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ON AHLFORS' IMAGINARY SCHWARZIAN

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Abstract. We study geometric aspects of the imaginary Schwarzian $S_2 f$ for curves in 3-space, as introduced by Ahlfors in [1]. We show that $S_2 f$ points in the direction from the center of the osculating sphere to the point of contact with the curve. We also establish an important law of transformation of $S_2 f$ under Möbius transformations. We finally study questions of existence and uniqueness up to Möbius transformations of curves with given real and imaginary Schwarzians. We show that curves with the same generic imaginary Schwarzian are equal provided they agree to second order at one point, while prescribing in addition the real Schwarzian becomes an overdetermined problem.

1. Introduction

In [1] the author introduces two differential operators for smooth parametrized curves in $\mathbf{R}^n$ that represent generalizations of the real and imaginary parts of the Schwarzian derivative

$$Sf = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

of a locally injective analytic mapping f. A parametrized curve $f\colon I \to \mathbf{R}^n$ defined on an interval $I \subset \mathbf{R}$ is said to be regular if $f' \neq 0$. For such curves the real and imaginary Schwarzians are given by

$$S_1 f = \frac{\langle f', f'''\rangle}{|f'|^2} - 3\frac{\langle f', f''\rangle^2}{|f'|^4} + \frac{3}{2}\frac{|f''|^2}{|f'|^2}, \tag{1.1}$$

and

$$S_2 f = \frac{f' \wedge f'''}{|f'|^2} - 3\frac{\langle f', f''\rangle}{|f'|^4} f' \wedge f''. \tag{1.2}$$

Here, for $\vec{a}, \vec{b} \in \mathbf{R}^n$, $\vec{a} \wedge \vec{b}$ is the antisymmetric bivector with components $(\vec{a} \wedge \vec{b})_{ij} = a_i b_j - a_j b_i$ and norm $(\sum_{i<j}(a_i b_j - a_j b_i)^2)^{1/2}$. Ahlfors indicated that he was led to these seemingly esoteric definitions by a direct identification of $\mathrm{Re}\{z\overline{\zeta}\}$ with the inner product $\langle z, \zeta\rangle$ of the 2-dimensional vectors z, ζ and the far from obvious identification of $\mathrm{Im}\{z\overline{\zeta}\}$ with the corresponding $\zeta \wedge z$ based on the fact that $(\mathrm{Im}\{z\overline{\zeta}\})^2 = |\zeta \wedge z|^2$. The operators S_1, S_2 can be expressed in terms of the geometric quantities commonly associated with a curve [7]. If $v = |f'|$ denotes speed and k the curvature, then one can see that

$$S_1 f = (v'/v)' - \frac{1}{2}(v'/v)^2 + \frac{1}{2}v^2 k^2, \tag{1.3}$$

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where the derivatives are with respect to the variable $x \in I$. For the imaginary Schwarzian one obtains

$$S_2 f = vk'(\hat{t} \wedge \hat{n}) + v^2 k\tau(\hat{t} \wedge \hat{b}),$$

where τ is the torsion and $\hat{n}, \hat{b}$ are the unit normal and binormal satisfying

$$\hat{t}' = kv\hat{n}$$

and

$$\hat{n}' = -kv\hat{n} + v\tau\hat{b}.$$

In [7] it was shown that the complex number

$$S_1 f + i|S_2 f|$$

has a natural interpretation in terms of the osculating sphere to the curve and is invariant under Möbius transformations of $\mathbf{R}^n$. Thus, $S_1 f$ and $|S_2 f|$ are separately invariant. It was also shown there that questions regarding the injectivity of f depend just on $S_1 f$, a result that has found several applications in the study of curves [2, 3], as well as in the conformal immersion of planar domains into higher dimensional euclidean space [4, 5, 6] (see also [9, 10]).

The purpose of the present paper is to investigate the role of the S_2 operator in 3 dimensions, where the wedge product can be interpreted as the vector product $\times$. Based on this identification we may rewrite $S_2 f$ as

$$S_2 f = vk'\hat{b} - v^2 k\tau\hat{n}. \tag{1.4}$$

In light of the classical formulas for the center of the osculating sphere, it will become evident that $S_2 f$ points in the direction from the center of the osculating sphere to the point on the curve. In particular, S_2 is normal to the osculating sphere at the point of contact with the curve. With R the radius of this sphere, it will follow that the quantity $Rv^2k^2|\tau|$ is Möbius invariant. Also, planes and spheres can be characterized as the only surfaces on which all curves have imaginary Schwarzian normal to the surface, while generic surfaces may allow for isolated points where this condition is met. For an arbitrary Möbius transformation T we will establish the important transformation rule

$$S_2(T \circ f) = \pm \frac{1}{|DT|} DT(S_2 f), \tag{1.5}$$

with $\pm$ depending on whether T preserves the balls bounded by the respective osculating spheres, or not.

In the final section, we will address the issue of existence and uniqueness of curves with given Schwarzians. Since for any curve f the tangent direction $\hat{t}$ is always orthogonal to $S_2 f$, we see that a curve is planar if and only if $S_2 f$ does not change direction. For such curves, uniqueness up to Möbius transformation follows from the classical theory and the above identification of $S_1 f, S_2 f$ with the complex number $S_1 f + i|S_2 f|$. For non-planar curves, the complex number $S_1 f + i|S_2 f|$ is far from determining the curve up to Möbius transformations. For an imaginary Schwarzian of non-constant direction, two curves with equal S_2 that agree to second order at a given point, will be the same. Also, the S_1 operator of such a curve will be uniquely determined. This is perhaps unexpected since, for example, spherical curves may have been thought as equivalent to planar curves, where S_2 alone does not determine the curve up to Möbius transformations. Our results imply that prescribing generically S_1 and S_2 becomes an overdetermined problem. In the same vein, prescribing S_2

and requiring that the curve be arc-length parametrized is also overdetermined. The analysis will rely on deriving a two-by-two system of differential equation in the speed of the curve and an appropriate angle of its tangent direction in the plane orthogonal to S_2, that will correspond to the curve having the prescribed imaginary Schwarzian. An analysis of the solutions to this system will also give existence of curves with prescribed S_2 of non-constant direction. Whether or not two curves with equal real and imaginary Schwarzian are the same without any assumptions on an initial condition remains an open question.

2. The geometry of S_2

It is well known from differential geometry that at each point a smooth curve in space admits a sphere to which it agrees to order three. When $\tau \neq 0$ then the center C is found to be

$$C = P + \frac{1}{k}\hat{n} - \frac{k'}{k^2\tau}\hat{b},$$

see, e.g., [11]. When $\tau = 0$ the osculating sphere reduces to the osculating plane and C is to be interpreted as the point at infinity. Here $'$ denotes differentiation with respect to the arc-length parameter s and P is the point of contact on the curve. We begin with a simple result that follows from this.

Proposition 2.1. *Let $f\colon I \to \mathbf{R}^3$ be a smooth parametrized curve with $f' \neq 0$. Then $S_2 f$ points in the direction from the center of the osculating sphere to the point of contact on the curve. In particular, $S_2 f$ is normal to the osculating sphere at the point of contact with f. If R denotes the radius of the osculating sphere, then $Rv^2k^2|\tau|$ is a Möbius invariant quantity.*

Proof. Let $f\colon I \to \mathbf{R}^3$ be a smooth parametrized curve with $f' \neq 0$, and recall formula (1.4) for $S_2 f$ where $'$ denotes differentiation with respect to the parameter of the curve, say x. Since $d/dx = v d/ds$ we see that

$$C = P - \frac{1}{v^2k^2\tau}S_2 f, \tag{2.1}$$

and it follows that $S_2 f$ points in the direction from the center C to the point P on the curve. In particular, $S_2 f$ is normal to the osculating sphere at the point of contact. Finally, since

$$R = |P - C| = \frac{|S_2 f|}{v^2k^2|\tau|},$$

we conclude that $Rv^2k^2|\tau|$ is Möbius invariant because of the invariance of $|S_2 f|$. □

If f represents a planar curve then we see from (1.4) that $S_2 f$ will be parallel to the binormal vector because $\tau = 0$, thus $S_2 f$ will be normal to the plane. Similarly, for a spherical curve f, $S_2 f$ will be normal to the sphere because of the previous proposition. No other surfaces will meet this condition for all curves lying on it, as the next result states. We include a geometric proof even though it will also follow from the characterization of a single point on a surface on which the condition is met.

Theorem 2.2. *Let $S \subset \mathbf{R}^3$ be a (connected) smooth surface on which all curves have imaginary Schwarzian normal to S. Then S is part of a plane or a sphere.*

Proof. If two surfaces have the condition, then the curve of intersection must have imaginary Schwarzian normal to both surfaces, and will therefore vanish. It follows from (1.4) that k is constant and that $\tau = 0$, so that such an intersection must be a piece of a straight line or a circle. For $P \in S$ we consider the normal sections obtained by intersecting S with planes containing the normal vector, which will be (pieces of) lines or circles. If S is not a plane, then at most one of the normal sections is a line, and it is not difficult to see that the curves obtained by intersecting S with planes parallel to the tangent plane at S will render circles only in the case when all normal sections are circles of the same radius, meaning that near P, the surface S is a piece of a sphere. □

We conclude this section with a local version of this theorem.

Theorem 2.3. *Let $P \in S \subset \mathbf{R}^3$ be a point on a smooth surface with the property that every curve on S through P has imaginary Schwarzian at P normal to S. Then P is umbilic and the second fundamental form of S has vanishing covariant derivative at S.*

Proof. We choose local coordinates near P so that, up to order three, S is represented by $z = ax^2 + by^2 + Ax^3 + Bx^2y + Cxy^3 + Dy^3$. We will show that $a = b$ and that $A = B = C = D = 0$. The argument presented in the proof Theorem 2.2 shows that the normal sections of S at P must have vanishing imaginary Schwarzian at that point. Since these sections are already planar, we see from (1.4) that $k' = 0$ at that point. It is easy to see that the planar section $x \to (x, 0, ax^2 + Ax^3)$ has $k'(0) = 0$ exactly when $A = 0$. Similarly, we find that $D = 0$. By considering the normal section corresponding to $y = \pm x$ we find that $B = C = 0$.

In order to show that $a = b$ we consider an arc-length parametrized curve f given by $(x(s), y(s), z(s))$ on the surface with $x(0) = y(0) = z(0) = 0$. If $\hat{t}(s), \hat{n}(s), \hat{b}(s)$ denote the unite tangent, normal and binormal to the curve, and $\hat{N}(s)$ the normal to the surface along the curve, then we have $\hat{N} \cdot \hat{t} = 0$, hence $\hat{N}' \cdot \hat{t} + k\hat{N} \cdot \hat{n} = 0$. One more derivative gives

$$k'\hat{N} \cdot \hat{n} + k\hat{N}' \cdot \hat{n} + k\hat{N} \cdot (-k\hat{t} + \tau\hat{b}) = -k\hat{N} \cdot \hat{n} - \hat{N}'' \cdot \hat{t},$$

or

$$\hat{N} \cdot (k'\hat{n} + k\tau\hat{b}) = -2k\hat{N}' \cdot \hat{n} - \hat{N}'' \cdot \hat{t}\,.$$

Observe that $k'\hat{n} + k\tau\hat{b}$ corresponds to $S_2 f$ rotated in ninety degrees in the $\hat{n}, \hat{b}$ plane, hence at $s = 0$ we have $\hat{N} \cdot (k'\hat{n} + k\tau\hat{b}) = 0$. Therefore at $s = 0$

$$\hat{N}'' \cdot \hat{t} = -2k\hat{N}' \cdot \hat{n}. \tag{2.2}$$

Since $\hat{N}'(0) = -2(ax'(0), by'(0), 0)$, $k\hat{n}(0) = (x''(0), y''(0), z''(0))$ and $\hat{N}''(0) = c\hat{N(0)}$ $-2(ax''(0), y''(0), 0)$, we obtain from (2.2) that

$$ax'(0)x''(0) + by'(0)y''(0) = 0\,.$$

By considering a curve with $x''(0), y''(0) \neq 0$ it follows from $x'(0)x''(0) + y'(0)y''(0) = 0$ that $a = b$. This finishes the proof of the theorem. □

Theorem 2.4. *If T is a Möbius transformation and f a regular curve, then*

$$S_2(T \circ f) = \pm\frac{1}{|DT|}DT(S_2 f),$$

with $\pm$ depending on whether T preserves the balls bounded by the respective osculating spheres, or not.

Proof. Let Σ be the osculating sphere to f at a given point P on the curve f. Then $S_2 f$ at P is normal to Σ and points outward. Since $T(\Sigma)$ is the osculating sphere to $T \circ f$ at $Q = T(P)$, we see that $S_2(T \circ f)$ is normal to $T(\Sigma)$ at Q, again pointing outward. But

$$\frac{1}{|DT|} DT(S_2 f)$$

evaluated at P is also normal to $T(\Sigma)$ at Q and has norm $|S_2 f| = |S_2(T \circ f)|$. This shows that $S_2(T \circ f)$ will agree with $\frac{1}{|DT|} DT(S_2 f)$ when T preserves the regions bounded by osculating spheres, and will be its opposite when not. □

3. Existence and uniqueness

The purpose of this section is to investigate the question of existence and uniqueness up to Möbius transformations of parametrized curves with given real and imaginary Schwarzian derivatives. Contrary to the intuition stemming from two dimensions, generic curves with just equal imaginary Schwarzian will be the same provided they agree to second order at one point. Only when they are planar is it necessary to consider in addition the real Schwarzian. For example, two spherical curves, say on the same sphere, which have the same imaginary Schwarzian will be equal because the direction of the normal vector to a sphere uniquely determines the point on the sphere. The same conclusion will hold if two curves f, g on the same sphere satisfy $S_2 f = \mu S_2 g$ for some positive scalar function μ. For negative μ the curves must be antipodal.

For planar curves the classical theory provides existence and uniqness up to Möbius transformations. Say we work on the complex plane and identify $S_1 f, S_2 f$ with the complex number $Sf = S_1 f + i|S_2 f|$. Any other complex valued curve g with $Sg = Sf$ will be given as the integral of $(u')^{-2}$ of some solution of the complex linear equation $u'' + (1/2)(Sf)u = 0$. Different Möbius shifts of f correspond to different initial conditions for u and the initial point of the curve.

For curves in space the quantity $Sf = S_1 f + i|S_2 f|$ is far from determining the curve up to Möbius transformations, unless it is assumed that both curves are arc-length parametrized. For the latter, if $S_1 f = S_1 g$ for arc-length parametrized curves, then we see from (1.3) that $k_f = k_g$. If also $|S_2 f| = |S_2 g|$ it follows from (1.4) that $\tau_f = \pm\tau_g$. We conclude that the curves will differ at most by an isometry and a reflection (that changes the sign of the torsion). For curves not both parametrized by arc-length, we may consider, for example, an arc-length paramertized regular helix f with curvature and torsion $k_0, \tau_0 > 0$. On the same interval of parametrization we consider a planar curve g with speed v and curvature k. The equations $S_1 g = S_1 f$ and $|S_2 g| = |S_2 f|$ are expressed as

$$(v'/v)' - \frac{1}{2}(v'/v)^2 = \frac{1}{2}\left(k_0^2 - v^2k^2\right),$$
$$vk' = k_0 v_0,$$

which corresponds to a system of differential equation for v, k that allows for a solution on some subinterval containing a given initial point x_0 and initial data $v(x_0) > 0, v'(x_0), k(x_0)$. The resulting curve g cannot be a Möbius transformations of f since g is planar and f non-spherical.

In our study of existence and uniqueness of curves with given Schwarzian derivatives, we have been led to consider curves that are not necessarily parametrized by arc-length. On the one hand, for the question of existence, prescribing a parametrization by arc-length imposes the restriction that the real Schwarzian be positive (see

(1.3)), while a positive real Schwarzian does not necessarily force a parametrization by arc-length. On the other hand, for the question of uniqueness, a common reparametrization of two curves with equal Schwarzian derivatives can be assumed to yield an arc-length parametrization for only one of the curves, and would therefore be of limited use.

Let $S = S(x)$ be a given parametrized vector in space, and suppose that $S_2 f = S$ for some regular curve f. Let $v = |f'|$ be speed, and consider the frame given by $\hat{t}, \hat{n}, \hat{b}$, together with the curvature k and torsion τ. Since S is always orthogonal to $\hat{t}$, it follows that f is determined by v and the direction of $\hat{t}$ in the plane orthogonal to S. Let us write $S = \lambda\hat{T}$, $\lambda = |S|$, as the tangent vector to a curve with normal and binormal vectors $\hat{N}, \hat{B}$. When S is parallel to a given fixed direction, we will think of $\hat{N}$ and $\hat{B}$ as constant on the plane normal to S. We seek to determine $\theta = \theta(x)$ and $v = v(x)$ so that the curve f with

$$\hat{t} = \cos\theta\,\hat{N} + \sin\theta\,\hat{B}$$

and velocity v satisfies $S_2 f = S$. Since f is not necessarily parametrized by arc-length, we will have $\hat{t}' = kv\hat{n}$ and $\hat{n}' = -vk\hat{t} + v\tau\hat{b}$. Similarly, for the curve with tangent vector S we write

$$\hat{T}' = \lambda\kappa\hat{N}, \quad \hat{N}' = -\lambda\kappa\hat{T} + \lambda\eta\hat{B},$$

where κ, η stand for the curvature and torsion of the curve with tangent S. With this we find that

$$\hat{t}' = vk\hat{n} = -\lambda\kappa\cos\theta\,\hat{T} + (\theta' + \lambda\eta)\hat{d}, \tag{3.1}$$

where $\hat{d} = \hat{T}\times\hat{t} = -\sin\theta\,\hat{N} + \cos\theta\,\hat{B}$ is the unit vector in the plane orthogonal to S that is also orthogonal to $\hat{t}$. The orientation is so that $\hat{t}\times\hat{d} = \hat{T}$. By differentiating (3.1) we find that

$$\begin{aligned}(vk)'\hat{n} + vk\left[-vk\hat{t} + v\tau\hat{b}\right] &= -(\lambda\kappa\cos\theta)'\hat{T} - (\lambda\kappa)^2\cos\theta\hat{N} + (\theta' + \lambda\eta)'\hat{d} \\ &\quad + (\theta' + \lambda\eta)\left[\lambda\kappa\sin\theta\,\hat{T} - (\theta' + \lambda\eta)\cos\theta\,\hat{b}\right].\end{aligned}$$

After elimination of the components in $\hat{t}$ we find the somewhat simpler equation

$$\begin{aligned}(vk)'\hat{n} + v^2k\tau\hat{b} &= [\lambda\kappa(\theta' + \lambda\eta)\sin\theta - (\lambda\kappa\cos\theta)']\,\hat{T} \\ &\quad + \left[(\theta' + \lambda\eta)' + (\lambda\kappa)^2\sin\theta\cos\theta\right]\hat{d}.\end{aligned} \tag{3.2}$$

We rewrite the left hand side as

$$(vk)'\hat{n} + v^2k\tau\hat{b} = vk'\hat{n} + v^2k\tau\hat{b} + v'k\hat{n} = \lambda\hat{d} + (v'/v)\left[-\lambda\kappa\cos\theta\,\hat{T} + (\theta' + \lambda\eta)\hat{d}\right],$$

where we have used that $vk'\hat{n} + v^2k\tau\hat{b}$ is equal to S rotated ninety degrees in the plane orthogonal to $\hat{t}$, together with equation (3.1) for $vk\hat{n}$. With this we equate components in $\hat{T}$ and $\hat{d}$ in (3.2) to obtain a pair of differential equations in v and θ of the form

$$(v'/v)\lambda\kappa\cos\theta = A(\theta, \theta'), \tag{3.3}$$

$$\theta'' = (v'/v)(\theta' + \lambda\eta) + B(\theta), \tag{3.4}$$

where in the terms

$$A(\theta,\theta') = (\lambda\kappa\cos\theta)' - \lambda\kappa(\theta' + \lambda\eta)\sin\theta, \tag{3.5}$$

$$B(\theta) = \lambda - (\lambda\eta)' - (\lambda\kappa)^2\sin\theta\cos\theta \tag{3.6}$$

we have suppressed the dependence in the quantities λ, κ, η.

Our analysis of existence and uniqueness will be based on the above system of equations for v, θ, and we will assume that both $\lambda, \kappa \neq 0$. On segments where $\lambda = 0$ we will have that $S = S_2 = 0$, so that $k = \tau = 0$ and the possible curve f lies on a line. Similarly, on segments where $\kappa = 0$ we will have that the curve with tangent vector S is rectilinear, and therefore f is reduced to the known planar case. We shall say that $S_2 f$ is of *generic type* if λ, κ never vanish.

Uniqueness. Let f be a given parametrized curve for which $S = S_2 f$ satisfies $\lambda, \kappa \neq 0$ on an interval I. We will show that knowing $S_2 f$ determines v, θ provided one is given an initial 2-jet $f(x_0) = P$, $f'(x_0) = v_0\hat{t}_0$, $f''(x_0) = v_0'\hat{t}_0 + v_0k_0\hat{n}_0 = (v_0'/v_0)f'(x_0) + v_0k_0\hat{n}_0$. Indeed, from the jet we can find the values of $v_0, v_0'/v_0, k_0$ and the directions $\hat{t}_0, \hat{n}_0$. We therefore know θ_0 and also θ_0', this last from equation (3.1). Observe that there are no solutions to the system of equations (3.3), (3.4) with $\cos\theta = 0$ on a subinterval $J \subset I$. If so, then $A(\theta,\theta') = 0$ from (3.3), which from (3.5) gives $\lambda^2\kappa\eta = 0$ on J, hence $\eta = 0$ there. Then $B(\theta) = \lambda$ from (3.6), which leads to a contradiction from equation (3.4). This shows that $E = \{x \in I\colon \cos\theta(x) = 0\}$ is a closed set with empty interior. From equation (3.5) we see that

$$v'/v = \frac{A(\theta,\theta')}{\lambda\kappa\cos\theta}, \tag{3.7}$$

so that the right-hand side becomes a regular function on all of I. In other words, for $x \in E$ the numerator $A(\theta,\theta')$ must also vanish in order for the right-hand side in (3.7) to remain regular. With this, then (3.4) becomes a second order equation for θ that has a unique solution for the given initial data θ_0, θ_0'. The function θ then gives v'/v uniquely, and hence v. We therefore know f', and thus f from the initial point P_0.

It is interesting that in this case the operator $S_1 f$ will be determined by $S_2 f$. Indeed, we recall equation (1.3)

$$S_1 f = (v'/v)' - \frac{1}{2}(v'/v)^2 + \frac{1}{2}v^2k^2,$$

where v'/v is determined from (3.7) and v^2k^2 can be found from (3.1).

We summarize this as follows.

Theorem 3.1. *Let f, g be two regular parametrized curves with $S_2 f = S_2 g$ on an interval I on which $\lambda, \kappa \neq 0$. If f and g agree up to order two at some point, then $f = g$ on I. The S_1 derivative of such curves will be determined by the S_2 derivative.*

Existence. Let $S = \lambda\hat{T}$ be a given nonwhere vanishing vector, parametrized on an interval I. We seek a curve f for which $S_2 f = S$. As before, and in order to leave the case of a planar curve f, we will assume that the curvature κ of the curve with tangent vector S is also nowhere vanishing on I. We seek to determine v, θ so that equations (3.3) and (3.4) hold. For a given initial value x_0 of the parameter, we set v_0, θ_0, θ_0' with $\cos\theta_0 \neq 0$. Note that prescribing these initial conditions amounts to prescribing a 2-jet of f at x_0 for which $f'(x_0)$ is not parallel to $\hat{B}$. Near x_0, we express v'/v as in equation (3.7) and solve the function θ from the second order equation obtained from (3.4). If the resulting function $\cos\theta$ does not vanish on I, then v'/v will be regular on I, and so will be f. If, on the contrary, there exists $x_1 \in I$, say $x_1 > x_0$, for which $\cos\theta = 0$ for the first time, then f will be regular up to but not necessarily including x_1. It may remain regular through x_1 if $A(\theta, \theta')$ also vanishes at x_1 to the appropriate order, as mentioned earlier when discussing uniqueness. In general, equation (3.4) may become singular, and our analysis for the moment falls short of being able to describe all possible (singular) configurations that may arise at the endpoint.

Theorem 3.2. *Let $S = S_2 = \lambda\hat{T}$ be of generic type on an interval I. Then for each $x_0 \in I$ and each 2-jet of f at x_0 with $f'(x_0)$ not parallel to $\hat{B}$, there exists a maximal subinterval $J \subset I$ containing x_0 and a curve f defined on J with $S_2 f = S$.*

Because the S_1 operator can de determined from the S_2 operator of generic type and initial conditions, we have the next theorem.

Theorem 3.3. *Prescribing the real Schwarzian S_1 and an imaginary Schwarzian S_2 of generic type for a curve is an overdetermined problem.*

Proof. We have seen that prescribing an imaginary Schwarzian of generic type determines the real Schwarzian provided we know an initial set of values for v, θ, θ'. In other words, there exists a mapping $\Phi\colon \mathbf{R}^3 \to C^0(I)$ taking initial data into the real Schwarzian within the space of all continuous functions on the interval I. The mapping Φ will be largely not onto, making the problem overdetermined. □

In the same vein we state.

Theorem 3.4. *Prescribing an imaginary Schwarzian S_2 of generic type for an arc-length parametrized curve is an overdetermined problem.*

Proof. If f is parametrized by arc-length, then $v = 1$ and $v' = 0$. This forces $A(\theta, \theta') = 0$, which is a first order equation for θ, which in addition must satisfy $\theta'' = B(\theta)$ from (3.4). This is clearly overdetermined. □

It would be interesting to understand under what conditions or to what extent the above mapping Φ is injective. If so, curves with equal real Schwarzian and equal imaginary Schwarzian of generic type would be identical.

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