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WILD EXAMPLES OF COUNTABLY RECTIFIABLE SETS

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Abstract. We study the geometry of sets based on the behavior of the Jones function, $J_E(x) = \int_0^1 \beta^1_{E;2}(x,r)^2 \frac{dr}{r}$. We construct two examples of countably 1-rectifiable sets in $\mathbf{R}^2$ with positive and finite $\mathcal{H}^1$-measure for which the Jones function is nowhere locally integrable. These examples satisfy different regularity properties: one is connected and one is Ahlfors regular. Both examples can be generalized to higher-dimension and co-dimension.

1. Introduction

In his solution to the Analyst's Traveling Salesman Problem [Jon90], Jones introduced a local gauge of flatness which has been generalized by David and Semmes [DS91] to higher dimensions. These families of local gauges of flatness are called the Jones β-numbers, and they have come to dominate the landscape in quantitative techniques relating rectifiability, potential theory, and boundedness of singular integrals. See, for example the landmark book [DS93].

For a set $E \subset \mathbf{R}^d$, $1 \le p < \infty$, and an integer $1 \le n \le d-1$, we write $\mu = \mathcal{H}^n \llcorner E$ and define the Jones β-numbers as follows,

$$\beta^n_{E;p}(x,r) = \left(\inf_{\substack{L \subset \mathbf{R}^d \\ \text{an } n\text{-plane}}} \int_{B(x,r)} \left(\frac{\operatorname{dist}(y,L)}{r} \right)^p \frac{d\mu(y)}{r^n} \right)^{\frac{1}{p}}. \tag{1.1}$$

We also write $\beta^n_{\mu;p}(x,r)$ for $\beta^n_{E;p}(x,r)$, when $\mu = \mathcal{H}^n \llcorner E$ is understood. If $p = \infty$, the β-numbers are defined in terms of the sup-norm instead of the L^∞-norm. Various notions of "rectifiability" have been studied over the years and are frequently characterized by β-numbers. We introduce them from most to least regular. The original notion is for 1-dimensional sets in $E \subset \mathbf{R}^d$. It is said that E is rectifiable if E can be contained in a curve of finite length. Thanks to [Jon90] in dimension $d = 2$ and [Oki92] for dimension $d \ge 3$, the following characterization is known:

$$E \subset \mathbf{R}^d \text{ is (finitely) rectifiable} \iff \int_E \int_0^\infty \beta^1_{E;\infty}(x,r)^2 \frac{dr}{r}\, d\mathcal{H}^1(x) < \infty. \tag{1.2}$$

In addition to generalizing the Jones β-numbers, [DS91] also introduced the notion of uniform rectifiability. A set $E \subset \mathbf{R}^d$ is said to be Ahlfors n-regular if there exists $0 < c < C < \infty$ such that $cr^n \le \mathcal{H}^n(E \cap B(x,r)) \le Cr^n$ for all $x \in E$ and all $0 < r < \operatorname{diam}(E)$. An n-Ahlfors regular $E \subset \mathbf{R}^d$ is said to be uniformly n-rectifiable if there exist finite constants $\theta, \Lambda > 0$ such that for all $x \in E$ and all $0 < r < \operatorname{diam}(E)$

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there is a Lipschitz mapping $g\colon B(0,r) \subset \mathbf{R}^n \to \mathbf{R}^d$ with $\mathrm{Lip}(g) \leq \Lambda$ such that $\mathcal{H}^n(E \cap B(x,r) \cap g(B(0,r))) \geq \theta r^n$.

In [DS91] the authors show that an n-Ahlfors regular set $E \subset \mathbf{R}^d$, is n-uniformly rectifiable if and only if the Jones β-numbers satisfy the following Carleson condition for some $1 \leq p < \frac{2n}{n-2}$,

$$C^n_{E;p}(x,R) := \int_{B(x,R)} \int_0^R \beta^n_{E;p}(y,r)^2 \frac{\mathrm{d}r}{r} \,\mathrm{d}\mu(y) \leq cR^n \quad \text{for all } x \in E,\ R > 0. \tag{1.3}$$

A set $E \subset \mathbf{R}^d$ is said to be countably n-rectifiable if there are Lipschitz maps $f_i\colon \mathbf{R}^n \to \mathbf{R}^d$ with $i = 1, 2, \ldots$, such that

$$\mathcal{H}^n(\mathbf{R}^d \setminus \textstyle\bigcup_i f_i(\mathbf{R}^n)) = 0.$$

Recently, Tolsa [Tol15] and Azzam and Tolsa [AT15] show, as a special case, that E is countably n-rectifiable if and only if

$$J^n_E(x,1) = \int_0^1 \beta^n_{E;2}(x,r)^2 \frac{\mathrm{d}r}{r} < \infty \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in E, \tag{1.4}$$

where $J^n_E(x,1)$ is the Jones function at x and scale 1. See [Mat95] for more about countably n-rectifiable sets and also [Paj97] and [BS16] for more about identifying countably n-rectifiable sets and measures via β-numbers.

In this paper, we show that sets which satisfy (1.4) can fail to satisfy (1.3) as dramatically as possible. We show this through two examples in the plane. The first, Theorem 1.1, is connected and the second, Theorem 1.2, is Ahlfors regular.

Theorem 1.1. *There exists a rectifiable curve (of finite length), $K_0 \subset \mathbf{R}^2$, such that for $\mu = \mathcal{H}^1 \llcorner K_0$, for any $x \in K_0$, and any $\delta > 0$*

$$\int_{B_\delta(x)} \int_0^\delta \beta^1_{K_0;2}(y,r)^2 \frac{\mathrm{d}r}{r} \,\mathrm{d}\mu(y) = \infty.$$

The set K_0 arises from unions of modifications of approximations to snowflake-like sets. Since K_0 is a rectifiable curve, by the Analyst's Traveling Salesman Theorem, i.e., (1.2), it follows

$$\int_{\mathbf{R}^2} \int_0^\infty \beta^1_{K_0,\infty}(y,r)^2 \frac{\mathrm{d}r}{r} \,\mathrm{d}\mu(y) < \infty,$$

which indicates that K_0 fails to be Ahlfors upper-regular at generic points.

We note that this example also gives rise to a curve of finite length (see Remark 2.10) which has classical tangents nowhere. This is in contrast to the well-known theorem that simple rectifiable curves have tangents almost everywhere, see [Fal86], and also demonstrates the necessity of the "simple" assumption in the main theorem of [CW16], which states that σ-finite simple curves have classical tangents on a set of positive measure.

Theorem 1.2. *There is a 1-Ahlfors regular, countably 1-rectifiable set A_0 contained in the unit cube in $\mathbf{R}^2$ such that for $\mu = \mathcal{H}^1 \llcorner A_0$, for every $x \in A_0$, and for every $\delta > 0$,*

$$\int_{B_\delta(x)} \int_0^\delta \beta^1_{A_0;2}(y,r)^2 \frac{\mathrm{d}r}{r} \,\mathrm{d}\mu(y) = \infty.$$

The set A_0, whose construction was initially motivated by the machinery introduced in [Tol17], is created from scaled unions of approximations to the 4-corner

Cantor set. Ultimately the presentation was simpler using the framework of self-similar sets.

Remark 1.3. These examples can be used to create higher-dimensional ones by taking Cartesian products with finite intervals. That is, if $A \in \{K_0, A_0\}$ for any positive integer $n < d$, define $E' = A \times [0,1]^{n-1} \subset \mathbf{R}^{n+1}$. Embedding E' into the first $(n+1)$-dimensions of $\mathbf{R}^d$ preserves the properties of A. In particular, it is standard that defining β-numbers over cubes (with sides parallel to the axes in $\mathbf{R}^d$) instead of balls leads to an equivalent definition of the β-numbers. Consequently finiteness of $C^n_{E;2}(x,R)$ is equivalent to the finiteness of $C^1_{A;2}(x',R)$ where x' is the orthogonal projection of x into $\mathbf{R}^2$.

2. Proof of Theorem 1.1

To construct a 1-rectifiable set, K_0, that is connected (hence Ahlfors lower-regular) for which the Jones function is locally non-integrable, we modify approximations to the Koch snowflake. This set will not be Ahlfors upper regular, i.e., $\mathcal{H}^1(B(x,r) \cap K_0) \le Cr$ fails for all C and some (x,r). We begin with an informal description of the technical construction which follows.

The construction splits into two parts. First, build a "base set" E_∞ which satisfies $C_{E_\infty}(0,\delta) = +\infty$. The base set E_∞ is designed from modified approximations of the Koch snowflake, see Definition 2.2 and subsequent discussion. The goal is to build the connected base set E_∞ so that within the triadic strips $[3^{-i}, 3^{-(i-1)}] \times \mathbf{R}$ the set E_∞ looks like successive approximations to the Koch snowflake which arise from more iterations of the "bumping process". See Figures 1–3 for example sets that could be scaled and set on the triadic intervals $[3^{-1}, 2\cdot 3^{-1}], [3^{-2}, 2\cdot 3^{-2}], [3^{-3}, 2\cdot 3^{-3}]$, as in Figure 4, to begin creating the base set E_∞. After doing this infinitely many times, and taking care to balance the number of corners with the shallowness of the corners, we create a connected set with finite length such that the infinitely many "bumps" in any neighborhood of the origin give $C_{E_\infty}(0,\delta) = +\infty$ for all $0 < \delta$.

After we have constructed the base set E_∞, we build the desired final set K_0. Roughly speaking, this happens by iteratively adding scaled copies of E_∞ in a dense way along E_∞ itself.

For the remainder of this paper, we only consider $E \subset \mathbf{R}^2$ and the β-numbers when $p = 2$. As such, we write β_E, β_μ, C_E, and C_μ in place of $\beta^1_{E;2}, \beta^1_{\mu;2}, C^1_{E;2}$, and $C^1_{\mu;2}$, see (1.1), (1.3). Moreover, for any set $L \subset \mathbf{R}^2$ we write $B_r(L) = \{x\colon \operatorname{dist}(x,L) < r\}$ and $B_r = B_r(\{0\})$.

We begin by stating two basic properties of the Jones β-numbers. The first, often called "doubling" despite our choice to scale by a factor of 3, controls how fast the β-numbers can shrink by relating the β-numbers at comparable scales. The second shows how the β-numbers behave under rescaling.

Proposition 2.1. *Let $E \subset \mathbf{R}^2$ have $\dim_{\mathcal{H}}(E) = 1$.*

(1) *For any ball $B_r(y) \subset B_{3r}(x)$,*

$$\beta_E(y,r)^2 \le 3^3 \beta_E(x,3r)^2$$

(2) *The β-numbers have the following scaling property. If $E^{z,t} = tE + z$ then $\beta_{E^{z,t}}(x,r)^2 = \beta_E\left(\frac{x-z}{t}, \frac{r}{t}\right)^2$. Consequently, $C_{E^{z,t}}(z,r) = tC_E(0,t^{-1}r)$.*

Next, for the reader's convenience we record some facts about and give a construction of the standard approximations to the Koch snowflake.

Definition 2.2. Let $I \subset \mathbf{R}^2$ be a line segment, and fix $0 < \alpha < \pi/2$. Define $P(I)$ as the set which results from the following operation:

1. Divide I into three equal subintervals, $I_{\text{left}} \cup I_{\text{center}} \cup I_{\text{right}}$.
2. Over the middle interval, I_{center}, construct an isosceles triangle with angles α and base I_{center}.
3. Delete I_{center}, the base of the isosceles triangle.

We define

$$S(I) = \overline{P(I) \setminus I}, \tag{2.1}$$

and call $S(I)$ the *bump*. If q_I is the orthogonal projection onto the line containing I and $q_I^{\perp}$ is the orthogonal projection onto $I^{\perp}$, then $\text{height}(S(I)) = \text{diam}\{q_I^{\perp}(S(I))\}$ and $\text{width}(S(I)) = \text{diam}\{\pi_I(S(I))\} = \frac{1}{3}\mathcal{H}^1(I)$. We shall abuse our notation slightly by saying that for a collection of line segments, E, the set $P(E)$ is obtained by applying P to each maximal line segment contained in E.

If $I = [0,1] \times \{0\}$ and $\alpha = \frac{\pi}{3}$, the standard approximations to the Koch snowflake are given by $\{P^k(I)\}_{k=1}^{\infty}$, where P^k denotes applying P iteratively k times. We emphasize a few properties about deformations under the operation P.

Proposition 2.3. *For any finite line segment $I \subset \mathbf{R}^2$ and positive integer n,*

$$\text{height}(S(I)) = \frac{\tan(\alpha)}{6}|I|, \tag{2.2}$$

$$\mathcal{H}^1(S(I)) = \frac{\sec(\alpha)}{3}|I|, \tag{2.3}$$

$$\mathcal{H}^1(P^n(E)) = \left(\frac{\sec(\alpha) + 2}{3}\right)^n \mathcal{H}^1(E). \tag{2.4}$$

When $\tau = \frac{1}{20}\min\left\{\frac{\tan(\alpha)}{6}, \frac{1}{3}\right\}$, there exists $c_0 = c(\alpha)$ such that for all lines L

$$\mathcal{H}^1\left(S(I) \setminus B_\tau(L)\right) \geq c_0 \mathcal{H}^1(S(I)). \tag{2.5}$$

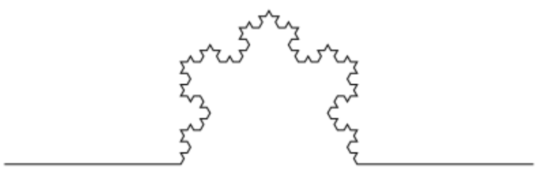

Figure 1. The set $\mathcal{P}_3(I)$ when $\alpha = \pi/3$. Note: despite the least iterations, this has more length than the following two images.

Figure 2. The set $\mathcal{P}_4(I)$ when $\alpha = \pi/9$.

Figure 3. The set $\mathcal{P}_5(I)$ when $\alpha = \pi/27$.

Figure 4. Zoomed in and truncated picture of the 3rd approximation to a base set, made by placing the sets from Figures 1–3 in their appropriate triadic intervals.

Proof. (2.2) and (2.3) follow from planar geometry. The $n = 1$ case for (2.4) follows by adding back in the unchanged intervals I_{left} and I_{right}, which have total length $\frac{2}{3}|I|$. The geometric nature of the definition of P allows us to then iterate this to achieve (2.4).

To verify (2.5) we proceed by contradiction. Suppose no such constant c_0 exists. Then, there exists a sequence of lines intersecting $S(I)$ such that

$$\mathcal{H}^1\left(S(I) \setminus B_\tau(L_i)\right) < 2^{-i}\mathcal{H}^1(S(I)).$$

After passing to a subsequence, L_i converge to some line L with the property that $\mathcal{H}^1\left(S(I) \setminus B_\tau(L)\right) = 0$. Since $S(I)$ is connected, this implies $S(I) \subset B_{2\tau}(L)$. However, this contradicts the fact that $2\tau \leq \frac{1}{10}\min\{\text{height}(S(I)), \text{width}(S(I))\}$. □

Definition 2.4. Define $\mathcal{P}_j$ to be the set operation defined on line-segments by

$$\mathcal{P}_j(I) = P^{j-1}(S(I)) \bigcup \left(I \setminus I_{\text{center}}\right),$$

recalling the definition of $S(I)$ can be found in (2.1). Loosely speaking, for any line segment, I, $\mathcal{P}_j(I)$ is the set that replaces the center of I with a jth approximation of the Koch curve.

Remark 2.5. The Hausdorff distance between two compact sets $A, F \subset \mathbf{R}^2$ is defined by

$$\text{dist}_{\mathcal{H}}(A, F) := \max\left\{\sup_{y\in A}\inf_{x\in F}|x-y|, \sup_{y\in F}\inf_{x\in A}|x-y|\right\}.$$

In addition to metrizing the collection of all non-empty compact sets, the Hausdorff distance generates a topology on the collection of non-empty compact sets that is complete, since $\mathbf{R}^2$ is complete.

Corollary 2.6. *For any line segment $I \subset \mathbf{R}^2$ and positive integer n*

$$\mathcal{H}^1\left(\mathcal{P}_n(I)\right) = \frac{2}{3}|I| + \left(\frac{\sec(\alpha)+2}{3}\right)^{n-1}\frac{\sec(\alpha)}{3}|I|. \tag{2.6}$$

Moreover, if $\alpha \leq \pi/3$,

$$\text{dist}_{\mathcal{H}}(I, P^n(I)) \leq \frac{\tan(\alpha)}{12}|I|. \tag{2.7}$$

Proof. Equations (2.3) and (2.4) verify (2.6). Indeed,

$$\mathcal{H}^1\left(P^{n-1}(S(I))\right) = \left(\frac{\sec(\alpha)+2}{3}\right)^{n-1}\mathcal{H}^1(S(I)) = \left(\frac{\sec(\alpha)+2}{3}\right)^{n-1}\frac{\sec(\alpha)}{3}|I|.$$

The restriction to $\alpha \leq \pi/3$ ensures the longest line segment of $P^i(I)$ has length at most 3^{-i}. Consequently, (2.2) guarantees

$$\text{dist}_{\mathcal{H}}(P^n(I), I) \leq \sum_{i=1}^{n}\text{dist}_{\mathcal{H}}(P^i(I), P^{i-1}(I)) \leq \sum_{i=1}^{n}3^{-i}\text{height}(S(I)) \leq \frac{\tan(\alpha)}{12}|I|. \quad \square$$

We now define a sequence of sets E_k which will be instrumental in defining the "base set" E_∞ in our construction.

Definition 2.7. Now, we let n be a natural number to be chosen later and $E_0 = I = [0,1] \times \{0\}$. We define $E_1 = \mathcal{P}_n(I)$. For $k \geq 2$ inductively define

$$E_k = \mathcal{P}_{kn}\left(\left[0, 3^{-(k-1)}\right] \times \{0\}\right) \bigcup \left(\left\{\left[3^{-(k-1)}, 1\right] \times \mathbf{R}\right\} \cap E_{k-1}\right). \tag{2.8}$$

Notably, for all integers j the operation $\mathcal{P}_j$ applied to $[0, 3^{-(k-1)}] \times \{0\}$ leaves the segment $[0, 3^{-k}] \times \{0\}$ untouched. Consequently, the sequence of sets $\{E_k\}$ are defined by replacing the "next" triadic interval with a scaled approximation of the Koch snowflake. The fact that each triadic strip $[3^{-k}, 3^{-(k-1)}] \times \mathbf{R}$ is only modified once in the sequence of sets E_k is ensures the Hausdorff dimension of the final set remains 1.

Lemma 2.8. (Base set) *Fix $\alpha \le \pi/3$ and any integer n satisfying*[1]

$$3^{-1}\left(\frac{\sec(\alpha)+2}{3}\right)^{n} < 1 < 3^{-1}\left(\frac{\sec(\alpha)+2}{3}\right)^{2n}. \tag{2.9}$$

Then the sequence of sets E_k from (2.8) *converge to a compact and connected Borel set E_∞ in the Hausdorff topology on compact subsets. Furthermore, E_∞ satisfies:*

(1) $\mathcal{H}^1(E_\infty) < \infty$
(2) *For all $\delta > 0$, $C_{E_\infty}(0, \delta) = +\infty$.*

Proof. Note that (2.7) ensure that $\operatorname{dist}_{\mathcal{H}}(E_{k+1}, E_k) \sim 3^{-k}$. In particular, $\{E_k\}$ is a Cauchy sequence in the Hausdorff topology. Hence, the existence of the limiting compact set E_∞ follows from completeness of the Hausdorff topology on compact sets, see Remark 2.5.

By construction each E_k is connected. In fact, since $E_k \cap \overline{B_{3^{-k}}} = [0, 3^{-k}] \times \{0\}$ it follows that $E_k \setminus B_{3^{-k}}(0)$ is connected for each k. Connectedness of E_∞ now follows since $E_\infty \setminus B_{3^{-k}} = E_k \setminus B_{3^{-k}}$. This demonstrates that outside every neighborhood of the origin E_∞ is connected. Consequently, E_∞ is connected.

To see that E_∞ has finite length we write the $\mathcal{H}^1$-measure of E_k as the measure of E_k outside $B_{3^{1-k}}$ plus the measure of E_k inside the ball $B_{3^{1-k}}$. The two key observations being $E_k \setminus B_{3^{1-k}} = E_{k-1} \setminus B_{3^{1-k}}$ and $\mathcal{H}^1(E_{k-1} \setminus B_{3^{1-k}}) = \mathcal{H}^1(E_{k-1}) - 3^{1-k}$. Indeed, (2.6) and these observations imply,

$$\begin{aligned}\mathcal{H}^1(E_k) &= \mathcal{H}^1(E_k \cap B_{3^{1-k}}) + \mathcal{H}^1(E_{k-1} \setminus B_{3^{1-k}}(0)) \\ &= \frac{2}{3}|[0, 3^{1-k}] \times \{0\}| + 3^{1-k}\left(\frac{\sec(\alpha)+2}{3}\right)^{nk-1}\frac{\sec(\alpha)}{3} + \left(\mathcal{H}^1(E_{k-1}) - 3^{-k}\right),\end{aligned}$$

or, equivalently,

$$\mathcal{H}^1(E_k) - \mathcal{H}^1(E_{k-1}) = 3^{-k}\left[\left(\frac{\sec(\alpha)+2}{3}\right)^{nk}\sec(\alpha) - 1\right].$$

Since $\mathcal{H}^1(E_0) = 1$, iteration yields

$$\mathcal{H}^1(E_k) = 1 + \sum_{i=1}^{k} 3^{-i}\left[\left(\frac{\sec(\alpha)+2}{3}\right)^{ni}\sec(\alpha) - 1\right]. \tag{2.10}$$

In particular, $\lim_{k\to\infty}\mathcal{H}^1(E_k) < \infty$ whenever n satisfies the lower bound from (2.9). Moreover $\mathcal{H}^1(E_\infty) = \lim_{k\to\infty}\mathcal{H}^1(E_k)$ since for all $j \ge k$,

$$\mathcal{H}^1\left(E_j \Delta E_k\right) \le 2\sum_{i=k+1}^{\infty} 3^{-i}\left(\frac{\sec(\alpha)+2}{3}\right)^{ni}\sec(\alpha),$$

which decays to zero as $k \to \infty$. Hence, (2.10) holds for E_∞ and $0 < \mathcal{H}^1(E_\infty) < \infty$.

[1] Note that for instance, $\alpha = \pi/3$ and $n \in \{2, 3\}$ satisfies (2.9).

It only remains to show $C_{E_\infty}(0,\delta) = +\infty$ for all $\delta > 0$. To this end, we first note that when $r = r(n,\alpha) = 3^{-1}\left(\frac{\sec(\alpha)+2}{3}\right)^n$,

$$\mathcal{H}^1(E_\infty \cap B_{3^{-k}}(0)) = 3^{-k} + \sec(\alpha)\frac{r^{k+1}}{1-r} - \frac{3^{-(k+1)}}{1-3^{-1}}. \tag{2.11}$$

Indeed, by (2.10) and the trick used to prove (2.10)

$$\mathcal{H}^1(E_\infty \cap B_{3^{-k}}(0)) = 3^{-k} + \sum_{i=k+1}^{\infty} 3^{-i}\left[\sec(\alpha)\left(\frac{\sec(\alpha)+2}{3}\right)^{ni} - 1\right].$$

Claim. *With τ as in Proposition 2.3 and $\alpha \le \pi/3$, there exists a constant c_1 and integer j_0 independent of k such that for any line L, and all k such that $nk-1-j_0 \ge 0$,*

$$\mathcal{H}^1\left(\left(E_\infty \setminus B_{\frac{\tau}{2\cdot 3^k}}(L)\right) \cap B_{3^{-k}}\right) \ge c_1 3^{-k}\left(\frac{\sec(\alpha)+2}{3}\right)^{nk-1-j_0}. \tag{2.12}$$

Proof of Claim. Writing $I' = [0,1]\times\{0\}$, we will in fact scale by 3^k and show the stronger result that

$$\mathcal{H}^1\left(\left(\mathcal{P}_{nk}(I') \setminus B_{\frac{\tau}{2\cdot 3^0}}(L)\right) \cap B_{3^0}\right) \ge c_1 3^0\left(\frac{\sec(\alpha)+2}{3}\right)^{nk-1-j_0}|I'|.$$

To do so, we find a line segment $J \subset S(I') \setminus B_\tau(L)$ such that J has an endpoint in common with one of the two line segments of $S(I')$ and $|J| = 3^{-j_0}\mathcal{H}^1(S(I'))/2$, where j_0 to be chosen later is independent of L. This specific choice of length and endpoint ensure that $P^{nk-1-j_0}(J) \subset \mathcal{P}_{nk}(I')$. Moreover, the choice of j_0 will both guarantee that $|J|$ is large enough and that $P^{nk-1-j_0}(J)$ remains outside of $B_{\tau/2}(L)$, hence verifying the claim.

To find J, we note that the simple shape of $S(I')$ guarantees that $S(I') \setminus B_\tau(L)$ has at most 4 maximal line segments. Hence, there exists a maximal line segment $K_L \subset S(I') \setminus B_\tau(L)$ with $\mathcal{H}^1(K_L) \ge \frac{1}{4}\mathcal{H}^1\left(S(I') \setminus B_\tau(L)\right)$. If K_L is parallel to L let x_L denote either endpoint of K_L. Otherwise, let x_L denote the unique endpoint of K_L that is not contained in $\overline{B_\tau(L)}$. Define J to be the unique subset of K_L of length $3^{-j_0}\frac{\sec(\alpha)}{6}|I'|$ with endpoint x_L. Now, define j_0 as the smallest integer such that

$$3^{-j_0} < \min\left\{\frac{c_0}{4}, \left(\frac{\tan(\alpha)}{12}\cdot\frac{\sec(\alpha)}{6}|I'|\right)^{-1}\frac{\tau}{2}\right\},$$

where c_0 is as in Proposition 2.3. The first condition ensures that $J \subset K_L$ and (2.5) guarantees that the first constraint on j_0 is independent of L and k. The second constraint combined with (2.3) and (2.7) ensure that $\mathrm{dist}_{\mathcal{H}}(P^{nk-1-j_0}(J), J) \le \frac{\tau}{2}$. Moreover, choosing j_0 to be the smallest admissible integer guarantees that $|J| = 3^{-j_0}\frac{\sec(\alpha)}{6}|I'| \ge c'|I'|$ where c' is independent of L and k. Finally, (2.4) completes the proof of the Claim since

$$\mathcal{H}^1\left(\mathcal{P}_{nk}(I') \setminus B_{\tau/2}(L)\right) \ge \mathcal{H}^1(P^{nk-1-j_0}(J)) \ge c_1\left(\frac{\sec(\alpha)+2}{3}\right)^{nk-1-j_0}|I'|,$$

where c_1 depends only on α.

Whenever $nk-1-j_0 \geq 0$, (2.12) implies

$$\begin{aligned}\beta_{E_\infty}(0,3^{-k})^2 &\geq \frac{1}{3^{-k}}\left(\frac{\frac{\tau}{2\cdot 3^k}}{3^{-k}}\right)^2\left(c_1 3^{-k}\left(\frac{\sec(\alpha)+2}{3}\right)^{nk-1-j_0}\right)\\ &= c_2\left(\frac{\sec(\alpha)+2}{3}\right)^{nk}\end{aligned} \tag{2.13}$$

Fix $\delta > 0$ and any integer k_δ such that $3^{-k_\delta} < \delta$ and $nk_\delta - 1 - j_0 \geq 0$. Then, with $\mu = \mathcal{H}^1 \llcorner E_\infty$, repeated applications of Proposition 2.1, (2.13), and (2.11) yield

$$\begin{aligned}\int_{B_\delta(0)}\int_0^\delta \beta_\mu(x,r)^2\frac{dr}{r}d\mu(x) &\geq \ln(3)3^{-2}\sum_{k=k_\delta}^\infty \mu(B_{3^{-(k+2)}})\beta_\mu(0,3^{-(k+2)})^2\\ &\geq \ln(3)3^{-2}\sum_{k=k_\delta}^\infty\left(3^{-k}+\sec(\alpha)\frac{r^{k+1}}{1-r}-\frac{3^{-(k+1)}}{1-3^{-1}}\right)\left(c_2\left(\frac{2+\sec(\alpha)}{3}\right)^{nk}\right).\end{aligned}$$

Due to the lower bound in (2.9), this sum diverges if and only if

$$\sum_{k=k_\delta}^\infty\left[\sec(\alpha)\frac{r^{k+1}}{1-r}-\frac{1}{3^{k+1}-3^k}\right]\left(\frac{2+\sec(\alpha)}{3}\right)^{nk} = \sum_{k=k_\delta}^\infty\left[\sec(\alpha)\frac{3^k r^{2k+1}}{1-r}-\frac{r^k}{3-1}\right]$$

diverges. Since the lower bound in (2.9) ensures $r < 1$, this diverges if and only if $\sum_{k=k_\delta}^\infty (3r^2)^k$ diverges which is equivalent to the upper bound in (2.9). □

Theorem 2.9. *There exists a connected set, $K_0 \subset \mathbf{R}^2$ of finite $\mathcal{H}^1$-measure such that for any $x \in K_0$ and $\delta > 0$*

$$C_{K_0}(x,\delta) = \infty.$$

Proof of Theorem 1.1. Let $\{r_i\}_{i=1}^\infty$ be a sequence of positive numbers such that $\sum_i r_i \leq 1$. Let $E^{x,r} \subset \mathbf{R}^2$ be the set $E^{x,r} = rE_\infty + x$. We construct K_0 as the union of a countable collection of nested sets $\{\Gamma_i\}$.

Let $\Gamma_0 = E_\infty$. Now, let $\{x_{1,j}\}_{j=1}^{N_1}$ be a maximal 2^{-1-1}-separated net in Γ_0. Let

$$\Gamma_1 = \Gamma_0 \cup \bigcup_{j=1}^{N_1} E^{x_{1,j},\frac{r_1}{N_1}}.$$

Suppose that we have defined Γ_{i-1}, some positive integers $\{N_\ell\}_{\ell=1}^{i-1}$ and a collection of points $\{x_{\ell,j} \in \Gamma_{i-2} \mid 1 \leq \ell \leq i-1, 1 \leq j \leq N_\ell\}$ that form a maximal $2^{-(i-1)-1}$-separated net for Γ_{i-2}. Then choose $N_i \in \mathbf{N}$ and points $\{x_{i,j}\}_{1\leq j\leq N_i} \subset \Gamma_{i-1}$ so that $\{x_{\ell,j} \in \Gamma_{i-1} \mid 1 \leq \ell \leq i, 1 \leq j \leq N_\ell\}$ is a maximal 2^{-i-1}-separated net in Γ_{i-1}. Then define Γ_i by

$$\Gamma_i = \Gamma_{i-1} \cup \left(\bigcup_{j=1}^{N_j} E^{x_{i,j}\,\frac{r_i}{N_i}}\right).$$

We claim that $K_0 = \bigcup_{i=0}^\infty \Gamma_i$ is the desired set. First note that since each Γ_i is countably rectifiable, then K_0 is countably rectifiable. Moreover, $\{x_{i,j}\}_{j=1}^{N_i} \subset \Gamma_{i-1}$ for all i ensures K_0 inherits connectivity from E_∞. Furthermore, since $\{\Gamma_i\}$ is a nested sequence increasing to K_0 and $\sum_i r_i \le 1$,

$$\mathcal{H}^1(K_0) = \mathcal{H}^1\left(E_\infty \cup \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{N_i} E^{x_{i,j},\frac{r_i}{N_i}}\right) \le \mathcal{H}^1(E_\infty)\left(1+\sum_{i=1}^{\infty} r_i\right) \le 2\mathcal{H}^1(E_\infty).$$

It only remains to show that for $x \in K_0$ and $\delta > 0$ that $C_{K_0}(x,\delta) = \infty$. To this end, fix $x \in K_0$, and $\delta > 0$. By definition of K_0, there exists ℓ_0 such that $x \in \Gamma_{\ell_0}$. Then, by the net property of the points $\{x_{i,j}\}$, it follows that for $\ell - 1 \ge \ell_0$ large enough that $2^{-\ell-1} < \delta/4$, there exists $i \le \ell$ with $x_{i,j} \in \Gamma_{\ell-1} \cap B(x,\delta/2) \subset K_0 \cap B(x,\delta/2)$. Writing $\mu = \mathcal{H}^1 \llcorner K_0$ and $\mu_{i,j} = \mathcal{H}^1 \llcorner E^{x_{i,j},\frac{r_i}{N_i}}$ it follows from monotonicity of the integral that

$$\int_{B_\delta(x)}\int_0^\delta \beta_{K_0;2}(y,r)^2\frac{dr}{r}\,\mathrm{d}\mu(y) \ge \int_{B_{\delta/2}(x_{i,j})}\int_0^{\delta/2} \beta_{\mu_{i,j};2}(y,r)\frac{dr}{r}\,\mathrm{d}\mu_{i,j}(y), \tag{2.14}$$

or equivalently $C_{K_0}(x,\delta) \ge C_{\mu_{i,j}}(x_{i,j},\delta/2)$. Recalling that $E^{z,t} = tE_\infty + z$, we use (2.14), Proposition 2.1(2), and Lemma 2.8 to conclude

$$C_{K_0}(x,\delta) \ge C_{E^{x_{i,j},\frac{r_i}{N_i}}}\left(x_{i,j},\frac{\delta}{2}\right) = \frac{r_i}{N_i}C_{E_\infty}\left(0,\frac{\delta N_i}{2r_i}\right) = \infty.$$

Since $x \in K_0$ and $\delta > 0$ are arbitrary this finishes the proof. □

Remark 2.10. Since K_0 from Theorem 1.1 is connected, $\mathcal{H}^1(\overline{K_0}) = \mathcal{H}^1(K_0) < \infty$ and $\overline{K_0}$ is compact, see [Sch07, Lemma 3.4, 3.5]. Thus $\overline{K_0}$ is a rectifiable curve by Ważewski's theorem, see [Sch07, Lemma 3.7] or [AO17, Theorem 4.4].

The authors thank Matthew Badger for pointing out that $\overline{K_0}$ coincides with the Hausdorff-limit of $\{\Gamma_i\}$. So, Gołab's semi-continuity theorem and Ważewski's theorem suffice to ensure $\overline{K_0}$ is a rectifiable curve. See, for instance, [AO17] or [Fal86] for relevant theorem statements.

3. Proof of Theorem 1.2

The unique compact set fixed by the iterated function system (IFS),

$$\{F_{i,j}\colon \mathbf{R}^2 \to \mathbf{R}^2 | F_{i,j}(E) = 2^{-2}(E + (i,j)), \quad i,j \in \{0,3\}\}$$

is called the 4-corner Cantor set, $\mathcal{C}$. The 4-corner Cantor set is an Ahlfors regular set with positive and finite $\mathcal{H}^1$-measure and is purely unrectificable. That is, $\mathcal{H}^1\big(\mathcal{C} \cap f(\mathbf{R})\big) = 0$ for all Lipschitz functions $f\colon \mathbf{R} \to \mathbf{R}^2$.

Typically, one approximates the 4-corner Cantor set by beginning with the "initial set" $[0,1]^2$ in their iteration scheme. However, our motivation for the construction of A_0 arises from considering the initial set $[0,1) \times \{0\}$. Beginning with a 1-dimensional set allows every approximating set to have positive and finite $\mathcal{H}^1$-measure. This is also critical to produce estimates on the β-numbers of each approximation.

The general strategy for producing the desired set A_0 in Theorem 1.2 is as follows. We produce a base set Σ_0 such that within successive tetradic strips $[2^{-2i}, 2^{-2i+2}] \times \mathbf{R}$ the set $\Sigma_0 \cap \big([2^{-2i}, 2^{-2i+2}] \times \mathbf{R}\big)$ is a scaled version of a higher-iteration approximation to the 4-corner Cantor set. This allows for precise control on the β-numbers in every neighborhood of the origin. Then, following the strategy for Theorem 1.1 we carefully iterate this set "on itself" in a dense way, taking care to preserve Ahlfors regularity.

3.1. Approximations to the 4-corner Cantor set. Consider the following sequence of approximations to the 4-corner cantor set, by sets of positive and finite $\mathcal{H}^1$-measure. Let $E_0 = [0,1) \times \{0\}$ and inductively define

$$E_k = \sum_{(i,j)\in\{0,3\}^2} p_{ij} + 2^{-2} E_{k-1} \quad \text{where } \; p_{ij} = \left(\frac{i}{2^2}, \frac{j}{2^2}\right). \tag{3.1}$$

The word similarity is used to refer to any mapping that can be written as a composition of scalings, rotations, reflections, and translations. Throughout the rest of the paper, we say that two sets are similar if one is the image of the other by a similarity. In reality the similarities we discuss can always be written as a scaling and translation, as in (3.1).

We let Δ denote the collection of tetradic half-open cubes in $\mathbf{R}^2$, that is

$$\Delta = \{[a2^{-2k}, (a+1)2^{-2k}) \times [b2^{-2k}, (b+1)2^{-2k}) \mid a, b, k \in \mathbf{Z}\}.$$

For some $Q \in \Delta$, we let $\ell(Q)$ denote the sidelength of Q. We partition the tetradic cubes into cubes of fixed sidelength by defining $\Delta^i = \{Q \in \Delta \mid \ell(Q) = 2^{-2i}\}$.

In general, for a set $E \subset \mathbf{R}^2$ we respectively denote the *length of* E and the *height of* E by

$$\ell(E) = \operatorname{diam}\{\pi_x(E)\} \quad \text{and} \quad h(E) = \operatorname{diam}\{\pi_y(E)\}$$

where π_x and π_y denote the orthogonal projection onto the horizontal and vertical axes. In particular, for a cube Q with axis-parallel sides, this notion of length coincides with the cubes sidelength. Hence no confusion with the earlier convention that $\ell(Q)$ is the sidelength of Q will arise.

Definition 3.1. (Clusters and sub-clusters) Any set which is similar to any E_k or $E_k \cup [0,1) \times \{0\}$ for $k \in \mathbf{N}$ will be called a *cluster*. Moreover, for fixed $k \in \mathbf{N}$, we will call E_k the 0*th sub-cluster of* E_k and the 2^{2k} line segments that make up E_k are called the k*th -subclusters of* E_k. For $\ell \in \{1, \dots, k-1\}$, the $2^{2\ell}$-sets contained in E_k which are similar to $E_{k-\ell}$ are called the ℓ*th sub-clusters of* E_k.

Definition 3.2. (Root points) We associate to each cluster and each cube a root point. The *root point* of a cluster E is the lower-most and left-most point in the cluster. Since a sub-cluster is itself a cluster, the notion of a root point extends to sub-clusters. For a cluster E, we let x_E denote its root point. For a tetradic cube $Q \in \Delta$ we let x_Q denote the lower-most and left-most point of Q and call x_Q *the root point of* Q.

Proposition 3.3. *For fixed non-negative integer* k*, the set* E_k *has the following properties.*

1. *Each* E_k *is a finite union of* 2^{2k} *intervals each of length* 2^{-2k}*. In particular,* $\mathcal{H}^1(E_k) = 1$ *and* E_k *is countably* 1*-rectifiable. Moreover, each connected component* I *of* E_k *has* $\partial I \subset \ell(I)\mathbf{Z}^2 = 2^{-2k}\mathbf{Z}^2$ *and consequently is contained in a line* $\mathbf{R} \times \{a2^{-2k}\}$ *for some* $a \in \mathbf{N}_0$.
2. *If* $j \geq 0$ *is an integer and if* $Q \in \Delta^j$ *is such that* $Q \cap E_k$ *is non-empty, then*

$$Q \cap E_k = \begin{cases} x_Q + [0, \ell(Q)) \times \{0\}, & j \geq k, \\ x_Q + 2^{-2j} E_{k-j}, & j \leq k. \end{cases} \tag{3.2}$$

3. *Each* E_k *is Ahlfors regular with regularity constant independent of* k.
4. *For* $0 \leq j \leq k$ *an integer, each* j*th subcluster of* E_k *has* $\mathcal{H}^1$*-measure* 2^{-2j}.

(5) *For* $1 \le j \le k$ *an integer, the* j*th subclusters of* E_k *are* $2 \cdot 2^{-2j}$*-separated horizontally and at least* $2 \cdot 2^{-2j}$*-separated vertically. In fact, they are* $\big(3 - \frac{3}{4}\sum_{i=1}^{k-j} 2^{-2i}\big) \cdot 2^{-2j}$*-separated vertically.*

(6) *If* $J \subset E_k$ *is a connected component, then* J *is a vertical distance of* $3 \cdot 2^{-2k}$ *from the nearest connected component* J' *of* E_k*.*

(7) *There exists a universal constant* $c > 0$ *such that if* $k \ge 2$ *and* $\mu_k = \mathcal{H}^1 \llcorner E_k$*, then for all* $x \in E_k$*,*

$$\int_{6\cdot 2^{-2k}}^{1} \beta_{\mu_k}(x,r)^2 \frac{dr}{r} \ge c(k-2)$$

Proof. (1) follows immediately from (3.1) since each $p_{ij} \in 2^{-2}\mathbf{Z}^2$.

To see (2), we first note that the case $j = 0$ is clear for any $k \in \mathbf{N}$. Further, the case $k = 0$ is clear for all $j \in \mathbf{N}$. To procede inductively suppose that (3.2) holds for all $k \in \mathbf{N}$ when $j = \ell - 1$. We will show it holds for all $k \in \mathbf{N}$ when $j = \ell$. Indeed, suppose that $Q \in \Delta^\ell$ has non-empty intersection with E_ℓ. Let x_Q be the root of Q. Choose $p \in \{p_{ij}\}_{(i,j)\in\{0,3\}^2}$ such that $Q \subset p + [0, 2^{-2})^2$. Then, $4(Q \cap E_k - p) = (4Q - 4p) \cap (4E_k - 4p) = \tilde{Q} \cap E_{k-1}$ where $\tilde{Q} := 4Q - 4p \in \Delta^{\ell-1}$. By the inductive assumption,

$$\tilde{Q} \cap E_{k-1} = \begin{cases} x_{\tilde{Q}} + [0, \ell(\tilde{Q})) \times \{0\}, & \ell - 1 \ge k-1, \\ x_{\tilde{Q}} + 2^{-2(i-1)} E_{(\ell-1)-(i-1)}, & \ell - 1 \le k-1. \end{cases}$$

Translating and scaling this back to what this means about $Q \cap E_k$ verifies the induction.

(3) follows from (1) and (2) since these imply that $\frac{\mathcal{H}^1(Q\cap E_k)}{\ell(Q)} = 1$ for tetradic cubes Q with $\ell(Q) \le 1$ that intersect E_k. This suffices since any ball contains a tetradic cube of comparable sidelength and is contained in 4^2 tetradic cubes of comparable sidelength.

(4) is equivalent to showing that E_k is made of 2^{2k} intervals, each of length 2^{-2k}.

(5) The horizontal separation is verified by an argument similar to the vertical separation. For the vertical separation, we only verify that the vertical separation is at least $2 \cdot 2^{-2j}$. Indeed, this follows since E_ℓ is contained in the horizontal strips $\mathbf{R} \times [0, 1/4] \cup [3/4, 1]$ for all ℓ. Then, the scaling from (3.1) ensures that the jth subclusters, which arise by applying (3.1) j times to the sets E_{k-j} are vertically $2 \cdot 2^{-2j} = \frac{1}{2}2^{-2(j-1)}$-separated. The reason the height-bound can be improved, is because the jth subclusters are actually contained in smaller strips. See for instance, E_1, where the first subclusters are contained in lines, and E_2 where the first subclusters are contained in the strips $\mathbf{R} \times [0, \frac{3}{16}] \cup [\frac{12}{16}, \frac{15}{16}]$.

(6) follows from the fact that vertically-closest connected components in E_k come from the connected components of E_1 which are $3 \cdot 2^{-2}$ separated. After being scaled by 2^{-2} in (3.1) another $(k-1)$ times the separation is reduced to a distance of $3 \cdot 2^{-2k}$ as claimed. This coincides with the precise formula in (5) and could be considered as a base case for induction on j for the interested reader.

(7) Throughout the proof of (7), we fix integers $1 \le j < k$ and $k \ge 2$.

Claim 1. *For all* $x \in E_k$ *there exists some* $x' \in E_j$ *with*

$$\operatorname{dist}(x, x') \le 2^{-2j}. \tag{3.3}$$

Proof of Claim 1. Note that the scaling in (3.1) ensures that for some ℓ, we know that every $x \in E_{\ell+1}$ is within a distance $3 \cdot 2^{-2(\ell+1)}$ of a point in E_ℓ. Iterating verifies the claim by showing for $x \in E_k$ there exists $x' \in E_j$ such that

$$\operatorname{dist}(x, x') \leq \sum_{\ell=j+1}^{k} 3 \cdot 2^{-2\ell} \leq 3 \sum_{\ell=j+1}^{\infty} 2^{-2\ell} = 4 \cdot 2^{-2(j+1)}.$$

Claim 2. *There exists c independent of j such that for all $5 \cdot 2^{-2j} \leq r \leq 11 \cdot 2^{-2j}$ and all $x' \in E_j$,*

$$\beta^1_{\mu_j;2}(x', r)^2 \geq c.$$

Proof of Claim 2. Let $J \subset E_j$ be the connected component containing x'. By (4)-(6) of this proposition, it follows that for $r \geq 5 \cdot 2^{-2j} = \sqrt{(3 \cdot 2^{-2j})^2 + (4 \cdot 2^{-2j})^2}$, the ball $B_r(x')$ contains J and 3 other connected components of E_j. Consequently, there are two horizontal lines, L^u and L^d, such that $B_r(x') \cap \left(L^u \cup L^d\right)$ contains at least 4 connected components of E_j. Part (1) of this proposition ensures,

$$\min\{\mu_j\left(L^u \cap B_r(x')\right), \mu_j\left(L^d \cap B_r(x')\right)\} \geq 2 \cdot 2^{-2j}. \tag{3.4}$$

Moreover, part (6) ensures that the distance between L^u and L^d is $3 \cdot 2^{-2j}$, which combined with (3.4) forces that any line L satisfies,

$$\mu_j\left(\left\{y \in B_r(x') \middle| \operatorname{dist}(y, L) \geq 3 \cdot 2^{-2j-1}\right\}\right) \geq 2 \cdot 2^{-2j}. \tag{3.5}$$

Finally, recalling $5 \cdot 2^{-2j} \leq r \leq 11 \cdot 2^{-2j}$, (3.5) implies

$$\inf_L \int_{B_r(x')} \left(\frac{\operatorname{dist}(y, L)}{r}\right)^2 \frac{d\mu_j(y)}{r} \geq \left(\frac{\frac{3 \cdot 2^{-2j}}{2}}{r}\right)^2 \left(\frac{2 \cdot 2^{-2j}}{r}\right) \geq c$$

which verifies Claim 2.

Claim 3. *There exists c' such that for all $x \in E_k$ and all integers $1 \leq j < k$ and ρ such that $6 \cdot 2^{-2j} \leq \rho \leq 12 \cdot 2^{-2j}$,*

$$\beta^1_{\mu_k;2}(x, \rho)^2 \geq c'. \tag{3.6}$$

Proof of Claim 3. Claim 1 ensures that for all $5 \cdot 2^{-2j} \leq r \leq 11 \cdot 2^{-2j}$ there exists $x' \in E_j$ such that $B_r(x') \subset B_\rho(x)$. As in Claim 2, fix lines L^d and L^u such that $B_r(x) \cap \left(L^u \cup L^d\right)$ contains at least 4 connected components of E_j. Choose a so that $L^d = \mathbf{R} \times \{a\}$ and $L^u = \{a + (0, 3 \cdot 2^{-2j})\} + \mathbf{R} \times \{0\}$. Moreover, suppose the left-most connected component of L^u has right-most endpoint with x-value equal to c_1. Define $L_v = \{c_1 + 2^{-2j}\} \times \mathbf{R}$ and $L_h = a + 2^{-2j}$. By Proposition 3.3(5,6), the neighborhoods $N_v = B_{2^{-2j}}(L_v)$ and $N_h = B_{2^{-2j}}(L_h)$ are disjoint from E_ℓ for all $\ell \geq j$. See Figure 5.

Consequently, for any line L the nieghborhood $B_{2^{-2j-1}}(L)$ can intersect at most 4 of the "quadrants" made by the neighborhoods of N_v and N_L. Making a generous estimate since the ball may cut-off part of one of the quadrants in Figure 5, we conclude

$$\mu_k\left(\{y \in B_r(x') \mid \operatorname{dist}(y, L) \geq 2^{-2j-2}\}\right) \geq 2^{-2j-2} \tag{3.7}$$

where the measure-bound comes Proposition 3.3(1). Since $B_r(x') \subset B_\rho(x)$ and $1 \leq \frac{\rho}{r} \leq C < \infty$, Claim 3 follows from (3.7) analogously to how Claim 2 followed from (3.5).

Finally, we verify (7) because

$$\int_{6\cdot 2^{-2j}}^{1} \beta_{\mu_k}(x,\rho)^2 \frac{d\rho}{\rho} \geq \sum_{j=2}^{k} \int_{6\cdot 2^{-2j}}^{11\cdot 2^{-2j}} c' \frac{d\rho}{\rho} = c(k-2). \tag{3.8}$$

□

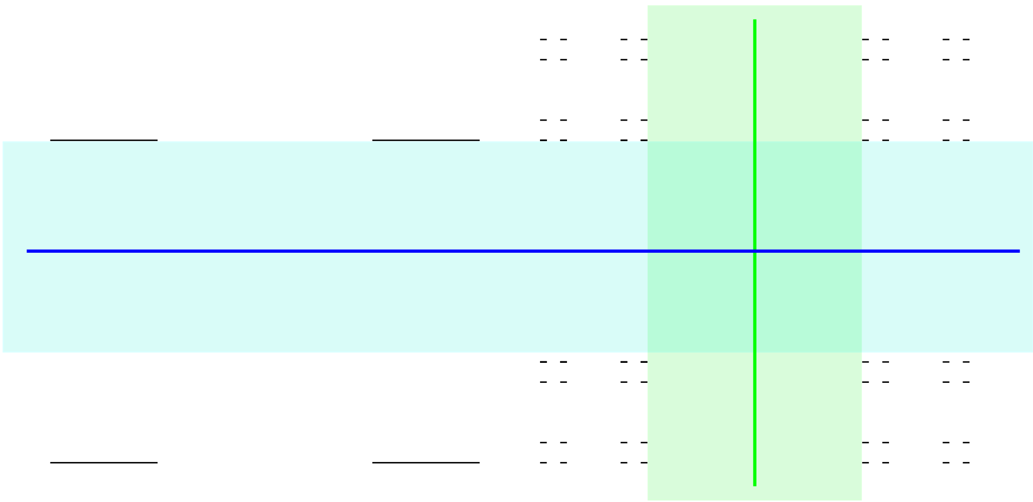

Figure 5. When $j = k-2$, the picture displays a subclusters of equal length for E_j and E_k on the left and right respectively. In E_k, the line L_v and its neighborhood N_v are in green, whereas the line L_h and its neighborhood N_h are drawn where it would pass through both E_j and E_k.

We construct the base set Σ_0 from approximations to the 4-corner Cantor set by first defining

$$E(n) := (2^{-2n}, 0) + 2^{-2n} E_{2^{2n}} \quad \text{and} \quad \Sigma_0 := \bigcup_n E(n) \cup \big([0,1) \times \{0\}\big). \tag{3.9}$$

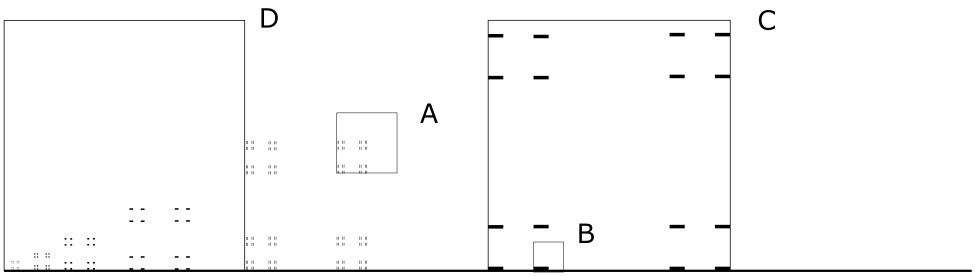

Figure 6. Here we see Σ_0 and several examples of $Q \in \Delta$. The cube D illustrates the first case in Equation (3.10). The cube A illustrates an example of the second case in Equation (3.10). The cubes B and C illustrate examples of the last case in Equation (3.10).

Proposition 3.4. *Σ_0 has the following properties.*

(1) $0 < \mathcal{H}^1(\Sigma_0) < \infty$ *and Σ_0 is countably 1-rectifiable.*
(2) *If $j \geq 0$ is an integer and $Q \in \Delta^j$ is such that $Q \cap \Sigma_0 \neq \emptyset$, then*

$$Q \cap \Sigma_0 = \begin{cases} \Sigma_0 \cap [0, \ell(Q))^2, & x_Q = (0,0), \\ x_Q + 2^{-2j} E_k \text{ for some } k\ , & x_Q \neq (0,0) \text{ and } \pi_y(x_Q) \neq 0, \\ x_Q + 2^{-2j} E_k \cup [0, \ell(E_k)) \times \{0\}, & x_Q \neq (0,0) \text{ and } \pi_y(x_Q) = 0. \end{cases} \tag{3.10}$$

(3) $C_{\Sigma_0}(0,\delta) = +\infty$ *for all $\delta > 0$.*

Proof. (1) Σ_0 has positive and finite mass due to Proposition 3.3(1) and the geometric scaling in (3.9). It is also the countable union of countably 1-rectifiable sets by Proposition 3.3(1).

(2) The case when $x_Q = (0,0)$ is clear. Suppose $x_Q \neq (0,0)$. There exists unique a, b such that

$$x_Q = \left(a2^{-2j}, b2^{-2j}\right). \tag{3.11}$$

If $j = 0$, $Q \cap \Sigma_0 \neq \emptyset$, and $\Sigma_0 \subset [0,1)^2$ forces $a = b = 0$. Therefore, $j \geq 1$. Since $h(E_{2^{2n}}) < \ell(E_{2^{2n}})$ and the $E(n)$ only use a translation in the positive horizontal direction of $E_{2^{2n}}$ and a homogeneous scaling, it follows that $\Sigma_0 \cap Q \neq \emptyset$ implies $0 \leq b < a$ so that $a \geq 1$. Since, $\ell(Q) = 2^{-2j}$ it follows that $a2^{-2j} \geq \ell(Q)$. Comparing the translation and scaling sizes in (3.1), $a \geq 2^{2j}\ell(Q)$ implies

$$\Sigma_0 \cap Q = \begin{cases} Q \cap E(n), & b \geq 1, \\ Q \cap \left(E(n) \cup [0, \ell(E(n))) \times \{b\}\right), & b = 0, \end{cases} \tag{3.12}$$

for some specific $n \leq j$. For simplicity of writing, assume we are in the first case. Then, $2^{2n}(Q \cap E(n) - (2^{-2n}, 0)) = (2^{2n}(Q - (2^{-2n}, 0))) \cap E_{2^{2n}}$ or equivalently

$$Q \cap E(n) = (2^{-2n}, 0) + 2^{-2n}\left(2^{2n}\left(Q - \left(2^{-2n}, 0\right)\right) \cap E_{2^{2n}}\right). \tag{3.13}$$

In light of (3.13), it follows that (3.2) implies the 2nd case of (3.10) since $2^{2n}(Q - (2^{-2n}, 0)) \in \Delta^{j-n}$ and $n \leq j$. Analogously the $b = 0$ case corresponds to the 3rd case of (3.10).

(3) Fix $\delta > 0$. Choose N large enough that $11 \cdot 2^{-2N} < \delta/2$. In particular, for all $n \geq N$, $E(n) \subset B_\delta(0)$. Then, with $\mu = \mathcal{H}^1 \llcorner \Sigma_0$ and $\mu_n = \mathcal{H}^1 \llcorner E(n)$, it follows from Proposition 3.3 (1,7), Proposition 2.1 (2), and the scaling in (3.9) that

$$C_{\Sigma_0}(0, \delta) \geq \sum_{n \geq N} \int_{E(n)} \int_0^{2^{-2n}} \beta^1_{\mu_n;2}(x, r)^2 \frac{dr}{r} \, \mathrm{d}\mu_n(x) \geq \sum_{n \geq N} c(2^{2n} - 2)\mathcal{H}^1(E(n)),$$

which diverges and completes the proof. □

We wish to iterate Σ_0 densely along itself while being careful to maintain Ahlfors upper- and lower-regularity. This is attained by scaling, and being careful where we iterate.

Definition 3.5. (Tail points) We say a point y is a *tail point of* E if $0 < \mathcal{H}^1(E) < \infty$ and there exists a tetradic number r and $\delta > 0$ such that

$$y + r\Sigma_0 \cap B_\delta \subseteq E.$$

Note, if $y \in B_\delta(x)$ is a tail point of a set E, then $C_E(x, \delta) \equiv \infty$. See Claim 1 from the proof of Theorem 1.2.

Definition 3.6. (Iterative construction) Let Σ_0 be as above. Supposing that Σ_{i-1} has been defined, we define a (possibly empty) special collection of tetradic points,

$$D^i = \left\{ x \in 2^{-2i}\mathbf{Z}^2 \,\middle|\, \left(x + [0, 2^{-2i})^2\right) \cap \Sigma_{i-1} = x + [0, 2^{-2i}) \times \{0\} \right\}, \tag{3.14}$$

and define Σ_i by

$$\Sigma_i = \Sigma_{i-1} \bigcup \left\{ \bigcup_{x \in D^i} x + 2^{-8i}\Sigma_0 \right\}. \tag{3.15}$$

Define,

$$A_0 = \bigcup_{j \in \mathbf{N}} \Sigma_j. \tag{3.16}$$

Proposition 3.7. *The sets $\{\Sigma_j\}_{j=0}^{\infty}$ and $\{D^j\}_{j=1}^{\infty}$ as in Definition 3.6 have the following properties:*

1. *$\Sigma_{j-1} \subset \Sigma_j$ for all $j \geq 1$.*
2. *Σ_j is contained in countably many horizontal line segments with tetradic heights.*
3. *There are infinitely many j so that D^j is non-empty.*
4. *If I is a connected component of Σ_j then $\partial I \subset \ell(I)\mathbf{Z}^2$.*
5. *Σ_j contains no connected component of length at least 2^{-2j} that contain no tail point.*

Proof. Indeed, (1) follows from (3.15).

(2) Follows by induction. For Σ_0 it follows from Proposition 3.3 (1) combined with the scaling in (3.9). For general Σ_j induction holds due to the fact that each scaled copy of Σ_0 in (3.15) has a tail point on the dyadic lattice D^i which is coarser than the tetradic scaling factor of Σ_0.

(3) follows from (2). (5) follows from (4) and the definition of D^j in (3.14).

(4) If I is a connected component of Σ_j then there exists $y \in D^i$ some $i \leq j$ such that I is a connected component of $y + 2^{-8i}\Sigma_0$. But then, $2^{8i}(I - y)$ is a connected component of Σ_0. Since $y \in 2^{-2i}\mathbf{Z}^2$, Propositions 3.3(1) and 3.4(2) ensure $\partial\left(2^{8i}(I-y)\right) \in 2^{8i}\ell(I)\mathbf{Z}^2$ which verifies (4). □

Definition 3.8. (Associated cubes) Any cluster (or subcluster) E has associated to it the dyadic cube $Q_E = x_E + [0, \ell(E))^2$. In particular, by Proposition 3.3 (5) it follows that if clusters E, E' are disjoint with $\ell(E) = \ell(E')$, then $Q_E, Q_{E'}$ are disjoint cubes. Moreover, for some cluster E, the root point of Q_E and the root point of E coincide.

Definition 3.9. We associate to the base set Σ_0 the following family of cubes

$$\mathcal{Q}_{\Sigma_0} = \left\{[0, 2^{-2i})^2 \colon i \geq 0\right\} \cup \left\{Q_E \colon E \text{ is a subcluster of } E(n) \subset \Sigma_0, n \geq 1\ \right\} \tag{3.17}$$

By similarity, for any $y \in D^i$ we associate to $y + 2^{-8i}\Sigma_0$ the family of cubes

$$\mathcal{Q}_y = \left(y + 2^{-8i}\mathcal{Q}_{\Sigma_0}\right) \bigcup \left(y + \{[0, 2^{-2k})^2 : i \leq k\}\right). \tag{3.18}$$

We will let

$$\mathcal{Q} = \bigcup_{i \geq 0} \bigcup_{y \in D^i} \mathcal{Q}_y \tag{3.19}$$

which we stratify by scale in the following sense

$$\mathcal{Q}^i = \{Q \in \mathcal{Q} \mid \ell(Q) = 2^{-2i}\} \tag{3.20}$$

and we enumerate the elements $\mathcal{Q}^i$ so that

$$\mathcal{Q}^i = \{Q^i_j\}_{j=1}^{N(i)}. \tag{3.21}$$

Finally, for $Q \in \mathcal{Q}$ and any positive integer ℓ we let $\mathcal{C}_\ell(Q) = \{Q' \in \mathcal{Q} \mid Q' \subset Q,\ \ell(Q') = 2^{-2\ell}\ell(Q)\}$, and call $\mathcal{C}_\ell(Q)$ the *ℓth descendent cubes of Q*.

Lemma 3.10. *For all $i \geq 0$ and all cubes, $Q^i_j \in \mathcal{Q}^i$, $\Sigma_i \cap Q^i_j$ is similar to one of the following:*

(1) $\left(2^{-2k}\Sigma_0 \cup [0,1) \times \{0\}\right) \cap [0,1)^2$ *for some integer k.*
(2) $E \cap Q_E$ *for some sub-cluster $E \subset E(n)$ for some integer $n \geq 1$*

This follows immediately from the explicit definition of cubes.

Lemma 3.11. *$\mathcal{Q}^j \subset \Delta^j$ and for all $Q \in \Delta^j$, then either $Q \cap \Sigma_j = \emptyset$ or $Q \in \mathcal{Q}_j$.*

This follows from an induction argument similar to the proofs of Propositions 3.3 (1) and 3.4 (2). The key observation in the induction is that the scaling in (3.15) ensures that all tail points added in the jth stage have root points in tetradic lattices that are coarser than the length of the scaled copy of Σ_0 being added.

Corollary 3.12. *The cubes $\mathcal{Q}$ have the following nice properties:*

(1) *Each collection $\mathcal{Q}_i$ is a disjoint collection of cubes, and for any $Q \in \mathcal{Q}$ and any integer $\ell \geq 0$, $\mathcal{C}_\ell(Q)$ is a disjoint collection of subcubes of Q.*
(2) *For all non-negative integers i and j,*

$$\Sigma_i \subseteq \bigcup\nolimits_{Q \in \mathcal{Q}_j} Q \tag{3.22}$$

(3) *In particular, for any $Q_0 \in \mathcal{Q}_i$*

$$\Sigma_i \cap Q_0 = \Sigma_i \bigcap \left(\bigcup\nolimits_{Q \in \mathcal{C}_1(Q)} Q\right) \tag{3.23}$$

Proof of Theorem 1.2. By Lemma 3.4 (1), Σ_0 is 1-rectifiable, and A_0 is a countable union of scaled translations of Σ_0 so A_0 is 1-rectifiable .

Next, we show that A_0 is 1-Ahlfors regular. Indeed, it suffices to show that there exists $0 < c \leq C < \infty$ independent of i such that for for any $j \geq 0$, $Q \in \Delta^j$, and $Q \cap A_0 \neq \emptyset$,

$$c\ell(Q) \leq \mathcal{H}^1(Q \cap A_0) \leq C\ell(Q). \tag{3.24}$$

We do this by showing similar bounds for $\frac{\mathcal{H}^1(Q \cap \Sigma_j)}{\ell(Q)}$ for cubes $Q \in \Delta^j$ that intersect Σ_j, and then proving that not too much additional mass is added to the cube Q.

Due to Lemma 3.11 the condition that $Q \in \Delta^j$ and $Q \cap A_j \neq \emptyset$ is equivalent to $Q \in \mathcal{Q}_j$. Since $Q \in \mathcal{Q}_j$ Lemma 3.10 characterizes what $Q \cap \Sigma_j$ looks like and we conclude

$$\ell(Q) \leq \mathcal{H}^1(Q \cap \Sigma_j) \leq 3\ell(Q), \tag{3.25}$$

by considering each of the three cases in Lemma 3.10. Indeed, each cube either contains its entire bottom portion, or contains a cluster E with $\ell(E) = \ell(Q)$. In either case this implies the lower bound in (3.25). On the other hand, we know that a rough upper-bound is to assume that $Q \cap \Sigma_j$ contains a cluster with a line segment at the bottom, and contains Σ_0 scaled by 2^{-2k}, then by Proposition 3.3, the upper bound in (3.25) follows.

It remains to show that (3.25) implies (3.24). Due to Proposition 3.7 (1), the lower-bound in (3.24) is inherited directly from (3.25). The upper-bound follows with the additional observation that for $\ell \geq j$,

$$\mathcal{H}^1\left(Q \cap \Sigma_{\ell+1} \setminus \Sigma_\ell\right) \leq \#|D_{\ell+1}| 2^{-8(\ell+1)} \mathcal{H}^1\left(\Sigma_0\right) \leq 2^{-4(\ell+1)} \mathcal{H}^1(\Sigma_0).$$

Summing over $\ell \geq j$ verifies (3.24). It is a standard argument to go from Ahlfors regularity in tetradic/dyadic cubes to in balls, see for instance the brief description in the proof of Proposition 3.3(3). Since the cubes in $\mathcal{Q}$ are all the tetradic cubes with non-empty intersection with A_0, we have regularity in tetradic cubes.

Finally, to see that $C_{A_0}(x,\delta) = \infty$ it suffices to show the following claim.

Claim 1. *If $x \in A_0$ and $\delta > 0$, then there is a tail point in $A_0 \cap B_{\delta/2}(x)$.*

Briefly assuming that Claim 1 holds, the fact that $C_{A_0}(x,\delta) = \infty$ for all $x \in A_0$ and $\delta > 0$ follows since if y is the tail point in $B_{\delta/2}(x)$ then, by Proposition 3.4 (3) and monotonicity of integrals of non-negative functions:

$$C_{A_0}(x,\delta) \geq C_{A_0}(y,\delta/2) \geq C_{\Sigma_0}(0,\epsilon_y) = \infty,$$

where $\epsilon_y > 0$ is some scale dependent on which D^i the tail point y is in.

To verify Claim 1, fix x and δ as in the claim. Adopting the convention that $\Sigma_{-1} = \emptyset$ fix i_0 such that $x \in \Sigma_{i_0} \setminus \Sigma_{i_0-1}$. Choose k to be the smallest natural number such that $\operatorname{diam}\left(2^{-8k}\Sigma_0\right) \leq \delta/4$.

Case 1. $B_{\delta/4}(x) \cap \Sigma_k$ contains a tail. Since $\Sigma_k \subset A_0$ in this case the claim holds.

Case 2. Otherwise, choose $k_0 \geq k$ such that

$$\begin{cases} \left(\Sigma_{k_0-1} \setminus \Sigma_k\right) \cap B_{\delta/4}(x) = \emptyset, \\ \left(\Sigma_{k_0} \setminus \Sigma_k\right) \cap B_{\delta/4}(x) \neq \emptyset, \end{cases}$$

that is k_0 is the first stage after k where something new is added to the ball $B_{\delta/4}(x)$. The way something new is added to the ball $B_{\delta/4}(x)$ in the k_0th stage is if there exists y such that,

$$\{y + 2^{-8k_0}\Sigma_0\} \cap \{\Sigma_{k_0} \cap B_{\delta/4}(x)\} \neq \emptyset.$$

But then, y is a tail point of Σ_{k_0} and consequently of A_0. By our choice of k, we conclude

$$|x - y| < \operatorname{diam}(2^{-4k_0}\Sigma_0) + \delta/4 \leq \delta/2.$$

Hence the tail point y is indeed in $B_{\delta/2}(x)$. So, by Proposition 2.1(2)

$$C_{A_0}(x,\delta) \geq C_{A_0}(y,\delta/2) \geq cC_{\Sigma_0}(0,\delta') = \infty.$$

This completes the theorem. □

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