

RECTIFIABILITY OF $\text{RCD}(K, N)$ SPACES VIA δ -SPLITTING MAPS

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Abstract. In this note we give simplified proofs of rectifiability of $\text{RCD}(K, N)$ spaces as metric measure spaces and lower semicontinuity of the essential dimension, via δ -splitting maps. The arguments are inspired by the Cheeger–Colding theory for Ricci limits and rely on the second order differential calculus developed by Gigli and on the convergence and stability results by Ambrosio–Honda.

Introduction

In the last years the study of $\text{RCD}(K, N)$ metric measure spaces has undergone a fast development. After the introduction of the *curvature-dimension* condition $\text{CD}(K, N)$ in the independent works [40, 41] and [36], the notion of $\text{RCD}(K, N)$ space was proposed in [26] as a finite-dimensional counterpart of $\text{RCD}(K, \infty)$, introduced in [3] (see also [2] for the case of σ -finite reference measure and [10] for the introduction of the *reduced curvature-dimension* condition $\text{CD}^*(K, N)$). In the infinite-dimensional case the equivalence of the original Lagrangian approach with an Eulerian one, based on the Bochner inequality, was studied in [4]. Then [25] established equivalence with the dimensional Bochner inequality for the so-called class $\text{RCD}^*(K, N)$ (see also [9]). Equivalence between $\text{RCD}^*(K, N)$ and $\text{RCD}(K, N)$ has been eventually achieved in [13] in the case of finite reference measure, closing the circle. Apart from smooth weighted Riemannian manifolds (with generalized Ricci tensor bounded from below), the $\text{RCD}(K, N)$ class includes Ricci limit spaces, whose study was initiated by Cheeger–Colding in the nineties [17, 18, 19] (see also the survey [15]), and Alexandrov spaces [38]. We refer the reader to [1] for an account about this quickly developing research area.

Many efforts have been recently aimed at understanding the structure theory of $\text{RCD}(K, N)$ spaces. After [37] by Mondino–Naber, we know that they are rectifiable as metric spaces and later, in the three independent works by Kell–Mondino, De Philippis–Marchese–Rindler and Gigli together with the second named author [23, 34, 31], the analysis was sharpened taking into account the behaviour of the reference measure and getting rectifiability as metric measure spaces. Moreover, in the recent [12], the first and the third named authors proved that $\text{RCD}(K, N)$ spaces have constant dimension, in the almost everywhere sense.

<https://doi.org/10.5186/aasfm.2021.4627>

2020 Mathematics Subject Classification: Primary 53C23.

Key words: Rectifiability, RCD space, tangent cone.

The development of this theory was inspired in turn by the results obtained for Ricci limit spaces in the seminal papers by Cheeger–Colding (see also [22] by Colding–Naber for constancy of dimension).

In the proofs given in [17, 19] a crucial role was played by (k, δ) -splitting maps:

Definition 0.1. Let $(X, \mathbf{d}, \mathbf{m})$ be an $\text{RCD}(-1, N)$ space. Let $x \in X$ and $\delta > 0$ be given. Then a map $u = (u_1, \dots, u_k): B_r(x) \rightarrow \mathbf{R}^k$ is said to be a (k, δ) -splitting map provided:

- i) $u_a: B_r(x) \rightarrow \mathbf{R}$ is harmonic and C_N -Lipschitz for every $a = 1, \dots, k$,
- ii) $r^2 \int_{B_r(x)} |\text{Hess}(u_a)|^2 \, \mathbf{d}\mathbf{m} \leq \delta$ for every $a = 1, \dots, k$,
- iii) $\int_{B_r(x)} |\nabla u_a \cdot \nabla u_b - \delta_{ab}| \, \mathbf{d}\mathbf{m} \leq \delta$ for every $a, b = 1, \dots, k$.

These maps provide approximations, in the integral L^2 -sense and up to the second order, of k independent coordinate functions in the Euclidean space. They were introduced in [16], in the study of Riemannian manifolds with lower Ricci curvature bounds.

Item ii) in the definition of δ -splitting maps is about smallness of the L^2 -norm of the Hessian, in scale invariant sense. In [17, 19] and in more recent works dealing with Ricci limits as [20], δ -splitting maps are built only at the level of the smooth approximating sequence, where there is a clear notion of Hessian available, the metric information they encode (ε -GH closeness to Euclidean spaces) is then passed to the limit.

Prior than [27], there was no notion of Hessian available in the RCD framework. This, together with the absence of smooth approximating sequences, motivated the necessity to find an alternative approach to rectifiability in [37, 23, 34, 31] with respect to the Cheeger–Colding theory. A new *almost splitting via excess* theorem was the main ingredient playing the role of the theory of δ -splitting maps in [37] while, studying the behaviour of the reference measure with respect to charts, a crucial role was played in both [23, 34, 31], by a recent and powerful result obtained by De Philippis–Rindler [24].

Nowadays we have at our disposal both a second order differential calculus on RCD spaces [27] and general convergence and stability results for Sobolev functions on converging sequences of $\text{RCD}(K, N)$ spaces [5, 6, 30]. In our previous paper [11] we exploited all these tools to prove rectifiability for reduced boundaries of sets of finite perimeter in this context. The study of [11] was devoted to the theory in codimension one, which required some additional ideas and technical efforts, but it was evident that similar arguments could provide more direct proofs of rectifiability for $\text{RCD}(K, N)$ spaces in the spirit of those in [17, 19].

Taking as a starting point existence of Euclidean tangents almost everywhere with respect to the reference measure, obtained by Gigli–Mondino–Rajala in [29], in this short note we provide the arguments to get uniqueness (almost everywhere) of tangents and rectifiability of $\text{RCD}(K, N)$ spaces as metric measure spaces via δ -splitting maps. Moreover, we recover via a different strategy the result about lower semicontinuity of the so called essential dimension proved firstly in [35]. After Section 1, dedicated to review some preliminaries and establish the basic tools about propagation of the δ -splitting property, the remaining Subsection 2.1, Subsection 2.3 and Subsection 2.4 are devoted to uniqueness of tangents and lower semicontinuity of the essential dimension, metric rectifiability and the behaviour of the reference measure under charts, respectively. In order to achieve the absolute continuity of the

reference measure with respect to the relevant Hausdorff measure we rely on [24], as in the previous approaches.

Acknowledgements. The authors wish to thank Luigi Ambrosio, Nicola Gigli, Andrea Mondino and Tapio Rajala for useful comments on an earlier version of this note. The second named author was partially supported by the Academy of Finland, projects 307333 and 314789. Part of this work was developed while the first and third named authors were visiting the Department of Mathematics and Statistics of the University of Jyväskylä: they wish to thank the institute for the excellent working conditions and the stimulating atmosphere. The authors are grateful to the anonymous reviewers whose detailed reports greatly improved the present note.

1. Preliminaries and notation

1.1. Differential calculus on metric measure spaces. For our purposes, a *metric measure space* is a triple $(X, \mathbf{d}, \mathbf{m})$, where (X, \mathbf{d}) is a proper metric space, while $\mathbf{m} \geq 0$ is a Radon measure on X such that $\mathbf{m}(X) \neq 0$. Given a Lipschitz function $f: X \rightarrow \mathbf{R}$, we will denote by $\text{lip}(f): X \rightarrow [0, +\infty)$ its *slope*, which is the function defined as

$$\text{lip}(f)(x) := \overline{\lim}_{y \rightarrow x} \frac{|f(x) - f(y)|}{\mathbf{d}(x, y)} \quad \text{for every accumulation point } x \in X$$

and $\text{lip}(f)(x) := 0$ elsewhere. Given any open set $\Omega \subseteq X$, we denote by $\text{LIP}_c(\Omega)$ the family of all Lipschitz functions $f: \Omega \rightarrow \mathbf{R}$ whose support is bounded and satisfies $\text{dist}(\text{spt}(f), X \setminus \Omega) > 0$.

1.1.1. Sobolev space. Following [14], we define the *Sobolev space* $H^{1,2}(X, \mathbf{d}, \mathbf{m})$ as

$$H^{1,2}(X, \mathbf{d}, \mathbf{m}) := \{f \in L^2(\mathbf{m}) \mid \text{Ch}(f) < +\infty\},$$

where the *Cheeger energy* $\text{Ch}: L^2(\mathbf{m}) \rightarrow [0, +\infty]$ is the convex, lower semicontinuous functional

$$\text{Ch}(f) := \inf \left\{ \underbrace{\liminf}_{n \rightarrow \infty} \int \text{lip}^2(f_n) \, \mathbf{d}\mathbf{m} \mid \begin{array}{l} (f_n)_n \subseteq L^2(\mathbf{m}) \text{ bounded Lipschitz,} \\ \lim_{n \rightarrow \infty} \|f_n - f\|_{L^2(\mathbf{m})} = 0 \end{array} \right\}.$$

It holds that $H^{1,2}(X, \mathbf{d}, \mathbf{m})$ is a Banach space if endowed with the norm $\|\cdot\|_{H^{1,2}(X, \mathbf{d}, \mathbf{m})}$, given by

$$\|f\|_{H^{1,2}(X, \mathbf{d}, \mathbf{m})} := \left(\|f\|_{L^2(\mathbf{m})}^2 + \text{Ch}(f) \right)^{1/2} \quad \text{for every } f \in H^{1,2}(X, \mathbf{d}, \mathbf{m}).$$

Given any $f \in H^{1,2}(X, \mathbf{d}, \mathbf{m})$, one can select a canonical object $|Df| \in L^2(\mathbf{m})$ —called the *minimal relaxed slope* of f —for which $\text{Ch}(f)$ admits the integral representation $\text{Ch}(f) = \int |Df|^2 \, \mathbf{d}\mathbf{m}$.

We have chosen to stress the dependence on the measure for the gradient and the other differential objects, here and in the sequel, to avoid confusion.

Given an open set $\Omega \subseteq X$, we define $H_{\text{loc}}^{1,2}(\Omega, \mathbf{d}, \mathbf{m})$ as the space of all those $f \in L^2_{\text{loc}}(\Omega)$ such that $\eta f \in H^{1,2}(X, \mathbf{d}, \mathbf{m})$ holds for every $\eta \in \text{LIP}_c(\Omega)$. Thanks to the locality property of the minimal relaxed slope, it makes sense to define $|Df| \in L^2_{\text{loc}}(\Omega)$ as

$$|Df| := |D(\eta f)| \quad \mathbf{m}\text{-a.e. on } \{\eta = 1\}, \quad \text{for any } \eta \in \text{LIP}_c(\Omega).$$

Finally, we define $H^{1,2}(\Omega, \mathbf{d}, \mathbf{m})$ as the space of all $f \in H_{\text{loc}}^{1,2}(\Omega, \mathbf{d}, \mathbf{m})$ such that $f, |Df| \in L^2(\Omega)$.

1.1.2. Tangent module. Whenever $H^{1,2}(X, \mathbf{d}, \mathbf{m})$ is a Hilbert space, we will say that $(X, \mathbf{d}, \mathbf{m})$ is *infinitesimally Hilbertian*. In this case, we recall from [27] that the *tangent module* $L^2(TX)$ and the corresponding *gradient* map $\nabla: H^{1,2}(X, \mathbf{d}, \mathbf{m}) \rightarrow L^2(TX)$ can be characterised as follows: $L^2(TX)$ is an $L^2(\mathbf{m})$ -normed $L^\infty(\mathbf{m})$ -module (in the sense of [28, Definition 1.3]) that is generated by $\{\nabla f: f \in H^{1,2}(X, \mathbf{d}, \mathbf{m})\}$, while ∇ is a linear map satisfying $|\nabla f| = |Df|$ \mathbf{m} -a.e. on X for all $f \in H^{1,2}(X, \mathbf{d}, \mathbf{m})$. The pointwise scalar product $L^2(TX) \times L^2(TX) \ni (v, w) \mapsto v \cdot w \in L^1(\mathbf{m})$,

$$v \cdot w := \frac{|v+w|^2 - |v|^2 - |w|^2}{2} \quad \text{for every } v, w \in L^2(TX),$$

is a symmetric bilinear form, as a consequence of the infinitesimal Hilbertianity assumption.

The dual module of $L^2(TX)$ is denoted by $L^2(T^*X)$ and called the *cotangent module* of X .

In the framework of weighted Euclidean spaces, we have another notion of tangent module at our disposal. Given a Radon measure $\nu \geq 0$ on \mathbf{R}^k , we denote by $L^2(\mathbf{R}^k, \mathbf{R}^k; \nu)$ the space of all $L^2(\nu)$ -maps from \mathbf{R}^k to itself. It turns out that $L^2(\mathbf{R}^k, \mathbf{R}^k; \nu)$ is an $L^2(\nu)$ -normed $L^\infty(\nu)$ -module generated by $\{\nabla f: f \in C_c^\infty(\mathbf{R}^k)\}$, where $\nabla f: \mathbf{R}^k \rightarrow \mathbf{R}^k$ stands for the ‘classical’ gradient of f .

1.1.3. Divergence and Laplacian. In the setting of infinitesimally Hilbertian spaces $(X, \mathbf{d}, \mathbf{m})$, one can consider the following notions of divergence and Laplacian:

- *Divergence.* We declare that $v \in L^2(TX)$ belongs to $D(\text{div})$ provided there exists a (uniquely determined) function $\text{div}(v) \in L^2(\mathbf{m})$ such that

$$\int \nabla f \cdot v \, \mathbf{d}\mathbf{m} = - \int f \, \text{div}(v) \, \mathbf{d}\mathbf{m} \quad \text{for every } f \in H^{1,2}(X, \mathbf{d}, \mathbf{m}).$$

- *Laplacian.* Given any open set $\Omega \subseteq X$, we declare that $f \in H^{1,2}(\Omega, \mathbf{d}, \mathbf{m})$ belongs to $D(\Omega, \Delta)$ provided there exists a (uniquely determined) function $\Delta f \in L^2(\Omega)$ such that

$$\int_{\Omega} \nabla f \cdot \nabla g \, \mathbf{d}\mathbf{m} = - \int_{\Omega} g \, \Delta f \, \mathbf{d}\mathbf{m} \quad \text{for every } g \in H_0^{1,2}(\Omega, \mathbf{d}, \mathbf{m}),$$

where $H_0^{1,2}(\Omega, \mathbf{d}, \mathbf{m})$ stands for the closure of $\text{LIP}_c(\Omega)$ in $H^{1,2}(X, \mathbf{d}, \mathbf{m})$. For brevity, we shall write $D(\Delta)$ in place of $D(X, \Delta)$.

The domains $D(\text{div})$ and $D(\Omega, \Delta)$ are vector subspaces of $L^2(TX)$ and $H^{1,2}(\Omega, \mathbf{d}, \mathbf{m})$, respectively. Moreover, the operators $\text{div}: D(\text{div}) \rightarrow L^2(\mathbf{m})$ and $\Delta: D(\Omega, \Delta) \rightarrow L^2(\Omega)$ are linear.

It can be readily checked that a given function $f \in H^{1,2}(X, \mathbf{d}, \mathbf{m})$ belongs to $D(\Delta)$ if and only if its gradient ∇f belongs to $D(\text{div})$. In this case, it also holds that $\Delta f = \text{div}(\nabla f)$.

1.2. RCD spaces. We assume the reader to be familiar with the language of $\text{RCD}(K, N)$ spaces and the notion of pointed measured Gromov–Hausdorff convergence (often abbreviated to pmGH).

We recall the following scaling property: if $(X, \mathbf{d}, \mathbf{m})$ is an $\text{RCD}(K, N)$ space, then $(X, \mathbf{d}/r, \lambda \mathbf{m})$ is an $\text{RCD}(r^2 K, N)$ space for any choice of $r, \lambda > 0$. Furthermore, there

exists a distance d_{pmGH} on the set (of isomorphism classes) of $\text{RCD}(K, N)$ spaces that metrises the pmGH-topology [30].

Remark 1.1. Any sequence (X_n, d_n, m_n, x_n) , $n \in \mathbf{N}$ of pointed $\text{RCD}(K, N)$ spaces such that $m_n(B_1(x_n))$ are uniformly bounded from below and from above by positive constants converges, up to the extraction of a subsequence, to some pointed $\text{RCD}(K, N)$ space (X, d, m, x) with respect to the pmGH-topology. This follows from a compactness argument due to Gromov and the stability of the $\text{RCD}(K, N)$ condition. ■

1.2.1. Test functions. Let (X, d, m) be an $\text{RCD}(K, N)$ space. A fundamental class of Sobolev functions on X is given by the algebra of *test functions* [27, 39]:

$$\text{Test}^\infty(X) := \left\{ f \in D(\Delta) \cap L^\infty(m) \mid |Df| \in L^\infty(m), \Delta f \in H^{1,2}(X, d, m) \cap L^\infty(m) \right\}.$$

With respect to the original definitions of the class $\text{Test}(X)$ we have added the assumption $\Delta f \in L^\infty$. Since RCD spaces enjoy the Sobolev-to-Lipschitz property, we know that any element of $\text{Test}^\infty(X)$ admits a Lipschitz representative. Moreover, it holds that $\text{Test}^\infty(X)$ is dense in $H^{1,2}(X, d, m)$ and that $\nabla f \cdot \nabla g \in H^{1,2}(X, d, m)$ for every $f, g \in \text{Test}^\infty(X)$.

Lemma 1.2. (Good cut-off functions [8, 37]) *Let (X, d, m) be an $\text{RCD}(K, N)$ space. Let $0 < r < R$ and $x \in X$. Then there exists $\eta \in \text{Test}^\infty(X)$ such that $0 \leq \eta \leq 1$ on X , the support of η is compactly contained in $B_R(x)$, and $\eta = 1$ on $B_r(x)$.*

We recall the notion of *Hessian* of a test function [27]: given $f \in \text{Test}^\infty(X)$, we denote by $\text{Hess}(f)$ the unique element of the tensor product $L^2(T^*X) \otimes L^2(T^*X)$ (cf. [27, Section 1.5]) such that

$$\begin{aligned} & 2 \int h \text{Hess}f(\nabla g_1 \otimes \nabla g_2) \, dm \\ &= - \int \nabla f \cdot \nabla g_1 \text{div}(h \nabla g_2) + \nabla f \cdot \nabla g_2 \text{div}(h \nabla g_1) + h \nabla f \cdot \nabla(\nabla g_1 \cdot \nabla g_2) \, dm \end{aligned}$$

holds for every $h, g_1, g_2 \in \text{Test}^\infty(X)$. The pointwise norm $|\text{Hess}(f)|$ of $\text{Hess}(f)$ belongs to $L^2(m)$.

Given an open set $\Omega \subseteq X$ and a function $f \in D(\Omega, \Delta)$, we say that f is *harmonic* if $\Delta f = 0$. Let us recall that, as it has been proved in [33], harmonic functions are locally Lipschitz in the interior of Ω . Under the assumption $\Delta f = 0$ we can define (the modulus of) its Hessian as follows:

$$(1.1) \quad \begin{aligned} |\text{Hess}(f)| &:= |\text{Hess}(\eta f)| \quad \mathbf{m}\text{-a.e. on } \{\eta = 1\}, \\ &\text{for every } \eta \in \text{Test}^\infty(X) \text{ with } \text{spt}(\eta) \subseteq \Omega. \end{aligned}$$

This way we obtain a well-defined function $|\text{Hess}(f)|: \Omega \rightarrow [0, +\infty)$, thanks to the locality property of the Hessian and the fact that $\eta f \in \text{Test}^\infty(X)$ for every η as in (1.1).

1.3. Splitting maps on RCD spaces. In this subsection we collect the main properties of δ -splitting maps that we will use in the sequel. Let us recall that their introduction in the study of spaces with lower Ricci curvature bounds dates back to [16] and that they have been extensively used in [17, 18, 19] and in more recent works on Ricci limits [21, 20]. Before the development of the second order calculus in [27], there was need for alternative arguments avoiding the use of the Hessian in

order to develop the structure theory of $\text{RCD}(K, N)$ spaces in [37]. In recent times (see [7, 11]) δ -splitting maps have come into play also in the RCD-theory thanks to [27] and the stability results of [5, 6, 30].

The results connecting δ -splitting maps with ε -isometries stated below are borrowed from [11]. Although being less local than those provided by the Cheeger–Colding theory, they are sufficient for our purposes and allow for more direct proofs via compactness. We refer to (2.1) below for the introduction of the normalized measure \mathbf{m}_x^r that plays a role in the following statements.

Definition 1.3. (Splitting map [11]) Let $(X, \mathbf{d}, \mathbf{m})$ be an $\text{RCD}(-1, N)$ space. Let $x \in X$ and $\delta > 0$ be given. Then a map $u = (u_1, \dots, u_k): B_r(x) \rightarrow \mathbf{R}^k$ is said to be a δ -splitting map provided:

- i) $u_a: B_r(x) \rightarrow \mathbf{R}$ is harmonic and C_N -Lipschitz for every $a = 1, \dots, k$,
- ii) $r^2 \int_{B_r(x)} |\text{Hess}(u_a)|^2 \, \mathbf{d}\mathbf{m} \leq \delta$ for every $a = 1, \dots, k$,
- iii) $\int_{B_r(x)} |\nabla u_a \cdot \nabla u_b - \delta_{ab}| \, \mathbf{d}\mathbf{m} \leq \delta$ for every $a, b = 1, \dots, k$.

It is worth clarifying that three conditions in definition 1.3 are not independent, whenever we work under the stronger assumption that $(X, \mathbf{d}, \mathbf{m})$ is $\text{RCD}(-\delta, N)$. Indeed in this case (ii) follows from (i) and (iii) by means of the *improved Bochner inequality* [32, Theorem 3.3]. Although it is redundant we prefer to keep condition (ii) in the definition for the sake of clarity and for consistency with the literature on Ricci limits.

Proposition 1.4. (From GH-isometry to δ -splitting [11]) *Let $N > 1$ be given. Then for any $\delta > 0$ there exists $\varepsilon = \varepsilon_{N, \delta} > 0$ such that the following property holds. If $(X, \mathbf{d}, \mathbf{m})$ is an $\text{RCD}(K, N)$ space, $x \in X$, $r > 0$ with $r^2|K| \leq \varepsilon$, and there is an $\text{RCD}(0, N - k)$ space $(Z, \mathbf{d}_Z, \mathbf{m}_Z, z)$ such that*

$$d_{\text{pmGH}}\left((X, \mathbf{d}/r, \mathbf{m}_x^r, x), (\mathbf{R}^k \times Z, \mathbf{d}_{\text{Eucl}} \times \mathbf{d}_Z, \mathcal{L}^k \otimes \mathbf{m}_Z, (0^k, z))\right) \leq \varepsilon,$$

then there exists a δ -splitting map $u: B_{5r}(x) \rightarrow \mathbf{R}^k$.

Proposition 1.5. (From δ -splitting to GH-isometry [11]) *Let $N > 1$ be given. Then for any $\varepsilon > 0$ there exists $\delta = \delta_{N, \varepsilon} > 0$ such that the following property holds. If $(X, \mathbf{d}, \mathbf{m})$ is an $\text{RCD}(K, N)$ space, $x \in X$, and there exist $r > 0$ and a map $u: B_r(x) \rightarrow \mathbf{R}^k$ such that $u: B_s(x) \rightarrow \mathbf{R}^k$ is a δ -splitting map for all $s < r$, then for any $(Y, \varrho, \mathbf{n}, y) \in \text{Tan}_x(X, \mathbf{d}, \mathbf{m})$ it holds that*

$$d_{\text{pmGH}}\left((Y, \varrho, \mathbf{n}, y), (\mathbf{R}^k \times Z, \mathbf{d}_{\text{Eucl}} \times \mathbf{d}_Z, \mathcal{L}^k \otimes \mathbf{m}_Z, (0^k, z))\right) \leq \varepsilon,$$

for some pointed $\text{RCD}(0, N - k)$ space $(Z, \mathbf{d}_Z, \mathbf{m}_Z, z)$.

Below we state and prove a result about propagation of the δ -splitting property at many locations with respect to the reference measure and at all scales. The proof is based on a standard maximal function argument.

Proposition 1.6. (Propagation of the δ -splitting property) *Let $N > 1$ be given. Then there exists a constant $C_N > 0$ such that the following property holds. If $(X, \mathbf{d}, \mathbf{m})$ is an $\text{RCD}(K, N)$ space and $u: B_{2r}(p) \rightarrow \mathbf{R}^k$ is a δ -splitting map for some $p \in X$, $r > 0$ with $r^2|K| \leq 1$, and $\delta \in (0, 1)$, then there exists a Borel set $G \subseteq B_r(p)$ such that $\mathbf{m}(B_r(p) \setminus G) \leq C_N \sqrt{\delta} \mathbf{m}(B_r(p))$ and*

$$u: B_s(x) \rightarrow \mathbf{R}^k \text{ is a } \sqrt{\delta}\text{-splitting map, for every } x \in G \text{ and } s \in (0, r).$$

Proof. Thanks to a scaling argument, it is sufficient to prove the claim for $r = 1$ and $|K| \leq 1$. Let us define $G \subseteq B_1(p)$ as $G := \bigcap_{a=1}^k G_a \cap \bigcap_{a,b=1}^k G_{a,b}$, where we set

$$G_a := \left\{ x \in B_1(p) \mid \sup_{s \in (0,1)} \int_{B_s(x)} |\text{Hess}(u_a)|^2 \, \mathbf{m} \leq \sqrt{\delta} \right\},$$

$$G_{a,b} := \left\{ x \in B_1(p) \mid \sup_{s \in (0,1)} \int_{B_s(x)} |\nabla u_a \cdot \nabla u_b - \delta_{ab}| \, \mathbf{m} \leq \sqrt{\delta} \right\}.$$

It holds that $u : B_s(x) \rightarrow \mathbf{R}^k$ is a $\sqrt{\delta}$ -splitting map for all $x \in G$ and $s \in (0, 1)$. To prove the claim, it remains to show that $\mathbf{m}(B_1(p) \setminus G_a), \mathbf{m}(B_1(p) \setminus G_{a,b}) \leq C_N \sqrt{\delta} \, \mathbf{m}(B_1(p))$ for all $a, b = 1, \dots, k$.

Given any $x \in B_1(p) \setminus G_a$, we can choose $s_x \in (0, 1)$ such that

$$\int_{B_{s_x}(x)} |\text{Hess}(u_a)|^2 \, \mathbf{m} > \sqrt{\delta}.$$

By Vitali covering lemma, we can find a sequence $(x_i)_i \subseteq B_1(p) \setminus G_a$ such that $\{B_{s_{x_i}}(x_i)\}_i$ are pairwise disjoint and $B_1(p) \setminus G_a \subseteq \bigcup_i B_{5s_{x_i}}(x_i)$. Therefore, it holds that

$$\begin{aligned} \mathbf{m}(B_1(p) \setminus G_a) &\leq \sum_{i \in \mathbf{N}} \mathbf{m}(B_{5s_{x_i}}(x_i)) \leq C_N \sum_{i \in \mathbf{N}} \mathbf{m}(B_{s_{x_i}}(x_i)) \\ &\leq \frac{C_N}{\sqrt{\delta}} \sum_{i \in \mathbf{N}} \int_{B_{s_{x_i}}(x_i)} |\text{Hess}(u_a)|^2 \, \mathbf{m} \\ &\leq \frac{C_N \mathbf{m}(B_2(p))}{\sqrt{\delta}} \int_{B_2(p)} |\text{Hess}(u_a)|^2 \, \mathbf{m} \leq C_N \sqrt{\delta} \, \mathbf{m}(B_1(p)), \end{aligned}$$

where we used the doubling property of \mathbf{m} , the defining property of s_{x_i} and the fact that u is a δ -splitting map on $B_2(p)$. An analogous argument shows that $\mathbf{m}(B_1(p) \setminus G_{a,b}) \leq C_N \sqrt{\delta} \, \mathbf{m}(B_1(p))$ for all $a, b = 1, \dots, k$, thus the statement is achieved. \square

2. Structure theory for RCD spaces

Given a pointed $\text{RCD}(K, N)$ space $(X, \mathbf{d}, \mathbf{m}, x)$ and a radius $r \in (0, 1)$, we define the normalised measure \mathbf{m}_r^x on X as

$$(2.1) \quad \mathbf{m}_r^x := \frac{\mathbf{m}}{C(x, r)}, \quad \text{where } C(x, r) := \int_{B_r(x)} \left(1 - \frac{\mathbf{d}(\cdot, x)}{r} \right) \, \mathbf{m}.$$

We shall indicate by $\text{Tan}_x(X, \mathbf{d}, \mathbf{m})$ the family of all *tangent cones* at x , i.e., those spaces $(Y, \varrho, \mathbf{n}, y)$ such that

$$\lim_{n \rightarrow \infty} \mathbf{d}_{\text{pmGH}}((X, \mathbf{d}/r_n, \mathbf{m}_{r_n}^x, x), (Y, \varrho, \mathbf{n}, y)) = 0$$

for some sequence $(r_n)_n \subseteq (0, 1)$ of radii with $r_n \searrow 0$. It follows from the scaling property of the RCD condition and Remark 1.1 that any tangent cone is a pointed $\text{RCD}(0, N)$ space.

Let us briefly recall the properties that we take as a starting point for our analysis of the structure theory of $\text{RCD}(K, N)$ spaces. The first one is a version of the *iterated tangent property* suited for this setting. Building upon this, in [29] it was proved that at almost every point there exists at least one Euclidean space in the tangent cone, on $\text{RCD}(K, N)$ spaces (see Theorem 2.2 below).

Theorem 2.1. (Iterated tangent property [29]) *Let (X, d, \mathbf{m}) be an $\text{RCD}(K, N)$ space. Then for \mathbf{m} -a.e. point $x \in X$ it holds that*

$$\text{Tan}_z(Y, \varrho, \mathbf{n}) \subseteq \text{Tan}_x(X, d, \mathbf{m}) \quad \text{for every } (Y, \varrho, \mathbf{n}, y) \in \text{Tan}_x(X, d, \mathbf{m}) \text{ and } z \in Y.$$

Theorem 2.2. (Euclidean tangents to RCD spaces [29]) *Let (X, d, \mathbf{m}) be an $\text{RCD}(K, N)$ space. Then for \mathbf{m} -a.e. point $x \in X$ there exists $k(x) \in \mathbf{N}$ with $k(x) \leq N$ such that*

$$(\mathbf{R}^{k(x)}, d_{\text{Eucl}}, c_{k(x)}\mathcal{L}^{k(x)}, 0^{k(x)}) \in \text{Tan}_x(X, d, \mathbf{m}),$$

where we set $c_k := \mathcal{L}^k(B_1(0^k))/(k + 1)$ for every $k \in \mathbf{N}$.

2.1. Uniqueness of tangent cones. Let (X, d, \mathbf{m}) be an $\text{RCD}(K, N)$ space. Then we define

$$\mathcal{R}_k := \left\{ x \in X \mid \text{Tan}_x(X, d, \mathbf{m}) = \{(\mathbf{R}^k, d_{\text{Eucl}}, c_k\mathcal{L}^k, 0^k)\} \right\}$$

for every $k \in \mathbf{N}$ with $k \leq N$. With a terminology borrowed from [17] and inspired by geometric measure theory, points in \mathcal{R}_k are called *k-regular* points of X . Moreover, given any point $x \in X$ and any $k \in \mathbf{N}$, we say that an element $(Y, \varrho, \mathbf{n}, y) \in \text{Tan}_x(X, d, \mathbf{m})$ *splits off a factor \mathbf{R}^k* provided

$$(2.2) \quad (Y, \varrho, \mathbf{n}, y) \cong (\mathbf{R}^k \times Z, d_{\text{Eucl}} \times d_Z, \mathcal{L}^k \otimes \mathbf{m}_Z, (0^k, z)).$$

Observe that, if (2.2) holds, $(Z, d_Z, \mathbf{m}_Z, z)$ has to be a pointed $\text{RCD}(0, N - k)$ metric measure space.

In [37] uniqueness of tangents (almost everywhere w.r.t. the reference measure \mathbf{m}) was proved together with rectifiability relying on a new δ -splitting via excess theorem (cf. [37, Theorem 6.7] and [37, Theorem 5.1]). Below we provide a simplified proof of uniqueness of tangents based on the same principle about propagation of regularity but more similar to the one given in [17] for Ricci limits. Part of the argument resembles the proof of the boundedness of the maximal function in L^2 on doubling metric measure spaces.

Theorem 2.3. (Uniqueness of tangents) *Let (X, d, \mathbf{m}) be an $\text{RCD}(K, N)$ space. Then it holds*

$$\mathbf{m}\left(X \setminus \bigcup_{k \leq N} \mathcal{R}_k\right) = 0.$$

Proof. Step 1. Fix any $k \in \mathbf{N}$ with $k \leq N$. We define the auxiliary sets $A_k, A'_k \subseteq X$ as follows:

- i) A_k is the set of all points $x \in X$ such that $(\mathbf{R}^k, d_{\text{Eucl}}, c_k\mathcal{L}^k, 0^k) \in \text{Tan}_x(X, d, \mathbf{m})$, but no other element of $\text{Tan}_x(X, d, \mathbf{m})$ splits off a factor \mathbf{R}^k .
- ii) A'_k is the set of all points $x \in X$ which satisfy $(\mathbf{R}^k, d_{\text{Eucl}}, c_k\mathcal{L}^k, 0^k) \in \text{Tan}_x(X, d, \mathbf{m})$ and $(\mathbf{R}^\ell, d_{\text{Eucl}}, c_\ell\mathcal{L}^\ell, 0^\ell) \notin \text{Tan}_x(X, d, \mathbf{m})$ for every $\ell \in \mathbf{N}$ with $\ell > k$.

Observe that $\mathcal{R}_k \subseteq A_k \subseteq A'_k$. The \mathbf{m} -measurability of the sets \mathcal{R}_k, A_k, A'_k can be proven adapting the proof of [37, Lemma 6.1]. It also follows from the very definition of the sets A'_k that $X \setminus \bigcup_{k \leq N} A'_k = \emptyset$.

Step 2. We aim to prove that $\mathbf{m}(A'_k \setminus A_k) = 0$. We argue by contradiction: suppose $\mathbf{m}(A'_k \setminus A_k) > 0$. Then we can find $x \in A'_k \setminus A_k$ where the iterated tangent property of Theorem 2.1 holds. Since $x \notin A_k$, there exists a pointed $\text{RCD}(0, N - k)$ space $(Y, \varrho, \mathbf{n}, y)$ with $\text{diam}(Y) > 0$ such that

$$(\mathbf{R}^k \times Y, d_{\text{Eucl}} \times \varrho, \mathcal{L}^k \otimes \mathbf{n}, (0^k, y)) \in \text{Tan}_x(X, d, \mathbf{m}).$$

Theorem 2.2 yields the existence of a point $z \in Y$ such that $(\mathbf{R}^\ell, \mathbf{d}_{\text{Eucl}}, c_\ell \mathcal{L}^\ell, 0^\ell) \in \text{Tan}_z(Y, \varrho, \mathbf{n})$, for some $\ell \in \mathbf{N}$ with $0 < \ell \leq N - k$. This implies that

$$(\mathbf{R}^{k+\ell}, \mathbf{d}_{\text{Eucl}}, c_{k+\ell} \mathcal{L}^{k+\ell}, 0^{k+\ell}) \in \text{Tan}_{(0^k, z)}(\mathbf{R}^k \times Y, \mathbf{d}_{\text{Eucl}} \times \varrho, \mathcal{L}^k \otimes \mathbf{n}).$$

Therefore, Theorem 2.1 guarantees that $(\mathbf{R}^{k+\ell}, \mathbf{d}_{\text{Eucl}}, c_{k+\ell} \mathcal{L}^{k+\ell}, 0^{k+\ell})$ belongs to $\text{Tan}_x(X, \mathbf{d}, \mathbf{m})$, which contradicts the fact that $x \in A'_k$. Consequently, we have proven that $\mathbf{m}(A'_k \setminus A_k) = 0$, as desired.

Step 3. In order to complete the proof of the statement, it suffices to show that

$$(2.3) \quad \mathbf{m}(B_R(p) \cap (A_k \setminus \mathcal{R}_k)) = 0 \quad \text{for every } p \in X \text{ and } R > 0.$$

Let $p \in X$ and $R, \eta > 0$ be fixed. Choose any $\delta \in (0, \eta)$ associated with η as in Proposition 1.5. Moreover, choose any $\varepsilon \in (0, 1/7)$ associated with δ^2 as in Proposition 1.4. Given a point $x \in A_k$, we can find $r_x \in (0, 1)$ such that $4r_x^2|K| \leq \varepsilon$ and

$$\mathbf{d}_{\text{pmGH}}\left((X, \mathbf{d}/(2r_x), \mathbf{m}_{2r_x}^x, x), (\mathbf{R}^k, \mathbf{d}_{\text{Eucl}}, c_k \mathcal{L}^k, 0^k)\right) \leq \varepsilon.$$

By applying Vitali covering lemma to the family $\{B_{r_x}(x) : x \in A_k \cap B_R(p)\}$, we obtain a sequence $(x_i)_i \subseteq A_k \cap B_R(p)$ such that $\{B_{r_{x_i}}(x_i)\}_i$ are pairwise disjoint and $A_k \cap B_R(p) \subseteq \bigcup_i B_{5r_{x_i}}(x_i)$. For any $i \in \mathbf{N}$, we know from Proposition 1.4 that there exists a δ^2 -splitting map $u^i: B_{10r_{x_i}}(x_i) \rightarrow \mathbf{R}^k$. Furthermore, by Proposition 1.6 there exists a Borel set $G_\eta^i \subseteq B_{5r_{x_i}}(x_i)$ such that $\mathbf{m}(B_{5r_{x_i}}(x_i) \setminus G_\eta^i) \leq C_N \delta \mathbf{m}(B_{5r_{x_i}}(x_i))$ and $u^i: B_s(x) \rightarrow \mathbf{R}^k$ is a δ -splitting map for every $x \in G_\eta^i$ and $s \in (0, 5r_{x_i})$. Hence, by Proposition 1.5, for any $x \in G_\eta^i$ the following property holds:

$$(2.4) \quad \begin{aligned} &\text{Given any element } (Y, \varrho, \mathbf{n}, y) \in \text{Tan}_x(X, \mathbf{d}, \mathbf{m}), \text{ there exists} \\ &\text{a pointed } \text{RCD}(0, N - k) \text{ space } (Z, \mathbf{d}_Z, \mathbf{m}_Z, z) \text{ such that} \\ &\mathbf{d}_{\text{pmGH}}\left((Y, \varrho, \mathbf{n}, y), (\mathbf{R}^k \times Z, \mathbf{d}_{\text{Eucl}} \times \mathbf{d}_Z, \mathcal{L}^k \otimes \mathbf{m}_Z, (0^k, z))\right) \leq \eta. \end{aligned}$$

Then let us define $G_\eta := \bigcup_i G_\eta^i$. Clearly, each element of G_η satisfies (2.4). Moreover, it holds

$$(2.5) \quad \begin{aligned} \mathbf{m}(B_R(p) \cap (A_k \setminus G_\eta)) &\leq \sum_{i \in \mathbf{N}} \mathbf{m}(B_{5r_{x_i}}(x_i) \setminus G_\eta^i) \leq C_N \delta \sum_{i \in \mathbf{N}} \mathbf{m}(B_{5r_{x_i}}(x_i)) \\ &\leq C_N \eta \sum_{i \in \mathbf{N}} \mathbf{m}(B_{r_{x_i}}(x_i)) \leq C_N \eta \mathbf{m}(B_{R+1}(p)). \end{aligned}$$

Now consider the Borel set $G := \bigcap_i \bigcup_j G_{1/2^{i+j}}$. It follows from (2.5) that $\mathbf{m}(B_R(p) \cap (A_k \setminus G)) = 0$. Moreover, let $x \in A_k \cap G$ and $(Y, \varrho, \mathbf{n}, y) \in \text{Tan}_x(X, \mathbf{d}, \mathbf{m})$ be fixed. Then by using (2.4) we can find a sequence $\{(Z_i, \mathbf{d}_{Z_i}, \mathbf{m}_{Z_i}, z_i)\}_i$ of pointed $\text{RCD}(0, N - k)$ spaces such that

$$(2.6) \quad (\mathbf{R}^k \times Z_i, \mathbf{d}_{\text{Eucl}} \times \mathbf{d}_{Z_i}, \mathcal{L}^k \otimes \mathbf{m}_{Z_i}, (0^k, z_i)) \xrightarrow{\text{pmGH}} (Y, \varrho, \mathbf{n}, y) \quad \text{as } i \rightarrow \infty.$$

Up to a not relabelled subsequence, we can suppose that $(Z_i, \mathbf{d}_{Z_i}, \mathbf{m}_{Z_i}, z_i) \rightarrow (Z, \mathbf{d}_Z, \mathbf{m}_Z, z)$ in the pmGH-topology, for some pointed $\text{RCD}(0, N - k)$ space $(Z, \mathbf{d}_Z, \mathbf{m}_Z, z)$. Consequently, (2.6) ensures that $(Y, \varrho, \mathbf{n}, y)$ is isomorphic to $(\mathbf{R}^k \times Z, \mathbf{d}_{\text{Eucl}} \times \mathbf{d}_Z, \mathcal{L}^k \otimes \mathbf{m}_Z, (0^k, z))$. Given that $x \in A_k$, we deduce that Z must be a singleton. In other words, we have proven that any element of $\text{Tan}_x(X, \mathbf{d}, \mathbf{m})$ is isomorphic to $(\mathbf{R}^k, \mathbf{d}_{\text{Eucl}}, c_k \mathcal{L}^k, 0^k)$, so that $x \in \mathcal{R}_k$. This shows that $A_k \cap G \subseteq \mathcal{R}_k$, whence the claim (2.3) follows. \square

By combining Theorem 2.3 with the properties of δ -splitting maps discussed in Section 1.3, we can give a direct proof of the following result, that was proved for the first time in [35]:

Theorem 2.4. *Let $(X, \mathbf{d}, \mathbf{m})$ be an $\text{RCD}(K, N)$ space. Let $k \in \mathbf{N}$, $k \leq N$ be the maximal number such that $\mathbf{m}(\mathcal{R}_k) > 0$. Then for any $x \in X$ and $\ell > k$ we have that no element of $\text{Tan}_x(X, \mathbf{d}, \mathbf{m})$ splits off a factor \mathbf{R}^ℓ . In particular, it holds that $\mathcal{R}_\ell = \emptyset$ for every $\ell > k$.*

Proof. First of all, we claim that for any given $\ell > k$ there exists $\varepsilon > 0$ such that

$$(2.7) \quad \mathbf{d}_{\text{pmGH}}\left(\left(\mathbf{R}^j, \mathbf{d}_{\text{Eucl}}, c_j \mathcal{L}^j, 0^j\right), \left(\mathbf{R}^\ell \times Z, \mathbf{d}_{\text{Eucl}} \times \mathbf{d}_Z, \mathcal{L}^\ell \otimes \mathbf{m}_Z, (0^\ell, z)\right)\right) > \varepsilon$$

for every $j \leq k$ and for every pointed $\text{RCD}(0, N - \ell)$ space $(Z, \mathbf{d}_Z, \mathbf{m}_Z, z)$. This can be easily checked arguing by contradiction.

We prove the main statement by contradiction: suppose there exist $x \in X$ and $\ell > k$ such that

$$(2.8) \quad \left(\mathbf{R}^\ell \times Z, \mathbf{d}_{\text{Eucl}} \times \mathbf{d}_Z, \mathcal{L}^\ell \otimes \mathbf{m}_Z, (0^\ell, z)\right) \in \text{Tan}_x(X, \mathbf{d}, \mathbf{m})$$

for some pointed $\text{RCD}(0, N - \ell)$ space $(Z, \mathbf{d}_Z, \mathbf{m}_Z, z)$. Consider $\varepsilon > 0$ associated with ℓ as in (2.7). Choose $\delta > 0$ associated with ε as in Proposition 1.5, then $\eta > 0$ associated with δ^2 as in Proposition 1.4. It follows from (2.8) that there is $r > 0$ such that $r^2|K| \leq \eta$ and

$$\mathbf{d}_{\text{pmGH}}\left(\left(X, \mathbf{d}/r, \mathbf{m}_r^x, x\right), \left(\mathbf{R}^\ell \times Z, \mathbf{d}_{\text{Eucl}} \times \mathbf{d}_Z, \mathcal{L}^\ell \otimes \mathbf{m}_Z, (0^\ell, z)\right)\right) \leq \eta.$$

Then Proposition 1.4 guarantees the existence of a δ^2 -splitting map $u: B_{5r}(x) \rightarrow \mathbf{R}^\ell$. Therefore, by Propositions 1.6 and 1.5 we know that there exists a Borel set $G \subseteq B_r(x)$ with $\mathbf{m}(G) > 0$ satisfying the following property: for any point $y \in G$, it holds that each element of $\text{Tan}_y(X, \mathbf{d}, \mathbf{m})$ is ε -close (with respect to the distance \mathbf{d}_{pmGH}) to some space that splits off a factor \mathbf{R}^ℓ . Given that $X \setminus (\mathcal{R}_1 \cup \dots \cup \mathcal{R}_k)$ has null \mathbf{m} -measure by Theorem 2.3, there must exist $y \in G$ and $j \leq k$ for which $(\mathbf{R}^j, \mathbf{d}_{\text{Eucl}}, c_j \mathcal{L}^j, 0^j)$ is the only element of $\text{Tan}_y(X, \mathbf{d}, \mathbf{m})$. Consequently, we have that

$$\mathbf{d}_{\text{pmGH}}\left(\left(\mathbf{R}^j, \mathbf{d}_{\text{Eucl}}, c_j \mathcal{L}^j, 0^j\right), \left(\mathbf{R}^\ell \times Z', \mathbf{d}_{\text{Eucl}} \times \mathbf{d}_{Z'}, \mathcal{L}^\ell \otimes \mathbf{m}_{Z'}, (0^\ell, z')\right)\right) \leq \varepsilon$$

for some pointed $\text{RCD}(0, N - \ell)$ space $(Z', \mathbf{d}_{Z'}, \mathbf{m}_{Z'}, z')$. This is in contradiction with (2.7). \square

Remark 2.5. (Constant dimension) We point out that the first and third named authors proved in [12] that any $\text{RCD}(K, N)$ space $(X, \mathbf{d}, \mathbf{m})$ has ‘constant dimension’, in the following sense: there exist a (unique) $k \in \mathbf{N}$, $k \leq N$ such that $\mathbf{m}(X \setminus \mathcal{R}_k) = 0$. The number k is called *essential dimension* of $(X, \mathbf{d}, \mathbf{m})$ and denoted by $\dim(X, \mathbf{d}, \mathbf{m})$. With this notation, Theorem 2.4 can be rephrased by saying that at no point of X an element of the tangent cone can split off a Euclidean factor of dimension bigger than $\dim(X, \mathbf{d}, \mathbf{m})$.

Actually, Theorem 2.4 above is an instance of a more general result that can be proved arguing in a similar manner: the essential dimension of $\text{RCD}(K, N)$ spaces is lower semicontinuous with respect to pointed measured Gromov–Hausdorff convergence.

This statement has been proved for the first time in [35, Theorem 4.10]. Below we just sketch how our techniques can provide a slightly more direct proof, still based

on the same ideas and on the theory of convergence of Sobolev functions on varying spaces developed in [5, 6, 30].

Let us point out that the result below is independent of [12] once the essential dimension of an $\text{RCD}(K, N)$ m.m.s. is understood as the maximal n for which $\mathbf{m}(\mathcal{R}_n) > 0$.

Theorem 2.6. *Let $(X_n, \mathbf{d}_n, \mathbf{m}_n, x_n)$ and $(X, \mathbf{d}, \mathbf{m}, x)$ be pointed $\text{RCD}(K, N)$ metric measure spaces and assume that $(X_n, \mathbf{d}_n, \mathbf{m}_n, x_n)$ converge to $(X, \mathbf{d}, \mathbf{m}, x)$ in the pointed measured Gromov–Hausdorff sense. Then*

$$\dim(X, \mathbf{d}, \mathbf{m}) \leq \liminf_{n \rightarrow \infty} \dim(X_n, \mathbf{d}_n, \mathbf{m}_n).$$

Proof. Let $k := \dim(X, \mathbf{d}, \mathbf{m})$. We need to prove that, for n sufficiently large, it holds $k \leq \dim(X_n, \mathbf{d}_n, \mathbf{m}_n)$.

Up to scaling of the distance \mathbf{d} on X , we can assume that $K \geq -1$ and by Proposition 1.4 we find $y \in X$ and a δ -splitting map $u: B_2(y) \rightarrow \mathbf{R}^k$. Arguing as in the proof of [11, Proposition 3.9], relying on the convergence and stability results of [6], we can find $1 < r < 2$, points $X_n \ni y_n \rightarrow y \in X$ and 2δ -splitting maps $u_n: B_r(y_n) \rightarrow \mathbf{R}^k$, for any n sufficiently large (it suffices to approximate the components of u in the strong $H^{1,2}$ -sense with harmonic functions).

Next, Proposition 1.6 provides sets $G_n \subset B_{r/2}(y_n)$ such that $\mathbf{m}_n(B_r(y_n) \setminus G_n) \leq C_N \sqrt{2\delta} \mathbf{m}_n(B_{r/2}(y_n))$ and

$$u_n: B_s(x) \rightarrow \mathbf{R}^k \text{ is a } \sqrt{2\delta}\text{-splitting map, for every } x \in G_n \text{ and } s \in (0, r/2),$$

for any n sufficiently large.

Now it suffices to choose δ such that $\sqrt{2\delta} \leq \delta_\epsilon$ given by Proposition 1.5 to get that, at any point in G_n , any tangent is ϵ -close to a space splitting a factor \mathbf{R}^k . Choosing ϵ small enough and arguing as in the proof of Theorem 2.4 above we obtain that $\dim(X_n, \mathbf{d}_n, \mathbf{m}_n) \geq k$ for sufficiently large n . \square

2.2. An alternative approach to uniqueness of tangent cones. It has been pointed out to us by one of the reviewers of the paper that there is an alternative strategy to prove uniqueness of tangents starting from a variant of Theorem 2.6 and not relying on Theorem 2.1 nor on Theorem 2.2. We briefly sketch the argument below.

For any $k \in \mathbf{N}$ let us denote by \mathcal{E}_k the set of all points $x \in X$ such that any tangent cone at x splits off a factor \mathbf{R}^k . Moreover, for any Borel set $A \subset X$ we shall indicate by $\dim_e(A)$ the maximum number k such that $\mathbf{m}(A \cap \mathcal{E}_k) > 0$. Then, under the same assumptions of Theorem 2.6 and with the same argument as in its proof we can get that

$$(2.9) \quad \liminf_{i \rightarrow \infty} \dim_e(B_i) \geq \dim_e(B)$$

whenever $B_i \subset X_i$, $B \subset X$ and $\chi_{B_i} \rightarrow \chi_B$ locally in L^1_{loc} strong. In particular for any Borel set $A \subset X$ and any Lebesgue point $x \in A$ it holds that

$$(2.10) \quad \dim_e(A) \geq \dim_e(Y) \quad \text{for any } (Y, \mathbf{d}_Y, \mathbf{m}_Y, y) \in \text{Tan}_x(X, \mathbf{d}, \mathbf{m}),$$

as an application of (2.9) to the sequence of rescaled spaces converging to the tangent cone and to the indicator function of A shows.

Next we claim that $\mathbf{m}(X \setminus \mathcal{E}_1) = 0$. In order to prove the claim we recall that, as shown in [29], \mathbf{m} -almost every point in X is an intermediate point of a geodesic and therefore any tangent cone at that point splits a line by the splitting theorem.

The next step aims at proving that if $A \subset X$ is Borel and $\dim_e(A) = k$, then $A \cap \mathcal{E}_k \sim A \cap \mathcal{R}_k$ where we indicate $B \sim C$ whenever $\mathbf{m}(B \Delta C) = 0$. If this is not the case we could find $x \in \mathcal{E}_k$ which is also a Lebesgue point of A and such that

$$(2.11) \quad (\mathbf{R}^k \times Z, \mathbf{d}_{\mathbf{R}^k} \times \mathbf{d}_Z, \mathcal{L}^k \times \mathbf{m}_Z, (0^k, z)) \in \text{Tan}_x(X, \mathbf{d}, \mathbf{m})$$

for some non-trivial pointed $\text{RCD}(0, N - k)$ space $(Z, \mathbf{d}_Z, \mathbf{m}_Z, z)$. Then the previous claim shows that the essential dimension of this tangent cone is at least $k + 1$, which contradicts the assumption $\dim_e(A) = k$ by (2.10).

Eventually, let us prove that $\mathbf{m}(X \setminus \bigcup_{k \leq N} \mathcal{R}_k) = 0$. Let $k_1 := \dim_e(X)$. Without loss of generality we can assume $\mathbf{m}(X \setminus \mathcal{R}_{k_1}) > 0$, otherwise the proof is completed. Let $k_2 = \dim_e(X \setminus \mathcal{R}_{k_1})$. Notice that $k_2 < k_1$. Indeed the inequality $k_2 \leq k_1$ is trivially verified and what we proved above shows that $\mathcal{E}_{k_2} \cap (X \setminus \mathcal{R}_{k_1}) \sim \mathcal{R}_{k_2} \cap (X \setminus \mathcal{R}_{k_1})$, which has positive measure. If $\mathbf{m}(X \setminus (\mathcal{R}_{k_2} \cup \mathcal{R}_{k_1})) = 0$ we conclude. Otherwise we can iteratively apply the argument again to get the sought conclusion.

2.3. Metric rectifiability of RCD spaces. Aim of this section is to exploit δ -splitting maps to show that $\text{RCD}(K, N)$ spaces are metrically rectifiable, in the following sense:

Definition 2.7. Given a metric measure space $(X, \mathbf{d}, \mathbf{m})$, $k \in \mathbf{N}$ and $\varepsilon > 0$, we say that a Borel set $E \subseteq X$ is $(\mathbf{m}, k, \varepsilon)$ -rectifiable provided there exists a sequence $\{(G_n, u_n)\}_n$, where $G_n \subseteq X$ are Borel sets satisfying $\mathbf{m}(E \setminus \bigcup_n G_n) = 0$ and the maps $u_n: G_n \rightarrow \mathbf{R}^k$ are $(1 + \varepsilon)$ -biLipschitz with their images.

Rectifiability of $\text{RCD}(K, N)$ spaces in the above sense was first proved in [37, Theorem 1.1]. Below we provide a different proof, more in the spirit of the Cheeger–Colding theory for Ricci limits (cf. [19]) and relying on the connection between δ -splitting maps and ε -isometries. Albeit the use of this tool, the skeleton of the proof is similar to the one of [37].

Lemma 2.8. *Let $N > 1$ be given. Then for any $\eta > 0$ there exists $\delta = \delta_{N, \eta} > 0$ such that the following property holds. If $(X, \mathbf{d}, \mathbf{m})$ is an $\text{RCD}(K, N)$ space and $u: B_r(x) \rightarrow \mathbf{R}^k$ is a δ -splitting map for some radius $r > 0$ with $r^2|K| \leq 1$ and some point $x \in X$ satisfying*

$$\mathbf{d}_{\text{pmGH}}((X, \mathbf{d}/r, \mathbf{m}_r^x, x), (\mathbf{R}^k, \mathbf{d}_{\text{Eucl}}, c_k \mathcal{L}^k, 0^k)) < \delta^2,$$

then it holds that $u: B_r(x) \rightarrow \mathbf{R}^k$ verifies

$$\left| |u(y) - u(z)| - \mathbf{d}(y, z) \right| \leq \eta r \quad \text{for every } y, z \in B_r(x).$$

Proof. Thanks to a scaling argument, it suffices to prove the statement for $r = 1$ and $|K| \leq 1$. We argue by contradiction: suppose there exist $\eta > 0$, a sequence of spaces $(X_n, \mathbf{d}_n, \mathbf{m}_n, x_n)$ and a sequence of maps $u^n: B_1(x_n) \rightarrow \mathbf{R}^k$, such that the following properties are satisfied.

- i) $(X_n, \mathbf{d}_n, \mathbf{m}_n)$ is an $\text{RCD}(K, N)$ space.
- ii) u^n is a $1/n$ -splitting map with $u^n(x_n) = 0^k$.
- iii) It holds that $\mathbf{d}_{\text{pmGH}}((X_n, \mathbf{d}_n, \mathbf{m}_n, x_n), (\mathbf{R}^k, \mathbf{d}_{\text{Eucl}}, c_k \mathcal{L}^k, 0^k)) \leq 1/n$.
- iv) there exist points $y_n, z_n \in B_1(x_n)$ such that

$$(2.12) \quad \left| |u^n(y_n) - u^n(z_n)| - \mathbf{d}_n(y_n, z_n) \right| > \eta.$$

Observe that item iii) guarantees that $(X_n, \mathbf{d}_n, \mathbf{m}_n, x_n) \rightarrow (\mathbf{R}^k, \mathbf{d}_{\text{Eucl}}, c_k \mathcal{L}^k, 0^k)$ in the pmGH-topology. Possibly taking a not relabelled subsequence, it holds that $u^n \rightarrow u^\infty$

strongly in $H^{1,2}$ on $B_1(0^k)$, for some limit map $u^\infty: B_1(0^k) \rightarrow \mathbf{R}^k$ (cf. [5, 6, 30] for the theory of convergence on varying spaces and [6, Theorem 4.2, Corollary 4.3, Theorem 4.4] for this specific conclusion). Moreover, thanks to the uniform Lipschitz continuity of the maps u_n , an Ascoli–Arzelà type argument implies that the convergence is also pointwise in the following sense: whenever $X_n \ni w_n \rightarrow w_\infty \in \mathbf{R}^k$, it holds $u^n(w_n) \rightarrow u^\infty(w_\infty)$.

We also deduce from item ii) above that $\text{Hess}(u_a^\infty) = 0$ and $\nabla u_a^\infty \cdot \nabla u_b^\infty = \delta_{ab}$ on $B_1(0^k)$ for all $a, b = 1, \dots, k$ (further details are discussed in the proof of [11, Proposition 3.7]), whence u^∞ is the restriction to $B_1(0^k)$ of an orthogonal transformation of \mathbf{R}^k . Indeed the vanishing of the Hessian, together with the condition $u^\infty(0) = 0$, yields the linearity of the components, while the orthogonality of the transformation comes from the orthogonality of the gradients, since the ambient space is Euclidean. This gives a contradiction since, by letting $n \rightarrow \infty$ in (2.12), we obtain that

$$\left| |u^\infty(y_\infty) - u^\infty(z_\infty)| - |y_\infty - z_\infty| \right| \geq \eta,$$

where $y_\infty, z_\infty \in B_1(0^k)$ stand for the limit points of $(y_n)_n$ and $(z_n)_n$, respectively (notice that $z_\infty \neq y_\infty$ as a consequence of (2.12) and (i) in Definition 1.3). \square

Let $(X, \mathbf{d}, \mathbf{m})$ be an $\text{RCD}(K, N)$ space. Let $k \in \mathbf{N}$ be such that $k \leq N$. Then we define

$$(\mathcal{R}_k)_{r,\delta} := \left\{ x \in \mathcal{R}_k \mid \mathbf{d}_{\text{pmGH}}((X, \mathbf{d}/s, \mathbf{m}_s^x, x), (\mathbf{R}^k, \mathbf{d}_{\text{Eucl}}, c_k \mathcal{L}^k, 0^k)) < \delta \text{ for every } s < r \right\}$$

for every $r, \delta > 0$. Observe that for any given $\delta > 0$ it holds that $(\mathcal{R}_k)_{r,\delta} \nearrow \mathcal{R}_k$ as $r \searrow 0$.

Theorem 2.9. (Rectifiability of RCD spaces) *Let $(X, \mathbf{d}, \mathbf{m})$ be an $\text{RCD}(K, N)$ space. Let $k \in \mathbf{N}$ be such that $k \leq N$. Then the k -regular set \mathcal{R}_k of X is $(\mathbf{m}, k, \varepsilon)$ -rectifiable for every $\varepsilon > 0$.*

Proof. We claim that for any $\varepsilon > 0$ there exists an $(\mathbf{m}, k, \varepsilon)$ -rectifiable set $G^\varepsilon \subset \mathcal{R}_k$ such that $\mathbf{m}(\mathcal{R}_k \setminus G^\varepsilon) < \varepsilon$. Notice that the statement follows from the claim above observing that

$$\mathbf{m}\left(\mathcal{R}_k \setminus \bigcup_{n=1}^\infty G^{\varepsilon/n}\right) = 0.$$

We now prove the claim in two steps relying on the general principle that a map which is an ε -isometry at any location and any sufficiently small scale is biLipschitz with its image.

Step 1. We claim that for any $\eta > 0$ there exists $\delta = \delta_{N,\eta} \in (0, 1)$ such that the following property holds: if $(X, \mathbf{d}, \mathbf{m})$ is an $\text{RCD}(K, N)$ space and $u: B_{5r}(p) \rightarrow \mathbf{R}^k$ is a δ -splitting map for some radius $r > 0$ satisfying $r^2|K| \leq 1$ and some point $p \in (\mathcal{R}_k)_{2r,\delta}$, then there exists a Borel set $G \subseteq B_r(p)$ such that $\mathbf{m}(B_r(p) \setminus G) \leq C_N \eta \mathbf{m}(B_r(p))$ and

$$(2.13) \quad \left| |u(x) - u(y)| - \mathbf{d}(x, y) \right| \leq \eta \mathbf{d}(x, y) \quad \text{for every } x, y \in (\mathcal{R}_k)_{2r,\delta} \cap G.$$

To prove it, choose any $\delta \in (0, \eta^2)$ so that $\sqrt{\delta}$ is associated with η as in Lemma 2.8. Now let us consider an $\text{RCD}(K, N)$ space $(X, \mathbf{d}, \mathbf{m})$ and a δ -splitting map $u: B_{5r}(p) \rightarrow \mathbf{R}^k$, for some $r > 0$ with $r^2|K| \leq 1$ and $p \in (\mathcal{R}_k)_{2r,\delta}$. By Proposition 1.6, we can find a Borel set $G \subseteq B_r(p)$ such that $\mathbf{m}(B_r(p) \setminus G) \leq C_N \eta \mathbf{m}(B_r(p))$ and $u: B_s(x) \rightarrow \mathbf{R}^k$ is a $\sqrt{\delta}$ -splitting map for all $x \in G$ and $s \in (0, 2r)$. Then Lemma 2.8 guarantees

that the map $u: B_s(x) \rightarrow \mathbf{R}^k$ is an ηs -GH isometry for every $x \in (\mathcal{R}_k)_{2r,\delta} \cap G$ and $s \in (0, 2r)$ (here we used the fact that $x \in (\mathcal{R}_k)_{2r,\delta} \subseteq (\mathcal{R}_k)_{s,\delta}$).

Fix any $x, y \in (\mathcal{R}_k)_{2r,\delta} \cap G$. Being $d(x, y) < 2r$, we know that the map $u: B_{d(x,y)}(x) \rightarrow \mathbf{R}^k$ is an $\eta d(x, y)$ -GH isometry, thus in particular $||u(x) - u(y)| - d(x, y)| \leq \eta d(x, y)$. This yields (2.13).

Step 2. Fix $\bar{x} \in X$, $R > 0$, $\varepsilon > 0$. We aim to build an $(\mathbf{m}, k, \varepsilon)$ -rectifiable set G satisfying $\mathbf{m}(B_R(\bar{x}) \cap \mathcal{R}_k \setminus G) < \varepsilon$. Note that this easily implies our claim.

Let $\eta < \varepsilon$ to be chosen later, $\delta = \delta_{N,\eta}$ according to Step 1, $\bar{\varepsilon} \in (0, \delta)$ associated to δ as in Proposition 1.4 and $r > 0$ satisfying $r^2|K| \leq 1$ and $\mathbf{m}(B_R(\bar{x}) \cap (\mathcal{R}_k \setminus (\mathcal{R})_{2r,\bar{\varepsilon}})) \leq \varepsilon/2$. By Vitali covering lemma, we find points $x_1, \dots, x_\ell \in B_R(\bar{x}) \cap (\mathcal{R}_k)_{2r,\bar{\varepsilon}}$ for which $\{B_{r/5}(x_i)\}_{i=1}^\ell$ are pairwise disjoint and $B_R(\bar{x}) \cap (\mathcal{R}_k)_{2r,\bar{\varepsilon}} \subseteq B_r(x_1) \cup \dots \cup B_r(x_\ell)$. Proposition 1.4 guarantees the existence of a δ -splitting map $u^i: B_{5r}(x_i) \rightarrow \mathbf{R}^k$ for every $i = 1, \dots, \ell$. Therefore Step 1 yields Borel sets $G_i \subseteq B_r(x_i)$ such that $\mathbf{m}(B_r(x_i) \setminus G_i) \leq C_N \eta \mathbf{m}(B_r(x_i))$ and $||u^i(x) - u^i(y)| - d(x, y)| \leq \eta d(x, y)$ for every $x, y \in (\mathcal{R}_k)_{2r,\bar{\varepsilon}} \cap G_i$, for every $i = 1, \dots, \ell$.

Since $\eta < \varepsilon$, we deduce that u^i is $(1 + \varepsilon)$ -biLipschitz with its image, up to slightly increasing ε , when restricted to $(\mathcal{R}_k)_{2r,\bar{\varepsilon}} \cap G_i$, whence $G := (\mathcal{R}_k)_{2r,\bar{\varepsilon}} \cap \bigcup_{i=1}^\ell G_i$ is $(\mathbf{m}, k, \varepsilon)$ -rectifiable. Observe that

$$\begin{aligned} \mathbf{m}\left((B_R(\bar{x}) \cap (\mathcal{R}_k)_{2r,\bar{\varepsilon}}) \setminus G\right) &\leq \sum_{i=1}^\ell \mathbf{m}(B_r(x_i) \setminus G_i) \leq C_N \eta \sum_{i=1}^\ell \mathbf{m}(B_r(x_i)) \\ &\leq C_N \eta \sum_{i=1}^\ell \mathbf{m}(B_{r/5}(x_i)) \leq C_N \eta \mathbf{m}(B_{R+1}(\bar{x})). \end{aligned}$$

Choosing $\eta > 0$ such that $C_N \eta \mathbf{m}(B_{R+1}(\bar{x})) < \varepsilon/2$ we get the sought conclusion. \square

2.4. Behaviour of the reference measure under charts. Aim of this subsection is to prove absolute continuity of the reference measure \mathbf{m} of an $\text{RCD}(K, N)$ metric measure space $(X, \mathbf{d}, \mathbf{m})$ with respect to the relevant Hausdorff measure. This result was first proved in the three independent works [23, 34, 31], heavily relying on [24]. The strategy of our proof is essentially taken from [31], the main technical simplification being that the charts providing rectifiability in our case are harmonic (indeed they are δ -splitting maps), while in [31] they were distance functions.

Let us introduce the notation we are going to use in this subsection. In order to avoid confusion, in this subsection we shall stress the dependence of the various differential objects appearing on the metric measure structure by indicating the reference measure as a subscript.

Let X, Y be Polish spaces. Fix a finite Borel measure $\mu \geq 0$ on X and a Borel map $\varphi: X \rightarrow Y$. We shall denote by φ_* the pushforward operator, which sends finite Borel measures on X into finite Borel measures on Y . Then we define

$$(2.14) \quad \text{Pr}_\varphi(f) := \frac{d\varphi_*(f\mu)}{d\varphi_*\mu} \quad \text{for every } f \in L^1(\mu),$$

where we adopted the usual notation of geometric measure theory for the density of a measure absolutely continuous with respect to another measure. The resulting map $\text{Pr}_\varphi: L^1(\mu) \rightarrow L^1(\varphi_*\mu)$ is linear and continuous. Given any $p \in (1, \infty]$, it holds that Pr_φ maps continuously $L^p(\mu)$ to $L^p(\varphi_*\mu)$. The *essential image* of a Borel set $E \subseteq X$ is defined as $\text{Im}_\varphi(E) := \{\text{Pr}_\varphi(\chi_E) > 0\} \subseteq Y$.

Proposition 2.10. (Differential of an \mathbf{R}^k -valued Lipschitz map) *Let (X, \mathbf{d}, μ) be an infinitesimally Hilbertian metric measure space such that μ is finite. Let $\varphi: X \rightarrow \mathbf{R}^k$ be a Lipschitz map. Then there exists a unique linear and continuous operator $\mathbf{D}_\varphi: L^2_\mu(TX) \rightarrow L^2(\mathbf{R}^k, \mathbf{R}^k; \varphi_*\mu)$ such that*

$$(2.15) \quad \int_F \nabla f \cdot \mathbf{D}_\varphi(v) \, d\varphi_*\mu = \int_{\varphi^{-1}(F)} \nabla_\mu(f \circ \varphi) \cdot v \, d\mu$$

$$\forall f \in C_c^\infty(\mathbf{R}^k), v \in L^2_\mu(TX), F \subseteq \mathbf{R}^k \text{ Borel.}$$

In particular, if $v \in D(\text{div}_\mu)$, then the distributional divergence of $\mathbf{D}_\varphi(v)$ is given by $\text{Pr}_\varphi(\text{div}_\mu(v))$. Moreover, if the map φ is biLipschitz with its image when restricted to some Borel set $E \subseteq X$ and $v_1, \dots, v_k \in L^2_\mu(TX)$ are independent on E , then the vectors $\mathbf{D}_\varphi(\chi_E v_1)(y), \dots, \mathbf{D}_\varphi(\chi_E v_k)(y)$ constitute a basis of \mathbf{R}^k for φ_μ -a.e. point $y \in \text{Im}_\varphi(E)$.*

Proof. Existence of the map \mathbf{D}_φ is proven in [31]: with the terminology used therein, it suffices to define $\mathbf{D}_\varphi := \iota \circ \text{Pr}_\varphi \circ d\varphi$. The fact that this map satisfies (2.15) follows from [31, Proposition 2.7] and the very definition of ι (we do not need to require properness of φ , as μ is a finite measure). Uniqueness of \mathbf{D}_φ follows from the fact that $\{\nabla f: f \in C_c^\infty(\mathbf{R}^k)\}$ generates $L^2(\mathbf{R}^k, \mathbf{R}^k; \varphi_*\mu)$. Now suppose $v \in D(\text{div}_\mu)$. Then for every $f \in C_c^\infty(\mathbf{R}^k)$ it holds that $f \circ \varphi \in H^{1,2}(X, \mathbf{d}, \mu)$, whence

$$\begin{aligned} \int \nabla f \cdot \mathbf{D}_\varphi(v) \, d\varphi_*\mu &\stackrel{(2.15)}{=} \int \nabla_\mu(f \circ \varphi) \cdot v \, d\mu = - \int f \circ \varphi \, \text{div}_\mu(v) \, d\mu \\ &= - \int f \, d\varphi_*(\text{div}_\mu(v)\mu) \stackrel{(2.14)}{=} - \int f \, \text{Pr}_\varphi(\text{div}_\mu(v)) \, d\varphi_*\mu. \end{aligned}$$

This shows that the distributional divergence of $\mathbf{D}_\varphi(v)$ is represented by $\text{Pr}_\varphi(\text{div}_\mu(v))$. Finally, the last claim of the statement follows from [31, Proposition 2.2] and [31, Proposition 2.10]. \square

Theorem 2.11. (Behaviour of \mathbf{m} under charts) *Let $(X, \mathbf{d}, \mathbf{m})$ be an $\text{RCD}(K, N)$ space. Consider a δ -splitting map $u: B_r(p) \rightarrow \mathbf{R}^k$ which is $(1 + \varepsilon)$ -biLipschitz with its image (for some $\varepsilon < 1/k$) when restricted to some compact set $K \subset B_r(p)$. Then it holds that*

$$u_*(\mathbf{m}|_K) \ll \mathcal{L}^k.$$

In particular, for any $k \in \mathbf{N}$, $k \leq N$, $\mathbf{m}|_{\mathcal{R}_k}$ is absolutely continuous with respect to the k -dimensional Hausdorff measure on (X, \mathbf{d}) .

Proof. First of all, fix a good cut-off function (in the sense of Lemma 1.2) $\eta: X \rightarrow \mathbf{R}$ for the pair $K \subset B_{r'}(p) \subset B_r(p)$, where $0 < r' < r$. Define $\mu := \mathbf{m}|_{B_r(p)}$ and $\varphi := \eta u: X \rightarrow \mathbf{R}^k$. Observe that the components $\varphi_1, \dots, \varphi_k$ of φ are test functions and $\varphi|_K$ is $(1 + \varepsilon)$ -biLipschitz with its image. Consider the differential $\mathbf{D}_\varphi: L^2_\mu(TX) \rightarrow L^2(\mathbf{R}^k, \mathbf{R}^k; \varphi_*\mu)$ defined in Proposition 2.10. Fix a sequence $(\psi_i)_i$ of compactly-supported, Lipschitz functions $\psi_i: X \rightarrow [0, 1]$ that pointwise converge to χ_K . We then set

$$v_a^i := \mathbf{D}_\varphi(\psi_i \nabla_\mu \varphi_a) \in L^2(\mathbf{R}^k, \mathbf{R}^k; \varphi_*\mu) \quad \text{for every } i \in \mathbf{N} \text{ and } a = 1, \dots, k.$$

Note that $\psi_i \nabla_\mu \varphi_a \in D(\text{div}_\mu)$ by the Leibniz rule for divergence and the fact that $\varphi_a \in D(\Delta_\mu)$, whence Proposition 2.10 ensures that the distributional divergence of each vector field v_a^i is an $L^2(\varphi_*\mu)$ -function. Hence, it holds that $\mathcal{I}_{ia} := v_a^i \varphi_*\mu$ is a

normal 1-current in \mathbf{R}^k (see [31, Corollary 2.12]). Note also that

$$\vec{\mathcal{I}}_{ia} = \chi_{\{|v_a^i|>0\}} \frac{v_a^i}{|v_a^i|} \quad \text{and} \quad \|\mathcal{I}_{ia}\| = |v_a^i| \varphi_* \mu \quad \text{for every } i \in \mathbf{N} \text{ and } a = 1, \dots, k.$$

Call A_i the set of $y \in \mathbf{R}^k$ such that $v_1^i(y), \dots, v_k^i(y)$ form a basis of \mathbf{R}^k . Since $(\varphi_* \mu)|_{A_i} \ll \|\mathcal{I}_{ia}\|$ holds for all $a = 1, \dots, k$, by applying [24, Corollary 1.12] we deduce that

$$(2.16) \quad (\varphi_* \mu)|_{A_i} \ll \mathcal{L}^k \quad \text{for every } i \in \mathbf{N}.$$

Now define $v_a := D_\varphi(\chi_K \nabla_\mu \varphi_a) \in L^2(\mathbf{R}^k, \mathbf{R}^k; \varphi_* \mu)$ for every $a = 1, \dots, k$. It can be readily checked that $\nabla_\mu \varphi_1, \dots, \nabla_\mu \varphi_k$ are independent on K (here the assumption $\varepsilon < 1/k$ plays a role), whence the vectors $v_1(y), \dots, v_k(y)$ are linearly independent for $\varphi_* \mu$ -a.e. $y \in \text{Im}_\varphi(K)$ by Proposition 2.10.

Furthermore, for any given $a = 1, \dots, k$, we can see (by using dominated convergence theorem) that $\psi_i \nabla_\mu \varphi_a \rightarrow \chi_K \nabla_\mu \varphi_a$ in $L^2_\mu(TX)$ as $i \rightarrow \infty$, thus $v_a^i \rightarrow v_a$ in $L^2(\mathbf{R}^k, \mathbf{R}^k; \varphi_* \mu)$ as $i \rightarrow \infty$ by continuity of D_φ . In particular, possibly passing to a not relabelled subsequence, we can assume that $\lim_i v_a^i(y) = v_a(y)$ for $\varphi_* \mu$ -a.e. $y \in \mathbf{R}^k$. This implies that $(\varphi_* \mu)(\text{Im}_\varphi(K) \setminus \bigcup_i A_i) = 0$, thus (2.16) yields $(\varphi_* \mu)|_{\text{Im}_\varphi(K)} \ll \mathcal{L}^k$. Since $\text{Im}_\varphi(K) = \{\text{Pr}_\varphi(\chi_K) > 0\}$ by definition, we conclude that

$$u_*(\mathbf{m}|_K) = \varphi_*(\mu|_K) = \frac{d\varphi_*(\chi_K \mu)}{d\varphi_* \mu} \varphi_* \mu = \text{Pr}_\varphi(\chi_K) \varphi_* \mu \ll \mathcal{L}^k.$$

Therefore, the first part of the statement is finally achieved.

The second part of the statement follows from the first one, the inner regularity of \mathbf{m} and (the proof of) Theorem 2.9. \square

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Received 22 January 2020 • Accepted 27 May 2020