Annales Fennici Mathematici
Volumen 46, 2021, 395–420

EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF SINGULAR AND QUASILINEAR ELLIPTIC PROBLEMS WITH CRITICAL EXPONENTIAL GROWTH

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Abstract. In this paper we use Galerkin method to investigate the existence of positive solution for a class of singular and quasilinear elliptic problems given by

$$\begin{cases} -\mathrm{div}(a_0(|\nabla u|^{p_0})|\nabla u|^{p_0-2}\nabla u) = \dfrac{\lambda_0}{u^{\beta_0}} + f_0(u),\ u>0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases}$$

and its version for systems given by

$$\begin{cases} -\mathrm{div}(a_1(|\nabla u|^{p_1})\ |\nabla u|^{p_1-2}\ \nabla u) = \dfrac{\lambda_1}{u^{\beta_1}} + f_1(v) & \text{in } \Omega,\\ -\mathrm{div}(a_2(|\nabla v|^{p_2})\ |\nabla v|^{p_2-2}\ \nabla v) = \dfrac{\lambda_2}{v^{\beta_2}} + f_2(u) & \text{in } \Omega,\\ u,v>0 & \text{in } \Omega,\\ u=v=0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^N$ is bounded smooth domain with $N \geq 3$ and for $i=0,1,2$ we have $2 \leq p_i < N$, $0 < \beta_i \leq 1$, $\lambda_i > 0$ and f_i are continuous functions. The hypotheses on the C^1-functions $a_i\colon \mathbf{R}^+ \to \mathbf{R}^+$ allow to consider a large class of quasilinear operators.

1. Introduction

In a celebrated paper in 1976 [39], Stuart considered the problem

$$L(u) = f(x,u) \ \text{ in } \Omega, \quad u = \phi(x) \ \text{ on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbf{R}^N$, $N \geq 2$, L be a second order linear elliptic operator and $f(x,p) \to \infty$ as $p \to 0$. Problems of this type are called singular and arise in the theory of heat conduction in electrically conducting materials. Moreover, they have wide application to physical models such as non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogenous, see [7] and [8]. Using the maximum principle Stuart establishes the existence of non-negative solutions of this problem and constructs iteration schemes that converge to a solution.

In 1997, Crandall, Rabinowitz, Tartar [14] go back to study this class of problems, where L is assumed to be a linear second order elliptic operator that satisfies a maximum principle. In the first part the existence of a classical solution continuous

https://doi.org/10.5186/aasfm.2021.4626
2020 Mathematics Subject Classification: Primary 35J20, 35J50; Secondary 58E05.
Key words: Galerkin method, Exponential growth, Trudinger–Moser inequality, Hardy–Sobolev inequality.

up to the boundary is proved by means of sub-supersolutions method. The second part of the paper is devoted to a detailed study of the continuity properties of a solution for special nonlinearities independent of x.

More recently, the version with $L = -\Delta$ and $f(x,u) = \frac{1}{u^\alpha} + \lambda|\nabla u|^p + \sigma$, where $\alpha > 0$, $\sigma \geq 0$, $0 < p \leq 2$, $\phi = 0$ was studied in [19] and [43]. The gradient in this equation is called convection term. The version without the convection term was studied in [37]. Another important results can be found in [4, 5, 9, 10, 13, 15, 17, 22, 28, 32]. The system versions were studied in [1, 12, 24, 30, 42].

In [20] Giacomoni, Prashanth and Sreenadh studied a problem with N-Laplacian such that the nonlinearity grows like $\exp(|t|^{N/N-1})$ at infinity and like $\frac{1}{t^\alpha}$ at the origin. A similar problem with the Laplacian operator in $\mathbf{R}^2$ was studied by Saoudi and Kratou in [35]. In [16] Dhanya, Prashanth, Sreenadh and Tiwari considered the singular case with critical exponential growth and discontinous nonlinearity. The inhomogeneous singular Neumann case was studied in [36]. The multiplicity results was considered in [33]. The version in $\mathbf{R}^N$ with N-Laplacian and critical exponential growth was studied in [2].

We are going back to our problem in order to enunciate the hypotheses on the functions a_i and f_i. More precisely, the hypotheses on functions $a_i\colon \mathbf{R}^+ \longrightarrow \mathbf{R}^+$ of C^1 class are the following:

(a_1) There exist constants $k_1, k_3, k_4 \geq 0$ and $k_2 > 0$, such that

$$k_1 t^{p_i} + k_2 t^N \leq a_i(t^{p_i}) t^{p_i} \leq k_3 t^{p_i} + k_4 t^N, \quad \text{for all } t \geq 0.$$

(a_2) The function

$$t \longmapsto a_i(t^{p_i}) t^{p_i - 2} \text{ is increasing.}$$

The functions $f_i\colon \mathbf{R} \longrightarrow \mathbf{R}$ are continuous satisfying the following properties:

(f_1) There exists $\alpha_0 > 0$ such that the exponential growth conditions at infinity are given by:

$$\lim_{t\to\infty} \frac{f_i(t)}{\exp\left(\alpha|t|^{\frac{N}{N-1}}\right)} = 0 \text{ for } \alpha > \alpha_0 \text{ and } \lim_{t\to\infty} \frac{f_i(t)}{\exp\left(\alpha|t|^{\frac{N}{N-1}}\right)} = \infty \text{ for } 0 < \alpha < \alpha_0.$$

(f_2) The growth condition at the origin:

$$\lim_{t\to 0^+} \frac{f_i(t)}{t^{p_i-1}} = 0.$$

(f_3) There exists $\gamma_i > N$ such that

$$f_i(t) \geq t^{\gamma_i - 1}, \quad \text{for all } t \geq 0.$$

Since we are looking for positive solution, in this paper we consider $f_i(t) = 0$ for all $t < 0$.

The purpose in the first part of this article is to prove the existence of solution for the following class of singular problems

$$(1.1) \qquad \begin{cases} -\mathrm{div}(a_0(|\nabla u|^{p_0})\, |\nabla u|^{p_0-2}\, \nabla u) = \dfrac{\lambda_0}{u^{\beta_0}} + f_0(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded smooth domain, $N \geq 3$, $2 \leq p_0 < N$ and $0 < \beta_0 \leq 1$, λ_0 are real parameters.

The main result in the first part is:

Theorem 1.1. *Assume that conditions (a_1)–(a_2) and (f_1)–(f_3) hold. Then, there exists $\lambda^* > 0$ such that the problem (1.1) has a positive weak solution for every $\lambda_0 \in (0, \lambda^*)$.*

In the second part of this article we study the following system

$$
\begin{cases}
-\operatorname{div}(a_1(|\nabla u|^{p_1})\ |\nabla u|^{p_1-2}\ \nabla u) = \dfrac{\lambda_1}{u^{\beta_1}} + f_1(v) & \text{in } \Omega,\\
-\operatorname{div}(a_2(|\nabla v|^{p_2})\ |\nabla v|^{p_2-2}\ \nabla v) = \dfrac{\lambda_2}{v^{\beta_2}} + f_2(u) & \text{in } \Omega,\\
u, v > 0 & \text{in } \Omega,\\
u = v = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{1.2}
$$

where $\Omega \subset \mathbf{R}^N$ is bounded smooth domain with $N \geq 3$ and for $i = 1, 2$ we have $2 \leq p_i < N$, $0 < \beta_i \leq 1$, $\lambda_i > 0$, $a_i\colon \mathbf{R}^+ \to \mathbf{R}^+$ are functions of C^1 class and $f_i\colon \mathbf{R} \to \mathbf{R}$ are continuous functions with exponential growth.

The main result of this second part is the following:

Theorem 1.2. *Assume that, for $i = 1, 2$, a_i satisfy (a_1)–(a_2) and f_i satisfy (f_1)–(f_3). Then, there exists $\lambda^* > 0$ such that the problem (1.2) has a positive weak solution for every $0 < \lambda_1 + \lambda_2 < \lambda^*$.*

We will give some examples of functions a_i in order to illustrate the degree of generality of the kind of problems studied here.

Example 1.1. Considering $a_i(t) = t^{\frac{N-p_i}{p_i}}$, we have that the function a_i satisfies the hypotheses (a_1)–(a_2) with $k_1 = k_3 = 0$ and $k_2 = k_4 = 1$. Hence, Theorems 1.1 and 1.2 are valid for the operator $-\Delta_N u$.

Example 1.2. Considering $a_i(t) = 1 + t^{\frac{N-p_i}{p_i}}$, we have that the function a_i satisfies the hypotheses (a_1)–(a_2) with $k_1 = k_2 = k_3 = k_4 = 1$. Hence, Theorems 1.1 and 1.2 are valid for the operator $-\Delta_{p_i} u - \Delta_N u$.

Problems with this operator come from a general reaction-diffusion system:

$$
u_t = \operatorname{div}[D(u)\nabla u] + c(x, u),
\tag{1.3}
$$

where $D(u) = (|\nabla u|^{p-2} + |\nabla u|^{N-2})$. This system has a wide range of applications in physics and related sciences, such as biophysics, plasma physics and chemical reaction design. In such applications, the function u describes a concentration, the first term on the right-hand side of (1.3) corresponds to the diffusion with a diffusion coefficient $D(u)$; whereas the second one is the reaction and relates to source and loss processes. Typically, in chemical and biological applications, the reaction term $c(x, u)$ is a polynomial of u with variable coefficients (see [11, 23, 29, 41]).

Beneath we present some other examples that are also interesting from mathematical point of view.

Example 1.3. Considering $a(t) = 1 + \frac{1}{(1+t)^{\frac{p_i-2}{p_i}}}$, we have that the function a satisfies the hypotheses $(a_1) - (a_2)$ with $k_1 = 1$, $k_3 = 2$, $k_4 = 0$ and $k_2 > 0$. Hence, Theorems 1.1 and 1.2 are valid for the operator

$$
-\operatorname{div}\left(|\nabla u|^{p_i-2}\nabla u + \frac{|\nabla u|^{p_i-2}\nabla u}{(1 + |\nabla u|^{p_i})^{\frac{p_i-2}{p_i}}}\right).
$$

Example 1.4. Considering $a_i(t) = 1 + t^{\frac{N-p}{p}} + \frac{1}{(1+t)^{\frac{p_i-2}{p_i}}}$, it follows that the function a satisfies the hypotheses $(a_1) - (a_2)$ with $k_1 = k_2 = k_4 = 1$ and $k_3 = 2$. Hence, Theorems 1.1 and 1.2 are valid for the operator

$$-\Delta_{p_i} u - \Delta_N u - \operatorname{div}\left(\frac{|\nabla u|^{p_i-2}\nabla u}{(1+|\nabla u|^{p_i})^{\frac{p_i-2}{p_i}}}\right).$$

Other combinations can be made with the functions presented in the examples above, generating very interesting elliptic problems from the mathematical point of view.

Below we list what we believe that are the main contributions of our paper:

1) In [13] and [12] were studied a singular problem and a singular system, respectively, with this general operator. But the nonlinearities have polynomial growth.
2) In [16], [20], [35] and [36] were studied the singular case with a nonlinearities with exponential growth. However, here we study problems with a general operator which brings some technical difficulties.
3) Here we use Galerkin method that was not used in the papers above cited.

The plan of the paper is the following: In Section 2 we recall some preliminary results for the scalar case. In Section 3 we study an auxiliary problem for the scalar case. We show existence of solution of the auxiliary problem in Section 4. In Section 5 we prove Theorem 1.1. In Section 6 we study an auxiliary problem for the system case. We show existence of solution of the auxiliary problem in Section 7. In Section 8 we prove Theorem 1.2

2. Preliminary results for the scalar case

Let us consider the Sobolev space $W_0^{1,N}(\Omega)$ endowed with the norm

$$\|u\|_{1,N} = \left(\int_\Omega |\nabla u|^N \, dx\right)^{\frac{1}{N}}.$$

We say that $u \in W_0^{1,N}(\Omega)$ is a weak solution of the problem (1.1) if $u > 0$ in Ω and it verifies

$$\int_\Omega a_0(|\nabla u|^{p_0})\ |\nabla u|^{p_0-2}\ \nabla u\ \nabla\phi \, dx - \lambda_0 \int_\Omega \frac{\phi}{u^{\beta_0}} \, dx - \int_\Omega f_0(u)\phi \, dx = 0,$$

for all $\phi \in W_0^{1,N}(\Omega)$. In this paper, we work with operator $T_i \colon W_0^{1,N}(\Omega) \to (W_0^{1,N}(\Omega))'$ such that

$$\langle T_i u_i, \phi_i\rangle = \int_\Omega a_i(|\nabla u_i|^{p_i})\ |\nabla u_i|^{p_i-2}\nabla u_i \nabla \phi_i \ dx.$$

A straightforward calculation shows that T_i is continuous. Furthermore, T_i is monotone and coercive, see [12, Lemma 1].

Firstly, we recall some important results due to Trudinger–Moser [31, 40] and Hardy–Sobolev [25]. A version of Trudinger–Moser inequality for systems can be found in [3].

Theorem 2.1. (Trudinger–Moser inequality) *For every $u \in W_0^{1,N}(\Omega)$ and $\alpha > 0$, then*

$$\exp\left(\alpha u^{\frac{N}{N-1}}\right) \in L^1(\Omega)$$

and there exists a constant $M > 0$ such that

$$\sup_{\|u\|_{1,N} \leq 1} \int_{\Omega} \exp\left(\alpha u^{\frac{N}{N-1}}\right) dx \leq M,$$

for every $\alpha \leq \alpha_N := N w_{N-1}^{\frac{1}{N-1}}$, where w_{N-1} is the $(N-1)$-dimensional measure of $(N-1)$ sphere.

Theorem 2.2. (Hardy–Sobolev inequality) *If $u \in C^1(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$ with $1 < p \leq N$, then $\dfrac{u}{Cd^{\tau}} \in L^r(\Omega)$, for $\frac{1}{r} = \frac{1}{p} - \frac{1-\tau}{N}$, $0 < \tau \leq 1$ and*

$$\left|\frac{u}{Cd^{\tau}}\right|_{L^r(\Omega)} \leq |\nabla u|_{L^p(\Omega)},$$

where $d(x) = \operatorname{dist}(x, \partial\Omega)$ and C is a positive constant which does not depend on x.

Our approach in the study of problem (1.1) and system (1.2) rests heavily on the following Weak Comparison Principle proved in [12, Lemma 2].

Lemma 2.1. *If Ω is a bounded domain and if $u_i, v_i \in W_0^{1,N}(\Omega)$ satisfy*

$$\begin{cases} -\operatorname{div}(a_i(|\nabla u_i|^{p_i})|\nabla u_i|^{p_i-2}\nabla u_i) \leq -\operatorname{div}(a_i(|\nabla v_i|^{p_i})|\nabla v_i|^{p_i-2}\nabla v_i) \text{ in } \Omega, \\ u_i \leq v_i \text{ on } \partial\Omega, \end{cases}$$

then $u_i \leq v_i$ a.e. in Ω.

We observe that, from (f_1)–(f_2), for all $\delta > 0$ and for all $\alpha > \alpha_0$, there exists $C_\delta > 0$ such that

$$|f_i(t)t| \leq \delta |t|^{p_i} + C_\delta |t|^{q_i} \exp\left(\alpha |t|^{\frac{N}{N-1}}\right), \tag{2.1}$$

for all $q_i \geq 0$. In this paper, we will use $q_i > N$.

3. Auxiliary problem for the scalar case

For each $\varepsilon > 0$, we consider the following auxiliary problem

$$\begin{cases} -\operatorname{div}(a_0(|\nabla u|^{p_0})|\nabla u|^{p_0-2}\nabla u) = \dfrac{\lambda_0}{(u+\varepsilon)^{\beta_0}} + f_0(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where the functions a_0 and f_0 satisfy the hypotheses of the Theorem 1.1.

To prove Theorem 1.1, we first show the existence of a solution for the problem (3.1). For this, we will use the Galerkin method together with the following fixed point theorem, see [38] and [27, Theorem 5.2.5].

Lemma 3.1. *Let $G\colon \mathbf{R}^d \longrightarrow \mathbf{R}^d$ be a continuous function such that $\langle G(\xi), \xi\rangle \geq 0$ for every $\xi \in \mathbf{R}^d$ with $|\xi| = r$ for some $r > 0$. Then, there exists $z_0 \in \overline{B}_r(0)$ such that $G(z_0) = 0$.*

The main result in this section is the following:

Lemma 3.2. *For each $0 < \varepsilon < 1$, there exists $\lambda^* > 0$ such that the problem (3.1) has a positive weak solution for every $\lambda_0 \in (0, \lambda^*)$.*

Proof. Let $B = \{e_1, e_2, \ldots, e_m, \ldots\}$ be a Schauder basis of $W_0^{1,N}(\Omega)$. For each $m \in \mathbf{N}$, define

$$W_m = [e_1, e_2, \ldots, e_m]$$

to be the finite-dimensional space generated by $\{e_1, e_2, \dots, e_m\}$. Note that the spaces $(W_m, \|\cdot\|_m)$ and $(\mathbf{R}^m, |\cdot|_s)$ are isometrically isomorphic by natural mapping

$$S\colon W_m \to \mathbf{R}^m$$

given by

$$u = \sum_{j=1}^{m} \xi_j e_j \mapsto S(u) = \xi = (\xi_1, \xi_2, \dots, \xi_m),$$

where

$$|\xi|_s = \sum_{j=1}^{m} |\xi_j| \quad \text{and} \quad \|u\|_m = \left(\int_\Omega |\nabla u|^m \, dx \right)^{\frac{1}{m}}.$$

Moreover,

$$\|u\|_m = |\xi|_s = |S(u)|_s. \tag{3.2}$$

For each $m \in \mathbf{N}$, define the function $G\colon \mathbf{R}^m \to \mathbf{R}^m$ such that

$$G(\xi) = G(\xi_1, \xi_2, \dots, \xi_m) = (G_1(\xi), G_2(\xi), \dots, G_m(\xi)),$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_m) \in \mathbf{R}^m$,

$$G_j(\xi) = \int_\Omega a_0(|\nabla u|^{p_0}) |\nabla u^{p_0 - 2} \nabla u \nabla e_j \, dx - \lambda_0 \int_\Omega \frac{e_j}{(u+\varepsilon)^{\beta_0}} \, dx - \int_\Omega f_0(u) e_j \, dx,$$

$j = 1, 2, \dots, m$ and $u = \sum_{j=1}^{m} \xi_j e_j \in W_m$. Therefore,

$$\langle G(\xi), \xi \rangle = \sum_{j=1}^{m} G_j(\xi) \xi_j = \int_\Omega a_0(|\nabla u|^{p_0}) |\nabla u|^{p_0} \, dx - \lambda_0 \int_\Omega \frac{u}{(u+\varepsilon)^{\beta_0}} \, dx - \int_\Omega f_0(u) u \, dx.$$

Note that

$$\int_\Omega \frac{u}{(u+\varepsilon)^{\beta_0}} \, dx \le |\Omega|. \tag{3.3}$$

Using (2.1) and Sobolev embedding, there exists positive constant C_1 such that

$$\int_\Omega f_0(u) u \, dx \le \delta C_1 \|u\|_{1,p_0}^{p_0} + C_\delta \int_\Omega |u|^{q_0} \exp\left(\alpha |u|^{\frac{N}{N-1}} \right) dx. \tag{3.4}$$

Now, from (a_1) we have

$$\begin{aligned} \int_\Omega a_0(|\nabla u|^{p_0}) |\nabla u|^{p_0} \, dx &\ge k_1 \int_\Omega |\nabla u|^{p_0} \, dx + k_2 \int_\Omega |\nabla u|^N \, dx \\ &= k_1 \|u\|_{1,p_0}^{p_0} + k_2 \|u\|_{1,N}^N. \end{aligned} \tag{3.5}$$

It follows from (3.3), (3.4) and (3.5) that

$$\langle G(\xi), \xi \rangle \ge k_2 \|u\|_{1,N}^N + (k_1 - \delta C_1) \|u\|_{1,p_0}^{p_0} - \lambda_0 |\Omega| - C_\delta \int_\Omega |u|^{q_0} \exp\left(\alpha |u|^{\frac{N}{N-1}} \right) dx. \tag{3.6}$$

Taking $\delta > 0$ sufficiently small such that $(k_1 - \delta C_1) > 0$, we can rewrite (3.6) as

$$\langle G(\xi), \xi \rangle \ge k_2 \|u\|_{1,N}^N - \lambda_0 |\Omega| - C_\delta \int_\Omega |u|^{q_0} \exp\left(\alpha |u|^{\frac{N}{N-1}} \right) dx. \tag{3.7}$$

Using Hölder's inequality with $s, s' > 1$ such that $\dfrac{1}{s} + \dfrac{1}{s'} = 1$, we get

$$C_\delta \int_\Omega |u|^{q_0} \exp\left(\alpha |u|^{\frac{N}{N-1}}\right) dx \leq C_\delta \left(\int_\Omega |u|^{q_0 s'} dx\right)^{\frac{1}{s'}} \left(\int_\Omega \exp\left(\alpha s |u|^{\frac{N}{N-1}}\right) dx\right)^{\frac{1}{s}}.$$

Since $q_0 > N$ and $s' > 1$, by Sobolev embedding there exists $\widetilde{C_1} > 0$ such that

$$C_\delta \int_\Omega |u|^{q_0} \exp\left(\alpha |u|^{\frac{N}{N-1}}\right) dx \leq C_\delta \widetilde{C_1} \|u\|_{1,N}^{q_0} \left(\int_\Omega \exp\left(\alpha s |u|^{\frac{N}{N-1}}\right) dx\right)^{\frac{1}{s}}. \tag{3.8}$$

Then, it follows (3.7) and (3.8) that

$$\langle G(\xi), \xi \rangle \geq k_2 \|u\|_{1,N}^N - \lambda_0 |\Omega| - C_\delta \widetilde{C_1} \|u\|_{1,N}^{q_0} \left(\int_\Omega \exp\left(\alpha s |u|^{\frac{N}{N-1}}\right) dx\right)^{\frac{1}{s}}.$$

Assume now that $\|u\|_{1,N} = r$ for some $r > 0$ to be chosen later. We have

$$\begin{aligned} \int_\Omega \exp\left(\alpha s |u|^{\frac{N}{N-1}}\right) dx &= \int_\Omega \exp\left(\alpha s \|u\|_{1,N}^{\frac{N}{N-1}} \left(\frac{|u|}{\|u\|_{1,N}}\right)^{\frac{N}{N-1}}\right) dx \\ &= \int_\Omega \exp\left(\alpha s r^{\frac{N}{N-1}} \left(\frac{|u|}{\|u\|_{1,N}}\right)^{\frac{N}{N-1}}\right) dx \end{aligned}$$

and in order to apply the Theorem 2.1, we impose that

$$r \leq \left(\frac{\alpha_N}{\alpha s}\right)^{\frac{N-1}{N}}.$$

Therefore, there exists $M > 0$ such that

$$\sup_{\|u\|_{1,N} \leq 1} \int_\Omega \exp\left(\alpha s r^{\frac{N}{N-1}} \left(\frac{|u|}{\|u\|_{1,N}}\right)\right) dx \leq M$$

and hence,

$$\langle G(\xi), \xi \rangle \geq k_2 r^N - \lambda_0 |\Omega| - C_\delta \widetilde{C_1} M^{1/s} r^q.$$

Now, it is necessary to choose r such that

$$k_2 r^N - C_\delta \widetilde{C_1} M^{1/s} r^q \geq \frac{k_2 r^N}{2},$$

in others words,

$$r \leq \left(\frac{k_2}{2 C_\delta \widetilde{C_1} M^{\frac{1}{s}}}\right)^{\frac{1}{q-N}}.$$

Thus, considering $r = \min\left\{\left(\dfrac{\alpha_N}{\alpha s}\right)^{\frac{N-1}{N}}, \left(\dfrac{k_2}{2 C_\delta \widetilde{C_1} M^{\frac{1}{s}}}\right)^{\frac{1}{q-N}}\right\}$ we get

$$\langle G(\xi), \xi \rangle \geq \frac{k_2 r^N}{2} - \lambda_0 |\Omega|.$$

Furthermore, choosing

$$\lambda^* = \frac{k_2 r^N}{4|\Omega|}$$

we obtain

$$\langle G(\xi), \xi \rangle > 0, \text{ for all } 0 < \lambda_0 < \lambda^*, \ \ \xi \in \mathbf{R}^m \text{ and } |\xi|_s = r.$$

By virtue of Lemma 3.1, for every $m \in \mathbf{N}$, there exists $y \in \mathbf{R}^m$ with $|y|_s \leq r < 1$ such that $G(y) = 0$. Thus, from (3.2) there exists $u_m \in W_m$ satisfying

$$\|u_m\|_{1,N} \leq r < 1, \text{ for every m } \in \mathbf{N} \tag{3.9}$$

such that

$$\int_\Omega a_0(|\nabla u_m|^{p_0})|\nabla u_m|^{p_0-2}\nabla u_m \nabla e_j\, dx = \lambda_0 \int_\Omega \frac{e_j}{(u_m+\varepsilon)^{\beta_0}}\, dx + \int_\Omega f_0(u_m)e_j\, dx, \tag{3.10}$$

$j = 1, 2, \ldots, m$. Multiplying the equality (3.10) by any constant σ_j, for each $j = 1, 2, \ldots, m$, and adding them, we conclude

$$\int_\Omega a_0(|\nabla u_m|^{p_0})|\nabla u_m|^{p_0-2}\nabla u_m \nabla \phi\, dx = \lambda_0 \int_\Omega \frac{\phi}{(u_m+\varepsilon)^{\beta_0}}\, dx + \int_\Omega f_0(u_m)\phi\, dx, \tag{3.11}$$

for all $\phi \in W_m$, which shows that u_m is an approximate weak solution of problem (3.1).

Since r does not depend on m and $W_m \subset W_0^{1,N}(\Omega)$, for all $m \in \mathbf{N}$, then (u_m) is a bounded sequence in $W_0^{1,N}(\Omega)$. Thus, for some subsequence, there exists $u \in W_0^{1,N}(\Omega)$ such that

$$\begin{cases} u_m \rightharpoonup u & \text{in } W_0^{1,N}(\Omega), \\ u_m \to u & \text{in } L^\theta(\Omega),\ \theta \geq 1, \\ u_m(x) \to u(x) & \text{a.e. in } \Omega, \\ |u_m(x)| \leq g(x) \in L^\theta(\Omega) & \text{a.e. in } \Omega,\ \theta \geq 1. \end{cases} \tag{3.12}$$

Fix $k \in \mathbf{N}$ and consider $m \geq k$, then $W_k \subset W_m$ and

$$\int_\Omega a_0(|\nabla u_m|^{p_0})|\nabla u_m|^{p_0-2}\nabla u_m \nabla \phi_k\, dx = \lambda_0 \int_\Omega \frac{\phi_k}{(u_m+\varepsilon)^{\beta_0}}\, dx + \int_\Omega f_0(u_m)\phi_k\, dx, \tag{3.13}$$

for all $\phi_k \in W_k$. Since $\phi_k \in W_k$, note that

$$\left| \frac{\phi_k}{(u_m+\varepsilon)^{\beta_0}} \right| \leq \frac{|\phi_k|}{\varepsilon^{\beta_0}} \in L^1(\Omega)$$

and by (3.12) we have

$$\frac{\phi_k}{(u_m(x)+\varepsilon)^{\beta_0}} \to \frac{\phi_k}{(u(x)+\varepsilon)^{\beta_0}} \quad \text{a.e. in } \Omega.$$

Therefore, we use [6, Theorem 4.2] to obtain that

$$\int_\Omega \frac{\phi_k}{(u_m+\varepsilon)^{\beta_0}}\, dx \to \int_\Omega \frac{\phi_k}{(u+\varepsilon)^{\beta_0}}\, dx. \tag{3.14}$$

Now, since f_0 is a continuous function, by (3.12) again we have

$$f_0(u_m(x))\phi_k \to f_0(u(x))\phi_k \quad \text{a.e. in } \Omega. \tag{3.15}$$

Using (2.1) we get

$$|f_0(u_m(x))\phi_k| \leq \delta |u_m(x)|^{p_0-1}|\phi_k| + C_\delta |u_m(x)|^{q_0-1} \exp\left(\alpha |u_m(x)|^{\frac{N}{N-1}}\right)|\phi_k|.$$

We will need to prove that the function $\widehat{g}\colon \mathbf{R} \to \mathbf{R}$ defined by

$$\widehat{g}(u_m(x)) := \delta |u_m(x)|^{p_0-1}|\phi_k| + C_\delta |u_m(x)|^{q_0-1} \exp\left(\alpha |u_m(x)|^{\frac{N}{N-1}}\right)|\phi_k|$$

satisfies

$$|f_0(u_m(x))\phi_k| \leq \widehat{g}(u_m(x)) \in L^1(\Omega). \tag{3.16}$$

It is sufficient to show that $\widehat{g}(u_m(x))$ is convergent in $L^1(\Omega)$. Indeed, since $2 \leq p_0 < N$, we invoke (3.12) to obtain

$$|u_m(x)|^{p_0-1}|\phi_k| \to |u(x)|^{p_0-1}|\phi_k| \quad \text{a.e. in } \Omega \tag{3.17}$$

and

$$|u_m(x)|^{p_0-1}|\phi_k| \leq g(x)^{p_0-1}|\phi_k| \in L^1(\Omega). \tag{3.18}$$

It follows from (3.17), (3.18) and [6, Theorem 4.2] that

$$\int_\Omega |u_m|^{p_0-1}|\phi_k|\, dx \to \int_\Omega |u|^{p_0-1}|\phi_k|\, dx. \tag{3.19}$$

Furthermore, from (3.12) again we get

$$|u_m(x)|^{q_0-1}\exp\left(\alpha|u_m(x)|^{\frac{N}{N-1}}\right) \to |u(x)|^{q_0-1}\exp\left(\alpha|u(x)|^{\frac{N}{N-1}}\right) \quad \text{a.e. in } \Omega. \tag{3.20}$$

Now, considering $s, s' > 1$ such that $\dfrac{1}{s} + \dfrac{1}{s'} = 1$, we use (3.12) and the fact that $q_0 > N$ to obtain

$$|u_m|^{q_0-1} \to |u|^{q_0-1} \quad \text{in } L^{s'}(\Omega). \tag{3.21}$$

Moreover, by (3.9) we have

$$\begin{aligned}\int_\Omega \exp\left(\alpha s|u_m(x)|^{\frac{N}{N-1}}\right) dx &= \int_\Omega \exp\left(\alpha s\|u_m(x)\|_{1,N}^{\frac{N}{N-1}}\left(\frac{|u_m(x)|}{\|u_m(x)\|_{1,N}}\right)^{\frac{N}{N-1}}\right) dx \\ &\leq \int_\Omega \exp\left(\alpha s r^{\frac{N}{N-1}}\left(\frac{|u_m(x)|}{\|u_m(x)\|_{1,N}}\right)^{\frac{N}{N-1}}\right) dx\end{aligned}$$

and applying the Theorem 2.1 we obtain

$$\int_\Omega \exp\left(\alpha s|u_m(x)|^{\frac{N}{N-1}}\right) dx \leq \int_\Omega \exp\left(\alpha_N\left(\frac{|u_m(x)|}{\|u_m(x)\|_{1,N}}\right)^{\frac{N}{N-1}}\right) dx \leq M. \tag{3.22}$$

Hence, by (3.21), (3.22) and Hölder's inequality we get

$$\begin{aligned}&\int_\Omega |u_m|^{q_0-1}\exp\left(\alpha|u_m|^{\frac{N}{N-1}}\right) dx \\ &\leq \left(\int_\Omega |u_m|^{(q_0-1)s'} dx\right)^{\frac{1}{s'}}\left(\int_\Omega \exp\left(\alpha s|u_m|^{\frac{N}{N-1}}\right) dx\right)^{\frac{1}{s}} \\ &\leq |u_m|^{q_0-1}_{L^{s'}(\Omega)} M^{\frac{1}{s}} = \overline{M}.\end{aligned} \tag{3.23}$$

We use (3.20), (3.23) and [26, Theorem 4.8] to conclude that

$$|u_m|^{q_0-1}\exp\left(\alpha|u_m|^{\frac{N}{N-1}}\right) \rightharpoonup |u|^{q_0-1}\exp\left(\alpha|u|^{\frac{N}{N-1}}\right). \tag{3.24}$$

It follows from (3.24) that

$$\int_\Omega |u_m|^{q_0-1}\exp\left(\alpha|u_m|^{\frac{N}{N-1}}\right)|\phi_k|\, dx \to \int_\Omega |u|^{q_0-1}\exp\left(\alpha|u|^{\frac{N}{N-1}}\right)|\phi_k|\, dx. \tag{3.25}$$

Therefore, by (3.19) and (3.25) we prove that

$$\int_\Omega \widehat{g}(u_m(x))\, dx \to \delta\int_\Omega |u(x)|^{p_0-1}|\phi_k|\, dx + C_\delta\int_\Omega |u(x)|^{q_0-1}\exp\left(\alpha|u(x)|^{\frac{N}{N-1}}\right)|\phi_k|\, dx,$$

which shows the identity (3.16).

Then, we use (3.15), (3.16) and [6, Theorem 4.2] to conclude that

$$\int_\Omega f_0(u_m)\phi_k\,dx \to \int_\Omega f_0(u)\phi_k\,dx. \tag{3.26}$$

The next step is to show that

$$\int_\Omega a_0\left(|\nabla u_m|^{p_0}\right)|\nabla u_m|^{p_0-2}\nabla u_m\nabla\phi_k\,dx \to \int_\Omega a_0\left(|\nabla u|^{p_0}\right)|\nabla u|^{p_0-2}\nabla u\nabla\phi_k\,dx. \tag{3.27}$$

To this end, we will use (a_2) and the same reasoning in [18, Lemma 2.4] to obtain

$$\begin{aligned}0 &\le C\|u_m-u\|_{1,N}\\ &\le \int_\Omega a_0\left(|\nabla u_m|^{p_0}\right)|\nabla u_m|^{p_0}\,dx - \int_\Omega a_0\left(|\nabla u_m|^{p_0}\right)|\nabla u_m|^{p_0-2}\nabla u_m\nabla u\,dx + o_n(1)\\ &= \lambda_0\int_\Omega \frac{u_m}{(u_m+\varepsilon)^{\beta_0}}\,dx + \int_\Omega f_0(u_m)u_m\,dx - \lambda_0\int_\Omega \frac{u}{(u_m+\varepsilon)^{\beta_0}}\,dx - \int_\Omega f_0(u_m)u\,dx\\ &= o_n(1),\end{aligned}$$

where

$$o_n(1) = \int_\Omega a_0\left(|\nabla u|^{p_0}\right)|\nabla u|^{p_0}\,dx - \int_\Omega a_0\left(|\nabla u|^{p_0}\right)|\nabla u|^{p_0-2}\nabla u_m\nabla u\,dx.$$

Hence,

$$\|u_m-u\|_{1,N} = o_n(1),$$

which implies that

$$u_m \to u \quad \text{in } W_0^{1,N}(\Omega). \tag{3.28}$$

Now, we know that the function defined by

$$E(u) = \int_\Omega a_0\left(|\nabla u|^{p_0}\right)|\nabla u|^{p_0-2}\nabla u\nabla\phi_k\,dx$$

is continuous. Then, we invoke this fact and (3.28) to get the convergence (3.27). Letting $m\to\infty$ in (3.13), we use (3.14), (3.26) and (3.27) to conclude that

$$\int_\Omega a_0(|\nabla u|^{p_0})|\nabla u|^{p_0-2}\nabla u\nabla\phi_k\,dx = \lambda_0\int_\Omega \frac{\phi_k}{(u+\varepsilon)^{\beta_0}}\,dx + \int_\Omega f_0(u)\phi_k\,dx, \tag{3.29}$$

for all $\phi_k\in W_k$.

Since $[W_k]_{k\in\mathbf{N}}$ is dense in $W_0^{1,N}(\Omega)$, we have

$$\phi_k\to\phi \quad \text{as} \quad k\to\infty.$$

Then,

$$\int_\Omega a_0(|\nabla u|^{p_0})|\nabla u|^{p_0-2}\nabla u\nabla\phi_k\,dx \to \int_\Omega a_0(|\nabla u|^{p_0})|\nabla u|^{p_0-2}\nabla u\nabla\phi\,dx, \tag{3.30}$$

$$\int_\Omega \frac{\phi_k}{(u+\varepsilon)^{\beta_0}}\,dx \to \int_\Omega \frac{\phi}{(u+\varepsilon)^{\beta_0}}\,dx, \tag{3.31}$$

and

$$\int_\Omega f_0(u)\phi_k\,dx \to \int_\Omega f_0(u)\phi\,dx. \tag{3.32}$$

Therefore, since $\phi\in W_0^{1,N}(\Omega)$ is arbitrary, it follows from (3.29) - (3.32) that

$$\int_\Omega a_0(|\nabla u|^{p_0})|\nabla u|^{p_0-2}\nabla u\nabla\phi\,dx = \lambda_0\int_\Omega \frac{\phi}{(u+\varepsilon)^{\beta_0}}\,dx + \int_\Omega f_0(u)\phi\,dx, \tag{3.33}$$

for all $\phi \in W_0^{1,N}(\Omega)$, which shows that u is a weak solution of the problem (3.1). Furthermore, $u > 0$ in Ω. In fact, since $f_0(t) = 0, \forall t < 0$, we use $\phi = u^-$ in (3.33) to obtain

$$\int_\Omega a_0(|\nabla u|^{p_0})|\nabla u^-|\,dx \le 0.$$

It follows from (a_1) and $k_2 > 0$ that

$$\int_\Omega |\nabla u^-|^N = \|u^-\|_{1,N}^N = 0,$$

which implies that $u^- = 0$ and then $u = u^+ \ge 0$. But thanks to the Harnack's inequality, see [21], $u > 0$ in Ω. □

4. Proof of the Theorem 1.1

For each $n \in \mathbf{N}$, let $\varepsilon = \dfrac{1}{n}$ and $u_{\frac{1}{n}} = u_n$ be, where u_n is a solution of auxiliary problem (3.1)

$$\begin{cases} -\mathrm{div}(a_0(|\nabla u_n|^{p_0})|\nabla u_n|^{p_0-2}\nabla u_n) = \dfrac{\lambda_0}{(u_n+\frac{1}{n})^{\beta_0}} + f_0(u_n) & \text{in } \Omega,\\ u_n > 0 & \text{in } \Omega,\\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

obtained by the Lemma 3.2. Note that, from (f_3) we get

$$\frac{\lambda_0}{(u_n+\frac{1}{n})^{\beta_0}} + f_0(u_n) \ge \frac{\lambda_0}{(u_n+1)^{\beta_0}} + |u_n|^{\gamma_0-1}.$$

Since the function $t \mapsto \dfrac{\lambda_0}{(t+1)^{\beta_0}} + t^{\gamma_0-1}$, for all $t \ge 0$, attains a positive minimum z. Then,

$$-\mathrm{div}\left(a_0(|\nabla u_n|^{p_0})|\nabla u_n|^{p_0-2}\nabla u_n\right) \ge z > 0 \quad \text{in } \Omega.$$

By virtue from Minty–Browder's Theorem [6, Theorem 5.15], we use the unique positive solution of the problem

$$\begin{cases} -\mathrm{div}\,(a_0(|\nabla v|^{p_0})|\nabla v|^{p_0-2}\nabla v) = z & \text{in } \Omega,\\ v > 0 & \text{in } \Omega,\\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

to obtain

$$\begin{cases} -\mathrm{div}\,(a_0(|\nabla u_n|^{p_0})|\nabla u_n|^{p_0-2}\nabla u_n) \ge -\mathrm{div}\,(a_0(|\nabla v|^{p_0})|\nabla v|^{p_0-2}\nabla v) & \text{in } \Omega,\\ u = v & \text{on } \partial\Omega. \end{cases}$$

Hence, we use the Lemma 2.1 to conclude that

$$u_n(x) \ge v(x) > 0 \ \text{ in } \Omega, \quad \forall n \in \mathbf{N}, \tag{4.2}$$

which implies that $u_n(x) \nrightarrow 0$, for each $x \in \Omega$.

Now, from (3.12) we get

$$u_m \rightharpoonup u_n \ \text{ in } W_0^{1,N}(\Omega) \ \text{ as } \ m \to +\infty$$

and it follows from (3.9) that

$$\|u_n\|_{1,N} \le \liminf_{m\to+\infty} \|u_m\|_{1,N} \le r < 1, \quad \text{for all } n \in \mathbf{N}.$$

Therefore, r does not depend on n, which shows that (u_n) is a bounded sequence. Thus, since $W_0^{1,N}(\Omega)$ is a reflective Banach space, for some subsequence, there exists $u \in W_0^{1,N}(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{in } W_0^{1,N}(\Omega), \\ u_n \to u & \text{in } L^\theta(\Omega),\ \theta \geq 1, \\ u_n(x) \to u(x) & \text{a.e. in } \Omega, \\ |u_n(x)| \leq g(x) \in L^\theta(\Omega) & \text{a.e. in } \Omega,\ \theta \geq 1. \end{cases} \tag{4.3}$$

Recall from (3.33) that

$$\begin{aligned} &\int_\Omega a_0(|\nabla u_n|^{p_0})|\nabla u_n|^{p_0-2}\nabla u_n \nabla \phi \, dx \\ &= \lambda_0 \int_\Omega \frac{\phi}{(u_n + \frac{1}{n})^{\beta_0}} \, dx + \int_\Omega f_0(u_n)\phi \, dx, \ \ \forall \phi \in W_0^{1,N}(\Omega). \end{aligned} \tag{4.4}$$

Since f_0 is a continuous function, by (4.3) we have

$$f_0(u_n(x))\phi \to f_0(u(x))\phi \ \text{ a.e. in } \Omega.$$

By same computation in (3.16), we obtain the function $\widehat{g}\colon \mathbf{R} \to \mathbf{R}$ such that

$$|f(u_n(x))\phi| \leq \widehat{g}(u_n(x))$$

such that $\widehat{g}(u_n(x))$ converges in $L^1(\Omega)$. Then, we use [6, Theorem 4.2] to conclude that

$$\int_\Omega f_0(u_n)\phi \, dx \to \int_\Omega f_0(u)\phi \, dx, \ \ \forall \phi \in W_0^{1,N}(\Omega). \tag{4.5}$$

Now, by same computation in (3.27), we have

$$\int_\Omega a_0(|\nabla u_n|^{p_0})|\nabla u_n|^{p_0-2}\nabla u_n \nabla \phi \, dx \to \int_\Omega a_0(|\nabla u|^{p_0})|\nabla u|^{p_0-2}\nabla u \nabla \phi \, dx, \tag{4.6}$$

for all $\phi \in W_0^{1,N}(\Omega)$. Note that, from (4.3) again we get

$$\frac{\phi}{\left(u_n(x) + \frac{1}{n}\right)^{\beta_0}} \to \frac{\phi}{u(x)^{\beta_0}} \quad \text{a.e. in } \Omega. \tag{4.7}$$

Moreover, by virtue from (4.1) and (a_1), we can argue as in [23] to obtain that $v \in C^1(\overline{\Omega})$. Consequently, from (4.2), for each $x \in \Omega$, it follows that $u_n(x) \geq v(x) > Cd(x) > 0$, where $d(x) = \operatorname{dist}(x, \partial\Omega)$ and C is a positive constant that does not depend on x. Thus,

$$\int_\Omega \frac{\phi}{(u_n(x) + \frac{1}{n})^{\beta_0}} \, dx \leq \int_\Omega \frac{\phi}{u_n(x)^{\beta_0}} \, dx \leq \int_\Omega \frac{\phi}{Cd(x)^{\beta_0}} \, dx.$$

Hence, we invoke Theorem 2.2 to obtain $\left|\frac{\phi}{Cd(x)^{\beta_0}}\right| \in L^r(\Omega)$ and $C_2 > 0$ such that

$$\int_\Omega \frac{\phi}{(u_n(x) + \frac{1}{n})^{\beta_0}} \, dx \leq C_2 \|\phi\|_{1,N}. \tag{4.8}$$

Therefore, by (4.7), (4.8) and [6, Theorem 4.2] we get

$$\int_\Omega \frac{\phi}{\left(u_n + \frac{1}{n}\right)^{\beta_0}} \, dx \to \int_\Omega \frac{\phi}{u^{\beta_0}} \, dx, \ \ \forall \phi \in W_0^{1,N}(\Omega). \tag{4.9}$$

Letting $n \to +\infty$ in (4.4), we use (4.5), (4.6) and (4.9) to conclude that

$$\int_\Omega a_0(|\nabla u|^{p_0})|\nabla u|^{p_0-2}\nabla u \nabla \phi \, dx = \lambda_0 \int_\Omega \frac{\phi}{u^{\beta_0}} \, dx + \int_\Omega f_0(u)\phi \, dx, \quad \forall \phi \in W_0^{1,N}(\Omega),$$

which proves that $u \in W_0^{1,N}(\Omega)$ is a weak solution for the problem (1.1).

5. Auxiliary problem for the case system

For each $\varepsilon > 0$, we consider the following auxiliary problem

$$\begin{cases} -\mathrm{div}(a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u) = \dfrac{\lambda_1}{(u+\varepsilon)^{\beta_1}} + f_1(v) & \text{in } \Omega, \\ -\mathrm{div}(a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v) = \dfrac{\lambda_2}{(v+\varepsilon)^{\beta_2}} + f_2(u) & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where the functions a_i and $f_i (i = 1, 2)$ satisfy the hypotheses of the Theorem 1.2.

The main result in this section is the following:

Lemma 5.1. *For each $0 < \varepsilon < 1$, there exists $\lambda^* > 0$ such that the problem (5.1) has a weak positive solution for every $0 < \lambda_1 + \lambda_2 < \lambda^*$, with $i = 1, 2$.*

Proof. Let $B = \{e_1, e_2, \ldots, e_m, \ldots\}$ be a Schauder basis of $W_0^{1,N}(\Omega)$. For each $m \in \mathbf{N}$, define

$$W_m = [e_1, e_2, \ldots, e_m]$$

to be the finite-dimensional space generated by $\{e_1, e_2, \ldots, e_m\}$.

For each $m \in \mathbf{N}$, we define the function $J \colon \mathbf{R}^{2m} \to \mathbf{R}^{2m}$ such that

$$J(\eta, \xi) = (F_1(\eta, \xi), F_2(\eta, \xi), \ldots, F_m(\eta, \xi), G_1(\eta, \xi), G_2(\eta, \xi), \ldots, G_m(\eta, \xi)),$$

where $(\eta, \xi) = (\eta_1, \eta_2, \ldots, \eta_m, \xi_1, \xi_2, \ldots, \xi_m) \in \mathbf{R}^{2m}$,

$$F_j(\eta, \xi) = \int_\Omega a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u \nabla e_j \, dx - \lambda_1 \int_\Omega \frac{e_j}{(u+\varepsilon)^{\beta_1}} \, dx - \int_\Omega f_1(v) e_j \, dx,$$

$j = 1, 2, \ldots, m$,

$$G_j(\eta, \xi) = \int_\Omega a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v \nabla e_j \, dx - \lambda_2 \int_\Omega \frac{e_j}{(v+\varepsilon)^{\beta_2}} \, dx - \int_\Omega f_2(u) e_j \, dx,$$

$j = 1, 2, \ldots, m$,

$$u = \sum_{j=1}^m \eta_j e_j \in W_m,$$

and

$$v = \sum_{j=1}^m \xi_j e_j \in W_m.$$

Moreover,

$$\|(u, v)\| = |(\eta, \xi)|_s. \tag{5.2}$$

Therefore,

$$\begin{aligned}\langle J(\eta,\xi),(\eta,\xi)\rangle &= \langle (F_1(\eta,\xi),F_2(\eta,\xi),\ldots,F_m(\eta,\xi),G_1(\eta,\xi),G_2(\eta,\xi),\ldots,G_m(\eta,\xi)),\\ &\quad (\eta_1,\eta_2,\ldots,\eta_m,\xi_1,\xi_2,\ldots,\xi_m)\rangle = \sum_{j=1}^m F_j(\eta,\xi)\eta_j + \sum_{j=1}^m G_j(\eta,\xi)\xi_j\\ &= \int_\Omega a_1(|\nabla u|^{p_1})|\nabla u|^{p_1}\,dx - \lambda_1\int_\Omega \frac{u}{(u+\varepsilon)^{\beta_1}}\,dx - \int_\Omega f_1(v)u\,dx\\ &\quad + \int_\Omega a_2(|\nabla v|^{p_2})|\nabla v|^{p_2}\,dx - \lambda_2\int_\Omega \frac{v}{(v+\varepsilon)^{\beta_2}}\,dx - \int_\Omega f_2(u)v\,dx.\end{aligned}$$

Note that

$$\int_\Omega \frac{u}{(u+\varepsilon)^{\beta_1}}\,dx \le |\Omega|, \tag{5.3}$$

and

$$\int_\Omega \frac{v}{(v+\varepsilon)^{\beta_2}}\,dx \le |\Omega| \tag{5.4}$$

Using (2.1), Young's inequality and Sobolev embedding, there exist positive constants $C_1, C_2, C_3, C_4, C_5, C_6, C_7$ and C_8 such that

$$\begin{aligned}\int_\Omega f_1(v)u\,dx &\le \delta_1 C_1\|v\|_{1,p_1}^{p_1} + \delta_1 C_2\|u\|_{1,p_1}^{p_1} + C_{\delta_1}C_3\int_\Omega |v|^{q_1}\exp\left(\alpha_1|v|^{\frac{N}{N-1}}\right)dx\\ &\quad + C_{\delta_1}C_4\int_\Omega |u|^{q_1}\exp\left(\alpha_1|v|^{\frac{N}{N-1}}\right)dx\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}\int_\Omega f_2(u)v\,dx &\le \delta_2 C_5\|u\|_{1,p_2}^{p_2} + \delta_2 C_6\|v\|_{1,p_2}^{p_2} + C_{\delta_2}C_7\int_\Omega |u|^{q_2}\exp\left(\alpha_2|u|^{\frac{N}{N-1}}\right)dx\\ &\quad + C_{\delta_2}C_8\int_\Omega |v|^{q_2}\exp\left(\alpha_2|u|^{\frac{N}{N-1}}\right)dx.\end{aligned} \tag{5.6}$$

Now, from (a_1) we have

$$\begin{aligned}\int_\Omega a_1(|\nabla u|^{p_1})|\nabla u|^{p_1}\,dx &\ge k_1\int_\Omega |\nabla u|^{p_1}\,dx + k_2\int_\Omega |\nabla u|^N\,dx\\ &= k_1\|u\|_{1,p_1}^{p_1} + k_2\|u\|_{1,N}^N\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}\int_\Omega a_2(|\nabla v|^{p_2})|\nabla v|^{p_2}\,dx &\ge k_1\int_\Omega |\nabla u|^{p_2}dx + k_2\int_\Omega |\nabla v|^N\,dx\\ &= k_1\|v\|_{1,p_2}^{p_2} + k_2\|v\|_{1,N}^N.\end{aligned} \tag{5.8}$$

It follows from (5.3)–(5.8) that

$$\begin{aligned}\langle J(\eta,\xi),(\eta,\xi)\rangle &\ge k_2(\|u\|_{1,N}^N + \|v\|_{1,N}^N) + (k_1-\delta_1 C_2)\|u\|_{1,p_1}^{p_1} + (k_1-\delta_2 C_6)\|v\|_{1,p_2}^{p_2}\\ &\quad - (\lambda_1+\lambda_2)|\Omega| - \delta_1 C_1\|v\|_{1,p_1}^{p_1} - C_{\delta_1}C_3\int_\Omega |v|^{q_1}\exp\left(\alpha_1|v|^{\frac{N}{N-1}}\right)dx\\ &\quad - C_{\delta_1}C_4\int_\Omega |u|^{q_1}\exp\left(\alpha_1|v|^{\frac{N}{N-1}}\right)dx - \delta_2 C_5\|u\|_{1,p_2}^{p_2}\\ &\quad - C_{\delta_2}C_7\int_\Omega |u|^{q_2}\exp\left(\alpha_2|u|^{\frac{N}{N-1}}\right)dx - C_{\delta_2}C_8\int_\Omega |v|^{q_2}\exp\left(\alpha_2|u|^{\frac{N}{N-1}}\right)dx.\end{aligned}$$

Note that, from $2 \le p_1, p_2 < N$ and Sobolev embedding, there exist $C_9, C_{10} > 0$ we have

$$\delta_1 C_1 \|v\|_{1,p_1}^{p_1} \le \delta_1 C_9 \|v\|_{1,N}^{p_1}$$

and

$$\delta_2 C_5 \|u\|_{1,p_2}^{p_2} \le \delta_2 C_{10} \|u\|_{1,N}^{p_2}.$$

Since $\|(u,v)\|_{1,N}^N = \|u\|_{1,N}^N + \|v\|_{1,N}^N$ and taking $\delta_1, \delta_2 > 0$ sufficiently small such that $(k_1 - \delta_1 C_2), (k_1 - \delta_2 C_6) > 0$ we obtain

$$\begin{aligned}\langle J(\eta,\xi),(\eta,\xi)\rangle \ge{}& k_2\|(u,v)\|^N - (\lambda_1+\lambda_2)|\Omega| - \delta_1 C_9\|(u,v)\|_{1,N}^{p_1} - \delta_2 C_{10}\|(u,v)\|_{1,N}^{p_2}\\ &- C_{\delta_1} C_3 \int_\Omega |v|^{q_1}\exp\left(\alpha_1|v|^{\frac{N}{N-1}}\right)dx - C_{\delta_1} C_4 \int_\Omega |u|^{q_1}\exp\left(\alpha_1|v|^{\frac{N}{N-1}}\right)dx\\ &- C_{\delta_2} C_7 \int_\Omega |u|^{q_2}\exp\left(\alpha_2|u|^{\frac{N}{N-1}}\right)dx - C_{\delta_2} C_8 \int_\Omega |v|^{q_2}\exp\left(\alpha_2|u|^{\frac{N}{N-1}}\right)dx.\end{aligned} \tag{5.9}$$

Using Hölder's inequality with $s, s' > 1$ such that $\dfrac{1}{s} + \dfrac{1}{s'} = 1$, we get

$$C_{\delta_1} C_3 \int_\Omega |v|^{q_1}\exp\left(\alpha_1|v|^{\frac{N}{N-1}}\right)dx \le C_{\delta_1} C_3\left(\int_\Omega |v|^{q_1 s'}dx\right)^{\frac{1}{s'}}\left(\int_\Omega \exp\left(\alpha_1 s|v|^{\frac{N}{N-1}}\right)dx\right)^{\frac{1}{s}},$$

$$C_{\delta_1} C_4 \int_\Omega |u|^{q_1}\exp\left(\alpha_1|v|^{\frac{N}{N-1}}\right)dx \le C_{\delta_1} C_4\left(\int_\Omega |u|^{q_1 s'}dx\right)^{\frac{1}{s'}}\left(\int_\Omega \exp\left(\alpha_1 s|v|^{\frac{N}{N-1}}\right)dx\right)^{\frac{1}{s}},$$

$$C_{\delta_2} C_7 \int_\Omega |u|^{q_2}\exp\left(\alpha_2|u|^{\frac{N}{N-1}}\right)dx \le C_{\delta_2} C_7\left(\int_\Omega |u|^{q_2 s'}dx\right)^{\frac{1}{s'}}\left(\int_\Omega \exp\left(\alpha_2 s|u|^{\frac{N}{N-1}}\right)dx\right)^{\frac{1}{s}}$$

and

$$C_{\delta_2} C_8 \int_\Omega |v|^{q_2}\exp\left(\alpha_2|u|^{\frac{N}{N-1}}\right)dx \le C_{\delta_2} C_8\left(\int_\Omega |v|^{q_2 s'}dx\right)^{\frac{1}{s'}}\left(\int_\Omega \exp\left(\alpha_2 s|u|^{\frac{N}{N-1}}\right)dx\right)^{\frac{1}{s}}.$$

Since $q_1, q_2 > N$ and $s' > 1$, by Sobolev embedding there exist $C_{11}, C_{12}, C_{13}, C_{14} > 0$ such that

$$C_{\delta_1} C_3 \int_\Omega |v|^{q_1}\exp\left(\alpha_1|v|^{\frac{N}{N-1}}\right)dx \le C_{\delta_1} C_{11}\|v\|_{1,N}^{q_1}\left(\int_\Omega \exp\left(\alpha_1 s|v|^{\frac{N}{N-1}}\right)dx\right)^{\frac{1}{s}},$$

$$C_{\delta_1} C_4 \int_\Omega |u|^{q_1}\exp\left(\alpha_1|v|^{\frac{N}{N-1}}\right)dx \le C_{\delta_1} C_{12}\|u\|_{1,N}^{q_1}\left(\int_\Omega \exp\left(\alpha_1 s|v|^{\frac{N}{N-1}}\right)dx\right)^{\frac{1}{s}},$$

$$C_{\delta_2} C_7 \int_\Omega |u|^{q_2}\exp\left(\alpha_2|u|^{\frac{N}{N-1}}\right)dx \le C_{\delta_2} C_{13}\|u\|_{1,N}^{q_2}\left(\int_\Omega \exp\left(\alpha_2 s|u|^{\frac{N}{N-1}}\right)dx\right)^{\frac{1}{s}}$$

and

$$C_{\delta_2} C_8 \int_\Omega |v|^{q_2}\exp\left(\alpha_2|u|^{\frac{N}{N-1}}\right)dx \le C_{\delta_2} C_{14}\|v\|_{1,N}^{q_2}\left(\int_\Omega \exp\left(\alpha_2 s|u|^{\frac{N}{N-1}}\right)dx\right)^{\frac{1}{s}}.$$

Then, it follows from (5.9) that

$$\begin{aligned}\langle J(\eta,\xi),(\eta,\xi)\rangle \ge{}& k_2\|(u,v)\|^N - (\lambda_1+\lambda_2)|\Omega| - \delta_1 C_9\|(u,v)\|_{1,N}^{p_1} - \delta_2 C_{10}\|(u,v)\|_{1,N}^{p_2}\\ &- C_{\delta_1} C_{11}\|(u,v)\|_{1,N}^{q_1}\left(\int_\Omega \exp\left(\alpha_1 s|v|^{\frac{N}{N-1}}\right)dx\right)^{\frac{1}{s}}\\ &- C_{\delta_1} C_{12}\|(u,v)\|_{1,N}^{q_1}\left(\int_\Omega \exp\left(\alpha_1 s|v|^{\frac{N}{N-1}}\right)dx\right)^{\frac{1}{s}}\end{aligned} \tag{5.10}$$

$$- C_{\delta_2}C_{13}\|(u,v)\|_{1,N}^{q_2}\left(\int_\Omega \exp\left(\alpha_2 s|u|^{\frac{N}{N-1}}\right)dx\right)^{\frac{1}{s}}$$
$$- C_{\delta_2}C_{14}\|(u,v)\|_{1,N}^{q_2}\left(\int_\Omega \exp\left(\alpha_2 s|u|^{\frac{N}{N-1}}\right)dx\right)^{\frac{1}{s}}.$$

Assume now that $\|(u,v)\|_{1,N} = r$ for some $r > 0$ to be chosen later. Then, we have

$$\begin{aligned}\int_\Omega \exp\left(\alpha_1 s|u|^{\frac{N}{N-1}}\right)dx &= \int_\Omega \exp\left(\alpha_1 s\|u\|_{1,N}^{\frac{N}{N-1}}\left(\frac{|u|}{\|u\|_{1,N}}\right)^{\frac{N}{N-1}}\right)dx\\ &\le \int_\Omega \exp\left(\alpha_1 s\|(u,v)\|^{\frac{N}{N-1}}\left(\frac{|u|}{\|u\|_{1,N}}\right)^{\frac{N}{N-1}}\right)dx\\ &= \int_\Omega \exp\left(\alpha_1 s r^{\frac{N}{N-1}}\left(\frac{|u|}{\|u\|_{1,N}}\right)^{\frac{N}{N-1}}\right)dx.\end{aligned}$$

Similarly,

$$\int_\Omega \exp\left(\alpha_2 s|v|^{\frac{N}{N-1}}\right)dx \le \int_\Omega \exp\left(\alpha_2 s r^{\frac{N}{N-1}}\left(\frac{|v|}{\|v\|_{1,N}}\right)^{\frac{N}{N-1}}\right)dx$$

and in order to apply the Theorem 2.1, we impose that

$$r \le \left(\frac{\alpha_N}{\alpha_1 s}\right)^{\frac{N-1}{N}} \text{ and } r \le \left(\frac{\alpha_N}{\alpha_2 s}\right)^{\frac{N-1}{N}}.$$

Therefore, there exist $M_1, M_2 > 0$ such that

$$\sup_{\|u\|_{1,N}\le 1}\int_\Omega \exp\left(\alpha_1 s r^{\frac{N}{N-1}}\left(\frac{|u|}{\|u\|_{1,N}}\right)\right)dx \le M_1$$

and

$$\sup_{\|v\|_{1,N}\le 1}\int_\Omega \exp\left(\alpha_2 s r^{\frac{N}{N-1}}\left(\frac{|v|}{\|v\|_{1,N}}\right)\right)dx \le M_2.$$

Hence, there exist $C_{15}, C_{16} > 0$ such that we can rewrite (5.10) as

$$\begin{aligned}&\langle J(\eta,\xi),(\eta,\xi)\rangle\\ &\ge k_2 r^N - (\lambda_1+\lambda_2)|\Omega| - \delta_1 C_9 r^{p_1} - \delta_2 C_{10} r^{p_2} - C_{\delta_1}C_{15}M_2^{1/s}r^{q_1} - C_{\delta_2}C_{16}M_1^{1/s}r^{q_2}.\end{aligned}$$

Now, it is necessary to choose r such that

$$\frac{k_2 r^N}{2} - C_{\delta_1}\widetilde{C_1}M_1^{1/s}r^{q_1} \ge \frac{k_2 r^N}{4}$$

and

$$\frac{k_2 r^N}{2} - C_{\delta_2}\widetilde{C_2}M_2^{1/s}r^{q_2} \ge \frac{k_2 r^N}{4},$$

in others words

$$r \le \left(\frac{k_2}{4C_{\delta_1}\widetilde{C_1}M_1^{\frac{1}{s}}}\right)^{\frac{1}{q_1-N}} \quad \text{and} \quad r \le \left(\frac{k_2}{4C_{\delta_2}\widetilde{C_2}M_2^{\frac{1}{s}}}\right)^{\frac{1}{q_2-N}}.$$

Considering

$$r = \min\left\{1, \left(\frac{\alpha_N}{\alpha_1 s}\right)^{\frac{N-1}{N}}, \left(\frac{\alpha_N}{\alpha_2 s}\right)^{\frac{N-1}{N}}, \left(\frac{k_2}{4C_{\delta_1}\widetilde{C_1}M_1^{\frac{1}{s}}}\right)^{\frac{1}{q_1-N}}, \left(\frac{k_2}{4C_{\delta_2}\widetilde{C_2}M_2^{\frac{1}{s}}}\right)^{\frac{1}{q_2-N}}\right\},$$

we get

$$\langle J(\eta,\xi),(\eta,\xi)\rangle \geq k_2 r^N - (\lambda_1+\lambda_2)|\Omega| - \delta_1 C_9 - \delta_2 C_{10}.$$

Thus, since $r>0$ is fixed and the last inequality is true for all $\delta_1,\delta_2>0$ then, there exists $\lambda^*>0$ such that

$$\langle J(\xi),\xi\rangle > 0\ ,\ \text{for all}\ \ \eta,\xi\in\mathbf{R}^m \text{ and } \ |(\eta,\xi)|_s = r, \quad \text{for all} \quad 0<\lambda_1+\lambda_2<\lambda^*.$$

By virtue of Lemma 3.1, for every $m\in\mathbf{N}$, there exists $(x,y)\in\mathbf{R}^{2m}$ with $|(x,y)|_s\leq r<1$ such that $J(x,y)=0$. Consequently, there exist $u_m,v_m\in W_m$ satisfying

$$\|(u_m,v_m)\|\leq r<1, \ \text{ for every m } \in\mathbf{N} \tag{5.11}$$

such that

$$\int_\Omega a_1(|\nabla u_m|^{p_1})|\nabla u_m|^{p_1-2}\nabla u_m\nabla\phi\,dx = \lambda_1\int_\Omega \frac{\phi}{(u_m+\varepsilon)^{\beta_1}}\,dx + \int_\Omega f_1(v_m)\phi\,dx \tag{5.12}$$

for all $\phi\in W_m$ and

$$\int_\Omega a_2(|\nabla v_m|^{p_2})|\nabla v_m|^{p_2-2}\nabla v_m\nabla\varphi\,dx = \lambda_2\int_\Omega \frac{\varphi}{(v_m+\varepsilon)^{\beta_2}}\,dx + \int_\Omega f_2(u_m)\varphi\,dx \tag{5.13}$$

for all $\varphi\in W_m$, which implies that (u_m,v_m) is an approximate weak solution of problem (5.1).

Since r does not depend on m and $W_m\subset W_0^{1,N}(\Omega)$, for all N, then (u_m) and (v_m) are bounded sequences in $W_0^{1,N}(\Omega)$. Thus, for some subsequence, there exist $u,v\in W_0^{1,N}(\Omega)$ such that

$$\begin{cases} u_m \rightharpoonup u & \text{in } W_0^{1,N}(\Omega),\\ u_m\to u & \text{in } L^\theta(\Omega),\ \theta\geq 1,\\ u_m(x)\to u(x) & \text{a.e. in } \Omega,\\ |u_m(x)|\leq g_1(x)\in L^\theta(\Omega) & \text{a.e. in } \Omega,\ \theta\geq 1\end{cases} \tag{5.14}$$

and

$$\begin{cases} v_m \rightharpoonup v & \text{in } W_0^{1,N}(\Omega),\\ v_m\to v & \text{in } L^\theta(\Omega),\ \theta\geq 1,\\ v_m(x)\to v(x) & \text{a.e. in } \Omega,\\ |v_m(x)|\leq g_2(x)\in L^\theta(\Omega) & \text{a.e. in } \Omega,\ \theta\geq 1.\end{cases} \tag{5.15}$$

Fix $k\in\mathbf{N}$ and consider $m\geq k$, then $W_k\subset W_m$ and

$$\begin{aligned}&\int_\Omega a_1(|\nabla u_m|^{p_1})|\nabla u_m|^{p_1-2}\nabla u_m\nabla\phi_k\,dx\\ &= \lambda_1\int_\Omega\frac{\phi_k}{(u_m+\varepsilon)^{\beta_1}}\,dx + \int_\Omega f_1(v_m)\phi_k\,dx,\ \ \forall\phi_k\in W_k\end{aligned} \tag{5.16}$$

and

$$\begin{aligned}&\int_\Omega a_2(|\nabla v_m|^{p_2})|\nabla v_m|^{p_2-2}\nabla v_m \nabla \varphi_k\,dx\\ &= \lambda_2 \int_\Omega \frac{\varphi_k}{(v_m+\varepsilon)^{\beta_2}} + \int_\Omega f_2(u_m)\varphi_k\,dx, \ \ \forall \varphi_k \in W_k.\end{aligned} \tag{5.17}$$

Since $\phi_k, \varphi_k \in W_m$, note that

$$\left|\frac{\phi_k}{(u_m+\varepsilon)^{\beta_1}}\right| \le \frac{|\phi_k|}{\varepsilon^{\beta_1}} \in L^1(\Omega)$$

and

$$\left|\frac{\varphi_k}{(v_m+\varepsilon)^{\beta_2}}\right| \le \frac{|\varphi_k|}{\varepsilon^{\beta_2}} \in L^1(\Omega).$$

By (5.14) and (5.15) we have

$$\frac{\phi_k}{(u_m(x)+\varepsilon)^{\beta_1}} \to \frac{\phi_k}{(u(x)+\varepsilon)^{\beta_1}} \quad \text{a.e. in } \Omega$$

and

$$\frac{\varphi_k}{(v_m(x)+\varepsilon)^{\beta_2}} \to \frac{\varphi_k}{(v(x)+\varepsilon)^{\beta_2}} \quad \text{a.e. in } \Omega.$$

Therefore, we use [6, Theorem 4.2] to obtain that

$$\int_\Omega \frac{\phi_k}{(u_m+\varepsilon)^{\beta_1}}\,dx \to \int_\Omega \frac{\phi_k}{(u+\varepsilon)^{\beta_1}}\,dx \tag{5.18}$$

and

$$\int_\Omega \frac{\varphi_k}{(v_m+\varepsilon)^{\beta_2}}\,dx \to \int_\Omega \frac{\varphi_k}{(v+\varepsilon)^{\beta_2}}\,dx. \tag{5.19}$$

Now, since f_i are continuous functions, by (5.14) and (5.15) again we have

$$f_1(v_m(x))\phi_k \to f_1(v(x))\phi_k \ \text{ a.e in } \Omega \tag{5.20}$$

and

$$f_2(u_m(x))\varphi_k \to f_2(u(x))\varphi_k \ \text{ a.e in } \Omega. \tag{5.21}$$

Using (2.1) we get

$$|f_1(v_m(x))\phi_k| \le \delta_1 |v_m(x)|^{p_1-1}|\phi_k| + C_{\delta_1}|v_m(x)|^{q_1-1}\exp\left(\alpha_1 |v_m(x)|^{\frac{N}{N-1}}\right)|\phi_k|$$

and

$$|f_2(u_m(x))\varphi_k| \le \delta_2 |u_m(x)|^{p_2-1}|\varphi_k| + C_{\delta_2}|u_m(x)|^{q_2-1}\exp\left(\alpha_2 |u_m(x)|^{\frac{N}{N-1}}\right)|\varphi_k|.$$

We will need to prove that the functions $\widehat{g_1}, \widehat{g_2}\colon \mathbf{R}\to\mathbf{R}$ defined by

$$\widehat{g_1}(v_m(x)) := \delta_1 |v_m(x)|^{p_1-1}|\phi_k| + C_{\delta_1}|v_m(x)|^{q_1-1}\exp\left(\alpha_1 |v_m(x)|^{\frac{N}{N-1}}\right)|\phi_k|$$

and

$$\widehat{g_2}(u_m(x)) := \delta_2 |u_m(x)|^{p_2-1}|\varphi_k| + C_{\delta_2}|u_m(x)|^{q_2-1}\exp\left(\alpha_2 |u_m(x)|^{\frac{N}{N-1}}\right)|\varphi_k|$$

satisfy

$$|f_1(v_m(x))\phi_k| \le \widehat{g_1}(v_m(x)) \in L^1(\Omega) \tag{5.22}$$

and

$$|f_2(u_m(x))\varphi_k| \le \widehat{g_2}(u_m(x)) \in L^1(\Omega). \tag{5.23}$$

It is sufficiently to show that $\widehat{g_1}(v_m(x))$ and $\widehat{g_2}(u_m(x))$ are convergent in $L^1(\Omega)$. We will prove only the first inequality, because the second follows of the same reasoning. Indeed, since $2 \le p_1 < N$ we invoke (5.15) to get

$$|v_m(x)|^{p_1-1}|\phi_k| \to |v(x)|^{p_1-1}|\phi_k| \quad \text{a.e. in } \Omega \tag{5.24}$$

and

$$|v_m(x)|^{p_1-1}|\phi_k| \le g_2(x)^{p_1-1}|\phi_k| \in L^1(\Omega). \tag{5.25}$$

It follows from (5.24), (5.25) and [1, Theorem 4.2] that

$$\int_\Omega |v_m|^{p_1-1}|\phi_k|\,dx \to \int_\Omega |v|^{p_1-1}|\phi_k|\,dx. \tag{5.26}$$

Furthermore, from (5.15) again we get

$$|v_m(x)|^{q_1-1}\exp\left(\alpha_1|v_m(x)|^{\frac{N}{N-1}}\right) \to |v(x)|^{q_1-1}\exp\left(\alpha_1|v(x)|^{\frac{N}{N-1}}\right) \quad \text{a.e. in } \Omega. \tag{5.27}$$

Now, considering $s, s' > 1$ such that $\dfrac{1}{s}+\dfrac{1}{s'} = 1$, we use (5.15) and the fact that $q_1 > N$ to obtain

$$|v_m|^{q_1-1} \to |v|^{q_1-1} \quad \text{in } L^{s'}(\Omega). \tag{5.28}$$

Moreover, by (5.11) we have

$$\begin{aligned}\int_\Omega \exp\left(\alpha_1 s|v_m(x)|^{\frac{N}{N-1}}\right)dx &= \int_\Omega \exp\left(\alpha_1 s\|v_m(x)\|_{1,N}^{\frac{N}{N-1}}\left(\frac{|v_m(x)|}{\|v_m(x)\|_{1,N}}\right)^{\frac{N}{N-1}}\right)dx\\ &\le \int_\Omega \exp\left(\alpha_1 s r^{\frac{N}{N-1}}\left(\frac{|v_m(x)|}{\|v_m(x)\|_{1,N}}\right)^{\frac{N}{N-1}}\right)dx\end{aligned}$$

and applying the Theorem 2.1 we obtain

$$\int_\Omega \exp\left(\alpha_1 s|v_m(x)|^{\frac{N}{N-1}}\right)dx \le \int_\Omega \exp\left(\alpha_N\left(\frac{|v_m(x)|}{\|v_m(x)\|_{1,N}}\right)^{\frac{N}{N-1}}\right)dx \le M_1. \tag{5.29}$$

Hence, by (5.28), (5.29) and Hölder's inequality we get

$$\begin{aligned}\int_\Omega |v_m|^{q_1-1}\exp\left(\alpha_1|v_m|^{\frac{N}{N-1}}\right)dx &\le \left(\int_\Omega |u_m|^{(q_1-1)s'}\,dx\right)^{\frac{1}{s'}}\left(\int_\Omega \exp\left(\alpha_1 s|v_m|^{\frac{N}{N-1}}\right)dx\right)^{\frac{1}{s}}\\ &\le |v_m|^{q_1-1}_{L^{s'}(\Omega)} M_1^{\frac{1}{s}} = \overline{M_1}.\end{aligned} \tag{5.30}$$

We use (5.27), (5.30) and [26, Theorem 4.8] to conclude that

$$|v_m|^{q_1-1}\exp\left(\alpha_1|v_m|^{\frac{N}{N-1}}\right) \rightharpoonup |v|^{q_1-1}\exp\left(\alpha_1|v|^{\frac{N}{N-1}}\right). \tag{5.31}$$

It follows from (5.31) that

$$\int_\Omega |v_m|^{q_1-1}\exp\left(\alpha_1|v_m|^{\frac{N}{N-1}}\right)|\phi_k|\,dx \to \int_\Omega |v|^{q_1-1}\exp\left(\alpha_1|v|^{\frac{N}{N-1}}\right)|\phi_k|\,dx. \tag{5.32}$$

Therefore, by (5.26) and (5.32) we prove that

$$\int_\Omega \widehat{g_1}(v_m(x))\,dx \to \delta_1\int_\Omega |v(x)|^{p_1-1}|\phi_k|\,dx + C_{\delta_1}\int_\Omega |v(x)|^{q_1-1}\exp\left(\alpha_1|v(x)|^{\frac{N}{N-1}}\right)|\phi_k|\,dx,$$

which shows the identity (5.22).

Then, we use (5.20), (5.21), (5.22), (5.23) and [6, Theorem 4.2] to conclude that

$$\int_\Omega f_1(v_m)\phi_k\,dx \to \int_\Omega f_1(v)\phi_k\,dx \tag{5.33}$$

and

$$\int_\Omega f_2(u_m)\varphi_k\,dx \to \int_\Omega f_2(u)\varphi_k\,dx. \tag{5.34}$$

The next step is to show that

$$\int_\Omega a_1\left(|\nabla u_m|^{p_1}\right)|\nabla u_m|^{p_1-2}\nabla u_m\nabla\phi_k\,dx \to \int_\Omega a_1\left(|\nabla u|^{p_1}\right)|\nabla u|^{p_1-2}\nabla u\nabla\phi_k\,dx \tag{5.35}$$

and

$$\int_\Omega a_2\left(|\nabla v_m|^{p_2}\right)|\nabla v_m|^{p_2-2}\nabla v_m\nabla\varphi_k\,dx \to \int_\Omega a_2\left(|\nabla v|^{p_2}\right)|\nabla v|^{p_2-2}\nabla u\nabla\varphi_k\,dx. \tag{5.36}$$

To this end, we will use (a_2) and the same reasoning in [18, Lemma 2.4] to obtain

$$\begin{aligned}
&0 \le C\|u_m-u\|_{1,N}\\
&\le \int_\Omega a_1\left(|\nabla u_m|^{p_1}\right)|\nabla u_m|^{p_1}\,dx - \int_\Omega a_1\left(|\nabla u_m|^{p_1}\right)|\nabla u_m|^{p_1-2}\nabla u_m\nabla u\,dx + o_n(1)\\
&= \lambda_1\int_\Omega \frac{u_m}{(u_m+\varepsilon)^{\beta_1}}\,dx + \int_\Omega f_1(v_m)u_m\,dx - \lambda_1\int_\Omega \frac{u}{(u_m+\varepsilon)^{\beta_1}}\,dx - \int_\Omega f_1(v_m)u = o_n(1),
\end{aligned}$$

where

$$o_n(1) = \int_\Omega a_1\left(|\nabla u|^{p_1}\right)|\nabla u|^{p_1}\,dx - \int_\Omega a_1\left(|\nabla u|^{p_1}\right)|\nabla u|^{p_1-2}\nabla u_m\nabla u\,dx$$

and

$$\begin{aligned}
&0 \le C\|v_m-v\|_{1,N}\\
&\le \int_\Omega a_2\left(|\nabla v_m|^{p_2}\right)|\nabla v_m|^{p_2}\,dx - \int_\Omega a_2\left(|\nabla v_m|^{p_2}\right)|\nabla v_m|^{p_2-2}\nabla v_m\nabla v\,dx + o_n(1)\\
&= \lambda_2\int_\Omega \frac{v_m}{(v_m+\varepsilon)^{\beta_2}}\,dx + \int_\Omega f_2(u_m)v_m\,dx - \lambda_2\int_\Omega \frac{v}{(v_m+\varepsilon)^{\beta_2}}\,dx - \int_\Omega f_2(u_m)v\,dx = o_n(1),
\end{aligned}$$

where

$$o_n(1) = \int_\Omega a_2\left(|\nabla v|^{p_2}\right)|\nabla v|^{p_2}dx - \int_\Omega a_2\left(|\nabla v|^{p_2}\right)|\nabla v|^{p_2-2}\nabla v_m\nabla v dx.$$

Hence,

$$\|u_m-u\|_{1,N} = o_n(1) \text{ and } \|v_m-v\|_{1,N} = o_n(1),$$

which implies that

$$u_m \to u \ \text{ in } W_0^{1,N}(\Omega) \tag{5.37}$$

and

$$v_m \to v \ \text{ in } W_0^{1,N}(\Omega). \tag{5.38}$$

Now, we know that the functions defined by

$$E_1(u) = \int_\Omega a_1\left(|\nabla u|^{p_1}\right)|\nabla u|^{p_1-2}\nabla u\nabla\phi_k\,dx$$

and

$$E_2(v) = \int_\Omega a_2\left(|\nabla v|^{p_2}\right)|\nabla v|^{p_2-2}\nabla v\nabla\varphi_k\,dx$$

are continuous. Then, by (5.37) and (5.38) we get the convergences (5.35) and (5.36).

Letting $m \to \infty$ in (5.16) and (5.17), we use (5.18), (5.19), (5.33), (5.34), (5.35) and (5.36) to conclude that

$$\int_\Omega a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u \nabla \phi_k \, dx = \lambda_1 \int_\Omega \frac{\phi_k}{(u+\varepsilon)^{\beta_1}} \, dx + \int_\Omega f_1(v)\phi_k \, dx, \tag{5.39}$$

and

$$\int_\Omega a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v \nabla \varphi_k \, dx = \lambda_2 \int_\Omega \frac{\varphi_k}{(v+\varepsilon)^{\beta_2}} \, dx + \int_\Omega f_2(u)\varphi_k \, dx, \tag{5.40}$$

for all $\phi_k, \varphi_k \in W_k$.

Since $[W_k]_{k\in\mathbf{N}}$ is dense in $W_0^{1,N}(\Omega)$, we have

$$\phi_k \to \phi \quad \text{as} \quad k \to \infty$$

and

$$\varphi_k \to \varphi \quad \text{as} \quad k \to \infty.$$

Then,

$$\int_\Omega a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u \nabla \phi_k \, dx \to \int_\Omega a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u \nabla \phi \, dx, \tag{5.41}$$

$$\int_\Omega a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v \nabla \varphi_k \, dx \to \int_\Omega a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v \nabla \varphi \, dx, \tag{5.42}$$

$$\int_\Omega \frac{\phi_k}{(u+\varepsilon)^{\beta_1}} \, dx \to \int_\Omega \frac{\phi}{(u+\varepsilon)^{\beta_1}} \, dx, \tag{5.43}$$

$$\int_\Omega \frac{\varphi_k}{(v+\varepsilon)^{\beta_2}} \, dx \to \int_\Omega \frac{\varphi}{(v+\varepsilon)^{\beta_2}} \, dx, \tag{5.44}$$

$$\int_\Omega f_1(v)\phi_k \, dx \to \int_\Omega f_1(v)\phi \, dx \tag{5.45}$$

and

$$\int_\Omega f_2(u)\varphi_k \, dx \to \int_\Omega f_2(u)\varphi \, dx. \tag{5.46}$$

Therefore, since $\phi, \varphi \in W_0^{1,N}(\Omega)$ are arbitrary, it follows from (5.39) - (5.46) that

$$\int_\Omega a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u \nabla \phi \, dx = \lambda_1 \int_\Omega \frac{\phi}{(u+\varepsilon)^{\beta_1}} \, dx + \int_\Omega f_1(v)\phi \, dx, \tag{5.47}$$

and

$$\int_\Omega a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v \nabla \varphi \, dx = \lambda_2 \int_\Omega \frac{\varphi}{(v+\varepsilon)^{\beta_2}} \, dx + \int_\Omega f_2(u)\varphi \, dx, \tag{5.48}$$

for all $\phi, \varphi \in W_0^{1,N}(\Omega)$, which shows that (u, v) is a weak solution of the problem (5.1). Furthermore, arguing as in the scalar case we conclude that $u, v > 0$ in Ω. □

6. Proof of the Theorem 1.2

For each $n \in \mathbf{N}$, let $\varepsilon = \frac{1}{n}$, $u_{\frac{1}{n}} = u_n$ and $v_{\frac{1}{n}} = v_n$ be, where (u_n, v_n) is a solution of auxiliary problem (5.1)

$$\begin{cases} -\mathrm{div}(a_1(|\nabla u_n|^{p_1})|\nabla u_n|^{p_1-2}\nabla u_n) = \dfrac{\lambda_1}{(u_n+\frac{1}{n})^{\beta_1}} + f_1(v_n) & \text{in } \Omega, \\ -\mathrm{div}(a_2(|\nabla v_n|^{p_2})|\nabla v_n|^{p_2-2}\nabla v_n) = \dfrac{\lambda_2}{(v_n+\frac{1}{n})^{\beta_2}} + f_2(u_n) & \text{in } \Omega, \\ u_n, v_n > 0 & \text{in } \Omega, \\ u_n, v_n = 0 & \text{on } \partial\Omega, \end{cases}$$

obtained by the Lemma 5.1. Note that, from (f_3) we obtain

$$-\mathrm{div}(a_1(|\nabla u_n|^{p_1})|\nabla u_n|^{p_1-2}\nabla u_n) \geq \frac{\lambda_1}{(u_n+v_n+1)^{\beta_1}} + |v_n|^{\gamma_1-1} \text{ in } \Omega$$

and as the function $t \mapsto \frac{\lambda_1}{(u_n+t+1)^{\beta_1}} + t^{\gamma_1-1}$, for all $t \geq 0$, attains a positive minimum z_1. Then,

$$-\mathrm{div}\left(a_1(|\nabla u_n|^{p_1})|\nabla u_n|^{p_1-2}\nabla u_n\right) \geq z_1 > 0 \text{ in } \Omega.$$

By virtue from Minty–Browder's Theorem [6, Theorem 5.15], we use the unique positive solution of the problem

$$\begin{cases} -\mathrm{div}\,(a_1(|\nabla w_1|^{p_1})|\nabla w_1|^{p_1-2}\nabla w_1) = z_1 & \text{in } \Omega, \\ w_1 > 0 & \text{in } \Omega, \\ w_1 = 0 & \text{on } \partial\Omega \end{cases} \tag{6.1}$$

to get that the solution w_1 bound from below the solution u_n. Therefore,

$$\begin{cases} -\mathrm{div}\,(a_1(|\nabla u_n|^{p_1})|\nabla u_n|^{p_1-2}\nabla u_n) \geq -\mathrm{div}\,(a_1(|\nabla w_1|^{p_1})|\nabla w_1|^{p_1-2}\nabla w_1) & \text{in } \Omega, \\ u_n = w_1 & \text{on } \partial\Omega. \end{cases}$$

Hence, we use the Lemma 2.1 to conclude that

$$u_n(x) \geq w_1(x) > 0 \text{ in } \Omega, \ \forall n \in \mathbf{N}. \tag{6.2}$$

Similarly, we prove that

$$v_n(x) \geq w_2(x) > 0 \text{ in } \Omega \ \forall n \in \mathbf{N}, \tag{6.3}$$

where w_2 satisfies

$$\begin{cases} -\mathrm{div}\,(a_2(|\nabla w_2|^{p_2})|\nabla w_2|^{p_2-2}\nabla w_2) = z_2 & \text{in } \Omega, \\ w_2 > 0 & \text{in } \Omega, \\ w_2 = 0 & \text{in } \partial\Omega \end{cases} \tag{6.4}$$

and z_2 is positive minimum of the function $t \mapsto \frac{\lambda_2}{(v_n+t+1)^{\beta_2}} + t^{\gamma_2-1}$.

Now, from (5.14) and (5.15) we get

$$u_m \rightharpoonup u_n \text{ in } W_0^{1,N}(\Omega) \text{ as } m \to +\infty$$

and

$$v_m \rightharpoonup v_n \text{ in } W_0^{1,N}(\Omega) \text{ as } m \to +\infty.$$

It follows from (5.11) that

$$\|u_n\|_{1,N} \leq \liminf_{m\to+\infty} \|u_m\|_{1,N} \leq \liminf_{m\to+\infty} \|(u_m, v_m)\| \leq r < 1, \text{ for all } n \in \mathbf{N}$$

and

$$\|v_n\|_{1,N} \le \liminf_{m\to+\infty} \|v_m\|_{1,N} \le \liminf_{m\to+\infty} \|(u_m, v_m)\| \le r < 1, \quad \text{for all } n \in \mathbf{N}.$$

Therefore, r does not depend on n, which shows that (u_n) and (v_n) are a bounded sequences. Thus, since $W_0^{1,N}(\Omega)$ is a reflective Banach space, for some subsequence, there exist $u, v \in W_0^{1,N}(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{in } W_0^{1,N}(\Omega), \\ u_n \to u & \text{in } L^\theta(\Omega),\ \theta \ge 1, \\ u_n(x) \to u(x) & \text{a.e. in } \Omega, \\ |u_n(x)| \le g_1(x) \in L^\theta(\Omega) & \text{a.e. in } \Omega,\ \theta \ge 1 \end{cases} \tag{6.5}$$

and

$$\begin{cases} v_n \rightharpoonup v & \text{in } W_0^{1,N}(\Omega), \\ v_n \to v & \text{in } L^\theta(\Omega),\ \theta \ge 1, \\ v_n(x) \to v(x) & \text{a.e. in } \Omega, \\ |v_n(x)| \le g_2(x) \in L^\theta(\Omega) & \text{a.e. in } \Omega,\ \theta \ge 1. \end{cases} \tag{6.6}$$

Recall from (5.47) and (5.48) that

$$\int_\Omega a_1(|\nabla u_n|^{p_1})|\nabla u_n|^{p_1-2}\nabla u_n \nabla\phi\, dx = \lambda_1 \int_\Omega \frac{\phi}{(u_n + \frac{1}{n})^{\beta_1}}\, dx + \int_\Omega f_1(v_n)\phi, \tag{6.7}$$

for all $\phi \in W_0^{1,N}(\Omega)\, dx$ and

$$\int_\Omega a_2(|\nabla v_n|^{p_2})|\nabla v_n|^{p_2-2}\nabla v_n \nabla\varphi\, dx = \lambda_2 \int_\Omega \frac{\varphi}{(v_n + \frac{1}{n})^{\beta_2}}\, dx + \int_\Omega f_2(u_n)\varphi\, dx, \tag{6.8}$$

for all $\varphi \in W_0^{1,N}(\Omega)$.

Since f_i are continuous functions, by (6.5) and (6.6), we have

$$f_1(v_n(x))\phi \to f_1(v(x))\phi \ \text{ a.e. in } \Omega$$

and

$$f_2(u_n(x))\varphi \to f_2(u(x))\varphi \ \text{ a.e. in } \Omega.$$

By same computation in (5.22) and (5.23), we obtain

$$|f_1(v_n(x))\phi| \le \widehat{g_1}(v_n(x))$$

and

$$|f_2(u_n(x))\varphi| \le \widehat{g_2}(u_n(x))$$

such that $\widehat{g_i}$ converge in $\in L^1(\Omega)$. Then, we use [6, Theorem 4.2] to conclude that

$$\int_\Omega f_1(v_n)\phi\, dx \to \int_\Omega f_1(v)\phi\, dx, \ \ \forall\phi \in W_0^{1,N}(\Omega) \tag{6.9}$$

and

$$\int_\Omega f_2(u_n)\varphi\, dx \to \int_\Omega f_2(u)\varphi, \ \ \forall\varphi \in W_0^{1,N}(\Omega)\, dx. \tag{6.10}$$

Now, by same computation in (5.35) and (5.36), we have

$$\int_\Omega a_1(|\nabla u_n|^{p_1})|\nabla u_n|^{p_1-2}\nabla u_n \nabla\phi\, dx \to \int_\Omega a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u \nabla\phi\, dx, \tag{6.11}$$

for all $\phi \in W_0^{1,N}(\Omega)$ and

$$\int_\Omega a_2(|\nabla v_n|^{p_2})|\nabla v_n|^{p_2-2}\nabla v_n \nabla\varphi dx \to \int_\Omega a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v\nabla\varphi dx, \tag{6.12}$$

for all $\varphi \in W_0^{1,N}(\Omega)$.

Note that, from (6.5) and (6.6) again we get

$$\frac{\phi}{\left(u_n(x)+\frac{1}{n}\right)^{\beta_1}} \to \frac{\phi}{u(x)^{\beta_1}} \quad \text{a.e. in } \Omega \tag{6.13}$$

and

$$\frac{\varphi}{\left(v_n(x)+\frac{1}{n}\right)^{\beta_2}} \to \frac{\varphi}{v(x)^{\beta_2}} \quad \text{a.e. in } \Omega. \tag{6.14}$$

Moreover, by virtue from (6.1), (6.4) and (a_1), we can argue as in [23] to obtain that $w_1, w_2 \in C^1(\overline{\Omega})$. Consequently, we use (6.1), (6.4) again to conclude

$$\frac{\partial w_1}{\partial\eta}, \frac{\partial w_2}{\partial\eta} < 0 \quad \text{on } \partial\Omega.$$

Then, for each $x \in \Omega$, it follows from (6.2) and (6.3) that

$$u_n(x) \geq w_1(x) > Cd(x) > 0 \quad \text{and} \quad v_n(x) \geq w_2(x) > Cd(x) > 0,$$

where $d(x) = \text{dist}(x, \partial\Omega)$ and C is positive constant that does not depend on x. Thus,

$$\int_\Omega \frac{\phi}{(u_n(x)+\frac{1}{n})^{\beta_1}}\,dx \leq \int_\Omega \frac{\phi}{u_n(x)^{\beta_1}}\,dx \leq \int_\Omega \frac{\phi}{Cd(x)^{\beta_1}}\,dx$$

and

$$\int_\Omega \frac{\varphi}{(v_n(x)+\frac{1}{n})^{\beta_2}}\,dx \leq \int_\Omega \frac{\varphi}{v_n(x)^{\beta_2}}\,dx \leq \int_\Omega \frac{\varphi}{Cd(x)^{\beta_2}}\,dx.$$

Hence, we invoke Theorem 2.2 to obtain $\left|\frac{\phi}{Cd(x)^{\beta_1}}\right|, \left|\frac{\varphi}{Cd(x)^{\beta_2}}\right| \in L^r(\Omega)$ and $C_{17}, C_{18} > 0$ such that

$$\int_\Omega \frac{\phi}{(u_n(x)+\frac{1}{n})^{\beta_1}}\,dx \leq C_{17}\|\phi\|_{1,N}. \tag{6.15}$$

and

$$\int_\Omega \frac{\varphi}{(v_n(x)+\frac{1}{n})^{\beta_2}}\,dx \leq C_{18}\|\varphi\|_{1,N}. \tag{6.16}$$

Therefore, by (6.13), (6.14), (6.15), (6.16) and [6, Theorem 4.2] we get

$$\int_\Omega \frac{\phi}{\left(u_n+\frac{1}{n}\right)^{\beta_1}}\,dx \to \int_\Omega \frac{\phi}{u^{\beta_1}}\,dx, \ \ \forall\phi \in W_0^{1,N}(\Omega) \tag{6.17}$$

and

$$\int_\Omega \frac{\varphi}{\left(v_n+\frac{1}{n}\right)^{\beta_2}}\,dx \to \int_\Omega \frac{\varphi}{v^{\beta_2}}\,dx, \ \ \forall\varphi \in W_0^{1,N}(\Omega). \tag{6.18}$$

Letting $n \to +\infty$ in (6.7) and (6.8), we use (6.9), (6.10), (6.11), (6.12), (6.17) and (6.18) to conclude that

$$\int_\Omega a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u\nabla\phi\,dx = \lambda_1\int_\Omega \frac{\phi}{u^{\beta_1}}\,dx + \int_\Omega f_1(v)\phi\,dx, \quad \forall\phi \in W_0^{1,N}(\Omega)$$

and

$$\int_\Omega a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v\nabla\varphi\,dx = \lambda_2\int_\Omega \frac{\varphi}{v^{\beta_2}}\,dx + \int_\Omega f_2(u)\varphi\,dx, \quad \forall\varphi\in W_0^{1,N}(\Omega),$$

which proves that $(u,v)\in W^{1,N}(\Omega)\times W^{1,N}(\Omega)$ is a weak solution of the problem (1.2).

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Received 27 February 2020 • Accepted 25 May 2020