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ANALYSIS OF SCHRÖDINGER MEANS

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Abstract. We study integral estimates of maximal functions for Schrödinger means.

1. Introduction

We shall study Schrödinger means $S_t f(x)$ defined by

$$S_t f(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{i\xi\cdot x} e^{it|\xi|^a} \hat{f}(\xi)\, d\xi, \quad x \in \mathbf{R}^n,\ t \geq 0.$$

Here $f \in L^2(\mathbf{R}^n)$, $n \geq 1$, $a > 0$, and $\hat{f}$ denotes the Fourier transform f, defined by

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} e^{-i\xi\cdot x} f(x)\, dx, \quad \xi \in \mathbf{R}^n.$$

We shall use Sobolev spaces $H_s = H_s(\mathbf{R}^n)$, $s \in \mathbf{R}$, with the norm given by

$$\|f\|_{H_s} = \left(\int_{\mathbf{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2\, d\xi \right)^{1/2}.$$

Let E denote a bounded set in $\mathbf{R}$. For $r > 0$ we let $N_E(r)$ denote the minimal number N of intervals I_l, $l = 1, 2, \dots N$, of length r such that $E \subset \bigcup_1^N I_l$. For f belonging to the Schwartz class $\mathscr{S}$ we introduce the maximal function

$$S_E^* f(x) = \sup_{t \in E} |S_t f(x)|, \quad x \in \mathbf{R}^n.$$

In Sjölin and Strömberg [4] we proved the following theorem.

Theorem A. *Assume that $n \geq 1$ and $a > 0$ and $s > 0$. If $f \in \mathscr{S}$, then one has*

$$\int_{\mathbf{R}^n} |S_E^* f(x)|^2\, dx \lesssim \left(\sum_{m=0}^{\infty} N_E(2^{-m}) 2^{-2ms/a} \right) \|f\|_{H_s}^2.$$

Here we write $A \lesssim B$ if there is a constant C such that $A \leq CB$. In the case $E = [0, 1]$ it is easy to see that Theorem A implies the estimate

$$\|S_E^* f\|_2 \lesssim \|f\|_{H_s}$$

if $s > a/2$.

Let $(t_k)_1^\infty$ be a sequence satisfying

$$(1) \qquad 1 > t_1 > t_2 > t_3 > \cdots > 0 \quad \text{and} \quad \lim_{k\to\infty} t_k = 0.$$

Set $A_j = \{t_k;\ 2^{-j-1} < t_k \leq 2^{-j}\}$ for $j = 1, 2, 3, \dots$. Let $\#A$ denote the number of elements in a set A. In [4] we used Theorem A to obtain the following results.

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Theorem B. *Assume that $n \geq 1$ and $a \geq 2s$ and $s > 0$, and $b < 2s/(a-2s)$. Assume also that*

$$\# A_j \lesssim 2^{bj} \quad \text{for } j = 1, 2, 3, \dots,$$

and that $f \in H_s$. Then

$$\lim_{k\to\infty} S_{t_k} f(x) = f(x) \tag{2}$$

almost everywhere.

Theorem C. *Assume that $n \geq 1$ and $a \geq 2s$ and $s > 0$, and that $\sum_1^\infty t_k^\gamma < \infty$, where $\gamma < 2s/(a-2s)$. If also $f \in H_s$, then (2) holds almost everywhere.*

Now let E denote a bounded set in $\mathbf{R}^{n+1}$ and let E_0 denote the projection of E onto the t-axis, i.e. $E_0 = \{t;\text{ there extists } x \in \mathbf{R}^n \text{ such that } (x,t) \in E\}$. For $a > 0$ define an a-cube in $\mathbf{R}^{n+1}$ with side $r > 0$ as an axis-parallell rectangular box with sidelength r^a in the t-direction and with sidelength r in the remaining directions $x_1, \dots, x_n$. Thus an a-cube in $\mathbf{R}^{n+1}$ has volume r^{n+a}. For $r > 0$ let $N_{E,a}(r)$ denote the minimal number N of a-cubes $Q_l, l = 1, 2, \dots, N$ of side r, such that $E \subset \bigcup_1^N Q_l$.

For $f \in \mathscr{S}$ we introduce the maximal function

$$S_E^* f(x) = \sup_{(y,t)\in E} |S_t f(x+y)|, \quad x \in \mathbf{R}^n.$$

We shall prove the following inequality.

Theorem 1. *Assume that $n \geq 1$ and $a > 0$ and $s > 0$. If $f \in \mathscr{S}$, then one has*

$$\int_{\mathbf{R}^n} |S_E^* f(x)|^2 \, dx \lesssim \left(\sum_{m=0}^{\infty} N_{E,a}(2^{-m}) 2^{-2ms} \right) \|f\|_{H_s}^2.$$

The following estimate follows directly.

Corollary 1. *Assume $n \geq 1$, $a > 0$, $s > 0$ and $f \in \mathscr{S}$. If*

$$\sum_{m=0}^{\infty} N_{E,a}(2^{-m}) 2^{-2ms} < \infty,$$

then

$$\|S_E^* f\|_2 \lesssim \|f\|_{H_s}. \tag{3}$$

We shall now describe a counter-example of of Sjögren and Sjölin [3]. Assume $a = 2$ and let $\gamma\colon \mathbf{R}^+ \to \mathbf{R}^+$ be strictly increasing. Then there exist a function $f \in H_{n/2}$ and a continuous function $u(x,t)$ in $\{(x,t);\, t > 0\}$ such that $u(x,t) = S_t f(x)$ and

$$\limsup_{\substack{(y,t)\to(x,0)\\ |y-x|<\gamma(t),\, t>0}} |u(y,t)| = \infty$$

for all $x \in \mathbf{R}^n$. This is generalized to a more general Schrödinger equation which contains the fractional Schrödinger equation ($a > 1$) by Johansson [2]. It follows that if $E = \{(y,t);\, 0 < t < 1,\ |y| < \gamma(t)\}$, then (3) does not hold for $s < n/2$.

Now let $\Gamma\colon [0,1] \to (\mathbf{R}^n)$ and let E be a subset of the graph of Γ that is

$$E \subset \{(\Gamma(t), t);\, 0 \leq t \leq 1\}.$$

Assume that

$$|\Gamma(t_1) - \Gamma(t_2)| \lesssim |t_1 - t_2|^\beta \quad \text{for } t_1, t_2 \in E_0, \tag{4}$$

where $0 < \beta$. Also set $a_1 = 1/\beta$ and $a_2 = \max(a, a_1)$. We shall first study the case when E is the graph of Γ. Then (4) holds for any $\beta > 1$ only if Γ is constant. It follows from a result of Cho, Lee, and Vargas [1] that if $n = 1$, $a = 2$, B is an interval of $\mathbf{R}$ and E is the graph of Γ, then

$$\|S_E^* f\|_{L^2(B)} \le C_B \|f\|_{H_s} \quad \text{if } s > \max(1/2 - \beta, 1/4), \text{ and } 0 < \beta \le 1.$$

We have the following result.

Theorem 2. *Assume $n \ge 1$, $a > 0$, $f \in \mathscr{S}$ and that E is the graph of Γ. Then* (3) *holds for $2s > a_2$.*

We shall then study the case when $E = \{(\Gamma(t_k), t_k);\ k = 1, 2, 3 \dots\}$, where the sequence $(t_k)_1^\infty$ satisfies (1). We have the following results.

Theorem 3. *Assume $n \ge 1$, $a > 0$, and $f \in \mathscr{S}$.*

(i) *In the case $2s > a_2$, then* (3) *holds.*
(ii) *In the case $2s = a_2$ assume that $\sum_1^\infty t_k^\gamma < \infty$ for some $\gamma > 0$. Then* (3) *holds.*
(iii) *In the case $2s < a_2$ assume that $\sum_1^\infty t_k^\gamma < \infty$ for some γ satisfying $\gamma < 2s/(a_2 - 2s)$. Then* (3) *holds.*

2. Proof of Theorem 1

If y and y_0 belong to to $\mathbf{R}^n$ we write $y = (y_1, \dots, y_n)$ and $y_0 = (y_{0,1}, \dots, y_{0,n})$. We shall give the proof of Theorem 1.

Proof. Assume $y_0 \in \mathbf{R}^n$, $t_0 \in \mathbf{R}$, $0 < r \le 1$ and let

$$E = \{(y, t) \in \mathbf{R}^{n+1};\ y_{0,j} \le y_j \le y_{0,j} + r \ \text{ for } 1 \le j \le n, \text{ and } t_0 \le t \le t_0 + r^a\}.$$

We have

$$S_t f(x + y) = c \int \mathrm{e}^{i\xi\cdot x} \mathrm{e}^{i\xi_1 y_1} \dots \mathrm{e}^{i\xi_n y_n} \mathrm{e}^{it|\xi|^a} \hat{f}(\xi)\, d\xi,$$

where $c = (2\pi)^{-n}$, and for $1 \le j \le n$ we write $\mathrm{e}^{i\xi_j y_j} = \Delta_j + \mathrm{e}^{i\xi_j y_{0,j}}$, where

$$\Delta_j = \mathrm{e}^{i\xi_j y_j} - \mathrm{e}^{i\xi_j y_{0,j}}.$$

We also write $\mathrm{e}^{it|\xi|^a} = \Delta_{n+1} + \mathrm{e}^{it_0|\xi|^a}$, where

$$\Delta_{n+1} = \mathrm{e}^{it|\xi|^a} - \mathrm{e}^{it_0|\xi|^a}.$$

Hence

$$S_t f(x + y) = c \int \mathrm{e}^{i\xi\cdot x} \left(\Delta_1 + \mathrm{e}^{i\xi_1 y_{0,1}}\right) \dots \left(\Delta_n + \mathrm{e}^{i\xi_n y_{0,n}}\right) \left(\Delta_{n+1} + \mathrm{e}^{it_0|\xi|^a}\right) \hat{f}(\xi)\, d\xi,$$

and it follows that $S_t f(x + y)$ is the sum of integrals of the form

$$c \int \mathrm{e}^{i\xi\cdot x} \left(\prod_{j\in D} \Delta_j\right) \left(\prod_{j\in B} \mathrm{e}^{i\xi_j y_{0,j}}\right) \Delta_{n+1} \hat{f}(\xi)\, d\xi, \tag{5}$$

or

$$c \int \mathrm{e}^{i\xi\cdot x} \left(\prod_{j\in D} \Delta_j\right) \left(\prod_{j\in B} \mathrm{e}^{i\xi_j y_{0,j}}\right) \mathrm{e}^{it_0|\xi|^a} \hat{f}(\xi)\, d\xi. \tag{6}$$

Here D and B ar disjoint subsets of $\{1, 2, 3, \dots, n\}$ and $D \cup B = \{1, 2, 3, \dots, n\}$. We denote the integrals in (5) by $S'_t f(x, y)$ and shall describe how they can be estimated. The same argument works also for integrals in (6).

For $j \in D$ we write

$$\Delta_j = i\xi_j \int_{y_{0,j}}^{y_j} e^{i\xi_j s_j}\, ds_j,$$

and we also write

$$\Delta_{n+1} = i|\xi|^a \int_{t_0}^{t} e^{i|\xi|^a s_{n+1}}\, ds_{n+1}.$$

Assuming $D=\{k_1,k_2,\dots,k_p\}$ we then have

$$S_t' f(x,y) = \int_{\mathbf{R}^n}\int_{y_{0,k_1}}^{y_{k_1}}\int_{y_{0,k_2}}^{y_{k_2}}\dots\int_{y_{0,k_p}}^{y_{k_p}}\int_{t_0}^{t} e^{i\xi\cdot x}\Bigl(\prod_{j\in D} i\xi_j e^{i\xi_j s_j}\Bigr)\Bigl(\prod_{j\in B} e^{i\xi_j y_{0,j}}\Bigr) \cdot i|\xi|^a e^{i|\xi|^a s_{n+1}}\hat f(\xi)\, ds_{k_1}\, ds_{k_2}\dots ds_{k_p}\, ds_{n+1}\, d\xi.$$

Changing the order of integration one then obtains

$$|S_t' f(x,y)| \le c\int_{y_{0,k_1}}^{y_{k_1}}\int_{y_{0,k_2}}^{y_{k_2}}\dots\int_{y_{0,k_p}}^{y_{k_p}}\int_{t_0}^{t}\left|\int_{\mathbf{R}^n} e^{i\xi\cdot x}\Bigl(\prod_{j\in D} i\xi_j e^{i\xi_j s_j}\Bigr)\Bigl(\prod_{j\in B} e^{i\xi_j y_{0,j}}\Bigr) \cdot i|\xi|^a e^{i|\xi|^a s_{n+1}}\hat f(\xi)\, d\xi\right| ds_{k_1}\, ds_{k_2}\dots ds_{k_p}\, ds_{n+1},$$

or

$$S_t' f(x,y) = \int_{y_{0,k_1}}^{y_{k_1}}\dots\int_{y_{0,k_p}}^{y_{k_p}}\int_{t_0}^{t} F_D(x; s_{k_1},\dots,s_{k_p},s_{n+1})\, ds_{k_1}\, ds_{k_2}\dots ds_{k_p}\, ds_{n+1},$$

where

$$F_D(x; s_{k_1},s_{k_2},\dots,s_{k_p},s_{n+1}) = c\int_{\mathbf{R}^n} e^{i\xi\cdot x}\Bigl(\prod_{j\in D} i\xi_j e^{i\xi_j s_j}\Bigr)\Bigl(\prod_{j\in B} e^{i\xi_j y_{0,j}}\Bigr) i|\xi|^a e^{i|\xi|^a s_{n+1}}\hat f(\xi)\, d\xi.$$

Then

$$\sup_{(y,t)\in E}|S_t' f(x,y)| \le \int_{y_{0,k_1}}^{y_{0,k_1}+r}\dots\int_{y_{0,k_p}}^{y_{0,k_p}+r}\int_{t_0}^{t_0+r^a} |F_D(x; s_{k_1},\dots,s_{k_p},s_{n+1})|\, ds_{k_1}\dots ds_{k_p}\, ds_{n+1}.$$

Invoking Minkowski's inequality and Plancherel's formula we then obtain

$$\begin{aligned}\Bigl(\int_{\mathbf{R}^n}\sup_{(y,t)\in E}|S_t' f(x,y)|^2\, dx\Bigr)^{1/2} &\le \int_{y_{0,k_1}}^{y_{0,k_1}+r}\dots\int_{y_{0,k_p}}^{y_{0,k_p}+r}\int_{t_0}^{t_0+r^a}\|F_D(\cdot; s_{k_1},\dots,s_{k_p},s_{n+1})\|_2\, ds_{k_1}\dots ds_{k_p}\, ds_{n+1}\\ &= c(2\pi)^{n/2}\int_{y_{0,k_1}}^{y_{0,k_1}+r}\dots\int_{y_{0,k_p}}^{y_{0,k_p}+r}\int_{t_0}^{t_0+r^a}\Bigl(\int_{\mathbf{R}^n}(\prod_{j\in D}|\xi_j|^2)|\xi|^{2a}|\hat f(\xi)|^2\, d\xi\Bigr)^{1/2} ds_{k_1}\dots ds_{k_p}\, ds_{n+1} \le r^p r^a A^p A^a\|f\|_2\end{aligned}$$

if $f\in L^2(\mathbf{R}^n))$ and supp $\hat f\subset B(0,A)$, where $A\ge 1$.

With similar arguments we get the estimate $r^pA^p\|f\|_2$ for the integrals in (6), and by summation of the all integrals of forms (5) or (6) we get

$$\|\sup_{(y,t)\in E}|S_t f(x+y)|\|_2 \le (1+rA)^n(1+r^aA^a)\|f\|_2. \tag{7}$$

Now let E be a set in $\mathbf{R}^{n+1}$ with the property that $E \subset \bigcup_1^N Q_l$ where the sets Q_l are a-cubes with side r with $rA \leq 1$ of the type we have just considered. One has

$$\sup_{(y,t)\in E} |S_t f(x+y)|^2 \leq \sum_1^N \sup_{(y,t)\in Q_l} |S_t f(x+y)|^2$$

and it follows from (7) that

$$\int_{\mathbf{R}^n} \sup_{(y,t)\in E} |S_t f(x+y)|^2 \, dx \leq 2^{2n+2} N \|f\|_2^2 \tag{8}$$

if $rA \leq 1$ and supp $\hat{f} \subset B(0, A)$.

Now let $f \in \mathscr{S}$. We write $f = \sum_{k=0}^{\infty} f_k$, where the functions f_k are defined in the following way. We set $\hat{f}_0(\xi) = \hat{f}(\xi)$ for $|\xi| \leq 1$ and $\hat{f}_0(\xi) = 0$ for $|\xi| > 1$. For $k \geq 1$ we let $\hat{f}_k(\xi) = \hat{f}(\xi)$ for $2^{k-1} < |\xi| \leq 2^k$ and $\hat{f}_k(\xi) = 0$ otherwise.

Choosing real numbers $g_k > 0, k = 0, 1, 2, \ldots,$ we have

$$\begin{aligned}
\sup_{(y,t)\in E} |S_t f(x+y)| &\leq \sum_{k=0}^{\infty} \sup_{(y,t)\in E} |S_t f_k(x+y)| \\
&= \sum_{k=0}^{\infty} g_k^{-1/2} \sup_{(y,t)\in E} |S_t f_k(x+y)| \, g_k^{1/2} \\
&\leq \left(\sum_{k=0}^{\infty} g_k^{-1} \sup_{(y,t)\in E} |S_t f_k(x+y)|^2 \right)^{1/2} \left(\sum_{k=0}^{\infty} g_k \right)^{1/2}
\end{aligned}$$

and invoking inquality (8) we also have

$$\begin{aligned}
&\int \sup_{(y,t)\in E} |S_t f(x+y)|^2 \, dx \\
&\leq \left(\sum_{k=0}^{\infty} g_k^{-1} \int \sup_{(y,t)\in E} |S_t f_k(x+y))|^2 \, dx \right) \left(\sum_{k=0}^{\infty} g_k \right) \\
&\leq \left(\sum_{k=0}^{\infty} g_k \right) \left(\sum_{k=0}^{\infty} g_k^{-1} \, 2^{2n+2} N_{E,a}(2^{-k}) \|f_k\|_2^2 \right).
\end{aligned}$$

Chosing $g_k = N_{E,a}(2^{-k}) 2^{-2ks}$ we conclude that

$$\begin{aligned}
\int \sup_{(y,t)\in E} |S_t f(x+y)|^2 \, dx &\leq 2^{2n+2} \left(\sum_{k=0}^{\infty} N_{E,a}(2^{-k}) 2^{-2ks} \right) \left(\sum_{k=0}^{\infty} 2^{2ks} \|f_k\|_2^2 \right) \\
&\lesssim \left(\sum_{k=0}^{\infty} N_{E,a}(2^{-k}) 2^{-2ks} \right) \|f\|_{H_s}^2,
\end{aligned}$$

and the proof of Theorem 1 is complete. □

3. Proof of Theorems 2 and 3

The following two lemmas follow easily from the definition of $N_{E,a}(r)$.

Lemma 1. *Assume that* $0 < r \leq 1$ *and* $0 < b < b_1$. *Then*

$$N_{E,b}(r) \leq N_{E,b_1}(r),$$

and

$$N_{E,b_1}(r) \lesssim r^{b-b_1} N_{E,b}(r).$$

Lemma 2. *Assume that E is a subset of the graph of Γ satisfying* (4) *with $\beta > 0$ and let $b \geq 1/\beta$. Then*

$$N_{E,b}(r) \lesssim N_{E_0}(r^b) \quad \textit{for } 0 < r \leq 1.$$

(E_0, Γ, β are defined and equation (4) *is in Section 1).*

We shall then give the proofs of Theorems 2 and 3.

Proof of Theorem 2. We have $a \leq a_2$. Invoking Lemma 1 and Lemma 2 we obtain

$$N_{E,a}(2^{-m}) \leq N_{E,a_2}(2^{-m}) \lesssim N_{E_0}(2^{-ma_2}) \leq 2^{ma_2} + 1$$

and

$$\sum_{m=0}^{\infty} N_{E,a}(2^{-m}) 2^{-2ms} < \infty$$

if $a_2 - 2s < 0$, i.e. $s > a_2/2$. Using Corollary 1 we conclude that

$$\|S_E^* f\|_2 \lesssim \|f\|_{H_s}$$

if $s > a_2/2$. □

We shall then prove Theorem 3.

Proof of Theorem 3. In the case $2s > a_2$ we can use the same argument as in the proof of Theorem 2 to prove that (3) holds. Then assume $2s \leq a_2$. We have $E_0 = \{t_k;\, k = 1, 2, 3, \dots\}$ and assuming $\sum_1^\infty t_k^\gamma < \infty$ one obtains $\#A_j \lesssim 2^{\gamma j}$. It then follows from Lemma 6 in [4] that $N_{E_0}(2^{-m}) \lesssim 2^{\gamma m/(\gamma+1)}$.

We have $a \leq a_2$. Applying Lemma 1 and Lemma 2 one obtains

$$N_{E,a}(2^{-m}) \leq N_{E,a_2}(2^{-m}) \lesssim N_{E_0}(2^{-ma_2}) \lesssim 2^{a_2 \gamma m/(\gamma+1)}.$$

It follows that

$$\sum_{m=0}^{\infty} N_{E,a}(2^{-m}) 2^{-2ms} < \infty$$

if $\gamma a_2/(\gamma+1) < 2s$, that is $a_2\gamma < 2s\gamma + 2s$. In the case $2s = a_2$ this holds for every $\gamma > 0$. In the case $2s < a_2$ one has to assume $\gamma < 2s/(a_2 - 2s)$. This completes the proof of Theorem 3. □

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