Annales Fennici Mathematici
Volumen 46, 2021, 523–526

MAXIMAL OPERATOR ON THE SPACE OF CONTINUOUS FUNCTIONS

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Abstract. We study the maximal operator on continuous functions in the setting of metric measure spaces. The boundedness is proven for metric measure spaces satisfying an annular decay property.

1. Introduction

It is well known that when $1 < p \leq \infty$, the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on $L^p(X, d, \mu)$, where (X, d, μ) is a doubling metric measure space (see e.g. [4]). Maximal operator has been also studied in different function spaces, e.g., Banach function spaces [6], Sobolev spaces [5], Lebesgue spaces with variable exponent [2], generalized Orlicz spaces [3].

More recently, Buckley proved [1] the following result:

Suppose that $0 < t, \delta \leq 1$. If (X, d, μ) satisfies the δ-annular decay property and μ is doubling, then $\mathcal{M}\colon C^{0,t}(X) \to C^{0,s}(X)$, where $s = \min(t, \delta)$.

On the other hand, if no annular decay property is assumed, then $\mathcal{M}f$ can fail to be continuous, even if $f \in C^{0,1}(X)$ (see [1, Example 1.4]). Nevertheless, we prove the following theorem about continuity of maximal operator on the space of continuous functions $C(X)$.

Theorem A. *Suppose that $0 < \delta \leq 1$, and that (X, d, μ) satisfies the δ-annular property. Then $\mathcal{M}\colon C(X) \to C(X)$ and the following estimate holds*

$$\|\mathcal{M}f\|_{C(X)} \leq \|f\|_{C(X)}.$$

The remainder of the paper is structured as follows. In Section 2, we introduce the notations and recall the definitions. The proof of Theorem A is contained in the last section.

2. Preliminaries

Let (X, d, μ) be a metric measure space equipped with a metric d and the Borel measure μ. We assume that the measure of every open nonempty set is positive and that the measure of every bounded set is finite. We say that μ is doubling if there exists a constant $C_\mu > 0$ such that $\mu(B(x, 2r)) \leq C_\mu \mu(B(x, r))$ for every ball $B(x, r)$.[1] We shall denote the average of integrable function f over the measurable set A in the following manner

$$\fint_A f \, d\mu = \frac{1}{\mu(A)} \int_A f \, d\mu.$$

https://doi.org/10.5186/aasfm.2021.4633
2020 Mathematics Subject Classification: Primary 42B25, 43A85.
Key words: Metric measure spaces, annular decay property, maximal operator.

[1]Constant C_μ is called the doubling constant of measure μ.

The maximal function $\mathcal{M}f$ of a locally integrable function $f\colon X \to \mathbf{R}$ is defined by

$$\mathcal{M}f(x) = \sup_{r>0} \fint_{B(x,r)} |f|\, d\mu.$$

Let us recall the notion of annular decay property [1]. Given $\delta \in (0,1]$, we say that the space (X,d,μ) satisfies the δ-annular decay property if there exists a constant $K \geq 1$ such that for all $x \in X$, $r > 0$, $0 < \epsilon < 1$, we have

$$\mu\left(B(x,r) \setminus B(x, r(1-\epsilon))\right) \leq K\epsilon^{\delta} \mu(B(x,r)).$$

One can easily convince oneself that $\mathbf{R}^n$ with the Lebesgue measure satisifies 1-annular decay property. Furthermore, if (X,d,μ) is a lenght metric measure space with a doubling measure μ, then X has the δ-annular decay property for some $\delta \in (0,1]$ dependent on a doubling constant of μ (see [1, Corollary 2.2]).

Finally, let (X,d) be a metric space, by $C(X)$ we denote the space of continuous functions on X such that the norm

$$\|f\|_{C(X)} = \sup_{x\in X} |f(x)|$$

is finite. Furthermore, for $s \in (0,1]$ we denote by $C^{0,s}(X)$ the Hölder space, i.e. the space of functions $f \in C(X)$ such

$$\|f\|_{C^{0,s}(X)} := \|f\|_{C(X)} + \sup_{x\neq y} \frac{|f(x)-f(y)|}{d(x,y)^s} < \infty.$$

3. Proof of the main result

We shall start with the following result.

Lemma 3.1. *If $f \in L^1_{\mathrm{loc}}(X)$, then $\mathcal{M}f$ is lower semicontinuous.*

Proof. Let $t \in \mathbf{R}$ and

$$L_t = \{x \in X \colon \mathcal{M}f(x) > t\}.$$

We shall prove that L_t is open. For this purpose we fix $x \in L_t$, then from the very definition of the maximal function, there exists r such that

$$t < \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f|\, d\mu.$$

Next, since $B(x,r) = \bigcup_{n=1}^{\infty} \overline{B}(x, r-1/n)$, we have[2]

$$\int_{B(x,r)} |f|\, d\mu = \lim_{n\to\infty} \int_{\overline{B}(x,r-1/n)} |f|\, d\mu.$$

Therefore, for sufficiently large n we have

$$t < \frac{1}{\mu(B(x,r))} \int_{\overline{B}(x,r-1/n)} |f|\, d\mu.$$

Moreover, let us observe that for $z \in B(x, 1/2n)$ we have

$$\overline{B}(x, r-1/n) \subset B(z, r-1/2n) \subset B(x,r).$$

Thus, finally, for $z \in B(x,1/2n)$

$$t < \frac{1}{\mu(B(z,r-1/2n))} \int_{B(z,r-1/2n)} |f|\, d\mu \leq \mathcal{M}f(z),$$

[2]The notation $\overline{B}(y,R) := \{z \in X \colon d(y,z) \leq R\}$ is used for closed balls.

and in this way we have proved that the set L_t is indeed open. □

Proof of Theorem A. Of course, the estimate is obvious. So, we only need to prove that for $f \in C(X)$, the maximal function $\mathcal{M}f$ is continuous. We can assume that $\|f\|_{C(X)} > 0$. By Lemma 3.1 we have that $\mathcal{M}f$ is lower semicontinuous. Therefore, we have to show that $\mathcal{M}f$ is upper semicontinuous. For this purpose we shall prove that for $t \in \mathbf{R}$ the set

$$U_t = \{x \in X : \mathcal{M}f(x) < t\}$$

is open. Let us fix $x_0 \in U_t$ and let $A := \mathcal{M}f(x_0)$, then for all $r > 0$

$$⨍_{B(x_0,r)} |f|\, d\mu \le A < t.$$

By continuity of f there exists $\sigma \in (0,1)$ such that

$$||f(x_0)| - |f(y)|| \le \frac{t-A}{4}, \quad \text{for } y \in B(x_0,\sigma). \tag{1}$$

Next, let us define

$$R := \frac{1}{2}\min\left(\frac{\sigma^2}{4}, \left(\frac{t-A}{2K2^\delta \|f\|_{C(X)}}\right)^{2/\delta}\right),$$

and we shall prove that $B(x_0,R) \subset U_t$. For this purpose we fix $x \in B(x_0,R)$. It is enought to prove that for $r > 0$

$$⨍_{B(x,r)} |f|\, d\mu < \frac{t+A}{2}. \tag{2}$$

We shall consider two cases: $r \ge \sqrt{d(x,x_0)}$ and $r < \sqrt{d(x,x_0)}$.

Case: $r \ge \sqrt{d(x,x_0)}$. Since

$$B(x,r) \subset B(x_0, r + d(x,x_0)) \subset B(x, r + 2d(x,x_0)),$$

we have

$$\begin{aligned}
⨍_{B(x,r)} |f|\, d\mu &= ⨍_{B(x,r)} |f| d\mu - ⨍_{B(x_0,r+d(x,x_0))} |f|\, d\mu + ⨍_{B(x_0,r+d(x,x_0))} |f|\, d\mu \\
&\le \left(\frac{1}{\mu(B(x,r))} - \frac{1}{\mu(B(x_0,r+d(x,x_0)))}\right)\int_{B(x,r)} |f|\, d\mu + ⨍_{B(x_0,r+d(x,x_0))} |f|\, d\mu \\
&\le \left(\frac{1}{\mu(B(x,r))} - \frac{1}{\mu(B(x,r+2d(x,x_0)))}\right)\int_{B(x,r)} |f|\, d\mu + A \\
&\le \left(\frac{\mu(B(x,r+2d(x,x_0))) - \mu(B(x,r))}{\mu(B(x,r+2d(x,x_0)))}\right)\|f\|_{C(X)} + A.
\end{aligned}$$

Next, by the δ-annular decay property and assumption $r \ge \sqrt{d(x,x_0)}$ we get

$$\begin{aligned}
⨍_{B(x,r)} |f|\, d\mu &\le K\left(\frac{2d(x,x_0)}{r + 2d(x,x_0)}\right)^\delta \|f\|_{C(X)} + A \\
&\le K\left(\frac{2d(x,x_0)}{\sqrt{d(x,x_0)} + 2d(x,x_0)}\right)^\delta \|f\|_{C(X)} + A \\
&\le K2^\delta d(x,x_0)^{\delta/2}\|f\|_{C(X)} + A \le K2^\delta R^{\delta/2}\|f\|_{C(X)} + A \\
&< (t-A)/2 + A = (t+A)/2.
\end{aligned}$$

Case: $r < \sqrt{d(x,x_0)}$. Let us observe that

$$B(x,r) \cup B(x_0, r + d(x,x_0)) \subset B(x_0,\sigma).$$

Indeed, for $y \in B(x,r)$ we have

$$\begin{aligned} d(y,x_0) &\le d(y,x) + d(x,x_0) < r + d(x,x_0) \\ &< \sqrt{d(x,x_0)} + d(x,x_0) < \sqrt{R} + R \le 2\sqrt{R} < \sigma. \end{aligned}$$

Moreover, since

$$\begin{aligned} r + d(x,x_0) &\le \sqrt{d(x,x_0)} + d(x,x_0) \\ &< \sqrt{R} + R \le 2\sqrt{R} < \sigma, \end{aligned}$$

we obtain $B(x_0, r + d(x,x_0)) \subset B(x_0,\sigma)$.

Next, by (1) we get

$$\begin{aligned} \fint_{B(x,r)} |f|\, d\mu &= \fint_{B(x,r)} (|f| - |f(x_0)|)\, d\mu - \fint_{B(x_0,r+d(x,x_0))} (|f| - |f(x_0)|)\, d\mu \\ &\quad + \fint_{B(x_0,r+d(x,x_0))} |f|\, d\mu \\ &\le \fint_{B(x,r)} ||f| - |f(x_0)||\, d\mu + \fint_{B(x_0,r+d(x,x_0))} ||f| - |f(x_0)||\, d\mu + A \\ &\le (t-A)/4 + (t-A)/4 + A = (t+A)/2. \end{aligned}$$

This finishes the proof of (2) and hence the whole proof of Theorem A is complete. □

Acknowledgements. Thanks to the referee for carefully reading this work.

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Received 11 May 2020 • Accepted 29 June 2020