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MULTIPLICATION ON UNIFORM λ-CANTOR SETS

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Abstract. Let C be the middle-third Cantor set. Define $C * C = \{x * y \colon x, y \in C\}$, where $* = +, -, \cdot, \div$ (when $* = \div$, we assume $y \neq 0$). Steinhaus [17] proved in 1917 that

$$C - C = [-1, 1], \quad C + C = [0, 2].$$

In 2019, Athreya, Reznick and Tyson [1] proved that

$$C \div C = \bigcup_{n=-\infty}^{\infty} \left[3^{-n}\frac{2}{3}, 3^{-n}\frac{3}{2}\right] \cup \{0\}.$$

In this paper, we give a description of the topological structure and Lebesgue measure of $C \cdot C$. We indeed obtain corresponding results on the uniform λ-Cantor sets.

1. Introduction

The middle-third Cantor set, denoted by C, is a celebrated set in fractal geometry. Many aspects of this set were analyzed [10, 11, 13]. In this paper, we shall investigate the arithmetic on the middle-third Cantor set. Arithmetic on the fractal sets has some connections to the geometric measure theory, dynamical systems, and number theory, see [2, 3, 4, 5, 6, 7, 8, 9, 12, 15, 16, 18, 19, 20] and references therein. However, relatively few papers investigate the structure of the fractal sets under some arithmetic operations. The theme of this paper is to explore this problem.

Let $C * C = \{x * y \colon x, y \in C\}$, where $* = +, -, \cdot, \div$ (when $* = \div$, we assume $y \neq 0$). Arithmetic on the middle-third Cantor set was pioneered by Steinhaus [17] who proved that

$$C + C = [0, 2], C - C = [-1, 1].$$

It is natural to ask the multiplication and division on C. For this question, recently, Athreya, Reznick and Tyson [1] proved that $17/21 \leq \mathcal{L}(C \cdot C) \leq 8/9$, where $\mathcal{L}$

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denotes the Lebesgue measure. Moreover, they also proved that

$$C \div C = \bigcup_{n=-\infty}^{\infty} \left[3^{-n}\frac{2}{3}, 3^{-n}\frac{3}{2}\right] \cup \{0\}.$$

Let f be a continuous function defined on an open set $U \subset \mathbf{R}^2$. Denote the continuous image by

$$f_U(C, C) = \{f(x, y) \colon (x, y) \in (C \times C) \cap U\}.$$

Jiang and Xi [14] proved that if $\partial_x f$, $\partial_y f$ are continuous on U, and there is a point $(x_0, y_0) \in (C \times C) \cap U$ such that one of the following conditions is satisfied,

$$1 < \left| \frac{\partial_y f|_{(x_0,y_0)}}{\partial_x f|_{(x_0,y_0)}} \right| < 3 \quad \text{or} \quad 1 < \left| \frac{\partial_x f|_{(x_0,y_0)}}{\partial_y f|_{(x_0,y_0)}} \right| < 3,$$

then $f_U(C, C)$ has a non-empty interior. In particular, $C \cdot C$ contains infinitely many intervals. To date, to the best of our knowledge, there are few results for the structure of $C \cdot C$. This is one of the main motivations of this paper. We shall give an answer to this question. Let K_λ be the uniform λ-Cantor set generated by the iterated function system (IFS) [13]

$$\{f_i(x) = \lambda x + i\delta\}_{i=0}^{m-1},$$

where $m \geq 2, 0 < \lambda < 1/m, g = \frac{1-m\lambda}{m-1}$, and $\delta = \lambda + g$. Suppose $\sigma = i_1 i_2 \cdots i_n \in \{0, 1, \cdots m-1\}^n$ with its length $|\sigma| = n \geq 0$. If $n = 0$, then $\sigma = \emptyset$. In this case, $f_\sigma([0, 1]) = [0, 1]$. For any $I = f_\sigma([0, 1])$, let $I^{(i)} = f_{\sigma i}([0, 1]), 0 \leq i \leq m-1$. Denote

$$I^{(i)} \cdot I^{(j)} = \{xy \colon x \in I^{(i)}, y \in I^{(j)}\}.$$

Define

$$\widehat{I} = \begin{cases} \bigcup_{0 \leq p < q} \left(I^{(p)} \cdot I^{(q)}\right) & \text{if } |\sigma| \geq 1, \\ \bigcup_{1 \leq p < q} \left(I^{(p)} \cdot I^{(q)}\right) & \text{if } \sigma = \emptyset. \end{cases}$$

We denote

$$\mathcal{A} = \{f_\sigma([0, 1]) \colon \sigma = i_1 i_2 \cdots i_n, i_1 \geq 1\} \cup \{[0, 1]\}.$$

Let

$$K_R = \bigcup_{i=1}^{m-1} f_i(K_\lambda).$$

Now we state the first result of this paper.

Theorem 1. *Let K_λ be the uniform λ-Cantor set. If $1/(m+1) \leq \lambda < 1/m$, then*

$$K_\lambda \cdot K_\lambda = \left(\bigcup_{n=0}^{\infty} \lambda^n (K_R \cdot K_R) \right) \bigcup \{0\},$$

where

$$K_R \cdot K_R = \left(\bigcup_{I \in \mathcal{A}} \widehat{I} \right) \bigcup \{x^2 : x \in K_R\}.$$

Remark 1. When $\lambda = 1/3, m = 2$, we give the structure of $C \cdot C$. With the help of a computer program, we are able to calculate the value of the Lebesgue measure of $C \cdot C$, which is about 0.80955.

Remark 2. The main ingredient of $K_R \cdot K_R$ is $\bigcup_{I\in\mathcal{A}} \widehat{I}$ as the Lebesgue measure of $\{x^2\colon x \in K_R\}$ is zero. We shall use this fact in the proof of next result. It is natural to consider that whether some points in $\{x^2\colon x \in K_R\}$ can be absorbed by $\bigcup_{I\in\mathcal{A}} \widehat{I}$. We can prove, for instance, that for $m = 2$, if $0.44 < \lambda < 1/2$, then

$$K_R \cdot K_R \setminus \bigcup_{I\in\mathcal{A}} \widehat{I} = \{(1-\lambda)^2, 1\}.$$

Equivalently, in this case

$$\bigcup_{I\in\mathcal{A}} \widehat{I} = ((1-\lambda)^2, 1).$$

The next result is to consider the function $\Phi(\lambda) = \mathcal{L}(K_\lambda \cdot K_\lambda)$. In terms of some basic results in analysis, we prove the following result.

Theorem 2. *If $\lambda \in [1/(m+1), 1/m)$, then the function $\mathcal{L}(K_\lambda \cdot K_\lambda)$ is a continuous function of λ.*

This paper is arranged as follows. In section 2, we give the proofs of main theorems. In section 3, we give some remarks and problems.

2. Proof of main results

In this section, we shall give the proofs of the main theorems.

2.1. Proof of Theorem 1. Let $E = [0, 1]$. For any $(i_1 \cdots i_n) \in \{0, 1, \cdots m-1\}^n$, we call $f_{i_1\cdots i_n}(E)$ a basic interval with length λ^n. Denote by E_n $(n \geq 1)$ the collection of all the basic intervals with length λ^n. Let $J = f_\sigma([0, 1]) \in E_n$ be a basic interval for some $\sigma \in \{0, 1, \cdots, m-1\}^n$. Denote $\widetilde{J} = \bigcup_{i=0}^{m-1} f_{\sigma i}([0, 1])$. Let $[A, B] \subset [0, 1]$, where A and B are the left and right endpoints of some basic intervals in E_k for some $k \geq 1$, respectively. A and B may not be in the same basic interval. Let F_k be the collection of all the basic intervals in $[A, B]$ with length $\lambda^k, k \geq k_0$ for some $k_0 \in \mathbf{N}^+$, i.e. the union of all the elements of F_k is denoted by $G_k = \bigcup_{i=1}^{t_k} I_{k,i}$, where $t_k \in \mathbf{N}^+$, $I_{k,i} \in E_k$ and $I_{k,i} \subset [A, B]$. Clearly, by the definition of G_n, it follows that $G_{n+1} \subset G_n$ for any $n \geq k_0$. Similarly, let M and N be the left and right endpoints of some basic intervals in E_k for some $k \geq k_0$. Denote by G'_k the union of all the basic intervals with length λ^k such that all of these basic intervals are subsets of $[M, N]$, i.e. $G'_k = \bigcup_{i=1}^{t'_k} I'_{k,i}$, where $t'_k \in \mathbf{N}^+$, $I'_{k,i} \in E_k$ and $I'_{k,i} \subset [M, N]$. Now, we state a key lemma of our paper.

Lemma 1. *Assume $F\colon \mathbf{R}^2 \to \mathbf{R}$ is a continuous function. Suppose A and B (M and N) are the left and right endpoints of some basic intervals in E_{k_0} for some $k_0 \geq 1$, respectively. Then $K_\lambda \cap [A, B] = \bigcap_{n=k_0}^{\infty} G_n$ ($K_\lambda \cap [M, N] = \bigcap_{n=k_0}^{\infty} G'_n$). Moreover, if for any $n \geq k_0$ and any basic intervals $I_1 \subset G_n$, $I_2 \subset G'_n$*

$$F(I_1, I_2) = F(\widetilde{I_1}, \widetilde{I_2}),$$

then $F(K_\lambda \cap [A, B], K_\lambda \cap [M, N]) = F(G_{k_0}, G'_{k_0})$.

Proof. Let $G_n = \bigcup_{i=1}^{t_n} I_{n,i}$ for some $t_n \in \mathbf{N}^+$, where $I_{n,i} \in E_n$ and $I_{n,i} \subset [A, B]$. Then by the construction of G_n, i.e. $G_{n+1} \subset G_n$ for any $n \geq k_0$, it follows that

$$K_\lambda \cap [A, B] = \bigcap_{n=k_0}^{\infty} G_n.$$

Analogously, we can prove that

$$K_\lambda \cap [M,N] = \bigcap_{n=k_0}^{\infty} G'_n.$$

By the continuity of F, we conclude that

$$F(K_\lambda \bigcap [A,B], K_\lambda \cap [M,N]) = \cap_{n=k_0}^{\infty} F(G_n, G'_n). \tag{2.1}$$

By virtue of the relation $G_{n+1} = \widetilde{G_n}$ ($G'_{n+1} = \widetilde{G'_n}$) and the condition in the lemma, we have

$$F(G_n, G'_n) = \bigcup_{1\le i\le t_n, 1\le j\le t'_n} F(I_{n,i}, I'_{n,j}) = \bigcup_{1\le i\le t_n, 1\le j\le t'_n} F(\widetilde{I_{n,i}}, \widetilde{I'_{n,j}})$$
$$= F\left(\bigcup_{1\le i\le t_n} \widetilde{I_{n,i}}, \bigcup_{1\le j\le t'_n} \widetilde{I'_{n,j}}\right) = F(G_{n+1}, G'_{n+1}).$$

Therefore, $F(K_\lambda \cap [A,B], K_\lambda \cap [M,N]) = F(G_{k_0}, G'_{k_0})$ follows immediately from identity (2.1) and $F(G_n, G'_n) = F(G_{n+1}, G'_{n+1})$ for any $n \ge k_0$. □

Recall that $\delta = \lambda + g$ and $g = \frac{1-m\lambda}{m-1}$ with $m \ge 2$.

Proposition 1. *Let $\lambda \in [\frac{1}{m+1}, \frac{1}{m})$. Suppose $I \in \mathcal{A}$ and $I^{(0)}, \cdots, I^{(m-1)}$ are basic subintervals of I from left to right such that $|I^{(i)}|/|I| = \lambda$ for all $0 \le i \le m-1$. Then for any $p < q$, we have*

$$(I^{(p)} \cap K_\lambda) \cdot (I^{(q)} \cap K_\lambda) = I^{(p)} \cdot I^{(q)}.$$

Proof. Given basic subintervals $I^* \subset I^{(p)}$ and $J^* \subset I^{(q)}$ with $I^* = [a, a+t]$ and $J^* = [b, b+t]$ ($b \ge a$), we suppose that $A^{(0)}, \cdots, A^{(m-1)}$ ($B^{(0)}, \cdots, B^{(m-1)}$) are basic subintervals of I^* (J^* respectively) from left to right.

For rectangles $R_1 = [a_1, b_1] \times [a_2, b_2]$ and $R_2 = [c_1, d_1] \times [c_2, d_2]$ with $a_1, a_2, c_1, c_2 \ge 0$, we denote by $R_1 \to R_2$ if $b_1 b_2 \ge c_1 c_2$. By Lemma 1, it suffices to verify that

$$\begin{aligned}
&(A^{(0)} \times B^{(0)}) \\
\to &(A^{(0)} \times B^{(1)}) \to (A^{(1)} \times B^{(0)}) \\
\to &(A^{(0)} \times B^{(2)}) \to (A^{(1)} \times B^{(1)}) \to (A^{(2)} \times B^{(0)}) \\
\to &(A^{(0)} \times B^{(3)}) \to (A^{(1)} \times B^{(2)}) \to (A^{(2)} \times B^{(1)}) \to (A^{(3)} \times B^{(0)}) \\
\cdots& \\
\to &(A^{(0)} \times B^{(m-1)}) \to (A^{(1)} \times B^{(m-2)}) \to \cdots \to (A^{(m-2)} \times B^{(1)}) \to (A^{(m-1)} \times B^{(0)}) \\
\to &(A^{(1)} \times B^{(m-1)}) \to (A^{(2)} \times B^{(m-2)}) \to \cdots \to (A^{(m-1)} \times B^{(1)}) \\
\cdots& \\
\to &(A^{(m-1)} \times B^{(m-1)}).
\end{aligned}$$

For example, please see Figure 1 when $m = 5$.

In fact, we need to verify

$$(A^{(i)} \times B^{(j)}) \to (A^{(i+1)} \times B^{(j-1)}), \tag{2.2}$$

and

$$(A^{(k)} \times B^{(0)}) \to (A^{(0)} \times B^{(k+1)}) \text{ and } (A^{(m-1)} \times B^{(i)}) \to (A^{(i+1)} \times B^{(m-1)}). \tag{2.3}$$

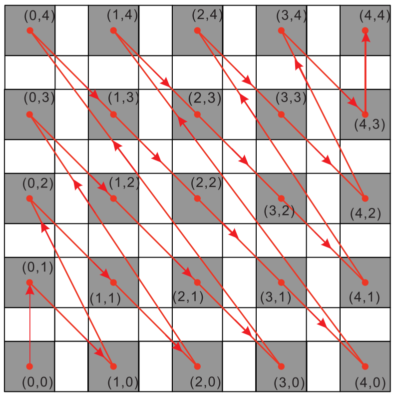

Figure 1. $m = 5$ and (i, j) represents rectangle $A^{(i)} \times B^{(j)}$.

(1) For inequality (2.2), let $(a, b) + (i, j)\delta t = (x, y)$, then $(a, b) + (i+1, j-1)\delta t = (x + \delta t, y - \delta t)$. We only need to show

$$l(x, y) = (x + \lambda t)(y + \lambda t) - (x + \delta t)(y - \delta t) \geq 0. \tag{2.4}$$

Note that $x \geq a$, $y \leq b + (1 - \lambda)t$ and

$$\frac{\partial l(x, y)}{\partial x} = \lambda t + \delta t > 0 \quad \text{and} \quad \frac{\partial l(x, y)}{\partial y} = -gt < 0.$$

We obtain that

$$l(x, y) \geq l(a, b + (1 - \lambda)t) = t\left(a\lambda + a\delta + t\lambda^2 + t\delta^2 - bg - gt + gt\lambda\right) = t \cdot u(\lambda, t).$$

Since $a \geq \delta$ and $b \leq 1$ we have

$$\begin{aligned} u(\lambda, t) &\geq \delta\lambda + \delta^2 + t\lambda^2 + t\delta^2 - g - gt + gt\lambda \\ &= t\left(\lambda^2 + g\lambda + \delta^2 - g\right) + (\delta^2 + \lambda\delta - g). \end{aligned}$$

By $\delta = \lambda + g$ and $g = \frac{1-m\lambda}{m-1}$, one can get

$$\lambda^2 + g\lambda + \delta^2 - g = \frac{m^2\lambda - m\lambda^2 - m + 2\lambda^2 - 3\lambda + 2}{(m-1)^2} = \frac{h(\lambda)}{(m-1)^2}.$$

Note that $\lambda < \frac{1}{m} \leq 1/2$, then

$$h'(\lambda) = (m^2 - 3) - 2\lambda\,(m - 2) \geq m(m-1) - 1 \geq 0.$$

Hence for any $\lambda \geq \frac{1}{m+1}$ and any $m \geq 2$,

$$\frac{h(\lambda)}{(m-1)^2} \geq \frac{h(\frac{1}{m+1})}{(m-1)^2} = \frac{1}{(m^2-1)^2}\left(m^2 - m + 1\right) > 0.$$

On the other hand, we obtain

$$\delta^2 + \lambda\delta - g = \frac{1}{(m-1)^2}\left((2-m)\,\lambda^2 + \left(m^2 - 3\right)\lambda + (2-m)\right) = \frac{1}{(m-1)^2}v(\lambda).$$

If $m = 2$, then $\delta^2 + \lambda\delta - g \geq 0$. If $m = 3$, then we note that the symmetric axis of the graph $w = v(\lambda)$ is

$$\lambda = \frac{m^2 - 3}{2(m-2)} \geq \frac{1}{2}\frac{m^2 - 4}{m - 2} = \frac{m+2}{2} \geq 2,$$

which implies

$$\inf_{\lambda\in[\frac{1}{m+1},\frac{1}{m})} v(\lambda) \geq \min\left(v(\frac{1}{m}), v(\frac{1}{m+1})\right) = \min\left(\frac{2\,(m-1)^2}{m^2}, \frac{m^2-m+1}{(m+1)^2}\right) > 0.$$

Then (2.4) follows.

(2) For inequality (2.3), we will need the following

Claim 1. *If $x_1+y_1 \geq x_2+y_2$ with $x_1<y_1$, $x_2<y_2$ and $x_1>x_2$, then we have $x_1y_1 \geq x_2y_2$.*

In fact, let $H(x)=x(c-x)$ with $c=x_1+y_1$, then we have

$$x_2y_2 \leq x_2(y_2+(x_1+y_1)-(x_2+y_2)) = H(x_2) < H(x_1)$$

since $x_2<x_1<c/2$.

Let (x_1,y_1) be the upper right corner of the rectangle $A^{(k)}\times B^{(0)}$, and (x_2,y_2) the lower left corner of $A^{(0)}\times B^{(k+1)}$, then

$$(x_1+y_1)-(x_2+y_2) = 2\lambda t - \delta t = t\frac{(2m-1)\lambda-1}{m-1} \geq 0$$

due to $\lambda \geq \frac{1}{m+1} \geq \frac{1}{2m-1}$. Using Claim 1, we have $x_1y_1 \geq x_2y_2$, i.e.,

$$(A^{(k)}\times B^{(0)}) \to (A^{(0)}\times B^{(k+1)}).$$

In the same way, we have $(A^{(m-1)}\times B^{(i)}) \to (A^{(i+1)}\times B^{(m-1)})$. □

The following lemma is from the fact

$$K_\lambda = \{0\}\cup\left(\bigcup_{n=0}^{\infty}\lambda^n(K_R)\right).$$

Lemma 2.

$$K_\lambda\cdot K_\lambda = \{0\}\cup\left(\bigcup_{n=0}^{\infty}\lambda^n(K_R\cdot K_R)\right).$$

Proof of Theorem 1. By Lemma 2, it remains to prove

$$K_R\cdot K_R = \left(\bigcup_{I\in\mathcal{A}}\widehat{I}\right)\cup\{x^2\colon x\in K_R\}.$$

By Proposition 1, it follows that

$$K_R\cdot K_R \supset \left(\bigcup_{I\in\mathcal{A}}\widehat{I}\right)\cup\{x^2\colon x\in K_R\}.$$

Conversely, for any $x,y\in K_R$, if $x=y$, then $xy\in\{x^2\colon x\in K_R\}$. If $x\neq y$, then there exists some basic interval, denoted by I, such that $x\in I^{(p)}, y\in I^{(q)}$ for $p\neq q$. Therefore, by Proposition 1, we have that $x\cdot y\in\bigcup_{I\in\mathcal{A}}\widehat{I}$. □

2.2. Proof of Theorem 2.

Lemma 3. *Let $\lambda\in[1/(m+1),1/m)$. For any $n\geq 1$, we have*

$$\mathcal{L}\left((K_R\cdot K_R)\setminus\bigcup_{I\in\mathcal{A},\mathrm{rank}(I)\leq k}\widehat{I}\right) \leq 3\frac{(m\lambda)^{k+1}}{1-m\lambda}.$$

Proof. First, note that for any basic interval $I = [a, a+t]$, by the definition of $\widehat{I}$, we have

$$\mathcal{L}(\widehat{I}) \leq (a+t)^2 - a^2 = t(2a+t) \leq 3t = 3\mathcal{L}(I).$$

Now we have the following estimation:

$$\mathcal{L}\left(K_R \cdot K_R \setminus \left(\bigcup\nolimits_{I\in\mathcal{A},\mathrm{rank}(I)\leq k} \widehat{I}\right)\right) \leq \mathcal{L}\left(\bigcup\nolimits_{I\in\mathcal{A},\mathrm{rank}(I)>k} \widehat{I}\right)$$
$$\leq \sum_{I\in\mathcal{A},\mathrm{rank}(I)>k} 3(m\lambda)^{k+1} = 3\frac{(m\lambda)^{k+1}}{1-m\lambda}. \qquad \square$$

Let

$$\phi_n(\lambda) = \mathcal{L}\left(\bigcup\nolimits_{k=0}^{n} \lambda^k \left(\bigcup\nolimits_{I\in\mathcal{A},\mathrm{rank}(I)\leq n} \widehat{I}\right)\right),$$

which is a continuous function for $\lambda \in [1/(m+1), 1/m)$. Lemma 3 implies the following result.

Lemma 4. *If $\alpha > 0$, then $\phi_n(\lambda)$ uniformly converges to $\Phi(\lambda) = \mathcal{L}(K_\lambda \cdot K_\lambda)$ for any $\lambda \in [1/(m+1), 1/m - \alpha)$.*

Proof. Note that

$$\Phi(\lambda) = \mathcal{L}\left(\bigcup\nolimits_{k=0}^{\infty} \lambda^k (K_R \cdot K_R)\right).$$

It follows from Lemma 3 that

$$\begin{aligned}
&|\Phi(\lambda) - \phi_n(\lambda)| \\
&\leq \left|\Phi(\lambda) - \mathcal{L}\left(\bigcup\nolimits_{k=0}^{n} \lambda^k (K_R \cdot K_R)\right)\right| + \left|\mathcal{L}\left(\bigcup\nolimits_{k=0}^{n} \lambda^k (K_R \cdot K_R)\right) - \phi_n(\lambda)\right| \\
&\leq \mathcal{L}\left(\bigcup\nolimits_{k=n+1}^{\infty} \lambda^k (K_R \cdot K_R)\right) + \sum_{k=0}^{n} \lambda^k 3\frac{(m\lambda)^{n+1}}{1-m\lambda} \\
&\leq \sum_{k=n+1}^{\infty} \lambda^k + 3\frac{1-\lambda^{n+1}}{1-\lambda}\frac{(1-m\alpha)^{n+1}}{1-m\lambda} \leq \frac{m^{-n-1}}{1-\lambda} + \frac{3}{1-\lambda}\frac{(1-m\alpha)^{n+1}}{1-m\lambda}.
\end{aligned}$$

Since λ is bounded, it follows that the last number of the above inequalities is only dominated by n. Therefore, the convergence is the uniform convergence. $\square$

Proof of Theorem 2. By the continuity of $\phi_n(\lambda)$ and Lemma 4, it follows that $\Phi(\lambda)$ is continuous. $\square$

3. Final remarks

From the graph of function $\mathcal{L}(K_\lambda \cdot K_\lambda)$ (Figure 2), we pose a natural question.

Conjecture 1. *The function $\mathcal{L}(K_\lambda \cdot K_\lambda)$ is an increasing function for any $\lambda \in [1/(m+1), 1/m)$.*

Here the graph is due to Python. In terms of Theorem 1, we use Python to approximate the graph of $\mathcal{L}(K_\lambda \cdot K_\lambda)$. Due to Figure 2, it is natural to have the above conjecture.

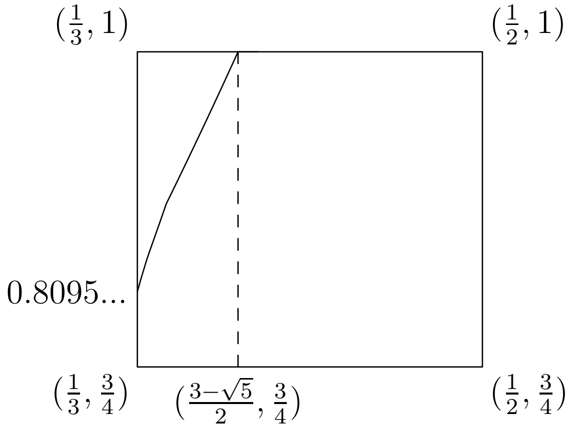

Figure 2. The graph of $\mathcal{L}(K_\lambda \cdot K_\lambda)$ with $m = 2$.

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