_{i+1}) - f(x_{i+2})| \leq \frac{1}{2}|f(x_i) - f(x_{i+1})|.$$

To prove this claim, for each $i \in \mathbf{N}$, take a point

$$z_{i+1} \in \mathbf{S}(x_{i+1}, d_G(x_{i+1})) \cap \mathbf{S}(x_i, d_G(x_i))$$

with $z_{i+1} \neq x_{i+2}$. Then

$$\frac{|f(x_{i+2}) - f(x_{i+1})|}{|f(z_{i+1}) - f(x_{i+1})|} \leq \eta\left(\frac{|x_{i+2} - x_{i+1}|}{|z_{i+1} - x_{i+1}|}\right) = \eta(1),$$

since f is η -quasisymmetric on $\overline{\mathbf{B}}(x_{i+1}, d_D(x_{i+1}))$. Also, we have

$$\frac{|f(x_{i+1}) - f(z_{i+1})|}{|f(x_{i+1}) - f(x_i)|} \leq \eta\left(\frac{|x_{i+1} - z_{i+1}|}{|x_{i+1} - x_i|}\right) \leq \eta(\varepsilon_{i+1}),$$

since f is η -quasisymmetric on $\overline{\mathbf{B}}(x_i, d_D(x_i))$ and $d_D(x_{i+1}) \leq \varepsilon_{i+1}d_D(x_i)$.

These ensure that

$$\begin{aligned} |f(x_{i+1}) - f(x_{i+2})| &\leq \eta(1)|f(x_{i+1}) - f(z_{i+1})| \\ &\leq \eta(1)\eta(\varepsilon)|f(x_i) - f(x_{i+1})| \\ &\leq \frac{1}{2}|f(x_i) - f(x_{i+1})|, \end{aligned}$$

and so, the claim is true.

Now, it follows from the claim that

$$|f(x_0) - f(x_{n+1})| \leq \sum_{i=0}^n |f(x_i) - f(x_{i+1})| \leq 2|f(x_0) - f(x_1)|,$$

and thus, we infer from the condition $\lim_{n \rightarrow \infty} x_n = a \in \partial D$ that

$$(3.8) \quad d_{D'}(f(x_0)) \leq |f(x_0) - f(a)| \leq 2|f(x_0) - f(x_1)|.$$

Moreover, we have that for any point $w \in \mathbf{S}(x_0, d_D(x_0))$,

$$|f(x_0) - f(x_1)| \leq \eta(1)|f(x_0) - f(w)|,$$

since f is η -quasisymmetric on the closed ball $\overline{\mathbf{B}}(x_0, d_D(x_0))$. Hence

$$(3.9) \quad |f(x_0) - f(x_1)| \leq \eta(1)d_{f(B_{x_0})}(f(x_0)).$$

Easily, (3.7) follows from (3.8) and (3.9), and so, the proof of Theorem 1.2 is complete. \square

3.3. Proof of Theorem 1.3. We start the proof with the following auxiliary result.

Lemma 3.5. *Let $D \subsetneq E$ and $D' \subsetneq E'$ be domains, and let $f: D \rightarrow D'$ be a homeomorphism. If f is η -quasisymmetric and (L, λ) -locally bilipschitz, then f is M -QH with $M = M(\eta, L, \lambda)$.*

Proof. To prove the theorem, by Theorem 1.1, we only need to show that f^{-1} is (L', λ') -locally bilipschitz. Since f is (L, λ) -locally bilipschitz, it is enough to check that there is a constant $\lambda' = \lambda'(\eta, \lambda)$ such that for $x' \in D'$,

$$(3.10) \quad f^{-1}(\mathbf{B}(x', \lambda'd_{D'}(x'))) \subset \mathbf{B}(x, \lambda d_D(x)),$$

where $x = f^{-1}(x')$.

To this end, let $\lambda' \in (0, \frac{1}{2})$ be such that

$$\eta'(\lambda') < \lambda,$$

where $\eta'(t) = \eta^{-1}(t^{-1})^{-1}$.

Fix $x' \in D'$, and let $x = f^{-1}(x')$. For any $y' \in \mathbf{B}(x', \lambda'd_{D'}(x'))$, set $y = f^{-1}(y')$. Since f is η -quasisymmetric, we know that the inverse map $f^{-1}: D' \rightarrow D$ is η' -quasisymmetric. Moreover, it follows from [15, Theorem 6.12] that f^{-1} can be extended to an η' -quasisymmetric mapping from $\overline{D'}$ to \overline{D} , which is still denoted by f^{-1} for simplicity.

For any $u \in \partial D$, we have

$$\frac{|x - y|}{|x - u|} \leq \eta' \left(\frac{|f(x) - f(y)|}{|f(x) - f(u)|} \right) \leq \eta'(\lambda') < \lambda.$$

Then the arbitrariness of $u \in \partial D$ implies

$$|x - y| < \lambda d_D(x).$$

Therefore, the relation (3.10) holds true. \square

Now, we are ready to prove Theorem 1.3. Assume that for each $x \in D$, the restriction $f|_{\mathbf{B}(x, d_D(x))}$ is both η -quasisymmetric and (L, λ) -locally bilipschitz. By Lemma 3.5, we see that $f|_{\mathbf{B}(x, d_D(x))}$ is M_0 -QH, where some $M_0 = M_0(L, \lambda, \eta)$. Thus Theorem 1.2 implies that $f: D \rightarrow D'$ is M -QH, and hence, the proof of Theorem 1.3 is complete. \square

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