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# ON A BERNOULLI-TYPE OVERDETERMINED FREE BOUNDARY PROBLEM

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**Abstract.** In this article we study a Bernoulli-type free boundary problem and generalize a work of Henrot and Shahgholian in [25] to $\mathcal{A}$-harmonic PDEs. These are quasi-linear elliptic PDEs whose structure is modelled on the $p$-Laplace equation for a fixed $1 < p < \infty$. In particular, we show that if $K$ is a bounded convex set satisfying the interior ball condition and $c > 0$ is a given constant, then there exists a unique convex domain $\Omega$ with $K \subset \Omega$ and a function $u$ which is $\mathcal{A}$-harmonic in $\Omega \setminus K$, has continuous boundary values 1 on $\partial K$ and 0 on $\partial\Omega$, such that $|\nabla u| = c$ on $\partial\Omega$. Moreover, $\partial\Omega$ is $C^{1,\gamma}$ for some $\gamma > 0$, and it is smooth provided $\mathcal{A}$ is smooth in $\mathbf{R}^n \setminus \{0\}$. We also show that the super level sets $\{u > t\}$ are convex for $t \in (0, 1)$.

## 1. Introduction and statement of main results

Classical Bernoulli free-boundary problems arise in electrostatics, fluid dynamics, optimal insulation, and electro chemistry. In the case of electrostatics, the task is to design an annular condenser consisting of a prescribed boundary component $\partial E$, and an unknown boundary component $\partial\Omega$ (where $\Omega \subset E$), such that the electric field $\nabla u$ is constant in magnitude on the surface $\partial\Omega$ of the second conductor (see [18, 19] for treatment of this problem and applications). This leads to the existence of the following *interior Bernoulli free-boundary* (will be denoted by **(IBFB)**) problem:

$$(1.1)\qquad (\mathbf{IBFB}) \begin{cases} -\Delta u = 0 & \text{in } E \setminus \overline{\Omega}, \\ u = 1 & \text{on } \partial E, \\ u = 0 & \text{on } \partial\Omega, \\ |\nabla u| = a & \text{on } \partial\Omega \end{cases}$$

for some given constant $a > 0$. The constraint $|\nabla u| = a$ is called *Bernoulli's law.*

Here Bernoulli's law $|\nabla u| = a$ should be understood in the following sense:

$$\liminf_{y \to x,\, y \in E \setminus \overline{\Omega}} |\nabla u(y)| = \limsup_{y \to x,\, y \in E \setminus \overline{\Omega}} |\nabla u(y)| = a \quad \text{for every} \quad x \in \partial\Omega.$$

The existence and uniqueness of this problem can be stated in the following manner: *is there a domain $\Omega$ with $\Omega \subset E$ and a potential $u\colon E \setminus \Omega \to \mathbf{R}$ satisfying* (1.2)*? If so, is the couple* $(\Omega, u)$ *unique?*

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The *exterior Bernoulli free-boundary problem* (**(EBFB)** problem for short) is defined in a similar fashion: *is there a couple* $(\Omega, u)$ *such that* $E \subset \Omega$ *and* (1.2) *below holds?*

$$\text{(EBFB)} \begin{cases} -\Delta u = 0 & \text{in } \Omega \setminus \overline{E}, \\ u = 1 & \text{on } \partial E, \\ u = 0 & \text{on } \partial\Omega, \\ |\nabla u| = a & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

In this paper our main goal is to generalize the work of [25] on the **(EBFB)** for $\mathcal{A}$-harmonic PDEs (see the definition below in (1.6)), proving existence and uniqueness of $(\Omega, u)$ and showing that $\partial\Omega$ is smooth. (see Theorem 1.4).

Regarding the existing literature, the existence of solutions to the **(EBFB)** problem was obtained by Alt and Caffarelli in [9] by variational methods, and by Beurling [12] using sub-super solution methods in the plane. The reader is also referred to results of Acker in [2, 1] concerning the uniqueness and monotonicity of this problem.

If we assume $E$ to be convex and require $\Omega$ to be convex as well, the question of existence and uniqueness of a pair satisfying (1.2) in the plane was answered affirmatively by Tepper in [35], by Hamilton in [22] both by using conformal mappings, and by Kawohl in [27], using different methods. In higher dimensions, convexity and uniqueness of $\Omega$ were shown by Henrot and Shahgholian in [24]. Under a convexity assumption on $E$, the **(EBFB)** problem was also studied by Henrot and Shahgholian in [25], where they proved existence of a pair $(\Omega, u)$ satisfying (1.2) without assuming $E$ to be bounded or regular. When $E$ is bounded, it was shown in [25] that the **(EBFB)** problem has a unique solution, and the same result was obtained independently by Acker and Meyer in [3].

Neither existence nor uniqueness is always true in the **(IBFB)** case. In the plane, existence of a pair $(\Omega, u)$ satisfying (1.1) was obtained by Lavrentèv in [30], Beurling in [12], and Daniljuk in [14]. A higher dimensional result was proved by Alt and Caffarelli in [9], and under certain assumptions Henrot and Shahgholian proved in [24] that the mean curvature of $\partial\Omega$ is positive for any connected component. In [24], it was shown that if the **(IBFB)** problem admits a solution and $E$ is convex, then $\Omega$ is also convex.

For further discussion of the problems we consider, we shall first introduce the $p$-Laplace equation:

$$\Delta_p u = \nabla \cdot \left(|\nabla u|^{p-2} \nabla u\right) = |\nabla u|^{p-4}[|\nabla u|^2 \Delta u + (p-2) \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j}].$$

Here $p$ is fixed with $1 < p < \infty$, $|\nabla u| = (u_{x_1}^2 + \ldots + u_{x_n}^2)^{1/2}$, and $\nabla \cdot$ is the divergence operator.

It is well-known that, in general, solutions of the p-Laplace equation do not enjoy second order derivatives in the classical sense, therefore solutions to these equations have to be understood as *weak solutions*. That is, given a bounded, connected open set $\Omega \subset \mathbf{R}^n$, $u$ is a *p-harmonic function* in $\Omega$ provided $u > 0$ in $\Omega$ and $u$ is in the *Sobolev space* $W^{1,p}(U)$ for each open set $U$ with $\bar{U} \subset \Omega$ and

$$\int_U |\nabla u|^{p-2} \langle \nabla u, \nabla \eta \rangle \, \mathrm{d}x = 0 \quad \text{whenever} \quad \eta \in W_0^{1,p}(U). \tag{1.3}$$

In the above paragraph $W^{1,p}(U)$ denotes the space of equivalence classes of functions $h$ with *distributional gradient* $\nabla h$ both of which are $p$ integrable in $U$, and $W_0^{1,p}$ denotes the closure of $C_0^\infty$ in the $W^{1,p}$ norm.

In [26], the $p$-Laplace operator was treated in the **(IBFB)** case:

$$\begin{cases} \Delta_p u = 0 & \text{in } E \setminus \Omega, \\ u = 1 & \text{on } \partial E, \\ u = 0 & \text{on } \partial\Omega, \\ |\nabla u| = a & \text{on } \partial\Omega. \end{cases}$$

Existence of a solution, and regularity (that is, $\partial\Omega$ is $C^{2,\alpha}$) were shown in that article.

The authors of [24] proved uniqueness and convexity of the **(EBFB)** problem if the Laplace operator is replaced by a general nonlinear operator $\mathcal{L}$ of the form $\mathcal{L} = \mathcal{L}u = \mathcal{F}(x, u, \nabla u, \nabla^2 u)$, if the operator $\mathcal{L}$ satisfies certain properties (see section 4 in that article).

In this article we consider the **(EBFB)** problem when the underlying PDE is the so called $\mathcal{A}$-*harmonic* PDE. We introduce this nonlinear elliptic equation in what follows.

**Definition 1.1.** Let $p, \alpha \in (1, \infty)$ be fixed and

$$\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_n) \colon \mathbf{R}^n \setminus \{0\} \to \mathbf{R}^n,$$

be such that $\mathcal{A} = \mathcal{A}(\eta)$ has continuous partial derivatives in $\eta_k$ for $k = 1, 2, \dots, n$ on $\mathbf{R}^n \setminus \{0\}$. We say that the function $\mathcal{A}$ belongs to the class $M_p(\alpha)$ if the following two conditions are satisfied whenever $\xi \in \mathbf{R}^n$ and $\eta \in \mathbf{R}^n \setminus \{0\}$:

(i) $\alpha^{-1}|\eta|^{p-2}|\xi|^2 \le \sum_{i,j=1}^n \frac{\partial \mathcal{A}_i}{\partial \eta_j}(\eta)\xi_i\xi_j \le \alpha|\eta|^{p-2}|\xi|^2,$

(ii) $\mathcal{A}$ is $p-1$ homogeneous, i.e., $\mathcal{A}(\eta) = |\eta|^{p-1}\mathcal{A}(\eta/|\eta|)$.

We set $\mathcal{A}(0) = 0$ and note that Definition 1.1 (i) and (ii) imply

$$\begin{aligned} c^{-1}(|\eta| + |\eta'|)^{p-2}\,|\eta - \eta'|^2 &\le \langle \mathcal{A}(\eta) - \mathcal{A}(\eta'), \eta - \eta' \rangle \\ &\le c|\eta - \eta'|^2(|\eta| + |\eta'|)^{p-2} \end{aligned} \tag{1.4}$$

whenever $\eta, \eta' \in \mathbf{R}^n \setminus \{0\}$. We will additionally assume that there exists $1 \le \Lambda < \infty$ such that

$$\left| \frac{\partial \mathcal{A}_i}{\partial \eta_j}(\eta) - \frac{\partial \mathcal{A}_i}{\partial \eta_j}(\eta') \right| \le \Lambda\,|\eta - \eta'||\eta|^{p-3} \tag{1.5}$$

whenever $0 < |\eta| \le 2|\eta'|$ and $1 \le i, j \le n$.

**Definition 1.2.** Given an open set $\Omega \subset \mathbf{R}^n$ and $\mathcal{A} \in M_p(\alpha)$, one says that $u$ is $\mathcal{A}$-harmonic in $\Omega$, and we write $\nabla \cdot \mathcal{A}(\nabla u) = 0$, provided $u > 0$ in $\Omega$, $u \in W^{1,p}(U)$ for each open set $U$ with $\bar{U} \subset \Omega$ and

$$\int \langle \mathcal{A}(\nabla u(x)), \nabla \eta(x) \rangle \, \mathrm{d}x = 0 \quad \text{whenever} \quad \eta \in W_0^{1,p}(U). \tag{1.6}$$

For more about PDEs of this type the reader is referred to [23]. Notice that when $\mathcal{A}(\eta) = \eta$ then (1.6) is the usual Laplace's equation, and when $\mathcal{A}(\eta) = |\eta|^{p-2}\eta$ then (1.6) becomes the $p$-Laplace equation.

**Definition 1.3.** We say that a set $K$ satisfies the *interior ball condition* if

$$\text{for each } x_0 \in \partial D, \text{ there is a ball } B(z,\delta) \subset D \text{ with } x_0 \in \overline{B(z,\delta)} \tag{1.7}$$

for some $\delta > 0$.

Our main goal is to generalize the work of Henrot and Shahgholian in [25] on the **(EBFB)** problem to $\mathcal{A}$-harmonic PDEs. In particular, we show that

**Theorem 1.4.** *Let $c > 0$ be a given constant and $K$ be a bounded convex domain satisfying the interior ball condition. Then there exist a unique convex domain $\Omega$ with $K \subset \Omega$ and function $u$ satisfying*

$$\begin{cases} \nabla \cdot \mathcal{A}(\nabla u) = 0 \text{ in } \Omega \setminus \bar{K}, \\ u \text{ has continuous boundary values } 1 \text{ on } \partial K \text{ and } 0 \text{ on } \partial\Omega, \\ \text{the superlevel sets } \{u > t\} \text{ are convex for every } t \in (0,1), \\ |\nabla u| = c \text{ on } \partial\Omega. \end{cases}$$

*Moreover, $\partial\Omega \in C^{1,\gamma}$ for some $\gamma > 0$. Furthermore we have that $\partial\Omega$ is smooth provided $\mathcal{A}$ is smooth.*

The plan of the paper is as follows. In section 2, we gather some known results concerning the regularity of $\mathcal{A}$-harmonic functions that are relevant for our work. We also show that the levels of $u$ are convex if $K$ is convex by adapting an idea of Lewis [31]. In section 3, using a method of Beurling (see also [25]), we prove the existence result in Theorem 1.4. Uniqueness in Theorem 1.4 will essentially follow from [24]. Finally the regularity result in Theorem 1.4 is obtained using ideas inspired by the work of Vogel in [37].

## 2. Notation and preparatory lemmas

Let $x = (x_1, \ldots, x_n)$ denote points in $\mathbf{R}^n$ and let $\overline{E}$, $\partial E$, be the closure and boundary of the set $E \subset \mathbf{R}^n$. Let $\langle \cdot, \cdot \rangle$ be the usual inner product in $\mathbf{R}^n$ and $|x|^2 = \langle x, x \rangle$. Let $d(E,F)$ denote the distance between the sets $E$ and $F$. Let $B(x,r)$ be the open ball centered at $x$ with radius $r > 0$ in $\mathbf{R}^n$ and $dx$ denote the Lebesgue $n$-measure in $\mathbf{R}^n$. Given $U \subset \mathbf{R}^n$ an open set and $q$ with $1 \leq q \leq \infty$, let $W^{1,q}(U)$ denote equivalence classes of functions $h\colon \mathbf{R}^n \to \mathbf{R}$ with distributional gradient $\nabla h = \langle h_{x_1}, \ldots, h_{x_n} \rangle$, both of which are $q$-integrable in $U$ with Sobolev norm

$$\|h\|^q_{W^{1,q}(U)} = \int_U (|h|^q + |\nabla h|^q)\, \mathrm{d}x.$$

Let $C_0^\infty(U)$ be the set of infinitely differentiable functions with compact support in $U$ and let $W_0^{1,q}(U)$ be the closure of $C_0^\infty(U)$ in the norm of $W^{1,q}(U)$.

In the sequel, $c$ will denote a positive constant $\geq 1$ (not necessarily the same at each occurrence), which may depend only on $p, n, \alpha, \Lambda$ unless otherwise stated. In general, $c(a_1, \ldots, a_n)$ denotes a positive constant $\geq 1$ which may depend only on $p, n, \alpha, \Lambda, a_1, \ldots, a_n$, which is not necessarily the same at each occurrence. By $A \approx B$ we mean that $A/B$ is bounded above and below by positive constants depending only on $p, n, \alpha, \Lambda$. Finally, in this section we will always assume that $1 < p < \infty$, and $r > 0$.

We next introduce the notion of the *Hausdorff measure*. To this end, let $\hat{r}_0 > 0$ be given, and let $0 < \delta < \hat{r}_0$ be fixed. Let $\operatorname{diam}(\cdot)$ denote the diameter of a set and let $E \subseteq \mathbf{R}^n$ be a given Borel set. For an arbitrary integer $k > 0$, we define the $(\delta, k)$-*Hausdorff content* of $E$ in the usual way:

$$\mathcal{H}^k_\delta(E) := \inf \left\{ \sum_i r_i^k \colon E \subset \bigcup_i B(x_i, r_i) \text{ with } r_i < \delta \right\}.$$

Here the infimum is taken over all possible covers $\{B(x_i, r_i)\}$ of $E$. Then the *Hausdorff $k$-measure* of $E$ is defined by

$$\mathcal{H}^k(E) := \lim_{\delta\to 0} \ \mathcal{H}^k_\delta(E). \tag{2.1}$$

**Lemma 2.1.** *Given $p$ with $1 < p < \infty$, assume that $\mathcal{A} \in M_p(\alpha)$ for some $\alpha > 1$. Let $u$ be an $\mathcal{A}$-harmonic function in $B(w, 4r)$. Then*

(a) $r^{p-n} \displaystyle\int_{B(w,r/2)} |\nabla u|^p \mathrm{d}x \leq c \max_{B(w,r)} u^p,$

(b) $\displaystyle\max_{B(w,r)} u \leq c \min_{B(w,r)} u.$

*Moreover, there exists $\gamma \in (0,1)$ depending on $p, n, \alpha$ such that if $x, y \in B(w, r)$, then*

(c) $|u(x) - u(y)| \leq c\,(|x-y|/r)^\gamma \displaystyle\max_{B(w,2r)} u.$

For a proof of Lemma 2.1 see [34].

**Lemma 2.2.** *Given $p$ with $1 < p < \infty$, assume that $\mathcal{A} \in M_p(\alpha)$ for some $\alpha > 1$. Let $u$ be an $\mathcal{A}$-harmonic function in $B(w, 4r)$. Then $u$ has a representative locally in $W^{1,p}(B(w,4r))$ with Hölder continuous partial derivatives in $B(w,4r)$ (also denoted $u$), and there exist $\beta \in (0,1]$ and $c \geq 1$ depending only on $p, n, \alpha$ such that if $x, y \in B(w, r)$, then*

$$\text{(i) } |\nabla u(x) - \nabla u(y)| \leq c(|x-y|/r)^\beta \max_{B(w,r)} |\nabla u| \leq c r^{-1} (|x-y|/r)^\beta u(w).$$

$$\text{(ii) } \int_{B(w,r)} \sum_{i,j=1}^n |\nabla u|^{p-2} |u_{x_i x_j}|^2 \,\mathrm{d}x \leq c r^{n-p-2} u(w). \tag{2.2}$$

*Moreover, if*

$$\gamma\, r^{-1} u \leq |\nabla u| \leq \gamma^{-1} r^{-1} u \quad \text{on} \quad B(w, 2r)$$

*for some $\gamma \in (0,1)$ and* (1.5) *holds then $u$ has Hölder continuous second partial derivatives in $B(w,r)$ and there exists $\theta \in (0,1), \bar{c} \geq 1$, depending only on the data and $\gamma$ such that if $x, y \in B(w, r/2)$, then*

$$\begin{aligned}
\Big[ \sum_{i,j=1}^n (u_{x_i x_j}(x) - u_{y_i y_j}(y))^2 \Big]^{1/2} &\leq \bar{c}(|x-y|/r)^\theta \max_{B(w,r)} \left( \sum_{i,j=1}^n |u_{x_i x_j}| \right) \\
&\leq \bar{c}^2 r^n (|x-y|/r)^\theta \left( \sum_{i,j=1}^n \int_{B(w,2r)} u^2_{x_i x_j} r dx \right)^{1/2} \\
&\leq \bar{c}^3 r^{-2} (|x-y|/r)^\theta u(w).
\end{aligned} \tag{2.3}$$

A proof of (2.2) can be found in [36]. Estimate (2.3) follows from (2.2), the added assumptions and Schauder type estimates (see [21]).

We will make use of following lemma when we rotate our coordinate system. A proof of it can be found in [32, Lemma 2.15].

**Lemma 2.3.** *Let $\Omega \subset \mathbf{R}^n$ be a domain and let $p$ with $1 < p < \infty$ be given. Let $\mathcal{A} \in M_p(\alpha)$ for some $\alpha > 1$ and $u$ be $\mathcal{A}$-harmonic in $\Omega$. If $F\colon \mathbf{R}^n \to \mathbf{R}^n$ is the composition of a translation and a dilation, then*

$$\hat{u}(z) = u(F(z)) \text{ is } \mathcal{A}\text{-harmonic in } F^{-1}(\Omega).$$

*Moreover, if $\tilde{F}\colon \mathbf{R}^n \to \mathbf{R}^n$ is the composition of a translation, a dilation, and a rotation then*

$$\tilde{u}(z) = u(F(z)) \text{ is } \hat{\mathcal{A}}\text{-harmonic in } \tilde{F}^{-1}(\Omega) \text{ for some } \hat{\mathcal{A}} \in M_p(\alpha).$$

In what follows we make observations that will be useful throughout the paper (see also [7, 8] for a similar computation). Define

$$\mathcal{L}(\eta,\xi) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left[b_{ij}(\eta)\xi_{x_j}\right], \quad \text{where } b_{ij}(\eta) = \frac{\partial \mathcal{A}_i}{\partial \eta_j}(\eta). \tag{2.4}$$

When $\eta = \nabla u$ we will write $\mathcal{L}(\eta,\xi) = \mathcal{L}_u\xi$, and when $\xi = \nabla w$ we will write $\mathcal{L}_u\xi = \mathcal{L}_u w$. We next show that if $u$ is $\mathcal{A}$-harmonic in $\Omega$, then $\xi = u$ and $\xi = u_{x_k}$ (for $k = 1,\dots,n$) are both weak solutions to $\mathcal{L}_u\xi = 0$.

We first see that if $u$ is $\mathcal{A}$-harmonic then

$$\mathcal{L}_u u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left[\frac{\partial \mathcal{A}_i}{\partial \eta_j}(\nabla u) u_{x_j}\right] = 0.$$

Indeed, using the $(p-1)$-homogeneity of $\mathcal{A}$ in Definition 1.1 we obtain

$$\mathcal{L}_u u = (p-1)\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\mathcal{A}_i(\nabla u) = (p-1)\nabla\cdot\mathcal{A}(\nabla u) = 0.$$

To show that $\mathcal{L}_u u_{x_k} = 0$ for $k = 1,\dots,n$, using Lemma 2.2 we first get $u \in W^{2,2}(\Omega)$. Then it follows that

$$\mathcal{L}_u u_{x_k} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left[\frac{\partial \mathcal{A}_i}{\partial \eta_j}(\nabla u) u_{x_k x_j}\right] = \frac{\partial}{\partial x_k}\nabla\cdot\mathcal{A}(\nabla u) = 0. \tag{2.5}$$

Note that above argument should be understood in the weak sense. Using these two observations and the structural assumptions on $\mathcal{A}$ from Definition 1.1 we also conclude that

$$\begin{aligned}\mathcal{L}_u(|\nabla u|^2) &= \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left[\frac{\partial \mathcal{A}_i}{\partial \eta_j}(\nabla u)(u_{x_1}^2 + \dots + u_{x_n}^2)_{x_j}\right] = 2\sum_{i,j,k=1}^{n} \frac{\partial}{\partial x_i}\left[\frac{\partial \mathcal{A}_i}{\partial \eta_j}(\nabla u) u_{x_k} u_{x_k x_j}\right] \\ &\overset{(2.5)}{=} 2\sum_{i,j,k=1}^{n} \frac{\partial \mathcal{A}_i}{\partial \eta_j}(\nabla u) u_{x_k x_i} u_{x_k x_j} \geq 2\alpha^{-1}|\nabla u|^{p-2}\sum_{i,j=1}^{n}(u_{x_i x_j})^2.\end{aligned}$$

Using this observation we conclude

$$\mathcal{L}_u(|\nabla u|^2) \geq c^{-1}|\nabla u|^{p-2}\sum_{i,j=1}^{n}(u_{x_i x_j})^2. \tag{2.6}$$

**Lemma 2.4.** *Let $\Omega$ be a domain, $K$ be a bounded, closed, convex set with $K \subset \Omega$, $1 < p < \infty$ and $\alpha > 1$ be given. Let $\mathcal{A} \in M_p(\alpha)$ and $u$ be $\mathcal{A}$-harmonic in*

*$\Omega \setminus K$ with $u = 1$ on $\partial K$. If $K$ satisfies the interior ball property then then there exists $c_* \geq 1$, depending only on $p, n, \alpha, r_0$ such that if $x \in \Omega \setminus K$*

$$\text{(2.7)} \qquad \begin{aligned} &\text{(a)} \quad c_* \langle \nabla u(x), z - x \rangle \geq u(x), \\ &\text{(b)} \quad c_*^{-1} \, |x|^{\frac{1-n}{p-1}} \leq |\nabla u(x)| \leq c_* \, |x|^{\frac{1-n}{p-1}} . \end{aligned}$$

*Proof.* A proof of this lemma can be found in [5] when $1 < p < n$ and in [6] when $n \leq p < \infty$. The proof uses Lemmas 2.2, 2.1, and 2.3. A barrier type argument is also used, as in [32, Section 2] and [8, Section 4]. We skip the details. □

Now let $u$ be a $\mathcal{A}$-harmonic function in $\Omega \setminus K$ where $K \subset \Omega$, it is continuous on $\mathbf{R}^n$ with $u \equiv 1$ on $K$ and $u \equiv 0$ on $\mathbf{R}^n \setminus \Omega$. The following lemma establishes the convexity of the superlevel sets $\{u > t\}$ when both $K$ and $\Omega$ are convex. We note that such a result plays a crucial role in the uniqueness assertion in Theorem 1.4. In the case of $p$-Laplacian, such a result was first established by Lewis in [31] following Gabriel's ideas as in [20]. Here we adapt the techniques in [31] and [5] to our situation. We also refer to the interesting paper [13] for a different proof in the case of the Laplacian.

**Lemma 2.5.** *Let $K \subset \Omega$ be such that $K, \Omega$ are convex and let $u$ be $\mathcal{A}$-harmonic in $\Omega \setminus K$, continuous on $\mathbf{R}^n$ with $u \equiv 1$ on $K$ and $u \equiv 0$ on $\mathbf{R}^n \setminus \Omega$. If $K$ satisfies interior ball condition then for each $t \in (0,1)$, the set $\{x \in \Omega \colon u(x) > t\}$ is convex.*

*Proof.* We note from (2.7) and Lemma 2.2 that

$$\text{(2.8)} \qquad \begin{cases} |\nabla u| \neq 0, \\ u \text{ has Hölder continuous second partial derivatives on compact subsets} \\ \text{of } \Omega. \end{cases}$$

Our proof of Lemma 2.5 is by contradiction, following the proof in [31, section 4]. We define for $\hat{x} \in \mathbf{R}^n$,

$$\mathbf{u}(\hat{x}) = \sup_{\substack{\hat{y}, \hat{z} \in \mathbf{R}^n \\ \hat{x} = \lambda \hat{y} + (1-\lambda)\hat{z}, \lambda \in [0,1]}} \min\{u(\hat{y}), u(\hat{z})\}.$$

Notice that $u \leq \mathbf{u}$, $\mathbf{u} \equiv 1$ in $K$ and $\mathbf{u} \equiv 0$ in $\mathbf{R}^n \setminus \Omega$. It suffices to show that $\mathbf{u} = u$. If that were not true, then from the convexity of $K$, continuity of $u$, and the fact that as $w \to w_0 \in \partial\Omega$, $u(w) \to 0$, we would conclude that there must exist $\epsilon > 0$, and $x_0 \in \Omega$ such that

$$\text{(2.9)} \qquad 0 < \mathbf{u}^{1+\epsilon}(x_0) - u(x_0) = \max_{\mathbf{R}^n} (\mathbf{u}^{1+\epsilon} - u).$$

For ease of writing we write $\mathbf{v} = \mathbf{u}^{1+\epsilon}$ and $v = u^{1+\epsilon}$. There exist $\lambda \in (0,1)$ and $y_0, z_0 \in \Omega \setminus \{x_0\}$ with

$$\text{(2.10)} \qquad x_0 = \lambda y_0 + (1-\lambda) z_0 \quad \text{and} \quad \mathbf{v}(x_0) = \min\{v(y_0), v(z_0)\}.$$

We first show that

$$\text{(2.11)} \qquad v(y_0) = v(z_0).$$

Assume for contradiction, for instance, that $v(y_0) < v(z_0)$. Since $u \leq \mathbf{u}$, $u(x_0) \leq u(y_0) = \mathbf{u}(x_0) < u(z_0)$. By continuity, if $z$ is in a small enough neighborhood of $z_0$, then $u(z) > u(y_0) + (u(z_0) - u(y_0))/2$. Since $|\nabla u| \neq 0$ in $\Omega$, we can choose $y'$ close enough to $y_0$ so that $u(y') > u(y_0)$ and also such that after connecting $y'$ and $x_0$ by a line, we can pick a corresponding $z'$ in the previous neighborhood of $z_0$. In this manner $\mathbf{u}(x_0) \geq \min\{u(y'), u(z')\} > u(y_0) = \mathbf{u}(x_0)$, a contradiction. Thus (2.11) is true.

Next we prove that

$$\xi = \frac{\nabla v(y_0)}{|\nabla v(y_0)|} = \frac{\nabla v(z_0)}{|\nabla v(z_0)|} = \frac{\nabla u(x_0)}{|\nabla u(x_0)|} \tag{2.12}$$

Indeed, let us show that

$$\frac{\nabla v(y_0)}{|\nabla v(y_0)|} = \frac{\nabla v(x_0)}{|\nabla v(x_0)|}.$$

Let $y$ not be on the line through $y_0$ and $z_0$ and be such that $\nabla v(y_0)\cdot(y-y_0) > 0$. Draw the line through $y$ and $z_0$ and denote by $x$ its intersection with the line originating at $x_0$ with direction $y - y_0$. One has that $v(y) > v(y_0)$ for $y$ close to $y_0$. Therefore $\tilde{v}(x_0) \leq \tilde{v}(x)$, for $y$ near $y_0$. From (2.9) we conclude $u(x) \geq u(x_0)$, for $y$ close to $y_0$, hence $\nabla u(x_0)\cdot(y-y_0) \geq 0$ whenever $\nabla v(y_0)\cdot(y-y_0) > 0$, showing that $\nabla u(x_0)$ and $\nabla v(y_0)$ point in the same direction.

To simplify our notation, let

$$A = |\nabla v(y_0)|, \quad B = |\nabla v(z_0)|, \quad C = |\nabla u(x_0)|, \quad a = |x_0 - y_0|, \quad b = |x_0 - z_0|.$$

Let $\eta \in \mathbf{S}^{n-1}$ be such that $\xi\cdot\eta > 0$. From (2.8) we can write

$$\begin{aligned} v(y_0+\rho\eta) &= v(y_0) + A_1\rho + A_2\rho^2 + o(\rho^2),\\ v(z_0+\rho\eta) &= v(z_0) + B_1\rho + B_2\rho^2 + o(\rho^2),\\ u(x_0+\rho\eta) &= u(x_0) + C_1\rho + C_2\rho^2 + o(\rho^2) \end{aligned} \tag{2.13}$$

as $\rho \to 0$. Also

$$A_1/A = B_1/B = C_1/C = \xi\cdot\eta,$$

where the coefficients and $o(\rho^2)$ depend on $\eta$. Given $\eta$ with $\xi\cdot\eta > 0$ and $\rho_1$ sufficiently small we see from (2.8) that the inverse function theorem can be used to obtain $\rho_2$ with

$$v\left(y_0 + \frac{\rho_1}{A}\eta\right) = v\left(z_0 + \frac{\rho_2}{B}\eta\right).$$

We conclude as $\rho_1 \to 0$ that

$$\rho_2 = \rho_1 + \frac{B}{B_1}\left(\frac{A_2}{A^2} - \frac{B_2}{B^2}\right)\rho_1^2 + o(\rho_1^2). \tag{2.14}$$

Now from geometry we see that $\lambda = \frac{b}{a+b}$ so

$$x = x_0 + \eta\frac{[\rho_1\frac{b}{A} + \rho_2\frac{a}{B}]}{a+b} = \lambda(y_0 + \frac{\rho_1}{A}\eta) + (1-\lambda)(z_0 + \frac{\rho_2}{B}\eta).$$

From this equality and Taylor's theorem for second derivatives we have

$$\begin{aligned} u(x) - u(x_0) &= C_1\left[\rho_1\frac{\lambda}{A} + \rho_2\frac{(1-\lambda)}{B}\right] + C_2\left[\rho_1\frac{\lambda}{A} + \rho_2\frac{(1-\lambda)}{B}\right]^2\\ &= C_1\rho_1\frac{(1-\lambda)A + \lambda B}{AB} + C_1\frac{(1-\lambda)}{B_1}\left(\frac{A_2}{A^2} - \frac{B_2}{B^2}\right)\rho_1^2\\ &\quad + C_2\rho_1^2\left(\frac{(1-\lambda)A + \lambda B)}{AB}\right)^2 + o(\rho_1^2). \end{aligned} \tag{2.15}$$

Now

$$v\left(y_0 + \frac{\rho_1}{A}\eta\right) - u(x) \leq \mathbf{v}(x) - u(x) \leq \mathbf{v}(x_0) - u(x_0) = v(y_0) - u(x_0).$$

Hence the mapping

$$\rho_1 \to v(y_0 + \frac{\rho_1}{A}\eta) - u(x)$$

has a maximum at $\rho_1 = 0$. Using the Taylor expansion for $v(y_0 + \frac{\rho_1}{A}\eta)$ in (2.13) and $u(x)$ in (2.15) we have

$$\begin{aligned} v\left(y_0 + \frac{\rho_1}{A}\eta\right) - u(x) = v(y_0) &+ \frac{A_1}{A}\rho_1 + \frac{A_2}{A^2}\rho_1^2 - u(x_0) \\ &- C_1\rho_1\frac{(1-\lambda)A + \lambda B}{AB} - \frac{C_1}{a+b}\frac{a}{B_1}\left(\frac{A_2}{A^2} - \frac{B_2}{B^2}\right)\rho_1^2 \\ &- C_2\rho_1^2\left(\frac{(1-\lambda)A+\lambda B}{AB}\right)^2 + o(\rho_1^2). \end{aligned}$$

Now from the calculus second derivative test, the coefficient of $\rho_1$ should be zero and the coefficient of $\rho_1^2$ should be non-positive. Hence combining terms we get

$$\frac{A_1}{A} = C_1\,\frac{(1-\lambda)A + \lambda B}{AB}$$

so taking $\eta = \xi$ we arrive first at

$$\frac{1}{C} = \frac{(1-\lambda)A+\lambda B}{AB} = \frac{(1-\lambda)}{B} + \frac{\lambda}{A}. \tag{2.16}$$

Second using (2.16) in the $\rho_1^2$ term we find that

$$0 \ge \frac{A_2}{A^2} - C_1\frac{(1-\lambda)}{B_1}\left(\frac{A_2}{A^2} - \frac{B_2}{B^2}\right) - \frac{C_2}{C^2}. \tag{2.17}$$

Using $C_1/B_1 = C/B$ and doing some arithmetic in (2.17) we obtain

$$0 \ge (1-K)\frac{A_2}{A^2} + K\frac{B_2}{B^2} - \frac{C_2}{C^2} \tag{2.18}$$

where

$$K = \frac{(1-\lambda)A}{(1-\lambda)A+\lambda B} < 1.$$

We now focus on (2.18) by writing $A_1, B_1, C_1$ in terms of derivatives of $u$ and $v$;

$$0 \ge \sum_{i,j=1}^{n}\left[\frac{(1-K)}{A^2}v_{x_ix_j}(y_0) + \frac{K}{B^2}v_{x_ix_j}(z_0) - \frac{u_{x_ix_j}(x_0)}{C^2}\right]\eta_i\eta_j. \tag{2.19}$$

From symmetry and continuity considerations we observe that (2.19) holds whenever $\eta \in \mathbf{S}^{n-1}$ Thus if

$$w(x) = -\frac{(1-K)}{A^2}v(y_0+x) - \frac{K}{B^2}v(z_0+x) + \frac{u_{x_ix_j}(x_0+x)}{C^2},$$

then the Hessian matrix of $w$ at $x = 0$ is positive semi-definite, i.e, $(w_{x_ix_j}(0))$ has non-negative eigenvalues. Also from (i) of Definition 1.1 we see that if

$$a_{ij} = \frac{1}{2}\left[\frac{\partial \mathcal{A}_i}{\partial \eta_j}(\xi)) + \frac{\partial \mathcal{A}_j}{\partial \eta_i}(\xi))\right], \quad 1 \le i,j \le n,$$

then $(a_{ij})$ is positive definite. From these two observations we conclude that

$$\operatorname{trace}\big(((a_{ij})\cdot(w_{x_ix_j}(0))\big) \ge 0. \tag{2.20}$$

To obtain a contradiction we observe from (1.6), the divergence theorem, (2.18), and $p-2$ homogeneity of partial derivatives of $\mathcal{A}_i$, that

$$\sum_{i,j=1}^{n} a_{ij}\, u_{x_j x_i} = |\nabla u|^{2-p} \sum_{i,j=1}^{n} \frac{\partial \mathcal{A}_i}{\partial \eta_j}(\nabla u) u_{x_j x_i} = 0 \quad \text{at } x_0, y_0, z_0. \tag{2.21}$$

Moreover, from the definition of $v$ we have

$$\begin{aligned} v_{x_i} &= (u^{1+\epsilon})_{x_i} = (1+\epsilon) u^{\epsilon} u_{x_i}, \\ v_{x_i x_j} &= (1+\epsilon)\epsilon u^{\epsilon-1} u_{x_i} u_{x_j} + (1+\epsilon) u^{\epsilon} u_{x_i x_j}. \end{aligned} \tag{2.22}$$

Using (2.21), (2.22), we find that

$$\begin{aligned} |\nabla u|^{p-2} \sum_{i,j=1}^{n} a_{ij}\, v_{x_j x_i} &= \sum_{i,j=1}^{n} \frac{\partial \mathcal{A}_i}{\partial \eta_j}(\nabla u)[(1+\epsilon)u^{\epsilon-1} u_{x_j} u_{x_i} + (1+\epsilon) u^{\epsilon} u_{x_j x_i}] \\ &= (1+\epsilon)\epsilon u^{\epsilon-1} \sum_{i,j=1}^{n} \frac{\partial \mathcal{A}_i}{\partial \eta_j}(\nabla u) u_{x_j} u_{x_i} + (1+\epsilon) u^{\epsilon} \sum_{i,j=1}^{n} \frac{\partial \mathcal{A}_i}{\partial \eta_j}(\nabla u) u_{x_j x_i} \\ &\geq \alpha^{-1}(1+\epsilon)\epsilon u^{\epsilon-1} |\nabla u|^{p-2} |\nabla u|^2 + 0 > 0 \end{aligned} \tag{2.23}$$

at points $y_0$ and $z_0$ ($\nabla u$ is also evaluated at these points). Using (2.21), (2.23), we conclude that

$$\text{trace}\big((a_{ij}) \cdot (w_{x_i x_j}(0))\big) = \sum_{i,j=1}^{n} a_{ij}\, w_{x_i x_j}(0) < 0. \tag{2.24}$$

Equations (2.24) and (2.20) contradict each other. The proof of Lemma 2.5 is now complete. □

## 3. Proof of Theorem 1.4

In this section we give a proof of Theorem 1.4 by a method of *Beurling*, inspired by Henrot and Shahgholian in [25]. To this end, Let $K$ be a convex domain and let $P_{x_0,a}$ denote the hyperplane in $\mathbf{R}^n$ passing through $x_0$ with the normal $a \neq 0$ pointing away from $K$.

A *supporting hyperplane to* $K$ *at boundary point* $x_0$ is a plane satisfying

$$P_{x,a} := \{x \colon a^T x = a^T x_0\}$$

where $a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in K$. By the supporting hyperplane theorem it is known that there exists a supporting hyperplane at every boundary point of a convex set $K$. Let $\Omega$ be another convex set containing $K$.

For each $x \in \partial K$ there exists a point $y_x \in \partial\Omega \cap \{z \colon a \cdot (z-x) > 0\}$ satisfying $a \cdot (y_x - x) = \max a \cdot (z-x)$, where maximum is taken over the set $\partial\Omega \cap \{z \colon a \cdot (z-x) > 0\}$.

We will work on convex ring domains. That is, let $D_1$ and $D_2$ be two convex domains satisfying $D_1 \subset \overline{D_1} \subset D_2$. We first need an auxiliary lemma.

**Lemma 3.1.** *Let $D_1, D_2$ be two convex domains with $D_1 \subset \overline{D_1} \subset D_2$. Let $u$ be $\mathcal{A}-$capacitary potential of $D_2 \setminus D_1$, that is,*

$$\begin{cases} \nabla \cdot \mathcal{A}(\nabla u) = 0 & \text{in } D_2 \setminus \overline{D_1}, \\ u = c_1 & \text{on } \partial D_1, \\ u = c_2 & \text{on } \partial D_2, \end{cases}$$

*where $c_1 > c_2 \geq 0$ are given constants. Then*

$$\limsup_{\substack{z\to x\\ z\in D_2\setminus\overline{D_1}}} |\nabla u(z)| \geq \limsup_{\substack{z\to y_x\\ z\in D_2\setminus\overline{D_1}}} |\nabla u(z)|$$

*for all $x \in \partial D_1$.*

*Proof.* Without loss of generality assume that $c_1 = 1$ and $c_2 = 0$ as we may using the the translation and dilation invariance of (1.6). Now let $x \in \partial D_1$ and also first assume that $\partial D_1$ is not $C^1$ at $x$. Note that locally near $x$, $u$ can be approximated by functions $u^\epsilon$, which are solutions to a uniformly elliptic PDE in non-divergence form with ellipticity bounds independent of $\epsilon$ (see [4, section 2.3]). This later fact follows from the structural assumptions on $\mathcal{A}$ as in (i) in Definition 1.1. Then it follows from [33] that there exists a barrier $v$ to such linear equations with $v(x) = u^\epsilon(x) = u(x)$, $v \leq u^\epsilon$ near $x$ and moreover $|\nabla v(x)| = \infty$. Thus it follows that $|\nabla u(x)| = \infty$. Likewise, if $\partial D_2$ is not $C^1$ at $y_x$, then similarly it follows from [33] that there exists an upper barrier $v$ such that $v(y_x) = u(y_x)$ and $v \geq u$ locally near $y_x$ and such that $|\nabla v(y_x)| = 0$. Then it follows that $|\nabla u(y_x)| = 0$, which gives the desired result. Thus in view of the above discussion, we may now restrict our attention to the case when $x, y_x$ are in the regular part of $\partial D_1$ and $\partial D_2$ respectively. Let $x \in \partial D_1$ be fixed and let $y_x$ be the associated point on $\partial D_2$ as described above. Let $P = P_{x,a}$ be a supporting plane at $x$ to $D_1$.

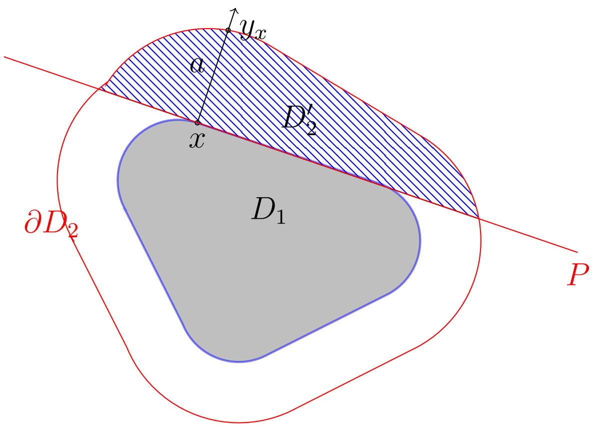


Figure 1. The supporting place $P = P_{x,a}$ and the domains $D_1$ and $D_2$.

Note that $D_1 \subset \{P < 0\}$ and let $D_2' := D_2 \cap \{P > 0\}$. By Lemma 2.3, we may assume that $P = \{x_n = 0\}$. Indeed, otherwise after a rotation we have $P = \{x_n = 0\}$; we first prove the present lemma for $\tilde{u}$ which is $\tilde{\mathcal{A}}$-harmonic for some $\tilde{\mathcal{A}} \in M_p(\alpha)$ and follow by transferring everything back to $u$. Hence assume $P = \{x_n = 0\}$ and define $v = u + \alpha x_n$, where

$$\alpha = \limsup_{\substack{z\to y_x\\ z\in D_2\setminus\overline{D_1}}} |\nabla u(z)| - \epsilon,$$

with $\epsilon > 0$ small. Since $\mathcal{L}u = \mathcal{L}v = 0$ in $D_2'$, $v$ attains its maximum on $\partial D_2'$. By the construction of $D_2'$, the maximum of $v$ is either at $x$ or $y_x$. If the maximum were at $y_x$, then

$$0 \leq \limsup_{\substack{z\to y_x\\ z\in D_2\setminus\overline{D_1}}} \frac{\partial v}{\partial x_n}(z) = -\limsup_{\substack{z\to y_x\\ z\in D_2\setminus\overline{D_1}}} |\nabla u(z)| + \alpha = -\epsilon < 0.$$

It follows that $v$ attains its maximum at $x$, hence

$$\begin{aligned} 0 &\leq -\limsup_{\substack{z\to x\\ z\in D_2\setminus\overline{D_1}}} \frac{\partial v}{\partial x_n}(z) = \limsup_{\substack{z\to x\\ z\in D_2\setminus\overline{D_1}}} |\nabla u(z)| - \alpha \\ &= \limsup_{\substack{z\to x\\ z\in D_2\setminus\overline{D_1}}} |\nabla u(z)| - \limsup_{\substack{z\to y_x\\ z\in D_2\setminus\overline{D_1}}} |\nabla u(z)| + \epsilon. \end{aligned}$$

We then conclude the validity of the lemma. □

We next show that if $D_1$ satisfies the so called *the interior ball property* as in (1.7) then the $\mathcal{A}$-capacitary function $u$ as above has bounded gradient.

**Lemma 3.2.** *Let $D_1, D_2$ be as in Lemma 3.1 and let $d_0 = \min d(\partial D_2, D_1)$. Assume also that $D_1$ satisfies the interior ball property as in* (1.7) *with constant $r_0$. Then there is a constant $M = M(d_0, r_0, n)$ such that*

$$|\nabla u| \leq M \text{ in } D_2 \setminus \overline{D_1}.$$

*Proof.* In view of (2.6), it is enough to show that $|\nabla u| \leq M$ on $\partial D_1 \cup \partial D_2$.

We first take care of points on $\partial D_2$. Without loss of generality take $c_1 = 1, c_2 = 0$ as (1.6) is invariant under translation and dilation. Let $x \in \partial D_2$ be fixed. By rotation, assume, by Lemma 2.3, that $x_n = 0$ is a supporting hyperplane to $\partial D_2$ at $x$ with $D_2 \subset \{x_n > 0\}$ and prove the present lemma for $\tilde{u}$ which is $\tilde{\mathcal{A}}$-harmonic for some $\tilde{\mathcal{A}} \in M_p(\alpha)$ and transfer the result back to $u$. Therefore, without loss of generality $x_n = 0$ is a supporting hyperplane. There exists a supporting hyperplane $x_n = d$ to $\partial D_1$ for which $D_1 \subset \{x_n > d\}$. Let $\tilde{D}_2 = D_2 \cap \{0 < x_n < d\}$ and let $\tilde{u} = x_n/d$. Basic comparison principle applied to positive weak solutions of $\mathcal{A}$-harmonic PDEs gives $u \leq \tilde{u}$ in $\tilde{D}_2$. This observation and the fact that $u(x) = \tilde{u}(x)$ implies

$$|\nabla u(x)| \leq |\nabla \tilde{u}(x)| \leq \frac{1}{d} \leq \frac{1}{d_0}.$$

This gives the desired results for points on $\partial D_2$.

In order to show the same estimates for points on $\partial D_1$, we proceed as follows. We first construct a barrier as we did in the proof of Lemma 2.4 and then we prove that Lemma 3.1 holds for $u_\epsilon$. Finally, using Lemmas 2.1 and 2.2 we conclude that Lemma 3.1 holds for $u$ as well. □

**3.1. A technique of Beurling.** In this subsection we give a brief introduction to a technique used by Beurling in [11] and in [25] as well. To this end, recall that $K$ is a convex domain and let

$$\mathcal{C} := \{\Omega \text{ convex bounded open subset of } \mathbf{R}^n \text{ with } K \subset \Omega\}.$$

Let $u_\Omega$ denote the $\mathcal{A}$-capacitary potential for $\Omega \setminus K$ whenever $\Omega \in \mathcal{C}$. Following [26], we also define

$$\begin{aligned} \mathcal{G} &:= \{\Omega \in \mathcal{C} \colon \liminf_{y\to x, y\in\Omega} |\nabla u_\Omega(y)| \geq c \text{ for all } x \in \partial\Omega\}, \\ \mathcal{G}_0 &:= \{\Omega \in \mathcal{C} \colon \liminf_{y\to x, y\in\Omega} |\nabla u_\Omega(y)| > c \text{ for all } x \in \partial\Omega\}, \\ \mathcal{B} &:= \{\Omega \in \mathcal{C} \colon \limsup_{y\to x, y\in\Omega} |\nabla u_\Omega(y)| \leq c \text{ for all } x \in \partial\Omega\}. \end{aligned}$$

In the language of Beurling, $\mathcal{G}$ is the collection of "subsolutions" and $\mathcal{B}$ is the collection of "supersolutions". Our aim is to show that $\mathcal{G} \cap \mathcal{B} \neq \emptyset$. To this end, we will make some observations.

**Lemma 3.3.** *$\mathcal{B}$ is closed under intersection. That is, if $\Omega_1, \Omega_2 \in \mathcal{B}$, then $\Omega_1 \cap \Omega_2 \in \mathcal{B}$.*

*Proof.* We will use the comparison principle for non-negative $\mathcal{A}$-harmonic functions. Let $u_{\Omega_i}$ for $i = 1, 2$ be $\mathcal{A}$-capacitary functions for $\Omega_i \in \mathcal{B}$. By the comparison principle, we have $u_{\Omega_1 \cap \Omega_2} \le \min\{u_{\Omega_1}, u_{\Omega_2}\}$ in $(\Omega_1 \cap \Omega_2) \setminus K$. Furthermore, $\partial(\Omega_1 \cap \Omega_2) \subset \partial\Omega_1 \cap \partial\Omega_2$, hence given $x \in \partial(\Omega_1 \cap \Omega_2)$ we can assume without loss of generality $x \in \partial\Omega_1$. Then $u_{\Omega_1}(x) = 0 = u_{\Omega_1 \cap \Omega_2}(x)$ and thereupon one concludes that

$$\limsup_{y \to x, y \in \Omega_1 \cap \Omega_2} |\nabla u_{\Omega_1 \cap \Omega_2}(y)| \le \limsup_{y \to x, y \in \Omega_1} |\nabla u_{\Omega_1}| \le c.$$

Therefore $\Omega_1 \cap \Omega_2 \in \mathcal{B}$. This finishes the proof of Lemma 3.3. □

Our next goal is to show the "stability" of $\mathcal{B}$.

**Lemma 3.4.** *Assume $K$ satisfies the interior ball property. Let $\Omega_1 \supset \Omega_2 \supset \ldots$ be a decreasing sequence of domains in $\mathcal{B}$. Let*

$$\Omega = \overline{\bigcap_n \Omega_n}^{\circ}.$$

*Assume $\Omega \in \mathcal{C}$. Then $\Omega \in \mathcal{B}$.*

*Proof.* Let $\Omega_1 \supset \Omega_2 \supset \ldots$ be a sequence of domains in $\mathcal{B}$ and let $\{u_k\}$ be a sequence of capacitary $\mathcal{A}$–harmonic functions for $\{\Omega_k\}$ respectively. Then $0 \le u_k \le 1$. Moreover, by Lemmas 2.1, 2.2, and 3.2 it follows that $\{u_k, \nabla u_k\}$ converges uniformly on compact subsets of $\Omega \setminus \overline{K}$ to $\{u, \nabla u\}$ where $u$ is a $\mathcal{A}$-harmonic function in $\Omega \setminus K$. The proof that $u$ is indeed the capacitary $\mathcal{A}$-harmonic function for $\Omega \setminus K$ essentially follows from the convergence of $\Omega_n$ to $\Omega$ in the Hausdorff distance sense and Lemma 2.4.

We next show that $\Omega \in \mathcal{B}$. To this end, let $M = \max_k(\sup |\nabla u_k|) < \infty$, by Lemma 3.2. Let $0 < \delta_k$ be such that $\delta_k \to 0$ as $k \to \infty$ and

$$\frac{|\nabla u_k|^2 - c^2}{M^2} - \frac{1}{k} \le 0 \quad \text{on } \{u_k = \delta_k\}.$$

Consider

$$\frac{u_k - \delta_k}{1 - 2\delta_k},$$

which is non-negative in $\{u_k > \delta_k\} \setminus \{u_k < 1 - \delta_k\}$ and has zero boundary values on $\{u_k = \delta_k\}$. Recall definition (2.4). By (2.6) applied to $\frac{|\nabla u_k|^2 - c^2}{M^2} - \frac{1}{k}$,

$$\mathcal{L}_{u_k}\left(\frac{|\nabla u_k|^2 - c^2}{M^2} - \frac{1}{k}\right) \ge 0 = \mathcal{L}_{u_k}\left(\frac{u_k - \delta_k}{1 - 2\delta_k}\right)$$

in $\{u_k > \delta_k\} \setminus \{u_k < 1 - \delta_k\}$. On the other hand, on $\{u_k = \delta_k\}$ we have

$$\frac{|\nabla u_k|^2 - c^2}{M^2} - \frac{1}{k} \le 0 = \frac{u_k - \delta_k}{1 - 2\delta_k}.$$

Furthermore, on $\{u_k = 1 - \delta_k\} \cap \{u_k \ge \delta_k\}$ we have

$$\frac{|\nabla u_k|^2 - c^2}{M^2} - \frac{1}{k} \le 1 = \frac{u_k - \delta_k}{1 - 2\delta_k}.$$

It follows that

$$\frac{|\nabla u_k|^2 - c^2}{M^2} - \frac{1}{k} \le \frac{u_k - \delta_k}{1 - 2\delta_k} \quad \text{in } \{u_k > \delta_k\} \setminus \{u_k < 1 - \delta_k\}. \tag{3.1}$$

Given $\epsilon > 0$ one can find a neighborhood $U_\epsilon$ of $\partial\Omega$ such that

$$u_k \le \epsilon \quad \text{in } U_\epsilon.$$

Letting $k \to \infty$ and using (3.1) we obtain

$$\frac{|\nabla u_k|^2 - c^2}{M^2} - \frac{1}{k} \to \frac{|\nabla u|^2 - c^2}{M^2} \le u \le \epsilon \quad \text{as } k \to \infty$$

uniformly on compact subsets of $U_\epsilon \cap \Omega$. By letting $\epsilon \to 0$ we conclude the proof of the Lemma. □

As a consequence of Lemma 3.4, we claim that if $\mathcal{G}_0$ is not empty and for $\Omega_0 \in \mathcal{G}_0$ the set $\{\tilde\Omega \in \mathcal{B}\colon \overline{\Omega_0} \subset \tilde\Omega\}$ is not empty, then there exists a domain $\Omega \in \{\tilde\Omega \in \mathcal{B}\colon \overline{\Omega_0} \subset \tilde\Omega\}$ with the property that

$$\text{if } \hat\Omega \in \{\tilde\Omega \in \mathcal{B}\colon \overline{\Omega_0} \subset \tilde\Omega\} \text{ and } \hat\Omega \subset \Omega, \text{ then } \hat\Omega = \Omega. \tag{3.2}$$

Such a domain $\Omega$ will be called *minimal element* in $\{\tilde\Omega \in \mathcal{B}\colon \overline{\Omega_0} \subset \tilde\Omega\}$. For simplicity, define

$$\mathcal{C}_0 := \{\tilde\Omega \in \mathcal{B}\colon \overline{\Omega_0} \subset \tilde\Omega\}.$$

To prove our claim, let

$$I = \bigcap_i \tilde\Omega_i \quad \text{where } \tilde\Omega_i \in \mathcal{C}_0.$$

Write $I = \bigcap_{i=1}^\infty \tilde\Omega_i$, with $\tilde\Omega_i \in \mathcal{C}_0$. Let $\Omega_1 = \tilde\Omega_1$ and $\Omega_{k+1} = \tilde\Omega_{k+1} \cap \Omega_k$ for $k = 2, 3, \ldots$. Then each $\Omega_k$ is convex and $\Omega_k \in \mathcal{B}$ by Lemma 3.3. Applying Lemma 3.4 to $\{\Omega_n\}$ we conclude that

$$\Omega = \overset{\circ}{\overline{\bigcap_n \Omega_n}} \in \mathcal{B}.$$

This finishes the proof of our claim.

We proceed by studying the behavior of capacitary $\mathcal{A}$-harmonic functions on extremal points of $\Omega$. To set the stage, let $\Omega$ be the minimal element in $\mathcal{C}_0$. A point $x \in \partial\Omega$ is called *extremal point* if there exists a supporting hyperplane to $\Omega$ touching $\partial\Omega$ at $x$ only. Let $E_\Omega$ denote the set of extremal points of $\Omega$.

**Lemma 3.5.** *Let $\Omega$ be a minimal element in the class $\mathcal{C}_0$ and let $x \in \overline{E_\Omega}$. Then*

$$\limsup_{\substack{y\to x\\ y\in\Omega}} |\nabla u_\Omega(y)| = c.$$

*Proof.* The proof will be a contradiction argument. To this end, suppose there exists $y_0 \in \overline{E_\Omega}$ with

$$\limsup_{\substack{y\to y_0\\ y\in\Omega}} |\nabla u_\Omega(y_0)| = c(1 - 4\tilde\alpha).$$

for some $\tilde\alpha > 0$. By the Hölder continuity of $\nabla u_\Omega$ there exists a neighborhood $\mathcal{N}$ of $\partial\Omega$ with $y_0 \in \mathcal{N}$ satisfying that

$$|\nabla u_\Omega(x)| \le c(1 - \tilde\alpha) \quad \text{for every } x \in \mathcal{N} \cap \Omega. \tag{3.3}$$

Assume that $y_0 \in E_\Omega$. Otherwise, we may choose a sequence in $E_\Omega$ converging to $y_0$ with the above property.

Let $d > 0$ and let $P_d$ be a plane such that $d(y_0, P_d) = d$ with $P_d \cap \Omega \subset \mathcal{N}$. Notice that without loss of generality, by Lemma 2.3 we may assume that $y_0 = 0$, $P_d = \{x_n = d\}$. Otherwise, we rotate our coordinate system, work with $\hat u$, which is $\hat{\mathcal{A}}$-harmonic for some $\hat{\mathcal{A}} \in M_p(\alpha)$, and at the end transfer everything back to $u_\Omega$.

Hence assume $y_0 = 0$, $P_d = \{x_n = d\}$, let $\epsilon > 0$ and define $\Omega_\epsilon = \Omega \setminus \{x_n \le \epsilon\}$. Assume $\epsilon$ is small enough so that $\Omega_0 \subset \Omega_\epsilon$. Let $u_\epsilon$ be the $\mathcal{A}$-capacitary function for $\Omega_\epsilon \setminus K$. As $u_\epsilon \le u_\Omega$ on $\partial\Omega_\epsilon$, by the comparison principle for non-negative $\mathcal{A}$-harmonic functions we have

$$0 \le u_\epsilon \le u_\Omega \ \text{ in } \ \Omega_\epsilon. \tag{3.4}$$

It follows that we have

$$\limsup |\nabla u_\epsilon| \le \limsup |\nabla u_\Omega| \le c \quad \text{on } \partial\Omega \cap \partial\Omega_\epsilon. \tag{3.5}$$

At the points where $\partial\Omega \cap \partial\Omega_\epsilon$ is not $C^1$ we claim that $|\nabla u_\epsilon| = 0$. This can be done as in [5, Section 7] by considering $\mathcal{A}(\eta, \delta)$ to obtain a uniformly elliptic equation in divergence form and $v_\epsilon^\delta$ which is $\mathcal{A}(\eta, \delta)$-harmonic in $\Omega_\epsilon$. Once again repeating following [5, Section 7], one concludes that $|\nabla v_\epsilon^\delta| = 0$ and thereupon letting $\delta \to 0$ we obtain our claim.

Using (3.3) and (3.4) we have

$$\max_{P_d \cap \Omega} u_\epsilon \le \max_{P_d \cap \Omega} u_\Omega \le d \sup_{\{0 \le x_n \le d\} \cap \Omega} |\nabla u_\Omega| \le d\, c(1 - \tilde{\alpha}). \tag{3.6}$$

Let now

$$v := u_\epsilon + \frac{d\, c(1 - \tilde{\alpha})}{d - \epsilon}(d - x_n)$$

and note that

$$\mathcal{L}_{u_\epsilon} v = \mathcal{L}_{u_\epsilon} u_\epsilon = 0 \ \text{ in } \ \Omega_\epsilon \cap \{x_n < d\}.$$

Thereupon we conclude that $v$ takes its maximum on the boundary of $\Omega_\epsilon \cap \{x_n < d\}$. Moreover, using (3.6) we obtain

$$v \le c(1 - \tilde{\alpha})d \ \text{ on } \ \partial(\Omega_\epsilon \cap \{x_n < d\}) \quad \text{and} \quad v = d\, c(1 - \tilde{\alpha}) \ \text{ on } \ P_\epsilon,$$

as

$$\partial(\Omega_\epsilon \cap \{x_n < d\}) \subset P_d \cup P_\epsilon \cup (\partial\Omega \cap \{\epsilon < x_n < d\})$$

Hence we have

$$0 \ge \frac{\partial v}{\partial x_n} = |\nabla u_\epsilon| - \frac{c(1 - \tilde{\alpha})d}{d - \epsilon} \ \text{ on } P_\epsilon.$$

By choosing $\epsilon \le \tilde{\alpha} d$, we obtain $|\nabla u_\epsilon| \le c$ on $P_\epsilon$. In view of this result and (3.5) we have $\Omega_\epsilon \in \mathcal{B}$ and by construction $\Omega_\epsilon \subset \Omega$. By (3.2) we conclude that $\Omega = \Omega_\epsilon$ which is a contradiction, hence the proof of Lemma 3.5 is complete. □

We next observe that if $\Omega$ is a minimal element in the class $\mathcal{C}_0$ and let $u_\Omega$ be the $\mathcal{A}$-capacitary function for $\Omega \setminus K$ then

$$|\nabla u_\Omega(x)| \ge c \quad \text{for all } \ x \in \Omega \setminus K. \tag{3.7}$$

To prove (3.7) we use the fact that for every $0 < t < 1$, $\{x \in \Omega \colon u_\Omega(x) > t\}$ is a convex set due to Lemma 2.5. The conclusion follows by applying Lemma 3.1 to $\{x \in \Omega \colon u_\Omega > t\}$ and $\Omega$, and using Lemma 3.5.

**3.2. Final Proof of Theorem 1.4.** We split the proof into two steps, existence of $\Omega$ and uniqueness of $\Omega$.

**3.2.1. Existence of $\Omega$.** In order to prove Theorem 1.4 we show that there exist domains $\Omega_0$ and $\Omega_1$ such that $\overline{\Omega_0} \in \mathcal{G}_0$ and $\Omega_1 \in \mathcal{B}$ with $\overline{\Omega_0} \subset \Omega_1$. Then from (3.2) there exists a minimal element $\Omega \in \mathcal{C}_0$ and by using (3.7) we have $\Omega \in \mathcal{G}$. In view of the definitions of $\mathcal{G}$ and $\mathcal{B}$, this would allow us to assert existence result of Theorem 1.4. Hence to finish the proof of existence, it remains to show the existence of $\Omega_0 \in \mathcal{G}_0$ and $\Omega_1 \in \mathcal{B}$ with $\overline{\Omega_0} \subset \Omega_1$.

*Existence of* $\Omega_1 \in \mathcal{B}$*:* For this, choose $R_0$ large enough so that $K \subset B_{R_0}$. Let $R > R_0$ large to be fixed below. Without loss of generality assume $0 \in K$. Let $u_R$ be the $\mathcal{A}$-capacitary function for $B(0,R) \setminus K$. Using (b) in Lemma 2.4 we can choose $R$ sufficiently large so that

$$|\nabla u_R(x)| \leq c_\star |x|^{\frac{1-n}{p-1}} = c_\star R^{\frac{1-n}{p-1}} \leq c.$$

Therefore, $\Omega_1 = B(0,R) \in \mathcal{B}$.

*Existence of* $\Omega_0 \in \mathcal{G}_0$*:* Let $R > 0$ be as above and $u_R$ be the $\mathcal{A}$-capacitary function for $B(0,R) \setminus K$. We first observe from the smoothness of $B(0,R)$ and Lemma 3.1 that there is a constant $C > 0$ and a neighbourhood $U$ of $\partial K$ such that

$$|\nabla u_R| \geq C \quad \text{in } U \setminus K.$$

For a given $t$, $0 < t < 1$, let $\Omega_t = \{x \in B(0,R) \colon u_R(x) > 1 - t\}$. Then the $\mathcal{A}$-capacitary function for $\Omega_t$ is

$$u_{\Omega_t}(x) = \frac{u_R - (1-t)}{t}.$$

By choosing $t$ sufficiently small we have on $\partial\Omega_t$

$$|\nabla u_{\Omega_t}| = \frac{|\nabla u_R|}{t} \geq \frac{C}{t} \geq c$$

Therefore, $\Omega_0 := \Omega_t \in \mathcal{G}_0$ and $\overline{\Omega_0} \subset \Omega_1$

In view of these two observations and our earlier remarks, the existence of $\Omega$ is done.

**3.2.2. Uniqueness of $\Omega$.** This will follow from [24], where uniqueness was shown for the Laplace equation, and nonlinear elliptic differential equations satisfying properties (i)–(iv) given below, by using the Lavrentèv principle. In order to make use of this result for nonlinear elliptic equations, one needs to have four conditions (see section 4 in [24]);

(3.8)
- (i) The PDE is weakly elliptic and satisfies the comparison principle.
- (ii) If $u$ is a solution, then rotations and translations are also solutions to some weakly elliptic PDE satisfying comparison principle.
- (iii) $u = x_n$ is a solution.
- (iv) If $\Omega$ and $K$ are both convex and if $u_\Omega$ is the $\mathcal{A}$-capacitary function for $\Omega \setminus K$, then superlevels of $u$ are convex; $\Omega_t = \{x \in \Omega \colon u(x) > t\}$ is convex.

Here (i) in (3.8) follows from the structural assumption on $\mathcal{A}$, (ii) follows from Lemma 2.3. Regarding (iii), it is clear that $u = x_n$ is $\mathcal{A}$-harmonic, and (iv) follows from Lemma 2.5.

**3.2.3. Proof of $\Omega \in C^{1,\gamma}$.** To obtain the $C^{1,\gamma}$ regularity of $\Omega$, one repeats the arguments of Vogel [37], which rely on the machinery of [9] and [10].

Furthermore, it follows from applying the Hodograph transform that if $\mathcal{A} \in C^\infty(\mathbf{R}^n \setminus \{0\})$, then $\partial\Omega \in C^\infty$, see [28, 29]. We notice that an interesting alternative method to obtain higher regularity has recently been done in [15], where the authors prove higher order boundary Harnack estimates. See also [17, 16] in the context of thin obstacle problems.

Now the proof of Theorem 1.4 is complete.

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