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RINGS AND BILIPSCHITZ MAPS IN NORMED SPACES

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Abstract. We define the geometric modulus $GM(A)$ of a ring A in a normed space E and show that a set-bounded homeomorphism $f\colon E \to E$ is bilipschitz if and only if $|GM(A) - GM(fA)| \le c$ for all rings $A \subset E$.

1. Introduction

1.1. Background. A domain A in the eudlidean space $\mathbf{R}^n$, $n \ge 2$, is a *ring domain* or a *ring* if its complement consists of two components, a bounded C_0 and an unbounded C_1. Let Γ_A be the family of all paths joining C_0 and C_1 in A, and let $\operatorname{mod}\Gamma_A$ denote the n-modulus of Γ_A. Then

$$M(A) = \left(\frac{\omega_{n-1}}{\operatorname{mod}\Gamma_A}\right)^{1/(n-1)}$$

is the *modulus* of the ring A. Here ω_{n-1} is $(n-1)$-measure of the unit sphere $S^{n-1} \subset \mathbf{R}^n$. In particular, the modulus of the spherical ring $A = B^n(b) \setminus \bar{B}^n(a)$ is $\log b/a$.

Rings and their moduli have turned out to be a useful tool in the theory of quasiconformal maps, and S. Rohde [Ro] showed in 1997 that also bilipschitz maps of $\mathbf{R}^n$ can be characterized in terms of rings. Indeed, a homeomorphism $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is α-bilipschitz iff there is a constant $c \ge 0$ such that

$$|M(A) - M(fA)| \le c$$

for all rings $A \subset \mathbf{R}^n$. Moreover, if f fixes two points, then α and c depend only on each other and n.

The purpose of this paper is to prove a similar result for normed spaces. The modulus of a path family is now no more available and we will replace the modulus $M(A)$ by a number $GM(A)$ defined by (1.3) below.

We thank the referee for pointing out that our results hold for all normed spaces, not only for Banach spaces.

1.2. Terminology and notation. Throughout this paper E will denote a real normed space with $\dim E \ge 2$ and norm written as $|x|$. A homeomorphism $f\colon E \to E$ is called *set-bounded* if the image fA and pre-image $f^{-1}A$ of every bounded set $A \subset E$ are bounded. If $\dim E < \infty$, then every homeomorphism $f\colon E \to E$ is set-bounded. A map $f\colon E \to E$ is α-bilipschitz, $\alpha \ge 1$, if

$$|x - y|/\alpha \le |fx - fy| \le \alpha|x - y|$$

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for all $x, y \in E$. We will give in 2.12 an example of a homeomorphism which is not set-bounded.

A domain $A \subset E$ is a *ring* if its complement consists of two components, a bounded C_0 and an unbounded C_1. The *geometric modulus* of a ring A is the number

$$GM(A) = \log\left(1 + \frac{2d(C_0, C_1)}{d(C_0)}\right). \tag{1.3}$$

Here $d(C_0)$ is the diameter of C_0 and $d(C_0, C_1)$ is the distance between C_0 and C_1. Observe that

$$GM(A) = \log\frac{b}{a}$$

for the spherical ring $A = B(b) \setminus \bar{B}(a)$ where $B(r)$ is the ball $\{x \in E\colon |x| < r\}$. Rohde [Ro, Lemma 2.1] made use of almost the same number

$$GR(A) = \log\left(1 + \frac{d(C_0, C_1)}{d(C_0)}\right)$$

for a ring A of $\mathbf{R}^n$ and showed that $|M(A) - GR(A)| \le C_n$ where C_n depends only on n. The proof was rather deep. Note that $GM(A)$ is between $GR(A)$ and $GR(A) + \log 2$.

If $f\colon E \to E$ is a set-bounded homeomorphism and if $x \in E$, $r > 0$, we write

$$L(x, r) = \sup\{|fy - fx|\colon |y - x| = r\},$$
$$\ell(x, r) = \inf\{|fy - fx|\colon |y - x| = r\}.$$

2. Results

2.1. Notation. We assume in this section that $f\colon E \to E$ is a set-bounded homeomorphism. Let $P_c(E)$, $c > 0$, be the family of set-bounded homeomorphisms $f\colon E \to E$ satisfying the condition

$$|GM(A) - GM(fA)| \le c \tag{2.2}$$

for all rings $A \subset E$. If, in addition, $f(0) = 0$ and there is a point $u \in E$ such that $|u| = |fu| = 1$, we write $f \in P_c^0(E)$.

We will prove the following three theorems:

2.3. Theorem. *If $f \in P_c(E)$ and if $x \in E$, $r > 0$, then*

$$\frac{L(x, r)}{\ell(x, r)} \le 3e^c.$$

2.4. Theorem. *If $f \in P_c(E)$ and if there are points $y, z \in E$ such that $|f(y) - f(z)| = |y - z| \ne 0$, then f is α-bilipschitz with $\alpha = 27e^{5c}$.*

Conversely:

2.5. Theorem. *If $f\colon E \to E$ is an α-bilipschitz homeomorphism, then*

$$|GM(A) - GM(fA)| \le 2\log\alpha$$

for all rings $A \subset E$.

Suppose that $x \in E$, $r > 0$. If $\dim E < \infty$, then, by compactness, there are points $a, b \in S(x, r)$ such that $|fa - fx| = L(x, r)$ and $|fb - fx| = \ell(x, r)$. If $\dim E = \infty$, we need the following simple result. Here $[a, b\rangle$ is the ray from a through b.

2.6. Lemma. *Let $\emptyset \neq F \subset E$ be a closed and bounded set and let $x \in E \setminus F$ and $\varepsilon > 0$. Then there are $a, b \in F$ such that*

$$|a - x| < d(x,F) + \varepsilon \quad \text{and} \quad [x,a) \cap F = \emptyset,$$
$$|b - x| > \sup\{|y - x| \colon y \in F\} - \varepsilon \quad \text{and} \quad ([x,b\rangle \setminus [x,b]) \cap F = \emptyset.$$

Proof. There is a point $a_1 \in F$ such that $|a_1 - x| < d(x,F) + \varepsilon$. Since $[x,a_1]$ is compact, there is $a \in F \cap [x,a_1]$ with $[x,a) \cap F = \emptyset$.

The point b is found similarly. □

The following result is trivial.

2.7. Lemma. *Let A and A^* be rings of E with complementary components C_0, C_1 and C_0^*, C_1^*. If $C_0^* \subset C_0$ and $C_1^* \subset C_1$, then $GM(A) \le GM(A^*)$.*

2.8. Proof of Theorem 2.3. Let $\varepsilon > 0$ be a small number, we will later let $\varepsilon \to 0$. We apply Lemma 2.6 with $F = \partial f B(x,r)$. There are points $a_0, a_1 \in S(x.r)$ such that

$$|fa_0 - fx| < \ell(x,r) + \varepsilon, \quad [fx, fa_0) \subset fB(x,r),$$
$$|fa_1 - fx| > L(x,r) - \varepsilon, \quad ([fx, fa_1\rangle \setminus [fx, fa_1]) \cap F = \emptyset.$$

Set

$$C_0' = [fx, fa_0], \quad C_1' = [fx, fa_1\rangle \setminus [fx, fa_1),$$
$$C_0 = f^{-1}C_0', \quad C_1 = f^{-1}C_1', \quad A = E \setminus (C_0 \cup C_1).$$

Then

$$GM(A) \le \log(1 + 2r/r) = \log 3.$$

Writing $L = L(x,r)$, $\ell = \ell(x,r)$, we have $d(C_0') = |fx - fa_0| < \ell + \varepsilon$.

To get a lower bound for $d(C_0', C_1')$, we assume that $y_0 \in C_0'$, $y_1 \in C_1'$. Then

$$\begin{aligned}|y_1 - y_0| &\ge |y_1 - fx| - |y_0 - fx| \\ &\ge |fa_1 - fx| - |fa_0 - fx| \ge L - \ell - 2\varepsilon.\end{aligned}$$

Hence $d(C_0', C_1') \ge L - \ell - 2\varepsilon$, and we obtain

$$GM(fA) \ge \log\left(1 + 2\frac{L - \ell - 2\varepsilon}{\ell + \varepsilon}\right).$$

As $\varepsilon \to 0$, this gives $GM(fA) \ge \log((2L - \ell)/\ell) \ge \log L/\ell$. Hence

$$c \ge GM(fA) - GM(A) \ge \log\frac{L}{\ell} - \log 3 = \log\frac{L}{3\ell}$$

and therefore $L/\ell \le 3e^c$. □

2.9. Proof of Theorem 2.4. Performing two auxiliary similarity mappings we may assume that $f \in P_c^0(E)$ (defined as in 2.1). Since $f\bar{B}(1)$ is bounded, we can choose, using 2.3, a number $R > 1$ such that $\ell(0,R) > L(0,1)$. We first show that

$$L(0,R) \le 9e^{3c}R. \tag{2.10}$$

Let A be the spherical ring $B(R)\setminus\bar{B}(1)$ and set $Q = 3e^c$. We have by Theorem 2.3

$$L(0,1) \le Q\ell(0,1), \quad \ell(0,R) \ge L(0,R)/Q.$$

Since $\ell(0,1) \le |fu| = 1$, Lemma 2.7 gives

$$GM(fA) \ge \log\frac{\ell(0,R)}{L(0,1)} \ge \log\frac{L(0,R)}{Q^2}.$$

Since $GM(fA) \le GM(A) + c = \log R + c$, we obtain

$$\log \frac{L(0,R)}{Q^2} \le \log R + c,$$

which implies (2.10).

Since $f^{-1} \in P_E^0(c)$, it suffices to show that f is α-Lipschitz. To get this, it suffices to show that

$$\mathrm{Lip}(x,f) = \limsup_{s\to 0} \frac{L(x,s)}{s} \le \alpha$$

for each $x \in E$, see e.g. [Fe, 2.2.7].

Let $x \in E$. Choose a large number R such that $R > |x|$ and $\ell(0,R) > L(0,1)$. Next choose s with $0 < s < R - |x|$, and let A be the ring $B(R) \setminus \bar{B}(x,s)$. Then

$$GM(A) = \log\left(1 + 2\frac{R - |x| - s}{2s}\right) = \log \frac{R - |x|}{s}.$$

Let C_0', C_1' be the complementary components of fA. We have

$$d(C_0') \ge 2\ell(x,s),$$
$$d(C_0', C_1') \le L(0,R) - |fx| - \ell(x,s) \le L(0,R) - \ell(x,s).$$

Hence

$$GM(fA) \le \log \frac{L(0,R)}{\ell(x,s)},$$

and we obtain

$$\log \frac{R - |x|}{s} \le GM(A) - GM(fA) + GM(fA) \le c + \log \frac{L(0,R)}{\ell(x,s)}.$$

This yields using (2.10) and Theorem 2.3

$$\frac{R - |x|}{s} \le e^c \frac{L(0,R)}{\ell(x,s)} \le \frac{9e^{4c}R}{L(x,s)/3e^c}$$

which implies that

$$\frac{L(x,s)}{s} \le \frac{27e^{5c}}{1 - |x|/R}.$$

As $R \to \infty$ and $s \to 0$, we obtain $\mathrm{Lip}(x,f) \le 27e^{5c} = \alpha$. □

2.11. Proof of Theorem 2.5. Observe that f is set-bounded as a bilipschitz map.

Let $A \subset E$ be a ring and let C_0, C_1 and C_0', C_1' be the complementary components of A and fA, respectively. Set

$$t = \frac{2d(C_0, C_1)}{d(C_0)}.$$

Since $d(C_0') \ge d(C_0)/\alpha$ and $d(C_0', C_1') \le \alpha d(C_0, C_1)$, we have

$$GM(A) = \log(1 + t), \quad GM(fA) \le \log(1 + \alpha^2 t),$$

and hence

$$GM(fA) - GM(A) \le \log g(t)$$

where $g(t) = (1 + \alpha^2 t)/(1 + t)$. Since $g(t)$ is increasing and $g(t) \to \alpha^2$ as $t \to \infty$, we have $g(t) \le \alpha^2$ for all t. A similar argument for the inverse map f^{-1} gives $|GM(A) - GM(fA)| \le 2\log\alpha$. □

2.12. Example. We give an example of a homeomorphism $f\colon E \to E$ which is not set-bounded. Let E be the Hilbert space l_2 of sequences $a = (a_1, a_2, \dots)$ such that $\sum_{i=1}^{\infty} |a_i|^2$ is finite.

Define now a map $f\colon E \to E$ by $f(a_1, a_2, \dots) = (f_1(a_1), f_2(a_2), \dots)$ where

$$\begin{aligned} f_i(t) &= t \ \text{ if } |t| \le 1, \\ &= it \ \text{ if } |t| \ge 2, \\ &= 1 + (2i-1)(t-1) \ \text{ if } 1 \le t \le 2, \\ &= -f_i(-t) \ \text{ if } -2 \le t \le -1. \end{aligned}$$

Each f_i is a homeomorphism of the real line such that $f_i(t) = f_i^{-1}(t) = t$ if $|t| < 1$. Given $a \in E$, there is i_a such that $|a_i| < 1$ if $i \ge i_a$. Thus $f_i(a_i) = a_i$ if $i \ge i_a$. Hence $\sum_{i \ge i_a} f(a_i)^2 = \sum_{i \ge i_a} a^2 < \infty$ and so f is a well-defined map of E. In addition, we see that f is bijective and $f^{-1} = (f_1^{-1}, f_2^{-1}, \dots)$.

Furthermore, both f and f^{-1} are continuous. To see this, fix $a = (a_1, a_2, \dots) \in E$. Let F_n consist of points $x \in E$ of the form $x = (0, \dots, 0, x_{n+1}, x_{n+2}, \dots)$ and let U_n be the subset of F_n such that each $|x_i| < 1$. There is n such that U_n is a neighbourhood of $(0, \dots, 0, a_{n+1}, a_{n+2}, \dots)$ in F_n. Let E_n consist of the sequences $(x_1, \dots, x_n, 0, \dots)$. Thus E is homeomorphic to the n-dimensional Euclidean space. Clearly, $f|E_n$ is continuous and so is $f|U_n$ as the the identity map of U_n. It follows that $f|E_n \times U_n$ is continuous and especially f is continuous at a. Similarly, f^{-1} is continuous at $f(a)$.

Set $e_i = (0, \dots, 0, 1, 0, \dots)$ (1 in the i-th place). Then f sends the bounded set $B = \{2e_i\}_{i>0}$ to the unbounded set $\{2ie_i\}_{i>0}$ and hence f is not set-bounded.

Note that if $g = (f_1, f_2^{-1}, f_3, f_4^{-1}, \dots)$, then both g and g^{-1} are homeomorphisms of E sending the bounded set B to an unbounded set.

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