Annales Fennici Mathematici
Volumen 46, 2021, 371–388

ON THE GLOBAL BEHAVIOR OF INVERSE MAPPINGS IN TERMS OF PRIME ENDS

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Abstract. The paper is devoted to the study of mappings with finite distortion, actively studied recently. For mappings whose inverse satisfy the Poletsky inequality, the results on boundary behavior in terms of prime ends are obtained. In particular, it was proved that the families of the indicated mappings are equicontinuous at the points of the boundary if a certain function determining the distortion of the modulus of families of paths under the mappings is integrable.

1. Introduction

In the theory of quasiconformal mappings, an important place is occupied by the results on their local and boundary behavior, see e.g. [Va_1, Theorem 19.2], [MRV, Theorem 3.17], [NP_1, Theorem 3.1], [NP_2, Theorem 3.1], [Cr, Theorem 8.9] and [MRSY_2, Theorem 3.1, Corollary 3.6]. Let us mention the following very important result, see [NP_1, Theorem 3.1].

Theorem. (Näkki–Palka) *Let $\mathfrak{F}$ be a family of K-quasiconformal mappings of a domain $D \neq \overline{\mathbf{R}^n}$ onto a domain D' and let either D or D' be quasiconformally collared on the boundary. Then $\mathfrak{F}$ is uniformly equicontinuous if and only if each $f \in \mathfrak{F}$ can be extended to a continuous mapping of $\overline{D}$ onto $\overline{D'}$ and $\inf_{\mathfrak{F}} q(f(A)) > 0$ for some continuum A in D.*

Here $q(f(A))$ denotes the chordal diameter of the set $f(A) \subset \overline{\mathbf{R}^n}$, see e.g. [$\mathrm{Va}_1$, Definition 12.1]. The main goal of this article is to prove results similar to the Näkki–Palka theorem and related to mappings with unbounded characteristic. In this case, we consider domains with a complex structure. We mean that in such domains, mappings may not even have continuous boundary extension in the Euclidean sense. However, under certain additional assumptions, such an extension holds in the sense of the so-called prime ends, to which the present studies relate. Similar studies have taken place in some of our earlier papers, see, for example, [SevSkv_1] and [Sev]. Unlike previous articles, the main attention here is paid to the behavior of homeomorphisms, the inverse of which satisfy Poletsky-type inequalities. For quasiconformal mappings, consideration of such mappings does not make sense, since the inverse of quasiconformal homeomorphisms belong to the same class. The situation

https://doi.org/10.5186/aasfm.2021.4630
2020 Mathematics Subject Classification: Primary 30C65; Secondary 32U20, 31B15.
Key words: Mappings with a finite and bounded distortion, boundary behaviour, prime ends.

changes drastically if the characteristic of mappings is unbounded. We will confirm what we said with one of the examples given at the end of this article.

Recall some definitions (see, for example, [KR$_1$] and [KR$_2$]). Let ω be an open set in $\mathbf{R}^k$, $k=1,\dots,n-1$. A continuous mapping $\sigma\colon \omega\to\mathbf{R}^n$ is called a *k-dimensional surface* in $\mathbf{R}^n$. A *surface* is an arbitrary $(n-1)$-dimensional surface σ in $\mathbf{R}^n$. A surface σ is called *a Jordan surface*, if $\sigma(x)\neq\sigma(y)$ for $x\neq y$. In the following, we will use σ instead of $\sigma(\omega)\subset\mathbf{R}^n$, $\overline{\sigma}$ instead of $\overline{\sigma(\omega)}$ and $\partial\sigma$ instead of $\overline{\sigma(\omega)}\setminus\sigma(\omega)$. A Jordan surface $\sigma\colon\omega\to D$ is called a *cut* of D, if σ separates D, that is $D\setminus\sigma$ has more than one component, $\partial\sigma\cap D=\varnothing$ and $\partial\sigma\cap\partial D\neq\varnothing$.

A sequence of cuts $\sigma_1,\sigma_2,\dots,\sigma_m,\dots$ in D is called *a chain*, if: (i) the set σ_{m+1} is contained in exactly one component d_m of the set $D\setminus\sigma_m$, wherein $\sigma_{m-1}\subset D\setminus(\sigma_m\cup d_m)$; (ii) $\bigcap_{m=1}^{\infty} d_m=\varnothing$.

Two chains of cuts $\{\sigma_m\}$ and $\{\sigma_k'\}$ are called *equivalent*, if for each $m=1,2,\dots$ the domain d_m contains all the domains d_k', except for a finite number, and for each $k=1,2,\dots$ the domain d_k' also contains all domains d_m, except for a finite number.

The *end* of the domain D is the class of equivalent chains of cuts in D. Let K be the end of D in $\mathbf{R}^n$, then the set $I(K)=\bigcap_{m=1}^{\infty}\overline{d_m}$ is called *the impression of the end* K. Throughout the paper, $\Gamma(E,F,D)$ denotes the family of all paths $\gamma\colon[a,b]\to\overline{\mathbf{R}^n}$ such that $\gamma(a)\in E$, $\gamma(b)\in F$ and $\gamma(t)\in D$ for every $t\in[a,b]$. In what follows, M denotes the modulus of a family of paths, and the element $dm(x)$ corresponds to the Lebesgue measure in $\mathbf{R}^n$, $n\geqslant 2$, see [Va$_1$]. Following [Na], we say that the end K is *a prime end*, if K contains a chain of cuts $\{\sigma_m\}$ such that $\lim_{m\to\infty}M(\Gamma(C,\sigma_m,D))=0$ for some continuum C in D (see Figure 1 for this).

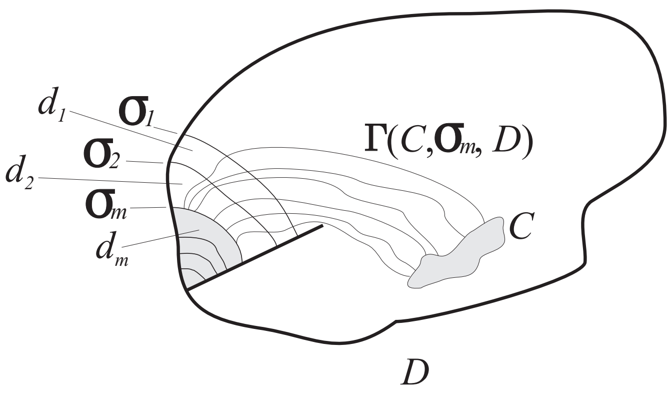

Figure 1. A prime end in some domain.

In the following, the following notation is used: the set of prime ends corresponding to the domain D, is denoted by E_D, and the completion of the domain D by its prime ends is denoted $\overline{D}_P$.

Consider the following definition, which goes back to Näkki [Na], see also [KR$_1$]–[KR$_2$]. We say that the boundary of the domain D in $\mathbf{R}^n$ is *locally quasiconformal*, if each point $x_0\in\partial D$ has a neighborhood U in $\mathbf{R}^n$, which can be mapped by a quasiconformal mapping φ onto the unit ball $\mathbf{B}^n\subset\mathbf{R}^n$ so that $\varphi(\partial D\cap U)$ is the intersection of $\mathbf{B}^n$ with the coordinate hyperplane.

Remark 1.1. There are various approaches related to the definition of prime ends. In particular, Näkki's work [Na, Sections 3, 4] uses a definition similar to ours, in which, however, cuts must be connected sets, and the domain must be divided by the cut into exactly two components. For this reason, we note that in domains with a locally quasiconformal boundary, the above definition of a prime end is equivalent to our (see [Na, Theorem 4.1] and [IS, Theorem 2.1]).

For a given set $E \subset \mathbf{R}^n$, we set $d(E) := \sup_{x,y\in E}|x-y|$. The sequence of cuts σ_m, $m = 1, 2, \ldots$, is called *regular,* if $\overline{\sigma_m} \cap \overline{\sigma_{m+1}} = \varnothing$ for $m \in \mathbf{N}$ and, in addition, $d(\sigma_m) \to 0$ as $m \to \infty$. If the end K contains at least one regular chain, then K will be called *regular.* We say that a bounded domain D in $\mathbf{R}^n$ is *regular*, if D can be quasiconformally mapped to a domain with a locally quasiconformal boundary whose closure is a compact in $\mathbf{R}^n$, and, besides that, every prime end in D is regular. Note that space $\overline{D}_P = D \cup E_D$ is metric, which can be demonstrated as follows. If $g : D_0 \to D$ is a quasiconformal mapping of a domain D_0 with a locally quasiconformal boundary onto some domain D, then for $x, y \in \overline{D}_P$ we put:

$$\rho(x, y) := |g^{-1}(x) - g^{-1}(y)|\,, \tag{1.2}$$

where the element $g^{-1}(x)$, $x \in E_D$, is to be understood as some (single) boundary point of the domain D_0. The specified boundary point is unique and well-defined by [IS, Theorem 2.1, Remark 2.1], cf. [Na, Theorem 4.1]. It is easy to verify that ρ in (1.2) is a metric on $\overline{D}_P$, and that the topology on $\overline{D}_P$, defined by such a method, does not depend on the choice of the map g with the indicated property.

We say that a sequence $x_m \in D$, $m = 1, 2, \ldots$, converges to a prime end of $P \in E_D$ as $m \to \infty$, if for any $k \in \mathbf{N}$ all elements x_m belong to d_k except for a finite number. Here d_k denotes a sequence of nested domains corresponding to the definition of the prime end P. Note that for a homeomorphism of a domain D onto D', the end of the domain D uniquely corresponds to some sequence of nested domains in the image under the mapping. See Figure 2 for an illustration.

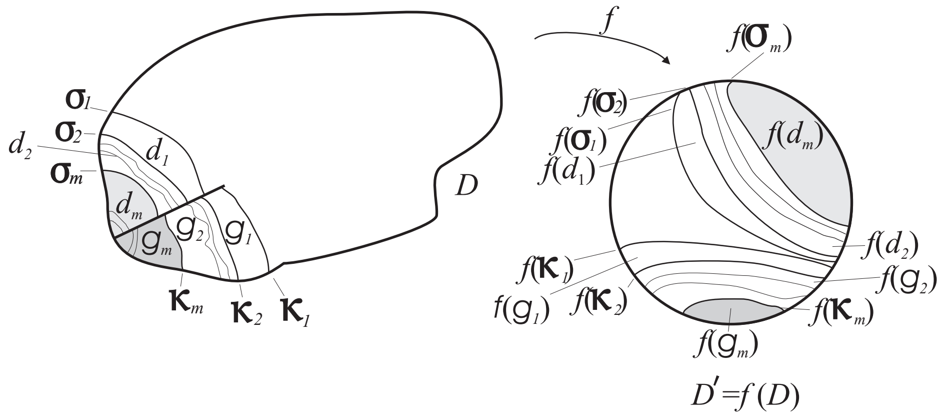

Figure 2. The correspondence between sequences of domains.

Everywhere below, unless otherwise stated, the boundary and the closure of a set are understood in the sense of an extended Euclidean space $\overline{\mathbf{R}^n}$. Let $x_0 \in \overline{D}$, $x_0 \neq \infty$,

$$S(x_0, r) = \{x \in \mathbf{R}^n \colon |x - x_0| = r\}, \quad S_i = S(x_0, r_i), \quad i = 1, 2,$$
$$A = A(x_0, r_1, r_2) = \{x \in \mathbf{R}^n \colon r_1 < |x - x_0| < r_2\}.$$

Let $Q \colon \mathbf{R}^n \to \mathbf{R}^n$ be a Lebesgue measurable function satisfying the condition $Q(x) \equiv 0$ for $x \in \mathbf{R}^n \setminus D$. The mapping $f \colon D \to \overline{\mathbf{R}^n}$ is called a *ring Q-mapping at the point* $x_0 \in \overline{D} \setminus \{\infty\}$, if the condition

$$M(f(\Gamma(S_1, S_2, D))) \leqslant \int_{A\cap D} Q(x) \cdot \eta^n(|x - x_0|)\, dm(x) \tag{1.3}$$

holds for all $0 < r_1 < r_2 < d_0 := \sup_{x\in D}|x - x_0|$ and each Lebesgue measurable function $\eta\colon (r_1, r_2) \to [0, \infty]$ such that

$$\int_{r_1}^{r_2} \eta(r)\, dr \geqslant 1. \tag{1.4}$$

The mapping of f is called a *ring Q-mapping in D*, if condition (1.3) is satisfied at every point $x_0 \in D$, and a *ring Q-mapping in $\overline{D}$*, if the condition (1.3) holds at every point $x_0 \in \overline{D}$. With regard to the definition of such mappings, we point to publications [RSY] and [MRSY$_2$]. The class of mappings satisfying relation (1.3) contains itself all conformal and quasiconformal mappings, as well as many mappings with finite distortion, see, for example, [Pol, Theorem 2], [Va$_2$, Theorem 3.1] and [MRSY$_1$, Theorems 4.6 and 6.10].

Let (X, d) and (X', d') be metric spaces with distances d and d', respectively. A family $\mathfrak{G}$ of mappings $g\colon X' \to X$ is said to be *equicontinuous at a point* $y_0 \in X'$, if for every $\varepsilon > 0$ there is $\delta = \delta(\varepsilon, y_0) > 0$ such that $d(g(y), g(y_0)) < \varepsilon$ for all $g \in \mathfrak{G}$ and $y \in X'$ with $d'(y, y_0) < \delta$. The family $\mathfrak{G}$ is *equicontinuous* if $\mathfrak{G}$ is equicontinuous at every point $y_0 \in X'$.

Everywhere below, unless otherwise stated, $d = \rho$ is one of the metrics in $\overline{D}_P$, defined by the relation (1.2), and $d' = q$ is a chordal metric defined by formula

$$q(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2}\sqrt{1 + |y|^2}}, \quad x \neq \infty \neq y, \quad q(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}}. \tag{1.5}$$

For a given set $E \subset \overline{\mathbf{R}^n}$, we set

$$q(E) := \sup_{x,y\in E} q(x, y). \tag{1.6}$$

The quantity $q(E)$ is called the *chordal diameter* of the set E. The boundary of the domain D is called *weakly flat at the point* x_0, if for every number $P > 0$ and every neighborhood U of this point there is a neighborhood V of point x_0 such that $M(\Gamma(E, F, D)) > P$ for any continua E and F, satisfying conditions $F \cap \partial U \neq \varnothing \neq F \cap \partial V$. The boundary of domain D is called *weakly flat* if it is such at each of its point.

For a given $\delta > 0$, domains $D \subset \mathbf{R}^n$ and $D' \subset \overline{\mathbf{R}^n}$, $n \geqslant 2$, a continuum $A \subset D$ and a Lebesgue measurable function $Q(x)\colon \mathbf{R}^n \to [0, \infty]$ such that $Q(x) \equiv 0$ for $x \notin D$, we denote by $\mathfrak{S}_{\delta,A,Q}(D, D')$ the family of all homeomorphisms h of D' onto D such that the mapping $f = h^{-1}$ satisfies the condition (1.3) in $\overline{D}$, while $q(f(A)) \geqslant \delta$. The following statement is true.

Theorem 1.7. *Suppose that D is regular, D' has a weakly flat boundary, and any component of $\partial D'$ is a non-degenerate continuum. If $Q \in L^1(D)$, then each map $h \in \mathfrak{S}_{\delta,A,Q}(D, D')$ has a continuous extension $\overline{h}\colon \overline{D'} \to \overline{D}_P$, in addition, $\overline{h}(D') = \overline{D}_P$, and the family $\mathfrak{S}_{\delta,A,Q}(\overline{D}_P, \overline{D'})$, consisting of all extended mappings $\overline{h}\colon \overline{D'} \to \overline{D}_P$, is equicontinuous in $\overline{D'}$.*

Remark 1.8. The possibility of continuous extension of a homeomorphism $h\colon D' \to D$ to the mapping $\overline{h}\colon \overline{D'} \to \overline{D}_P$ in Theorem 1.7 may be established similarly to [GRY, Theorem 6.1]; see also [SalSev, Theorem 2]. Since the proof of this result almost literally repeats the reasoning related to the mentioned publications, we will not give this proof in the present text.

2. Preliminaries

Recall that a *path* will be called a continuous mapping $\gamma\colon I\to \mathbf{R}^n$ of a segment, interval or half-interval $I\subset \mathbf{R}$ into n-dimensional Euclidean space $\mathbf{R}^n$. As usual, the set $|\gamma|=\{x\in\mathbf{R}^n\colon \exists\, t\in I\colon \gamma(t)=x\}$ is called the *locus* of a path $\gamma\colon I\to\mathbf{R}^n$. We say that the path γ lies in the domain D, if its locus belongs to this domain. We also say that paths γ_1 and γ_2 do not intersect each other if their loci do not intersect as sets in $\mathbf{R}^n$. By definition, a prime end $P\in E_D$ corresponds to a sequence of nested domains d_m, $m\geqslant 1$, and if $P\in D$, then we assume that P corresponds to a sequence of balls $B(P,r_m)$ with radii $r_m\to 0$ as $m\to\infty$, $r_m>0$, which lie in the domain of D with its closure. The following statement is true, see e.g. [SevSkv$_1$, Proposition 1].

Proposition 2.1. *Let $n\geqslant 2$, and let D be a domain in $\mathbf{R}^n$ that is locally connected on its boundary. Then every two pairs of points $a\in D, b\in \overline{D}$ and $c\in D, d\in\overline{D}$ can be joined by non-intersecting paths $\gamma_1\colon[0,1]\to\overline{D}$ and $\gamma_2\colon[0,1]\to\overline{D}$ so that $\gamma_i(t)\in D$ for all $t\in(0,1)$ and all $i=1,2$, while $\gamma_1(0)=a$, $\gamma_1(1)=b$, $\gamma_2(0)=c$ and $\gamma_2(1)=d$.*

The proof of the following statement completely repeats the proof of [Va$_1$, Theorem 17.10], and therefore is omitted.

Proposition 2.2. *Let $D\subset\mathbf{R}^n$ be a domain with a locally quasiconformal boundary, then the boundary of this domain is weakly flat. Moreover, the neighborhood of U in the definition of a locally quasiconformal boundary can be taken arbitrarily small, and in this definition $\varphi(x_0)=0$.*

The following statement points to the possibility of a "convenient" joining of the points of a regular domain by paths.

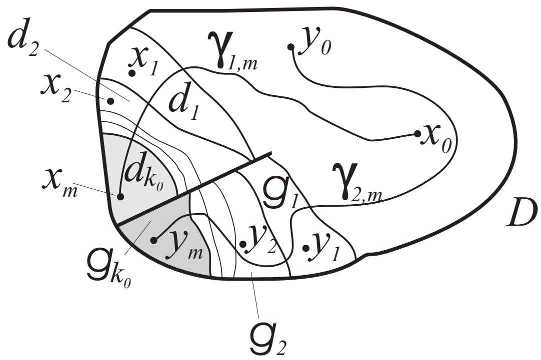

Figure 3. To the statement of Lemma 2.3.

Lemma 2.3. *Let $D\subset\mathbf{R}^n$, $n\geqslant 2$, be a regular domain, and let $x_m\to P_1$, $y_m\to P_2$ as $m\to\infty$, $P_1,P_2\in\overline{D}_P$, $P_1\ne P_2$. Suppose that d_m, g_m, $m=1,2,\ldots$, are sequences of descending domains, corresponding to P_1 and P_2, $d_1\cap g_1=\varnothing$, and $x_0,y_0\in D\setminus(d_1\cup g_1)$. Then there are arbitrarily large $k_0\in\mathbf{N}$, $M_0=M_0(k_0)\in\mathbf{N}$ and $0<t_1=t_1(k_0), t_2=t_2(k_0)<1$ for which the following condition is fulfilled: for each $m\geqslant M_0$ there are non-intersecting paths*

$$\gamma_{1,m}(t)=\begin{cases}\widetilde{\alpha}(t), & t\in[0,t_1],\\ \widetilde{\alpha_m}(t), & t\in[t_1,1],\end{cases}\qquad \gamma_{2,m}(t)=\begin{cases}\widetilde{\beta}(t), & t\in[0,t_2],\\ \widetilde{\beta_m}(t), & t\in[t_2,1],\end{cases}$$

such that:

1) $\gamma_{1,m}(0)=x_0$, $\gamma_{1,m}(1)=x_m$, $\gamma_{2,m}(0)=y_0$ *and* $\gamma_{2,m}(1)=y_m$;

2) $|\gamma_{1,m}| \cap \overline{g_{k_0}} = \varnothing = |\gamma_{2,m}| \cap \overline{d_{k_0}}$;

3) $\widetilde{\alpha_m}(t) \in d_{k_0+1}$ *for* $t \in [t_1, 1]$ *and* $\widetilde{\beta_m}(t) \in g_{k_0+1}$ *for* $t \in [t_2, 1]$ *(see Figure 3).*

Proof. Since, by condition, D is a regular domain, it can be mapped onto some domain with a locally quasiconformal boundary by (some) quasiconformal mapping $h\colon D \to D_0$. Note that the domain D_0 is locally connected on its boundary, which follows directly from the definition of local quasiconformality.

Note that, if P_1 and P_2 are different prime ends in D, then $h(P_1)$ and $h(P_2)$ are different prime ends in D_0. Indeed, let σ_m be a sequence of cuts corresponding to the prime end P_1. The fact that $h(\sigma_m)$ is also a cut of the domain D_0 is obvious, since h is a homeomorphism. Now we verify that the sequence $h(\sigma_m)$, $m = 1, 2, \ldots$, is a chain. The conditions (ii) and (iii) taken from the definition of a chain are obvious, since h is a homeomorphism. We now verify the condition (i): $\overline{h(\sigma_m)} \cap \overline{h(\sigma_{m+1})} = \varnothing$ for $m \in \mathbf{N}$. Suppose the contrary, namely, that $\overline{h(\sigma_m)} \cap \overline{h(\sigma_{m+1})} \neq \varnothing$ at least for one $m \in \mathbf{N}$. Then there is a point $x_0 \in \partial D_0$ such that $x_0 \in \overline{h(\sigma_m)} \cap \overline{h(\sigma_{m+1})}$. Proposition 2.2 implies that $M(\Gamma(h(\sigma_m), h(\sigma_{m+1}), D_0)) = \infty$. On the other hand, in view of the definition of the modulus of families of paths, $M(\Gamma(\sigma_m, \sigma_{m+1}, D)) \leqslant l_0^{-n} \cdot m(D) < \infty$, where $l_0 := \operatorname{dist}(\sigma_m, \sigma_{m+1}) > 0$ and $m(D)$ is a Lebesgue measure of D. Here it was also taken into account that the domain D is bounded, so that $m(D) < \infty$. Then, due to the quasiconformality of the mapping h, we have that $M(\Gamma(h(\sigma_m), h(\sigma_{m+1}), D_0)) \leqslant K \cdot M(\Gamma(\sigma_m, \sigma_{m+1}, D_0)) < \infty$, where $K < \infty$ is some constant. The resulting contradiction refutes the assumption that $\overline{h(\sigma_m)} \cap \overline{h(\sigma_{m+1})} \neq \varnothing$.

Thus, the chain of cuts $h(\sigma_m)$ defines some end $h(P_1)$. The fact that this end is prime also simply follows from the quasiconformality of the mapping h. Similarly, $h(P_2)$ is a prime end in D_0.

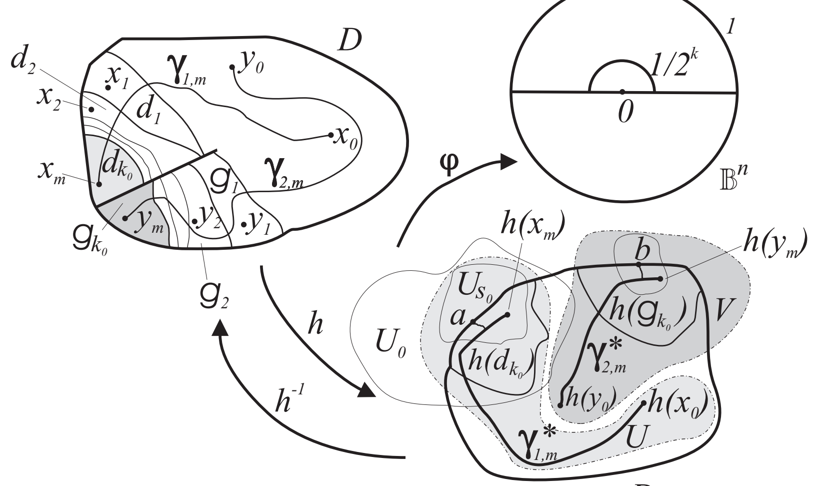

Figure 4. To the proof of Lemma 2.3.

Note that the impressions $I(h(P_1))$ and $I(h(P_2))$ of $h(P_1)$ and $h(P_2)$ are some different points a and b in ∂D_0 (see [IS, Lemma 2.2], cf. [Na, Theorem 4.1]). If P_1 or P_2 are inner points of D, then $h(P_1)$ (or $h(P_2)$) are inner points of D_0, which we denote by a or b, respectively. Since by assumption $x_0, y_0 \in D \setminus (d_1 \cup g_1)$, then, in particular, $P_1 \neq x_0 \neq P_2$, $P_1 \neq y_0 \neq P_2$. This implies that $a, b, h(x_0)$ and $h(y_0)$ are four different points in $\overline{D_0}$, at least two of which are inner points of D_0 (see Figure 4).

By Proposition 2.1, one can join pairs of points $a, h(x_0)$ and $b, h(y_0)$ by disjoint paths $\alpha\colon [0,1]\to\overline{D_0}$ and $\beta\colon [0,1]\to\overline{D_0}$ so that $|\alpha|\cap|\beta|=\varnothing$, $\alpha(t),\beta(t)\in D_0$ for all $t\in(0,1)$, $\alpha(0)=h(x_0)$, $\alpha(1)=a$, $\beta(0)=h(y_0)$ and $\beta(1)=b$. Since $\mathbf{R}^n$ is a normal topological space, the loci $|\alpha|$ and $|\beta|$ have non-intersecting open neighborhoods U, V such that

$$|\alpha|\subset U,\quad |\beta|\subset V. \tag{2.4}$$

Here two cases are possible: either $h(P_1)$ is a prime end in E_{D_0}, or a point in D_0. Let $h(P_1)$ be a prime end in E_{D_0}. Since $I(h(P_1))=a$, then there is a number $k_1\in\mathbf{N}$ such that $\overline{h(d_k)}\subset U$ for $k\geqslant k_1$. If $h(P_1)$ is a point of D, then there is also a number $k_1\in\mathbf{N}$ such that $\overline{h(d_k)}\subset U$ for all $k\geqslant k_1$, where $d_k:=B(P_1,r_k)$, $r_k\to 0$ as $k\to\infty$ and $r_k>0$. In either of these two cases, $\overline{h(d_k)}\subset U$ for $k\geqslant k_1$. Similarly, there is a number $k_2\in\mathbf{N}$ such that $\overline{h(g_k)}\subset V$ for all $k\geqslant k_2$. Then for $k_0:=\max\{k_1,k_2\}$ we obtain that

$$\overline{h(d_k)}\subset U,\quad \overline{h(g_k)}\subset V,\quad U\cap V=\varnothing,\quad k\geqslant k_0. \tag{2.5}$$

Since the sequence x_m converges to P_1 as $m\to\infty$, then $h(x_m)$ converges to a. Therefore, there is a number $m_1\in\mathbf{N}$ such that $h(x_m)\in h(d_{k_0+1})$ for $m\geqslant m_1$. Similarly, since y_m converges to P_2 as $m\to\infty$, then $h(y_m)$ converges to b. Therefore, there is a number $m_2\in\mathbf{N}$ such that $h(y_m)\in h(g_{k_0+1})$ for $m\geqslant m_2$. Put $M_0:=\max\{m_1,m_2\}$. Show that

$$|\alpha|\cap h(d_{k_0+1})\neq\varnothing,\quad |\beta|\cap h(g_{k_0+1})\neq\varnothing. \tag{2.6}$$

It suffices to establish the first of these relations, since the second relation can be proved similarly. If $a=h(P_1)$ is an inner point of D_0, then this inclusion is obvious. Now suppose that $h(P_1)$ is a prime end in E_{D_0}. Since the domain D_0 has a locally quasiconformal boundary, there is a sequence of spheres $S(0,1/2^k)$, $k=0,1,2,\ldots$, a decreasing sequence of neighborhoods U_k of the point a and some quasiconformal mapping $\varphi\colon U_0\to\mathbf{B}^n$, for which $\varphi(U_k)=B(0,1/2^k)$, $\varphi(\partial U_k\cap D_0)=S(0,1/2^k)\cap\mathbf{B}^n_+$, where $\mathbf{B}^n_+=\{x=(x_1,\ldots,x_n)\colon |x|<1, x_n>0\}$ (see the arguments given in the proof of [Na, Lemma 3.5]). Note that $U_k\cap D_0$ is a domain, since $U_k\cap D_0=\varphi^{-1}(B_+(0,1/2^k))$, $B_+(0,1/2^k)=\{x=(x_1,\ldots,x_n)\colon |x|<1/2^k, x_n>0\}$, and φ is a homeomorphism. In addition, the sequence of domains $U_k\cap D_0$ corresponds to some prime end, the impression of which is the point a, and the corresponding cuts are the sets $\sigma_k:=\partial U_k\cap D_0$. By [Na, Theorem 4.1] and Remark 1.1, the point $a\in D_0$ corresponds to exactly one prime end, therefore every domain $h(d_m)$ contains all domains $U_k\cap D_0$, except for a finite number, and vice versa. In particular, there is $s_0\in\mathbf{N}$ such that $U_k\cap D_0\subset h(d_{k_0+1})$ for all $k\geqslant s_0$. Since $a\in|\alpha|$, there is $t_1\in(0,1)$ such that $p:=\alpha(t_1)\in U_{s_0}\cap D_0$. But then also $p\in h(d_{k_0+1})$, since $U_{s_0}\cap D_0\subset h(d_{k_0+1})$. The first relation in (2.6) is proved. As we said above, the second relation may be proved in exactly the same way.

So, let $p:=\alpha(t_1)\in|\alpha|\cap h(d_{k_0+1})$. Fix $m\geqslant M_0$ and join the point p with the point $h(x_m)$ using the path $\alpha_m\colon [t_1,1]\to h(d_{k_0+1})$ so that $\alpha_m(t_1)=p$, $\alpha_m(1)=h(x_m)$, what is possible because $h(d_{k_0+1})$ is a domain. Set

$$\gamma^*_{1,m}(t)=\begin{cases}\alpha(t), & t\in[0,t_1],\\ \alpha_m(t), & t\in[t_1,1].\end{cases} \tag{2.7}$$

Note that the path $\gamma^*_{1,m}$ completely lies in U. Reasoning similarly, we have the point $t_2\in(0,1)$ and the point $q:=\beta(t_2)\in|\beta|\cap h(g_{k_0+1})$. Fix $m\geqslant M_0$ and join the point q with the point $h(y_m)$ using the path $\beta_m\colon [t_2, 1] \to h(g_{k_0+1})$ so that $\beta_m(t_2) = q$, $\beta_m(1) = h(y_m)$, that is possible, because $h(g_{k_0+1})$ is a domain. Set

$$\gamma^*_{2,m}(t) = \begin{cases} \beta(t), & t \in [0, t_2], \\ \beta_m(t), & t \in [t_2, 1]. \end{cases} \tag{2.8}$$

Note that the path $\gamma^*_{2,m}$ completely lies in V. Set

$$\gamma_{1,m} := h^{-1}(\gamma^*_{1,m}), \quad \gamma_{2,m} := h^{-1}(\gamma^*_{2,m}). \tag{2.9}$$

Note that the paths $\gamma_{1,m}$ and $\gamma_{2,m}$ satisfy all the conditions of Lemma 2.3 for $m \geqslant M_0$. In fact, by definition, these paths join the points x_m, x_0 and y_m, y_0, respectively. The paths $\gamma_{1,m}$ and $\gamma_{2,m}$ do not intersect, since their images under the mapping h belong to non-intersecting neighborhoods U and V, respectively.

Note also that $|\gamma_{1,m}| \cap \overline{g_{k_0}} = \varnothing$ for $m \geqslant M_0$. Indeed, if $x \in |\gamma_{1,m}| \cap \overline{g_{k_0}}$, then either $x \in |\gamma_{1,m}| \cap g_{k_0}$ or $x \in |\gamma_{1,m}| \cap \partial g_{k_0}$. In the first case, if $x \in |\gamma_{1,m}| \cap g_{k_0}$ then $h(x) \in |\gamma^*_{1,m}| \cap h(g_{k_0}) \subset U \cap h(g_{k_0})$, which is impossible due to the relation (2.5). In the second case, if $x \in |\gamma_{1,m}| \cap \partial g_{k_0}$, then there is a sequence $z_m \in g_{k_0}$ such that $z_m \to x$ as $m \to \infty$. Now $h(z_m) \to h(x)$ as $m \to \infty$ and, therefore, $h(x) \in \overline{h(g_{k_0})}$. At the same time, $h(x) \in U$, and this is impossible by virtue of relation (2.5). Thus, the relation $|\gamma_{1,m}| \cap \overline{g_{k_0}} = \varnothing$ for $m \geqslant M_0$ is established.

Similarly, $|\gamma_{2,m}| \cap \overline{d_{k_0}} = \varnothing$ for $m \geqslant M_0$. Finally, defining paths $\widetilde{\alpha}$, $\widetilde{\alpha}_m$, $\widetilde{\beta}$ and $\widetilde{\beta}_m$ by means of relations $\widetilde{\alpha} = h^{-1}(\alpha)$, $\widetilde{\alpha}_m = h^{-1}(\alpha_m)$, $\widetilde{\beta} = h^{-1}(\beta)$ and $\widetilde{\beta}_m = h^{-1}(\beta_m)$, we see that these paths correspond to the construction of $\gamma_{1,m}$ and $\gamma_{2,m}$, and also satisfy conditions 3) from the formulation of the lemma. Lemma 2.3 is proved. $\square$

Consider the family of paths joining $|\gamma_{1,m}|$ and $|\gamma_{2,m}|$ from the previous lemma. The following statement contains the upper estimate of the modulus of the transformed family of paths under the mapping f with the inequality (1.3).

Lemma 2.10. *Let $D \subset \mathbf{R}^n$, $n \geqslant 2$, be a regular domain in $\mathbf{R}^n$, and let $f\colon D \to \overline{\mathbf{R}^n}$ be a continuous map satisfying the estimate (1.3) at every point $x_0 \in \overline{D}$ and some $Q \in L^1(D)$. Then, under the conditions and notation of Lemma 2.3, it is possible to choose the sequence of domains d_m and the number k_0 in such a way that there exist a constant $0 < N = N(k_0, Q, D) < \infty$, independent of the parameter m and a mapping f, under which*

$$M(f(\Gamma(|\gamma_{1,m}|, |\gamma_{2,m}|, D))) \leqslant N, \qquad m \geqslant M_0 = M_0(k_0).$$

Proof. By [IS, Lemma 3.1], cf. [KR$_2$, Lemma 1], the prime end P_1 contains a chain of cuts σ_m lying on spheres $S(\overline{x_0}, r_m)$ such that $\overline{x_0} \in \partial D$ and $r_m \to 0$ as $m \to \infty$. Let d_m be a sequence of domains corresponding to cuts σ_m. Consider $M_0 = M_0(k_0)$ and paths $\gamma_{1,m}$ and $\gamma_{2,m}$ corresponding to this number.

Using the notation of Lemma 2.3, we put

$$\varepsilon_0 := \min\{\mathrm{dist}\,(|\widetilde{\alpha}|, \overline{g_{k_0}}), \mathrm{dist}\,(|\widetilde{\alpha}|, |\widetilde{\beta}|)\} > 0.$$

Now, consider covering of $|\widetilde{\alpha}|$ of the following type: $\bigcup_{x \in |\widetilde{\alpha}|} B(x, \varepsilon_0/4)$. Since $|\widetilde{\alpha}|$ is compact in D, there are $i_1, \ldots, i_{N_0}$ such that $|\widetilde{\alpha}| \subset \bigcup_{i=1}^{N_0} B(z_i, \varepsilon_0/4)$, where $z_i \in |\widetilde{\alpha}|$ for $1 \leqslant i \leqslant N_0$. Taking into account [Ku, Theorem 1.I.5.46], it is easy to verify that

$$\Gamma(|\widetilde{\alpha}|, |\gamma_{2,m}|, D) \subset \bigcup_{i=1}^{N_0} \Gamma(S(z_i, \varepsilon_0/4), S(z_i, \varepsilon_0/2), D).$$

Putting

$$\eta(t)=\begin{cases}4/\varepsilon_0, & t\in[\varepsilon_0/4,\varepsilon_0/2],\\ 0, & t\notin[\varepsilon_0/4,\varepsilon_0/2],\end{cases}$$

we observe that the function η satisfies relation (1.4). Then, by the definition of a ring Q-map in (1.3) and taking into account the semi-additivity of the modulus of families of paths, see [Va$_1$, Theorem 6.2], we obtain that

$$\begin{aligned}M(f(\Gamma(|\widetilde{\alpha}|,|\gamma_{2,m}|,D)))&\leqslant\sum_{i=1}^{N_0}M(f(\Gamma(S(z_i,\varepsilon_0/4),S(z_i,\varepsilon_0/2),D)))\\&\leqslant\frac{N_0 4^n\|Q\|_1}{\varepsilon_0^n},\quad m\geqslant M_0,\end{aligned}\tag{2.11}$$

where $\|Q\|_1=\int_D Q(x)\,dm(x)$. Further, by [Ku, Theorem 1.I.5.46], we obtain that

$$\Gamma(|\widetilde{\alpha_m}|,|\gamma_{2,m}|,D)\leqslant\Gamma(S(\overline{x_0},r_{k_1}),S(\overline{x_0},r_{k_2}),D).$$

Arguing as above, choosing an admissible function

$$\eta(t)=\begin{cases}1/(r_{k_0}-r_{k_0+1}), & t\in[r_{k_0+1},r_{k_0}],\\ 0, & t\notin[r_{k_0+1},r_{k_0}],\end{cases}$$

we obtain that

$$\begin{aligned}M(f(\Gamma(|\widetilde{\alpha_m}|,|\gamma_{2,m}|,D)))&\leqslant M(f(\Gamma(S(\overline{x_0},r_{k_0}),S(\overline{x_0},r_{k_0+1}),D)))\\&\leqslant\frac{\|Q\|_1}{(r_{k_0}-r_{k_0+1})^n},\quad m\geqslant M_0.\end{aligned}\tag{2.12}$$

Now note that

$$\Gamma(|\gamma_{1,m}|,|\gamma_{2,m}|,D)\subset\Gamma(|\widetilde{\alpha_m}|,|\gamma_{2,m}|,D)\cup\Gamma(|\widetilde{\alpha}|,|\gamma_{2,m}|,D).$$

In this case, from (2.11) and (2.12), taking into account the semi-additivity of the modulus of families of paths, we obtain that

$$M(f(\Gamma(|\gamma_{1,m}|,|\gamma_{2,m}|,D)))\leqslant\left(\frac{N_0 4^n}{\varepsilon_0^n}+\frac{1}{(r_{k_0}-r_{k_0+1})^n}\right)\|Q\|_1\,,\quad m\geqslant M_0.$$

The right side of the last relation does not depend on m, so that we can put $N:=\left(\frac{N_0 4^n}{\varepsilon_0^n}+\frac{1}{(r_{k_0}-r_{k_0+1})^n}\right)\|Q\|_1$. Lemma 2.10 is completely proved. □

The following statement indicates that for some wide class of mappings fixing the diameter of the image of a certain non-degenerate continuum, the image of this continuum cannot be close to the boundary of the corresponding domain under these mappings. Note that similar statements were previously known for quasiconformal mappings, see, for example, [Va$_1$, Theorems 21.13 and 21.14]. We may also point to our recent result on this, see [SevSkv$_1$, Lemma 2(v)] and [SevSkv$_2$, Lemma 4.1].

Lemma 2.13. *Let $n\geqslant 2$, let D be a regular domain in $\mathbf{R}^n$, and let D' be some domain in $\overline{\mathbf{R}^n}$. Suppose that D' has a weakly flat boundary, $Q\in L^1(D)$ and, moreover, any connected component of $\partial D'$ does not degenerate into a point. Let $f_m\colon D\to D'$ be a sequence of homeomorphisms of D onto D', satisfying the relation (1.3) in D with the same function Q. Suppose also that there is a continuum $A\subset D$ and a number $\delta>0$ such that $q(f_m(A))\geqslant\delta>0$ for all $m=1,2,\ldots$, where, as usual, $q(f_m(A))$ is defined by (1.6). Then there is $\delta_1>0$ such that*

$$q(f_m(A),\partial D')>\delta_1>0\quad\forall\, m\in\mathbf{N},$$

where $q(f_m(A), \partial D') = \inf_{x\in f_m(A), y\in\partial D'} q(x, y)$.

Proof. We carry out the proof by contradiction. Suppose that the conclusion of the lemma is not true. Then for each $k \in \mathbf{N}$ there is some number $m = m_k$ such that $q(f_{m_k}(A), \partial D') < 1/k$. Of course, we may assume that the sequence m_k increases on k. Since $\overline{\mathbf{R}^n}$ is compact, the set $\partial D'$ is also compact in extended Euclidean space. Note that the set $f_{m_k}(A)$ is compact as a continuous image of a compact set $A \subset D$ under the mapping f_{m_k}. In this case, there are elements $x_k \in f_{m_k}(A)$ and $y_k \in \partial D'$ such that $q(f_{m_k}(A), \partial D') = q(x_k, y_k) < 1/k$ (see Figure 5).

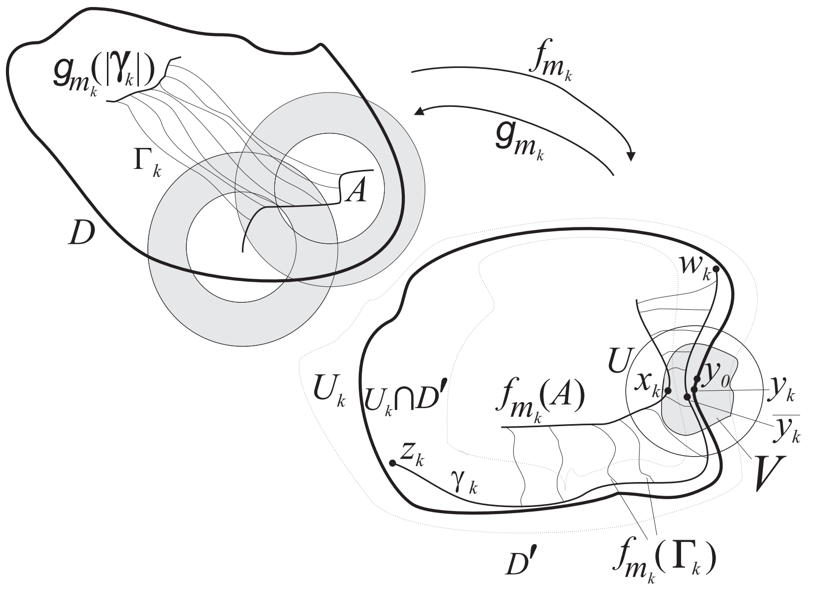

Figure 5. To the proof of Lemma 2.13.

Since $\partial D'$ is a compact set, we may assume that $y_k \to y_0 \in \partial D'$ as $k \to \infty$; then also

$$x_k \to y_0 \in \partial D', \quad k \to \infty.$$

Let K_0 be a connected component of the set $\partial D'$, containing y_0. Obviously, K_0 is a nondegenerate continuum in $\overline{\mathbf{R}^n}$. Since D' has a weakly flat boundary, the mapping $g_{m_k} := f_{m_k}^{-1}$ can be extended to a continuous mapping $\overline{g}_{m_k} \colon \overline{D'} \to \overline{D}_P$ (see Remark 1.8). Moreover, $\overline{g}_{m_k}$ is uniformly continuous on the set $\overline{D'}$ for every fixed k, because the mapping $\overline{g}_{m_k}$ is continuous on the compact set $\overline{D'}$. Let ρ be one of the metrics in $\overline{D}_P$, defined in (1.2), and let $g \colon D_0 \to D$ be a quasiconformal mapping of some domain D_0 with locally quasiconformal boundary corresponding to the definition of the metric ρ in (1.2). In this case, for each $\varepsilon > 0$ there is $\delta_k = \delta_k(\varepsilon) < 1/k$ such that

$$\rho(\overline{g}_{m_k}(x), \overline{g}_{m_k}(x_0)) < \varepsilon \quad \forall\, x, x_0 \in \overline{D'}, \quad q(x, x_0) < \delta_k, \quad \delta_k < 1/k. \tag{2.14}$$

Choose $\varepsilon > 0$ such that

$$\varepsilon < (1/2) \cdot \operatorname{dist}(\partial D_0, g^{-1}(A)), \tag{2.15}$$

where A is a continuum from the conditions of the lemma. Denote $B_q(x_0, r) = \{x \in \overline{\mathbf{R}^n} : q(x, x_0) < r\}$. For a given $k \in \mathbf{N}$, we set

$$B_k := \bigcup_{x_0 \in K_0} B_q(x_0, \delta_k), \quad k \in \mathbf{N}.$$

Since the set B_k is a neighborhood of the continuum K_0, due to [HK, Lemma 2.2] there is a neighborhood U_k of the set K_0 such that $U_k\subset B_k$ and $U_k\cap D'$ is connected. Without loss of generality, we may assume that U_k is an open set, so $U_k\cap D'$ is also path connected (see [MRSY$_3$, Proposition 13.1]). Let $q(K_0)=m_0$, where the chordal diameter $q(K_0)$ of the set K_0 is defined by the relation (1.6). In this case, there are $z_0, w_0\in K_0$ such that $q(K_0)=q(z_0,w_0)=m_0$. So, there are sequences $\overline{y_k}\in U_k\cap D'$, $z_k\in U_k\cap D'$ and $w_k\in U_k\cap D'$ such that $z_k\to z_0$, $\overline{y_k}\to y_0$ and $w_k\to w_0$ as $k\to\infty$. We may assume that

$$q(z_k,w_k)>m_0/2\quad \forall\, k\in \mathbf{N}. \tag{2.16}$$

Since the set $U_k\cap D'$ is path-connected, we can sequentially join the points z_k, $\overline{y_k}$ and w_k using some path γ_k in $U_k\cap D'$. As usual, we denote by $|\gamma_k|$ the locus of the path γ_k in the domain D'. Then $g_{m_k}(|\gamma_k|)$ is a compact set in the domain D. If $x\in|\gamma_k|$, then there is $x_0\in K_0$ such that $x\in B(x_0,\delta_k)$. Put $\omega\in A\subset D$. Since $x\in|\gamma_k|$ and, moreover, x is an inner point of the domain D', we can write here $g_{m_k}(x)$ instead of $\overline{g}_{m_k}(x)$. By the relations (2.14) and (2.15), as well as by the triangle inequality, we obtain that for sufficiently large $k\in\mathbf{N}$,

$$\begin{aligned}\rho(g_{m_k}(x),\omega)&\geqslant \rho(\omega,\overline{g}_{m_k}(x_0))-\rho(\overline{g}_{m_k}(x_0),g_{m_k}(x))\\ &\geqslant \operatorname{dist}(\partial D_0,g^{-1}(A))-(1/2)\cdot\operatorname{dist}(\partial D_0,g^{-1}(A))\\ &=(1/2)\cdot\operatorname{dist}(\partial D_0,g^{-1}(A))>\varepsilon,\end{aligned}\tag{2.17}$$

where $\operatorname{dist}(\partial D_0,g^{-1}(A)):=\inf_{x\in\partial D_0,y\in g^{-1}(A)}|x-y|$. Taking inf in (2.17) over all $x\in|\gamma_k|$ and $\omega\in A$, we obtain that

$$\rho(g_{m_k}(|\gamma_k|),A):=\inf_{x\in g_{m_k}(|\gamma_k|),y\in A}\rho(x,y)>\varepsilon,\quad \forall\, k=1,2,\ldots. \tag{2.18}$$

We now show that there exists $\varepsilon_1>0$ such that

$$\operatorname{dist}(g_{m_k}(|\gamma_k|),A)>\varepsilon_1,\quad \forall\, k=1,2,\ldots, \tag{2.19}$$

where dist, as usual, denotes the Euclidean distance between the sets $A,B\subset\mathbf{R}^n$. Indeed, let (2.19) be violated, then for the number $\varepsilon_l=1/l$, $l=1,2,\ldots$ there are $\xi_l\in|\gamma_{k_l}|$ and $\zeta_l\in A$ such that

$$|g_{m_{k_l}}(\xi_l)-\zeta_l|<1/l,\quad l=1,2,\ldots. \tag{2.20}$$

Without loss of generality, we may assume that the sequence k_l, $l=1,2,\ldots$, is increasing. Since A is compact, we may assume that the sequence ζ_l converges to $\zeta_0\in A$ as $l\to\infty$. By the triangle inequality and from (2.20) it follows that

$$|g_{m_{k_l}}(\xi_l)-\zeta_0|\to 0,\quad l\to\infty. \tag{2.21}$$

On the other hand, we recall that $\rho(g_{m_k}(x),\omega)=|g^{-1}(g_{m_k}(x))-g^{-1}(\omega)|$, where $g\colon D_0\to D$ is some quasiconformal mapping of D_0 onto D, see (1.2). In particular, g^{-1} is continuous in D, therefore, by the triangle inequality and (2.21), we obtain that

$$\begin{aligned}|g^{-1}(g_{m_{k_l}}(\xi_l))-g^{-1}(\zeta_l)|&\leqslant|g^{-1}(g_{m_{k_l}}(\xi_l))-g^{-1}(\zeta_0)|\\ &\quad+|g^{-1}(\zeta_0)-g^{-1}(\zeta_l)|\to 0,\quad l\to\infty.\end{aligned}\tag{2.22}$$

However, by definition ρ and from (2.22) it follows that

$$\rho(g_{m_{k_l}}(|\gamma_{k_l}|),A)\leqslant\rho(g_{m_{k_l}}(\xi_l),\zeta_l)=|g^{-1}(g_{m_{k_l}}(\xi_l))-g^{-1}(\zeta_l)|\to 0,\quad l\to\infty,$$

which contradicts (2.18). The resulting contradiction indicates the validity of (2.19).

We cover the continuum A with the help of balls $B(x, \varepsilon_1/4)$, $x \in A$. Since A is a compact set, we may assume that $A \subset \bigcup_{i=1}^{M_0} B(x_i, \varepsilon_1/4)$, $x_i \in A$, $i = 1, 2, \ldots, M_0$, $1 \leqslant M_0 < \infty$. By definition, M_0 depends only on A, in particular, M_0 does non depend on k. We set

$$\Gamma_k := \Gamma(A, g_{m_k}(|\gamma_k|), D). \tag{2.23}$$

Note that

$$\Gamma_k = \bigcup_{i=1}^{M_0} \Gamma_{ki}, \tag{2.24}$$

where Γ_{ki} consists of all paths $\gamma\colon [0,1] \to D$, belonging to the family Γ_k, such that $\gamma(0) \in B(x_i, \varepsilon_1/4)$ and $\gamma(1) \in g_{m_k}(|\gamma_k|)$. We now show that

$$\Gamma_{ki} > \Gamma(S(x_i, \varepsilon_1/4), S(x_i, \varepsilon_1/2), D). \tag{2.25}$$

Indeed, let $\gamma \in \Gamma_{ki}$, in other words, $\gamma : [0,1] \to D$, $\gamma(0) \in B(x_i, \varepsilon_1/4)$ and $\gamma(1) \in g_{m_k}(|\gamma_k|)$. By (2.19), $|\gamma| \cap B(x_i, \varepsilon_1/4) \neq \varnothing \neq |\gamma| \cap (D \setminus B(x_i, \varepsilon_1/4))$. Therefore, by [Ku, Theorem 1.I.5.46] there is $0 < t_1 < 1$ with the condition $\gamma(t_1) \in S(x_i, \varepsilon_1/4)$. We can assume that $\gamma(t) \notin B(x_i, \varepsilon_1/4)$ for $t > t_1$. Put $\gamma_1 := \gamma|_{[t_1,1]}$. By (2.19), $|\gamma_1| \cap B(x_i, \varepsilon_1/2) \neq \varnothing \neq |\gamma_1| \cap (D \setminus B(x_i, \varepsilon_1/2))$. Thus, by [Ku, Theorem 1.I.5.46] there is $t_1 < t_2 < 1$ with $\gamma(t_2) \in S(x_i, \varepsilon_1/2)$. We may assume that $\gamma(t) \in B(x_i, \varepsilon_1/2)$ for $t < t_2$. Put $\gamma_2 := \gamma|_{[t_1,t_2]}$. Then, the path γ_2 is a subpath of γ, which belongs to the family $\Gamma(S(x_i, \varepsilon_1/4), S(x_i, \varepsilon_1/2), D)$. Thus, the relation (2.25) is established.

Further reasoning is based, as before, on the successful choice of an admissible function η. Put

$$\eta(t) = \begin{cases} 4/\varepsilon_1, & t \in [\varepsilon_1/4, \varepsilon_1/2], \\ 0, & t \notin [\varepsilon_1/4, \varepsilon_1/2]. \end{cases}$$

Note that η satisfies (1.4) for $r_1 = \varepsilon_1/4$ and $r_2 = \varepsilon_1/2$. Then, according to the definition of a ring Q-homeomorphism at x_i, we obtain that

$$M(f_{m_k}(\Gamma(S(x_i, \varepsilon_1/4), S(x_i, \varepsilon_1/2)), D)) \leqslant (4/\varepsilon_1)^n \cdot \|Q\|_1 < c < \infty, \tag{2.26}$$

where c is some positive constant and $\|Q\|_1$ is L_1-norm of the function Q in D. By (2.24), (2.25) and (2.26), using the subadditivity of modulus, we obtain that

$$M(f_{m_k}(\Gamma_k)) \leqslant \frac{4^n M_0}{\varepsilon_1^n} \int_D Q(x)\, dm(x) \leqslant c \cdot M_0 < \infty. \tag{2.27}$$

Let us show that the estimate (2.27) contradicts the condition of the weak flatness of the boundary of the domain D'. Let $U := B_q(y_0, r_0) = \{y \in \overline{\mathbf{R}^n} \colon q(y, y_0) < r_0\}$, where $0 < r_0 < \min\{\delta/4, m_0/4\}$, δ is the number from the condition of the lemma and $q(K_0) = m_0$. (Here, as usual, $q(K_0)$ denotes the chordal diameter of the set $E = K_0$, defined by the formula (1.6)). Note that $|\gamma_k| \cap U \neq \varnothing \neq |\gamma_k| \cap (D' \setminus U)$ for sufficiently large $k \in \mathbf{N}$, since $q(|\gamma_k|) > m_0/2 > m_0/4$ by (2.16), in addition, $\overline{y_k} \in |\gamma_k|$ and $\overline{y_k} \to y_0$ as $k \to \infty$. Similarly, $f_{m_k}(A) \cap U \neq \varnothing \neq f_{m_k}(A) \cap (D' \setminus U)$. Since $|\gamma_k|$ and $f_{m_k}(A)$ are continua, we obtain that

$$f_{m_k}(A) \cap \partial U \neq \varnothing, \quad |\gamma_k| \cap \partial U \neq \varnothing, \tag{2.28}$$

see [Ku, Theorem 1.I.5.46]. For a given $P := c \cdot M_0 > 0$, where c and M_0 is from (2.27), let $V \subset U$ be a neighborhood of the point y_0, corresponding to the definition of a weakly flat boundary. Then we have that

$$M(\Gamma(E, F, D')) > c \cdot M_0 \tag{2.29}$$

for any continua $E, F \subset D'$ with $E \cap \partial U \neq \varnothing \neq E \cap \partial V$ and $F \cap \partial U \neq \varnothing \neq F \cap \partial V$. Observe that

$$f_{m_k}(A) \cap \partial V \neq \varnothing, \quad |\gamma_k| \cap \partial V \neq \varnothing \tag{2.30}$$

for sufficiently large $k \in \mathbf{N}$. Indeed, $\overline{y_k} \in |\gamma_k|$, $x_k \in f_{m_k}(A)$, where $x_k, \overline{y_k} \to y_0 \in V$ as $k \to \infty$. Therefore, $|\gamma_k| \cap V \neq \varnothing \neq f_{m_k}(A) \cap V$ for large $k \in \mathbf{N}$. In addition, we have that $q(V) \leqslant q(U) \leqslant 2r_0 < m_0/2$. By (2.16), $q(|\gamma_k|) > m_0/2$, therefore, $|\gamma_k| \cap (D' \setminus V) \neq \varnothing$. Thus, by [Ku, Theorem 1.I.5.46], $|\gamma_k| \cap \partial V \neq \varnothing$. Similarly, $q(V) \leqslant q(U) \leqslant 2r_0 < \delta/2$. Since $q(f_{m_k}(A)) > \delta$, we obtain that $f_{m_k}(A) \cap (D' \setminus V) \neq \varnothing$. By [Ku, Theorem 1.I.5.46], we have that $f_{m_k}(A) \cap \partial V \neq \varnothing$. Thus, relations (2.30) are established.

By (2.28), (2.29) and (2.30), we obtain that

$$M(\Gamma(f_{m_k}(A), |\gamma_k|, D')) > c \cdot M_0. \tag{2.31}$$

Note that $\Gamma(f_{m_k}(A), |\gamma_k|, D') = f_{m_k}(\Gamma(A, g_{m_k}(|\gamma_k|), D)) = f_{m_k}(\Gamma_k)$. Therefore, the relation (2.31) can be written as

$$M(\Gamma(f_{m_k}(A), g_{m_k}(|\gamma_k|), D)) = M(f_{m_k}(\Gamma_k)) > c \cdot M_0.$$

The relation obtained above contradicts the estimate (2.27). The resulting contradiction means that the above assumption $q(f_{m_k}(A), \partial D') < 1/k$ was incorrect. The proof of the lemma is complete. □

3. Proof of Theorem 1.7

For the continuous extension of the mapping $h \in \mathfrak{S}_{\delta,A,Q}(D, D')$ to the boundary of the domain D', see Remark 1.8. The equicontinuity of $\mathfrak{S}_{\delta,A,Q}(D, D')$ at inner points of D' is the result of [SevSkv$_2$, Theorem 1.1].

We show the equicontinuity $\mathfrak{S}_{\delta,A,Q}(\overline{D}_P, \overline{D'})$ on $\partial D'$. We carry out the proof by contradiction. Suppose there are a point $z_0 \in \partial D'$, a number $\varepsilon_0 > 0$, sequences $z_m \in \overline{D'}$ and $\overline{h}_m \in \mathfrak{S}_{\delta,A,Q}(\overline{D}_P, \overline{D'})$ such that $z_m \to z_0$ as $m \to \infty$ and, in addition,

$$\rho(\overline{h}_m(z_m), \overline{h}_m(z_0)) \geqslant \varepsilon_0, \quad m = 1, 2, \ldots, \tag{3.1}$$

where ρ is one of the metrics in $\overline{D}_P$, defined by the formula (1.2). Since $h_m = \overline{h}_m|_{D'}$ extends by continuity to the boundary of D', we may assume that $z_m \in D'$ and, in addition, there is another sequence $z'_m \in D'$, $z'_m \to z_0$ as $m \to \infty$, such that $\rho(h_m(z'_m), \overline{h}_m(z_0)) \to 0$ as $m \to \infty$. Then from (3.1) it follows that

$$\rho(h_m(z_m), h_m(z'_m)) \geqslant \varepsilon_0/2, \quad m \geqslant m_0. \tag{3.2}$$

Since the domain D is regular, the space $\overline{D}_P$ is compact. Therefore, we may assume that the sequences $h_m(z_m)$ è $\overline{h}_m(z_0)$ converge as $m \to \infty$ to some elements $P_1, P_2 \in \overline{D}_P$, $P_1 \neq P_2$. Let d_m and g_m be sequences of descending domains corresponding to prime ends P_1 and P_2, respectively. By [IS, Lemma 3.1], cf. [KR$_2$, Lemma 1], we may consider that the cuts σ_m corresponding to domains d_m, $m = 1, 2, \ldots$, belong to spheres $S(\overline{x_0}, r_m)$ so that $\overline{x_0} \in \partial D$ and $r_m \to 0$ as $m \to \infty$. Choose $x_0, y_0 \in A$ so that $x_0 \neq y_0$ and $x_0 \neq P_1 \neq y_0$, where the continuum $A \subset D$ is from conditions of Theorem 1.7. Without loss of generality, we may assume that $d_1 \cap g_1 = \varnothing$ and $x_0, y_0 \notin d_1 \cup g_1$.

By Lemmas 2.3 and 2.10, there are disjoint paths $\gamma_{1,m}\colon [0, 1] \to D$ and $\gamma_{2,m}$: $[0, 1] \to D$, the number $M_0 > 0$ and the number $N > 0$ such that $\gamma_{1,m}(0) = x_0$,

$\gamma_{1,m}(1) = h_m(z_m)$, $\gamma_{2,m}(0) = y_0$, $\gamma_{2,m}(0) = h_m(z'_m)$, wherein,

$$M(f_m(\Gamma_m)) \leqslant N, \quad m \geqslant M_0, \tag{3.3}$$

where $f_m := h_m^{-1}$, $\Gamma_m := \Gamma(|\gamma_{1,m}|, |\gamma_{2,m}|, D)$ (see Figure 6).

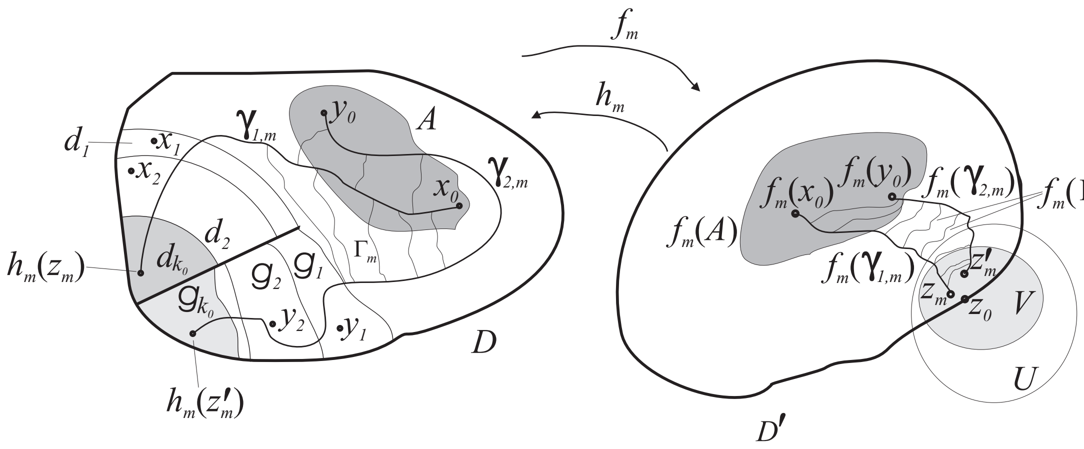

Figure 6. To the proof of Theorem 1.7.

On the other hand, by Lemma 2.13 there is a number $\delta_1 > 0$ such that $q(f_m(A), \partial D') > \delta_1 > 0$, $m = 1, 2, \ldots$. From this we obtain that

$$\begin{aligned} q(f_m(|\gamma_{1,m}|)) &\geqslant q(z_m, f_m(x_0)) \geqslant (1/2)\cdot q(f_m(A), \partial D') > \delta_1/2, \\ q(f_m(|\gamma_{2,m}|)) &\geqslant q(z'_m, f_m(y_0)) \geqslant (1/2)\cdot q(f_m(A), \partial D') > \delta_1/2 \end{aligned} \tag{3.4}$$

for large $m \in \mathbf{N}$. Choose a chordal ball $U := B_q(z_0, r_0)$, where $r_0 > 0$ and $r_0 < \delta_1/4$, and δ_1 is the number from relations (3.4). Note that $f_m(|\gamma_{1,m}|) \cap U \neq \varnothing \neq f_m(|\gamma_{1,m}|) \cap (D' \setminus U)$ for sufficiently large $m \in \mathbf{N}$, because $q(f_m(|\gamma_{1,m}|)) \geqslant \delta_1/2$ and $z_m \in f_m(|\gamma_{1,m}|)$, $z_m \to z_0$ as $m \to \infty$. Due to the same considerations $f_m(|\gamma_{2,m}|) \cap U \neq \varnothing \neq f_m(|\gamma_{2,m}|) \cap (D' \setminus U)$. Since $f_m(|\gamma_{1,m}|)$ and $f_m(|\gamma_{2,m}|)$ are continua, then by [Ku, Theorem 1.I.5.46]

$$f_m(|\gamma_{1,m}|) \cap \partial U \neq \varnothing, \quad f_m(|\gamma_{2,m}|) \cap \partial U \neq \varnothing. \tag{3.5}$$

For a fixed $P := N > 0$, where N is from (3.3), let $V \subset U$ be a neighborhood of the point z_0, corresponding to the definition of a weakly flat boundary, that is, such that for any continua $E, F \subset D'$ with $E \cap \partial U \neq \varnothing \neq E \cap \partial V$ and $F \cap \partial U \neq \varnothing \neq F \cap \partial V$ the inequality

$$M(\Gamma(E, F, D')) > N \tag{3.6}$$

holds. Note that for sufficiently large $m \in \mathbf{N}$

$$f_m(|\gamma_{1,m}|) \cap \partial V \neq \varnothing, \quad f_m(|\gamma_{2,m}|) \cap \partial V \neq \varnothing. \tag{3.7}$$

Indeed, $z_m \in f_m(|\gamma_{1,m}|)$ and $z'_m \in f_m(|\gamma_{2,m}|)$, where $z_m, z'_m \to z_0 \in V$ as $m \to \infty$ Therefore, $f_m(|\gamma_{1,m}|) \cap V \neq \varnothing \neq f_m(|\gamma_{2,m}|) \cap V$ for large $m \in \mathbf{N}$. In addition, $q(V) \leqslant q(U) = 2r_0 < \delta_1/2$ and since by (3.4) $q(f_m(|\gamma_{1,m}|)) > \delta_1/2$, then $f_m(|\gamma_{1,m}|) \cap (D' \setminus V) \neq \varnothing$. Then $f_m(|\gamma_{1,m}|) \cap \partial V \neq \varnothing$ (see [Ku, Theorem 1.I.5.46]). Similarly, $q(V) \leqslant q(U) = 2r_0 < \delta_1/2$ and, since by (3.4) $q(f_m(|\gamma_{2,m}|)) > \delta_1/2$, then $f_m(|\gamma_{2,m}|) \cap (D' \setminus V) \neq \varnothing$. Now, by [Ku, Theorem 1.I.5.46] we obtain that $f_m(|\gamma_{1,m}|) \cap \partial V \neq \varnothing$. Thus, (3.7) is proved.

According to (3.6) and taking into account (3.5) and (3.7), we obtain that

$$M(f_m(\Gamma_m)) = M(\Gamma(f_m(|\gamma_{1,m}|), f_m(|\gamma_{2,m}|), D')) > N,$$

which contradicts the inequality (3.3). The resulting contradiction indicates that the original assumption made in (3.1) is incorrect. The theorem is proved. □

4. Examples

Example 4.1. Let D be the unit square from which the sequence of segments $I_k = \{z = (x, y) \in \mathbf{R}^2 \colon x = 1/k,\ 0 < y < 1/2\}$, $k = 2, 3, \ldots$, is removed (see Figure 7).

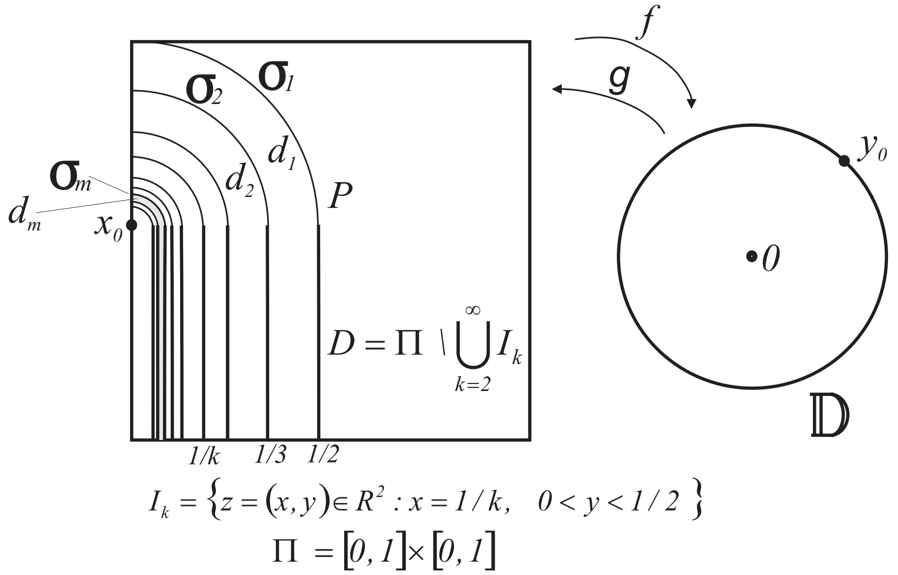

Figure 7. Illustration for Example 4.1.

Consider the prime end P in the domain D, formed by cuts

$$\sigma_m = \left\{ z = x_0 + \frac{e^{i\varphi}}{m+1},\ x_0 = (0, 1/2),\ 0 \leqslant \varphi \leqslant \pi/2 \right\}, \quad m = 1, 2, \ldots, .$$

It can be shown that the end P is really prime. According to the Riemann mapping theorem, there exists a conformal mapping g of the unit disk $\mathbf{D}$ onto the domain D and by the Caratheodory theorem, a prime end P corresponds to some point $y_0 \in \partial\mathbf{D}$ so that $C(f, y_0) = I(P)$, $f = g^{-1}$, see [CL, Theorem 9.4]. It follows that we may choose two sequences $z_k, w_k \in D$, $k = 1, 2, \ldots$, such that $z_k, w_k \to P$, $z_k \to z_0$ and $w_k \to w_0$ as $k \to \infty$, $z_0 \neq w_0$, while $f(z_k) \to y_0$ and $f(w_k) \to y_0$ as $k \to \infty$. Consequently, the mapping $f := g^{-1}$ does not have a continuous extension to the point y_0 in the pointwise sense, although g has a continuous extension $\overline{g} \colon \overline{\mathbf{D}} \to \overline{D}_P$.

Consider another auxiliary family of mappings. As is known, linear fractional automorphisms of the unit disk $\mathbf{D} \subset \mathbf{C}$ have the form

$$f(z) = e^{i\theta} \frac{z - a}{1 - \overline{a}z}, \quad z \in \mathbf{D}, \quad a \in \mathbf{D}, \quad \theta \in [0, 2\pi).$$

We set, for example, $\theta = 0$, $a = 1/n$, $n = 1, 2, \ldots$. In this case, consider the family of mappings $\widetilde{f}_n(z) = \frac{z-1/n}{1-z/n} = \frac{nz-1}{n-z}$. Let $\widetilde{A} = [0, 1/2]$. Then we obtain that $\widetilde{f}_n(0) = -1/n \to 0$ and $\widetilde{f}_n(1/2) = \frac{n-2}{2n-1} \to 1/2$ as $n \to \infty$. Thus, the mappings $\widetilde{f}_n$ satisfy the condition $q(\widetilde{f}_n(\widetilde{A})) \geqslant \delta$ say, with $\delta = 1/4$.

Now we put $f_n := \widetilde{f}_n \circ f$. Note that the mappings f_n are conformal; therefore, they satisfy the estimate (1.3) for $Q \equiv 1$ at each point $x_0 \in \overline{D}$, see [Pol, Theorem 1], cf. [MRSY$_1$, Theorems 4.6 and 6.10] and [MRSY$_3$, Theorems 8.1 and 8.6]. Note also that the mappings f_n satisfy all the conditions of Theorem 1.7, in particular, $q(f_n(A)) \geqslant \delta$ with $\delta = 1/4$, where $A := f^{-1}(\widetilde{A})$. Note that the unit disk $\mathbf{D}$ has a weakly flat boundary by [Va$_1$, Theorems 17.10 and 17.12]. By construction, mappings $g_n := f_n^{-1}$ do not even have a pointwise continuous extension to $\partial\mathbf{D}$ in particular, the family of these mappings is not equicontinuous in $\overline{D}$. Nevertheless, the extended family $\overline{g}_n\colon \overline{\mathbf{D}} \to \overline{D}_P$ is equicontinuous in terms of prime ends by Theorem 1.7.

Put now $\theta = 0$, $a = (n-1)/n$, $n = 1, 2, \dots$ and $\widetilde{f}_n^{-1}(z) = \frac{z-(n-1)/n}{1-z(n-1)/n} = \frac{nz-n+1}{n-nz+1}$. We set $f_n := \widetilde{f}_n \circ f$. It is easy to understand that the sequence $\widetilde{f}_n^{-1}$ is locally uniformly converges to a constant function -1 in the unit disk. On the other hand, we have the equality $\widetilde{f}_n^{-1}(1) = 1$, which immediately implies that the sequence $\widetilde{f}_n^{-1}$ is not equicontinuous at the point 1.

It follows that the sequence f_n^{-1} is also not equicontinuous at point 1. The reason for this is a violation of the requirement $q(f_n(A)) \geqslant \delta$.

Example 4.2. It is also easy to indicate a similar example of a family of mappings with unbounded characteristic. Let D be the domain constructed in Example 4.1. Then we put $f_1(z) = \frac{1}{e\sqrt{2}}|z - 1/2|$. Note that f_1 maps D onto a domain D_1 lying in the ball $B(0, 1/e)$. Now we put $f_2(z) := \frac{z}{|z|\log\frac{1}{|z|}}$ and $F(z) = (f_2 \circ f_1)(z)$. Using the technique outlined in the consideration of [MRSY$_3$, Proposition 6.3], we may establish that F is a ring Q-homeomorphism in $\overline{D}$ with $Q(z) = \log\frac{e\sqrt{2}}{|z-1/2|}$. One can also prove that $Q \in L^1(D)$. Note that D_1 is a simply connected domain, therefore, by the Riemann theorem, it is possible to map it onto the unit disk using some conformal mapping f_3.

Consider the family of mappings $\widetilde{f}_n(z) = \frac{z-1/n}{1-z/n} = \frac{nz-1}{n-z}$. We set $F_n(z) = (\widetilde{f}_n \circ f_3 \circ f_2 \circ f_1)(z)$. Let $\widetilde{A} = [0, 1/2]$. Then we obtain that $\widetilde{f}_n(0) = -1/n \to 0$ and $\widetilde{f}_n(1/2) = \frac{n-2}{2n-1} \to 1/2$ as $n \to \infty$. Thus, the mappings $\widetilde{f}_n$ satisfy the condition $q(\widetilde{f}_n(\widetilde{A})) \geqslant \delta$ say, with $\delta = 1/4$. In this case, the mappings F_n satisfy the condition $q(F_n(A)) \geqslant \delta$ say, with $\delta = 1/4$ and $A = (f_1^{-1} \circ f_2^{-1} \circ f_3^{-1})(\widetilde{A})$.

Since the modulus of families of paths does not change under conformal transformations, the mappings F_n are ring Q-maps in D, where $Q(z) = \log\frac{e\sqrt{2}}{|z-1/2|}$ (see [Va$_1$, Theorem 8.1]). The mappings $G_n = F_n^{-1}$ do not have a pointwise continuous extension to $\partial\mathbf{D}$, however, this extension is valid in the sense of prime ends. In addition, the family of extended mappings $\overline{G}_n\colon \overline{\mathbf{D}} \to \overline{D}_P$, $n = 1, 2, \dots$, is equicontinuous in $\overline{\mathbf{D}}$ by Theorem 1.7.

Example 4.3. As it was stated at the beginning of the article, let us give an example of a family of homeomorphisms, in which the transition to the inverse family takes us beyond the limits of the class of mappings under consideration. At the same time, it is also an example of a family that is equicontinuous in the closure of an domain, however, the inverse of it is not such. We will identify an isolated point of the boundary of a domain with a prime end; in this case, we will consider some descending sequence of spheres with a center at this point as cuts. Note that the example with minor modifications was taken from our earlier paper [SSI].

Let $n \geqslant 2$ and let $p \geqslant 1$ be a number, such that $n/p(n-1) < 1$. Put $\alpha \in (0, n/p(n-1))$. Let $\mathbf{B}^n = \{x \in \mathbf{R}^n\colon |x| < 1\}$. We define a sequence of mappings f_m

of $\mathbf{B}^n$ onto $B(0,2)$ in the following way:

$$f_m(x) = \begin{cases} \frac{1+|x|^\alpha}{|x|} \cdot x, & 1/m \leqslant |x| \leqslant 1, \\ \frac{1+(1/m)^\alpha}{(1/m)} \cdot x, & 0 < |x| < 1/m, \end{cases}$$

see Figure 8.

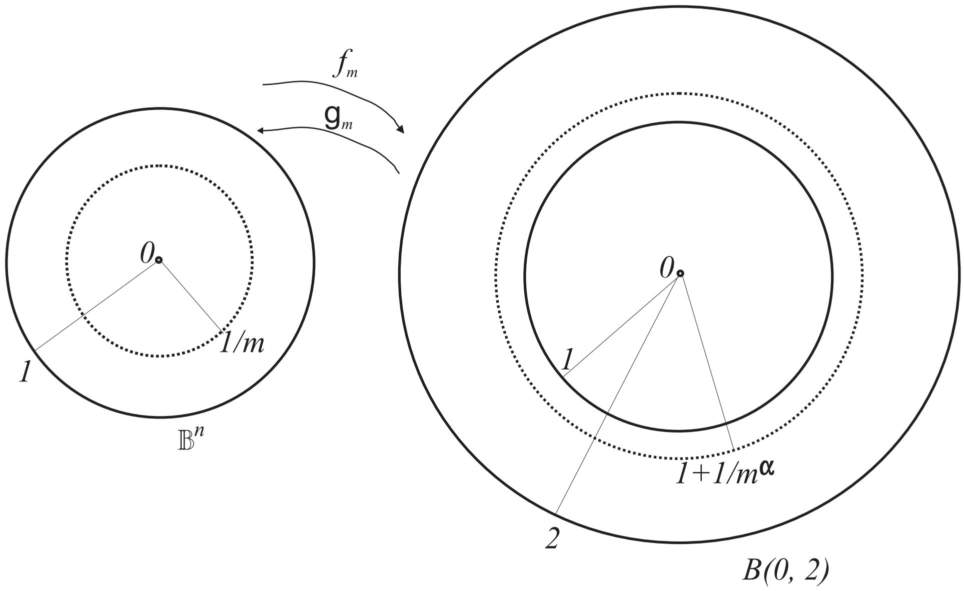

Figure 8. Illustration for Example 4.3.

Notice, that f_m satisfies (1.3) in $\overline{\mathbf{B}^n}$ for $Q = \left(\frac{1+|x|^\alpha}{\alpha|x|^\alpha}\right)^{n-1} \in L^p(\mathbf{B}^n)$ at every $x_0 \in \overline{\mathbf{B}^n}$. As above, this can be shown by the reasoning used in the consideration of [MRSY$_3$, Proposition 6.3]. By [Va$_1$, Theorems 17.10 and 17.12] $B(0,2)$ has a weakly flat boundary, cf. [Vu, Lemma 4.3]. Observe that f_m fixes an infinite number of points of the unit ball for all $m \geqslant 2$, so that the condition $q(f_m(A)) \geqslant \delta$ holds for some continuum $A \subset D$ and some $\delta > 0$. By direct computation, it can be established that the family $\overline{\mathfrak{G}} = \{\overline{g}_m\}_{m=1}^{\infty}$, $g_m := f_m^{-1}$, consisting of all extended mappings $\overline{g}_m \colon \overline{B(0,2)} \to \overline{\mathbf{B^n}}$, is equicontinuous in $\overline{B(0,2)}$.

Observe that the "inverse" family $F = \{\overline{f}_m\}_{m=1}^{\infty}$, $\overline{f}_m \colon \overline{\mathbf{B}^n} \to \overline{B(0,2)}$, is not equicontinuous in $\overline{\mathbf{B}^n}$. In fact, $|f_m(x_m) - f(0)| = 1 + 1/m^\alpha \not\to 0$ as $m \to \infty$, where $|x_m| = 1/m$. The reason for the last circumstance is that g_m are not ring Q-homeomorphism with some integrable Q in $B(0,2)$. Indeed, if the latter were true, then the family of mappings $F = \{f_m\}_{m=1}^{\infty}$ would be equicontinuous in $\overline{\mathbf{B}^n}$ by see [SSI, Theorem 1].

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Received 25 January 2020 • Accepted 22 May 2020