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BI-GEODESIC MAPPINGS BETWEEN PAIRS OF PANTS

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Abstract. It is proved that a bijection between the interiors of two pairs of pants is an isometry if it and its inverse preserve geodesics as subsets.

1. Introduction

In 1847 von Staudt [14] proved the Fundamental Theorem of Affine Geometry: A bijective self-mapping of a Euclidean space is affine if it maps lines to lines. Since then, von Staudt's theorem has been generalized to various settings by Beltrami, Levi–Civita, Weyl, Sinyukov, Mikeš, and others (see [2, 15, 9, 4, 8, 10, 6, 16, 12, 11, 1, 13]).

Recently, Shulkin and van Limbeek considered the generalization in the setting of manifolds and posed a new problem:

Problem 1. [13, Question 1.7] *Let M be a closed nonpositively curved manifold of dimension > 1. Suppose $f\colon M \to M$ is a bijection that maps geodesics to geodesics (as sets). Is f affine (i.e., smooth and preserves the Levi–Civita connection)?*

And they gave an affirmative answer for flat tori in [13]. Since any affine map between two closed negatively curved manifolds is homothetic (see [7, Theorem 5.2]), it is natural to ask:

Problem 2. *Let M, N be two compact negatively curved manifolds. Suppose $f\colon M \to N$ is a bijection that maps geodesics to geodesics (as sets). Is f homothetic? Here we say f is* homothetic *if there is a constant k such that the length $l(\alpha)$ of any arc $\alpha \subset M$ and the length $l(f(\alpha))$ of its image $f(\alpha) \subset N$ satisfy $l(f(\alpha)) = k \cdot l(\alpha)$.*

In this paper we deal with the case where M, N are two pairs of pants and establish:

Theorem 3. *Let P_1 and P_2 be two pairs of pants. Let S_1 and S_2 be the interiors of P_1 and P_2, respectively. Suppose $f\colon S_1 \to S_2$ is a bijection such that f and f^{-1} map geodesics to geodesics (as sets). Then f is an isometry.*

Since a geodesic in a hyperbolic surface with boundary is a boundary component if and only if it intersects another geodesic at most twice, by Theorem 3 we obtain:

Corollary 4. *Let P_1 and P_2 be two pairs of pants. Suppose $f\colon P_1 \to P_2$ is a bijection such that f and f^{-1} map geodesics to geodesics (as sets). Then f is an isometry.*

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We face the same difficulties as described in [13, Remark 1.5], that is, f cannot be lifted to the universal cover at once and the classical proof of the Fundamental Theorem of Affine Geometry does not work here. Our proof will be based on a characterization of simple geodesics in terms of intersections with geodesics.

2. Preliminaries

Here we recall some definitions and results from the theory of hyperbolic surfaces, see [3] or [5] for more details.

We say a subset of a Riemannian manifold is a *geodesic* if it is the image of a locally shortest curve which cannot be extended to any other locally shortest curve. A geodesic is called *closed* if it is the image of a closed curve, and otherwise it is called *non-closed*. We say a mapping between two Riemannian manifolds $f\colon M \to N$ is a *geodesic mapping* if it maps each geodesic in M onto some geodesic in N. We say a bijection $f\colon M \to N$ is a *bi-geodesic mapping* if f and f^{-1} are geodesic mappings.

A compact hyperbolic surface with totally geodesic boundary is called *a pair of pants* if it is homeomorphic to a disk with two holes. Throughout this article, we fix a pair of pants P. Let α, β, γ denote the three boundary components of P. The interior of P will be denoted by S. We say a subset of S *fills* S if it intersects every geodesic in S.

Let τ_1, τ_2 be two curves in S. We say τ_1 and τ_2 have the same *topological type* if some self-homeomorphism of S maps τ_1 to τ_2. We use the Klein disk model Δ for hyperbolic plane. In this model, geodesics are straight Euclidean segments.

The following theorem is essential to understand the behavior of geodesics in hyperbolic surfaces, though we will not mention it in the proofs since it is used too frequently.

Theorem 5. (Gauss–Bonnet) *In the hyperbolic plane, an n-gon with angles $\alpha_1, \cdots, \alpha_n$ has area $(n-2)\pi - (\alpha_1 + \cdots + \alpha_n)$.*

For example, it follows that the curves τ_1, τ_2, τ_3 illustrated in Figure 1 cannot be geodesics.

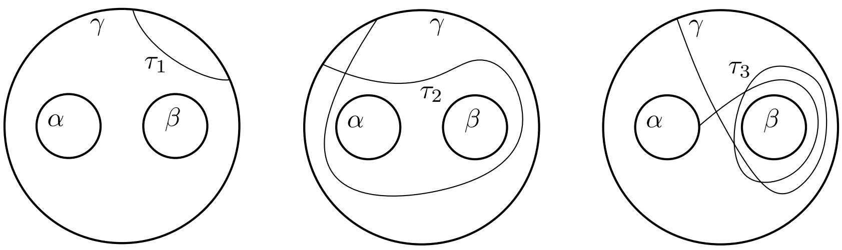


Figure 1. Since there is no 2-gon in the hyperbolic plane, it follows that τ_1 and τ_3 cannot be geodesics. To see that τ_2 cannot be a geodesic, we cut P along τ_2 and obtain a 3-gon, a 4-gon and a surface homeomorphic to a disk with two holes. If τ_2 is a geodesic, then the sum of the interior angles of the 4-gon is greater than 2π, contradicting the Gauss-Bonnet theorem.

3. A characterization of simple geodesics

First of all, we associate two *words* to every non-closed geodesic in S as follows.

As illustrated in the top left of Figure 2, let τ_a be the shortest geodesic from γ to α. We associate the words $p_\alpha p_\gamma$ and $p_\gamma p_\alpha$ to τ_a, and denote them by $\tau_a \approx p_\alpha p_\gamma$ and $\tau_a \approx p_\gamma p_\alpha$. Similarly we have $\tau_b \approx p_\alpha p_\beta$, $\tau_b \approx p_\beta p_\alpha$, $\tau_c \approx p_\beta p_\gamma$ and $\tau_c \approx p_\gamma p_\beta$.

Denote the connected components of $S - \tau_a \cup \tau_b \cup \tau_c$ by Q_1 and Q_2. Let $\tau \notin \{\tau_a, \tau_b, \tau_c\}$ be a non-closed geodesic in S. Let $r_\tau : (x_1, x_2) \to S$ be a smooth curve parametrized by arc length such that τ is the image of (x_1, x_2) under τ, where $x_1, x_2 \in [-\infty, \infty]$. Let I_τ be a subset of (x_1, x_2) such that

$$r_\tau(I_\tau) = \tau \cap (\tau_a \cup \tau_b \cup \tau_c).$$

Then the elements of I_τ can be listed as an increasing sequence

$$\cdots < s_{i-1} < s_i < s_{i+1} < \cdots .$$

For $s_i \in I_\tau$, set $t_i = a$ if r_τ goes across τ_a from Q_1 to Q_2 at $r_\tau(s_i)$, set $t_i = a^{-1}$ if r_τ goes across τ_a from Q_2 to Q_1 at $r_\tau(s_i)$. Similarly we have b, b^{-1}, c, c^{-1}; see the bottom left of Figure 2.

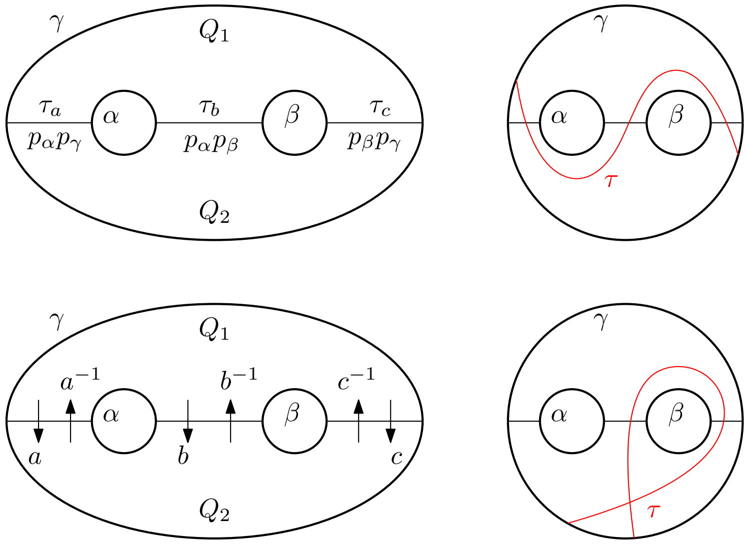


Figure 2. Words associated to geodesics.

We associate to τ a *word* $w_\tau = T_1 \cdots t_{i-1} t_i t_{i+1} \cdots T_2$ and write $\tau \approx w_\tau$, where T_1, T_2 are defined as follows. If $r_\tau(x)$ tends to a point in α as $x \to x_j$, we substitute $T_j = p_\alpha$. Similarly we have p_β and p_γ. If $r_\tau(x)$ does not tend to any point in ∂P as $x \to x_j$, we leave T_j empty. For example, we have $\tau \approx p_\gamma a b^{-1} c p_\gamma$ and $\tau \approx p_\gamma c^{-1} b a^{-1} p_\gamma$ in the top right of Figure 2, $\tau \approx p_\gamma c^{-1} b p_\gamma$ and $\tau \approx p_\gamma b^{-1} c p_\gamma$ in the bottom right of Figure 2, and $\tau \approx p_\alpha c a^{-1} (c b^{-1})^\infty$ and $\tau \approx (b c^{-1})^\infty a c^{-1} p_\alpha$ in Figure 3.

On the one hand, to simplify the notation, we write ∞_α instead of $(ab^{-1})^\infty$ and $(a^{-1}b)^\infty$, write ∞_β instead of $(bc^{-1})^\infty$ and $(b^{-1}c)^\infty$, and write ∞_γ instead of $(ac^{-1})^\infty$ and $(a^{-1}c)^\infty$. For example, we have $\tau \approx p_\alpha c a^{-1} c \infty_\beta$ in Figure 3. On the other hand, we regard $p_\alpha, p_\beta, p_\gamma$ as the ordinary endpoints of the geodesic, and $\infty_\alpha, \infty_\beta, \infty_\gamma$ as the ideal endpoints of the geodesic.

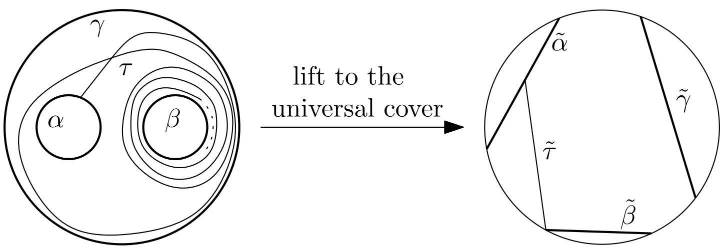
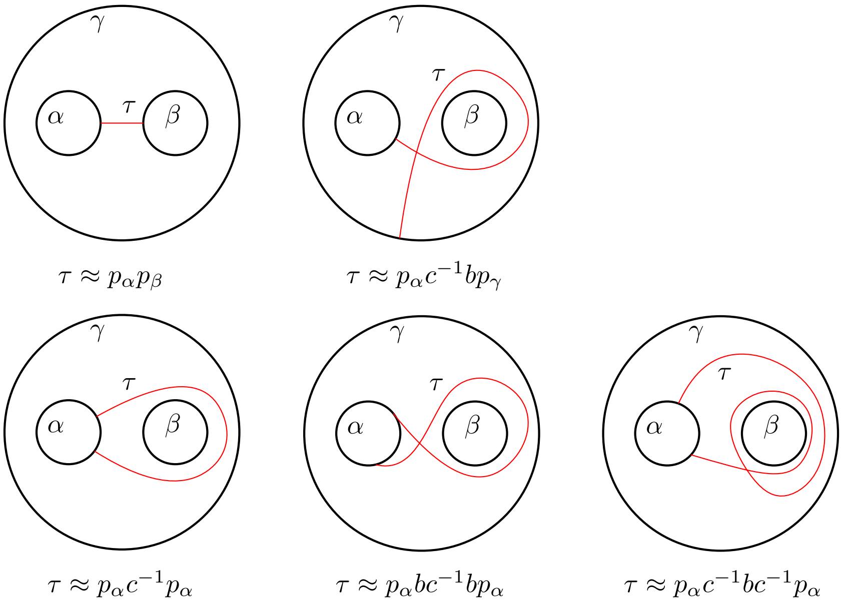


Figure 3. A geodesic $\tau \approx p_\alpha c a^{-1} c \infty_\beta$.

Remark 6. *Every non-closed geodesic is associated with two words. However, given an arbitrary word, there may be no geodesic associated with it. For example, any word in the form of "$\cdots ab \cdots$" is not associated to any geodesic. Nevertheless, geodesics associated with the same word are homotopic, simple geodesics associated with the same word are isotopic, and any word associated to a geodesic will be associated to a unique geodesic if the ordinary endpoints are given.*

Lemma 7. *Let τ be a geodesic in S. Then τ does not fill S if and only if τ has the same topological type as one of the following geodesics:*

(1) $\approx p_\alpha (c^{-1}b)^n p_\gamma$ *with exactly n self-intersections for some $n \geqslant 0$;*
(2) $\approx p_\alpha c^{-1} (bc^{-1})^n p_\alpha$ *with exactly n self-intersections for some $n \geqslant 0$;*
(3) $\approx p_\alpha b (c^{-1}b)^n p_\alpha$ *with exactly n self-intersections for some $n \geqslant 1$.*



Proof. To prove the "if" assertion, we only need to show that each geodesic listed in the lemma does not fill S.

We first check the case where

$$\tau \approx \infty_\alpha a c^{-1} a c^{-1} \infty_\beta = (ab^{-1})^\infty a c^{-1} a c^{-1} (bc^{-1})^\infty.$$

Notice that τ is a geodesic that has the same topological type as the geodesics $\approx p_\alpha c^{-1} b p_\gamma$ with one self-intersection. By cutting S along τ, we obtain an annulus and a 5-gon; see Figure 4. The 5-gon is convex, since p_1, p_2, p_3 are copies of p. In the annulus there is no geodesic disjoint from τ, but in the 5-gon there is exactly one geodesic connecting ∞_α to ∞_β that is disjoint from τ. Therefore, τ does not fill S.

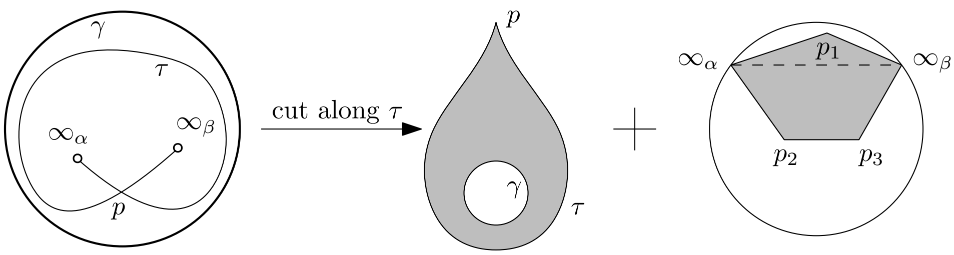


Figure 4. $\tau \approx \infty_\alpha a c^{-1} a c^{-1} \infty_\beta$.

Next we check the case where

$$\tau \approx p_\alpha c \infty_\beta = p_\alpha c(b^{-1}c)^\infty.$$

Notice that τ is a geodesic that has the same topological type as the simple geodesics $\approx p_\alpha p_\beta$. By cutting S along τ, we obtain an annulus whose boundary consists of γ and a geodesic triangle; see Figure 5. Let ρ be a geodesic from α to γ in the annulus. Then ρ is disjoint from τ. Therefore, τ does not fill S.

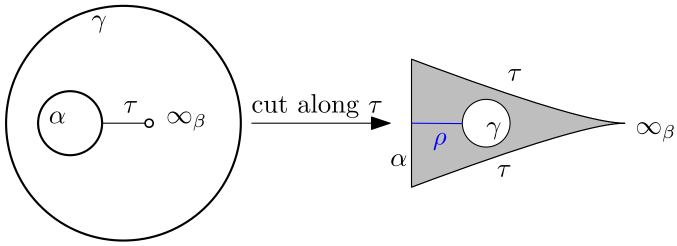


Figure 5. $\tau \approx p_\alpha c \infty_\beta$.

The other cases are similar. We check all the cases in Table 1.

topological type	without loss of generality we may assume $\tau \approx$	by cutting S along τ we obtain	where is the geodesic disjoint from τ
$p_\alpha(c^{-1}b)^n p_\gamma$	$p_\alpha p_\beta$	an annulus	in the annulus
	$p_\alpha c^{-1} b p_\gamma$	an annulus and a 7-gon	in the 7-gon
	$p_\alpha(c^{-1}b)^n p_\gamma$ $(n \geqslant 2)$	an annulus, a 3-gon, $(n-2)$ 4-gons and an 8-gon	in the 8-gon
$p_\alpha c^{-1}(bc^{-1})^n p_\alpha$	$p_\alpha c^{-1} p_\alpha$	two annuli	in the annuli
	$p_\alpha c^{-1} b c^{-1} p_\alpha$	two annuli and a 3-gon or 4-gon	in the annulus that has γ as one of its boundary components
	$p_\alpha c^{-1}(bc^{-1})^n p_\alpha$ $(n \geqslant 2)$	two annuli, a 3-gon, $(n-2)$ 4-gons and a 4-gon or 5-gon	
$p_\alpha b(c^{-1}b)^n p_\alpha$	$p_\alpha b c^{-1} b p_\alpha$	two annuli and a 3-gon	same as above
	$p_\alpha b(c^{-1}b)^n p_\alpha$ $(n \geqslant 2)$	two annuli, two 3-gon and $(n-2)$ 4-gons	

Table 1. The geodesics that do not fill S.

It remains to prove the "only if" assertion. Recall that the closed curve τ_8 illustrated in Figure 6 fills S. In fact, if τ contains two subsets that are circles homotopic to distinct boundary components of P, then τ fills S.

Suppose τ is a closed geodesic in S. Then τ is a geodesic with self-intersection, since the only simple closed geodesics in P are boundary components of P. Notice that there is no geodesic with self-intersection in a hyperbolic annulus. It follows that τ separates the boundary components of P. Thus every boundary component of P is homotopic to a circle contained in τ. Therefore τ fills S.

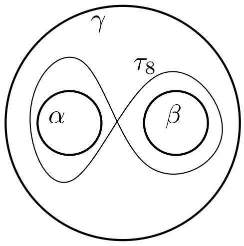


Figure 6. A closed curve that fills S.

A simple geodesic in S has the same topological type as a geodesic $\approx p_\alpha p_\beta$ or $p_\alpha c^{-1} p_\alpha$, so it does not fill S according to the "if" assertion we have just proved.

Finally, suppose τ is a non-closed geodesic in S that has self-intersections. Let $r_\tau\colon (x_1, x_2) \to S$ be a smooth curve parametrized by arc length such that τ is the image of (x_1, x_2) under τ, where $x_1, x_2 \in [-\infty, \infty]$. Let J_τ be the preimage of the set of self-intersections of τ under r_τ. Then there exist two distinct points $y_1, y_2 \in J_\tau$ such that:

- $r_\tau(y_1) = r_\tau(y_2)$;
- There is no point $z \in (y_1, y_2) \cap J_\tau$ such that $r_\tau(z) = r_\tau(y_1)$;
- There are no distinct points $z_1, z_2 \in (y_1, y_2) \cap J_\tau$ such that $r_\tau(z_1) = r_\tau(z_2)$.

In other words, the restriction of r_τ to $[y_1, y_2]$ is a circle. Without loss of generality, we may assume that this circle is freely homotopic to γ; see Figure 7.

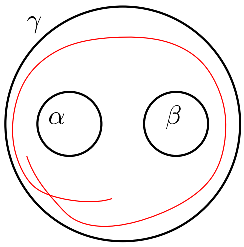


Figure 7. A self-intersection arises when τ wraps around γ.

Suppose τ does not fill S. Since τ has contained a circle homotopic to γ, it can not contain a circle homotopic to α or β. It follows that α and β are not separated by τ. Hence there exists an arc ρ homotopic to $\tau_b \approx p_\alpha p_\beta$ and disjoint from τ. Thus τ is contained in the annulus $S - \rho$. And all the possibilities of geodesics disjoint from ρ in the annulus must have the same topological type as one of the geodesics listed in the lemma. □

Lemma 8. *Let τ be a geodesic with self-intersection in S. Then there exists a geodesic $\rho \neq \tau$ such that every geodesic disjoint from τ is disjoint from ρ.*

Proof. If τ fills S, then no geodesic in S is disjoint from τ, and the condition is automatically satisfied. For the case that τ does not fill S, we need only check the geodesics listed in Lemma 7.

We first check the case where $\tau \approx p_\alpha (c^{-1}b)^2 p_\gamma$ is a geodesic with two self-intersections. By cutting S along τ, we obtain an annulus, a 3-gon and an 8-gon; see Figure 8. It follows that the union of all geodesics disjoint from τ is the 4-gon $D = p'_\alpha p_\gamma p'_\gamma p_\alpha$, and we can choose ρ to be a geodesic connecting p_α to β in $S - D$.

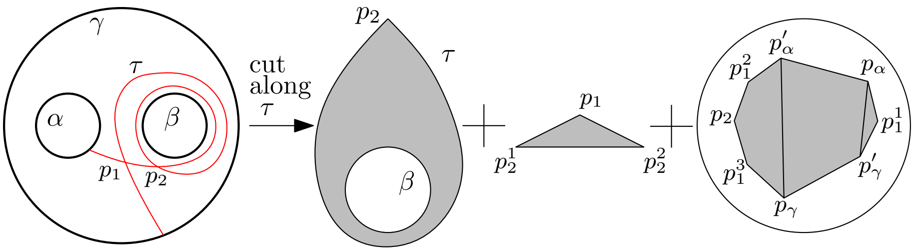


Figure 8. $\tau \approx p_\alpha(c^{-1}b)^2 p_\gamma$.

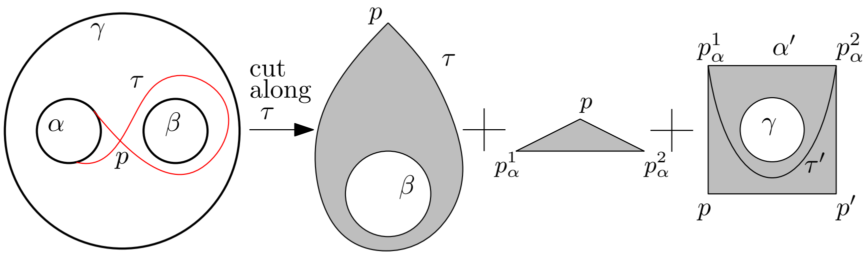


Figure 9. $\tau \approx p_\alpha bc^{-1}bp_\alpha$.

Next we check the case where $\tau \approx p_\alpha bc^{-1}bp_\alpha$. By cutting S along τ, we obtain a 3-gon and two annuli. The union D of all geodesics disjoint from τ is also an annulus. As illustrated in the right of Figure 9, the boundary components of D are γ and $\alpha' \cup \tau'$, where $\tau' \approx p_\alpha c p_\alpha$ is a geodesic having the same endpoints as τ. We can choose $\rho \approx p_\alpha p_\beta$ to be a geodesic in $S - D$.

The other cases are similar. We check all the cases in Table 2. □

topological type	without loss of generality we may assume $\tau \approx$	the union of all geodesics disjoint from τ is $D =$	a geodesic ρ in $S - D$
$p_\alpha(c^{-1}b)^n p_\gamma$	$p_\alpha c^{-1} b p_\gamma$	$p_\alpha p_\alpha p_\gamma p_\gamma$, a 4-gon in a 7-gon	connects p_α to β
	$p_\alpha(c^{-1}b)^n p_\gamma$ $(n \geqslant 2)$	$p_\alpha p_\alpha p_\gamma p_\gamma$, a 4-gon in an 8-gon	
$p_\alpha c^{-1}(bc^{-1})^n p_\alpha$	$p_\alpha c^{-1}(bc^{-1})^n p_\alpha$ $(n \geqslant 1)$	an annulus that has γ as one of its boundary components	
$p_\alpha b(c^{-1}b)^n p_\alpha$	$p_\alpha b(c^{-1}b)^n p_\alpha$ $(n \geqslant 1)$		

Table 2. The geodesics with self-intersection that do not fill S.

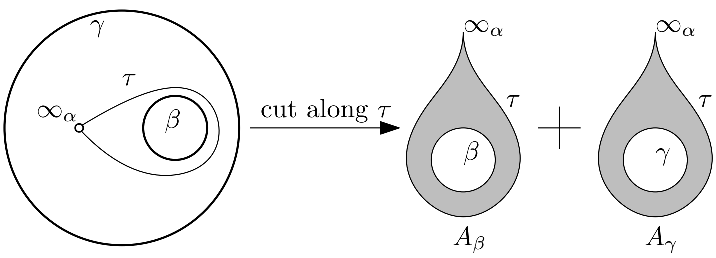


Figure 10. $\tau \approx \infty_\alpha ac^{-1}b\infty_\alpha$.

We can classify the simple geodesics in S as follows:

set	every element in the set		example
$G(\infty)$	has the same topological type as a geodesic $\approx p_\alpha c p_\alpha$	does not reach the boundary of P	$\infty_\alpha a c^{-1} b \infty_\alpha$
$G(p)$		reaches the boundary of P in both directions, i.e. of finite length	$p_\alpha c p_\alpha$
$G(p,p')$	has the same topological type as a geodesic $\approx p_\alpha p_\beta$	same as $G(p)$	$p_\alpha p_\beta$
$G(p,\infty')$		reaches the boundary of P in exactly one direction	$p_\alpha \infty_\beta$
$G(\infty,\infty')$		does not reach the boundary of P	$\infty_\alpha a c^{-1} \infty_\beta$

Table 3. Simple geodesics in S.

Theorem 9. *Let τ be a geodesic in S. Then τ is simple if and only if the union of all geodesics disjoint from τ intersects every geodesic in S except τ.*

Proof. The "if" assertion is just the contrapositive of Lemma 8, so it has already been proved.

Now assume τ is simple. We prove the "only if" assertion by checking all the cases in Table 3.

Suppose $\tau \in G(\infty)$. Without loss of generality we may assume $\tau \approx \infty_\alpha a c^{-1} b \infty_\alpha$. By cutting S along τ, we obtain two annuli A_β and A_γ; see Figure 10. In A_β the geodesic ρ illustrated in Figure 11 fills A_β. Thus we can find two geodesics disjoint from τ such that their union intersects every geodesic in S except τ.

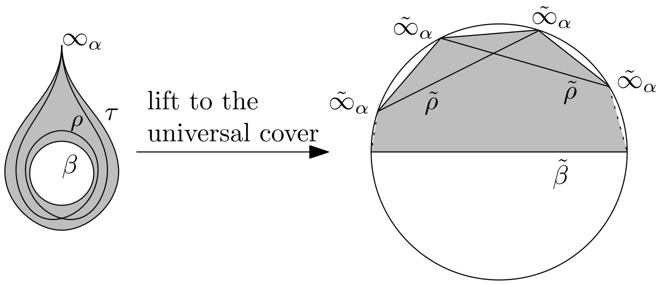


Figure 11.

Suppose $\tau \in G(p)$. Without loss of generality we may assume $\tau \approx p_\alpha c p_\alpha$ hits α at one point. By cutting S along τ, we obtain two annuli. Then the union of the geodesics ρ_1 and ρ_2 illustrated in Figure 12 intersects every geodesic in S except τ.

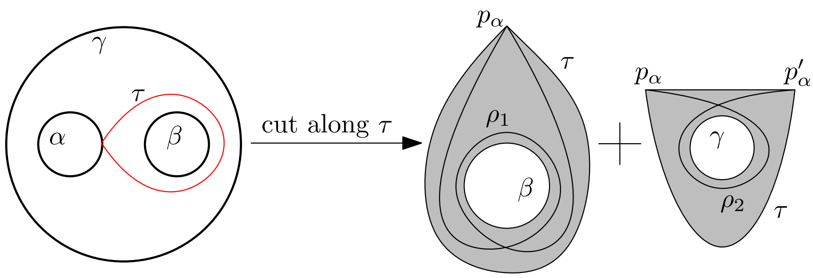


Figure 12. $\tau \approx p_\alpha c p_\alpha$.

Suppose $\tau \in G(p, p')$. Without loss of generality we may assume $\tau \approx p_\alpha p_\beta$. By cutting S along τ, we obtain an annulus. Then the union of the geodesics ρ_1 and ρ_2 illustrated in Figure 13 intersects every geodesic in S except τ.

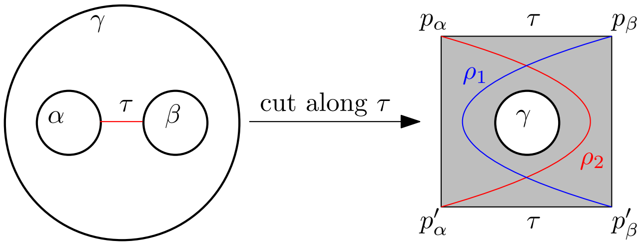


Figure 13. $\tau \approx p_\alpha p_\beta$.

Suppose $\tau \in G(p, \infty')$. Without loss of generality we may assume $\tau \approx p_\alpha \infty_\beta$. By cutting S along τ, we obtain an annulus. Then the geodesic ρ illustrated in Figure 14 intersects every geodesic in S except τ.

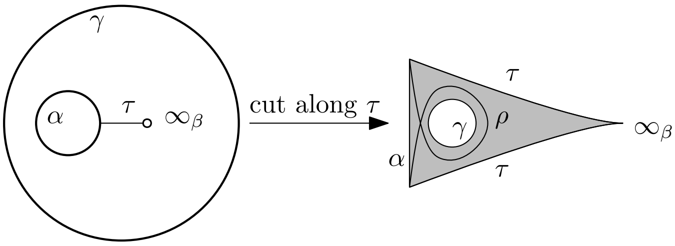


Figure 14. $\tau \approx p_\alpha \infty_\beta$.

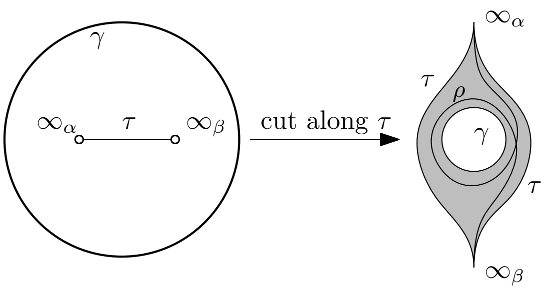


Figure 15. $\tau \approx \infty_\alpha a c^{-1} \infty_\beta$.

Suppose $\tau \in G(\infty, \infty')$. Without loss of generality we may assume $\tau \approx \infty_\alpha a c^{-1} \infty_\beta$. By cutting S along τ, we obtain an annulus. Then the geodesic ρ illustrated in Figure 15 intersects every geodesic in S except τ. □

Proposition 10. *Let τ be a simple geodesic in S. Then $\tau \in G(\infty, \infty') \cup G(p, \infty')$ if and only if there exists a geodesic ρ with self-intersection and disjoint from τ such that $\rho \cup \tau$ fills S.*

Proof. If $\tau \in G(\infty, \infty')$, without loss of generality, we may assume that $\tau \approx \infty_\alpha a c^{-1} \infty_\beta$. By cutting S along τ, we obtain an annulus. Then the geodesic ρ illustrated in Figure 15 is a desired geodesic.

If $\tau \in G(p, \infty')$, without loss of generality, we may assume $\tau = p_\alpha \infty_\beta$. By cutting S along τ, we obtain an annulus too. Then the geodesic ρ illustrated in Figure 14 is a desired geodesic.

It is easy to check that the other three kinds of simple geodesics do not satisfy the condition (see Figure 10, 12 and 13), and the proof is complete. □

Proposition 11. *Let τ be a simple geodesic in S. Then $\tau \in G(\infty) \cup G(p)$ if and only if there exist two geodesics ρ_1, ρ_2 with self-intersections such that ρ_1, ρ_2, τ are mutually disjoint.*

Proof. This is because $\tau \in G(\infty) \cup G(p)$ if and only if when we cut S along τ we obtain two annuli but not one. □

4. From bijection to isometry

Let $f\colon S \to S$ be a bijection such that f and f^{-1} map each geodesic onto some geodesic.

Theorem 9 implies that f maps each simple geodesic onto some simple geodesic. According to Proposition 10 and 11, $G(\infty, \infty') \cup G(p, \infty')$ and $G(\infty) \cup G(p)$ are invariant under f. It follows that $G(p, p')$ is also invariant under f.

Lemma 12. *The sets $G(p, p')$, $G(\infty, \infty')$, $G(p, \infty')$, $G(\infty)$ and $G(p)$ are invariant under f.*

Proof. If $\tau_1 \in G(\infty, \infty')$ and $\tau_2 \in G(p, \infty')$, then there exists $\rho \in G(p, p')$ that intersects τ_2 finitely many times. For example, if $\tau_2 \approx p_\alpha c \infty_\beta$, then we can choose $\rho \approx p_\alpha p_\gamma$. However, every element of $G(p, p')$ intersects τ_1 infinitely many times, a contradiction. Hence $G(\infty, \infty')$ and $G(p, \infty')$ are invariant under f.

If $\tau_1 \in G(\infty)$ and $\tau_2 \in G(p)$, then there exists $\rho \in G(p, p')$ that intersects τ_1 infinitely many times. For example, if $\tau_1 \approx \infty_\alpha a c^{-1} b \infty_\alpha$, then we can choose $\rho \approx p_\alpha p_\beta$. However, every element of $G(p, p')$ intersects τ_2 finitely many times, a contradiction. Hence $G(\infty)$ and $G(p)$ are invariant under f. □

We can classify the simple geodesics of finite length, up to homotopy with the boundary of P invariant, as follows:

set	$G(p_\alpha, p_\beta)$	$G(p_\alpha, p_\gamma)$	$G(p_\beta, p_\gamma)$	$G(p_\alpha)$	$G(p_\beta)$	$G(p_\gamma)$
the homotopy class of	$p_\alpha p_\beta$	$p_\alpha p_\gamma$	$p_\beta p_\gamma$	$p_\alpha c p_\alpha$	$p_\beta a p_\beta$	$p_\gamma b p_\gamma$

Lemma 13. *There exists a permutation $(\alpha', \beta', \gamma')$ of (α, β, γ) such that f maps $G(p_\alpha, p_\beta)$, $G(p_\alpha, p_\gamma)$, $G(p_\beta, p_\gamma)$ to $G(p_{\alpha'}, p_{\beta'})$, $G(p_{\alpha'}, p_{\gamma'})$, $G(p_{\beta'}, p_{\gamma'})$, respectively.*

Proof. Suppose $\tau_1 \in G(p_\alpha, p_\beta)$, $\tau_2 \in G(p_\beta, p_\gamma)$ and $\tau_3 \in G(p_\alpha, p_\gamma)$.

If $f(\tau_1), f(\tau_2), f(\tau_3) \in G(p_\alpha, p_\beta)$, then $\rho \approx p_\alpha \infty_\gamma$ intersects $f(\tau_1 \cup \tau_2 \cup \tau_3)$ finitely many times. But every element of $G(p, \infty')$ intersects $\tau_1 \cup \tau_2 \cup \tau_3$ infinitely many times, a contradiction.

If $f(\tau_1) \in G(p_\alpha, p_\beta)$ and $f(\tau_2), f(\tau_3) \in G(p_\alpha, p_\gamma)$, then $\rho \approx p_\alpha \infty_\beta$ intersects exactly one element of $\{f(\tau_1), f(\tau_2), f(\tau_3)\}$ infinitely many times. But every element of $G(p, \infty')$ intersects exactly two elements of $\{\tau_1, \tau_2, \tau_3\}$ infinitely many times, a contradiction.

Thus we conclude that $f(\tau_1)$, $f(\tau_2)$ and $f(\tau_3)$ belong to distinct sets. Hence f induces a permutation of $\{G(p_\alpha, p_\beta), G(p_\alpha, p_\gamma), G(p_\beta, p_\gamma)\}$, and the lemma follows. □

Suppose $\tau \in G(p_\alpha) \cup G(p_\beta) \cup G(p_\gamma)$. Then $\tau \in G(p_\alpha)$ if and only if τ intersects every element of $G(p_\beta, p_\gamma)$. Therefore f maps $G(p_\alpha)$ to $G(p_{\alpha'})$ by Lemma 13. Similarly, f must map $G(p_\beta)$ and $G(p_\gamma)$ to $G(p_{\beta'})$ and $G(p_{\gamma'})$, respectively.

Lemma 14. *Suppose $\tau \in G(p, p')$. Then for any ε-neighborhood $V(f(\tau))$ of $f(\tau)$ there is a neighborhood $U(\tau)$ of τ whose image under f is contained in $V(f(\tau))$.*

Proof. Without loss of generality we may assume $\tau \approx p_\alpha p_\beta$.

Let $\tau_2 \approx p_{\alpha'} p_{\gamma'}$, $\tau_3 \approx p_{\beta'} p_{\gamma'}$ be two geodesics such that $f(\tau)$, τ_2, τ_3 are mutually disjoint even in P. For any $\xi_2 \in G(p_{\alpha'}, p_{\gamma'})$ and $\xi_3 \in G(p_{\beta'}, p_{\gamma'})$, we set

$$L'(\xi_2, \xi_3) \coloneqq \{\rho \in G(p_{\alpha'}, p_{\beta'})\colon \rho, \xi_2, \xi_3 \text{ are mutually disjoint}\}.$$

For any set L of geodesics, we set $D(L) = \{x \in S\colon x \in l \text{ for some } l \in L\}$. Then $D(L'(\tau_2, \tau_3))$ is a closed domain that contains $f(\tau)$.

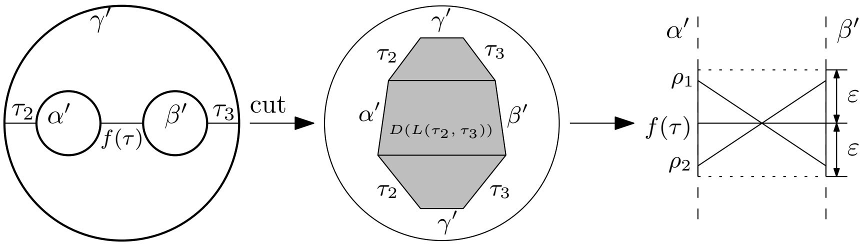


Figure 16.

Choose $\rho_1, \rho_2 \in G(p_{\alpha'}, p_{\beta'})$ in $V(f(\tau))$ as illustrated in Figure 16. Then choose $(\tau_{1,2}, \tau_{1,3}), (\tau_{2,2}, \tau_{2,3}) \in G(p_{\alpha'}, p_{\gamma'}) \times G(p_{\beta'}, p_{\gamma'})$ disjoint from $f(\tau) \cup \rho_1 \cup \rho_2$ such that

$$D\left(\bigcap_{i=1}^{2} L'(\tau_{i,2}, \tau_{i,3})\right) \subset V(f(\tau)).$$

For any $\xi_2 \in G(p_\alpha, p_\gamma)$ and $\xi_3 \in G(p_\beta, p_\gamma)$, we set

$$L(\xi_2, \xi_3) \coloneqq \{\rho \in G(p_\alpha, p_\beta)\colon \rho, \xi_2, \xi_3 \text{ are mutually disjoint}\}.$$

Now

$$\begin{aligned} U(\tau) &\coloneqq D\left(\bigcap_{i=1}^{2} L(f^{-1}(\tau_{i,2}), f^{-1}(\tau_{i,3}))\right) \\ &= D\left(\bigcap_{i=1}^{2} f^{-1}(L'(\tau_{i,2}, \tau_{i,3}))\right) = f^{-1}\left(D\left(\bigcap_{i=1}^{2} L'(\tau_{i,2}, \tau_{i,3})\right)\right) \end{aligned}$$

is a neighborhood of $\tau \cup f^{-1}(\rho_1) \cup f^{-1}(\rho_2)$, as desired. □

Recall that $\tau_a \approx p_\alpha p_\gamma$, $\tau_b \approx p_\alpha p_\beta$ and $\tau_c \approx p_\beta p_\gamma$ are the shortest geodesics joining the boundary components of P. By cutting S along $\tau_a \cup \tau_c$, we obtain an octagon, which we regard as a fundamental domain in the universal cover; see the shaded region in the middle of Figure 17.

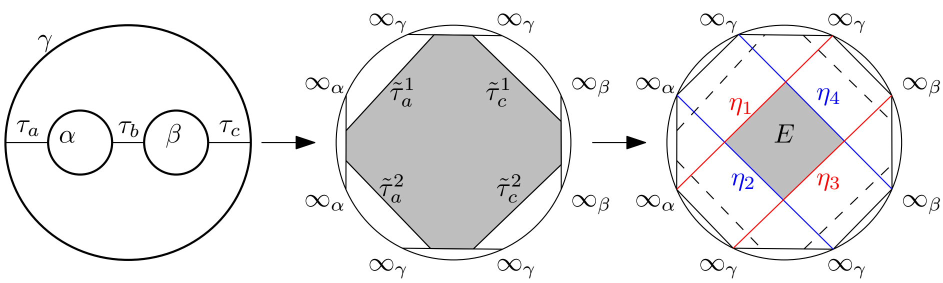


Figure 17. The symbols ∞_α, ∞_β and ∞_γ stand for the ideal endpoints of the lifts of α, β and γ, respectively.

As illustrated in the right of Figure 17, we take two geodesics η_1, η_2 connecting ∞_α to ∞_γ, and take two geodesics η_3, η_4 connecting ∞_β to ∞_γ. Then $\eta_1, \eta_2, \eta_3, \eta_4$ bound a quadrilateral E, which we regard as a closed domain in S.

Suppose $x \in S\backslash E$. It follows from the construction of E that there exist two geodesics $\tau_1, \tau_2 \in G(p, p')$ such that x is an intersection of τ_1 with τ_2. Notice that any neighborhood of x can be constructed as the intersection of a neighborhood of τ_1 with a neighborhood of τ_2. Since x is arbitrary, we conclude from Lemma 14 that $f\colon S\backslash E \to S\backslash f(E)$ is continuous. One can show similarly that $f^{-1}\colon S\backslash f(E) \to S\backslash E$ is continuous. Therefore $f\colon S\backslash E \to S\backslash f(E)$ is a homeomorphism.

Before proceeding further, we recall the following lemma and its proof for the convenience of the reader.

Lemma 15. *Let T_1, T_2 be two geodesic triangles in the hyperbolic plane. Suppose $h\colon T_1 \to T_2$ is a homeomorphism that maps each geodesic onto some geodesic. Then $h \in \mathrm{PGL}(3, \mathbf{R})$.*

Proof. By postcomposing with an element of $\mathrm{PGL}(3, \mathbf{R})$, we can assume that $T_1 = T_2$ and h fixes the vertices of T_1 and an interior point in T_1, namely A_1, A_2, A_3 and A_0 in Figure 18. Let A_4 be the intersection point of the edge A_2A_3 and the geodesic ray with initial point A_1 passing through A_0. Similarly we have A_5 and A_6. Note that A_4, A_5, A_6 are fixed points of h.

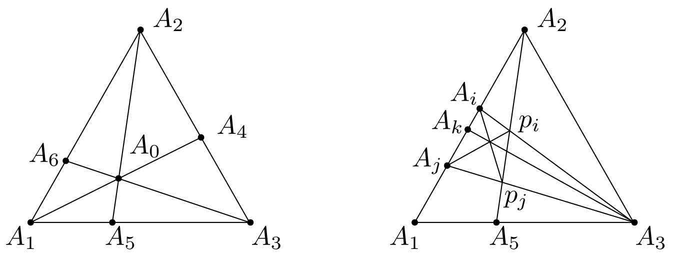


Figure 18. All intersection points are fixed points of h.

As illustrated in the right of Figure 18, suppose A_i and A_j are two distinct fixed points of h in the edge A_1A_2. Here we allow A_j to be A_1. Suppose the geodesic segment A_2A_5 intersects the geodesic segments A_iA_3 and A_jA_3 at p_i and p_j, respectively. Let A_k be the intersection point of the edge A_1A_2 and the geodesic ray with initial point A_3 passing through the intersection point of the geodesic segments A_ip_j and A_jp_i. Then A_k is a new fixed point of h.

Repeating the (similar) process, we can find more and more fixed points that ultimately are dense in the edges of T_1. Hence h must be the identity map on the edges of T_1, and finally $h\colon T_1 \to T_1$ is the identity map. □

Now consider the universal covering map $\pi\colon \widetilde{S} \to S$, where $\widetilde{S} \subset \Delta$. Let $\widetilde{E} = \pi^{-1}(E)$ and $\widetilde{f(E)} = \pi^{-1}(f(E))$. Then $f\colon S\backslash E \to S\backslash f(E)$ can be lifted to a homeomorphism $\widetilde{f}\colon \widetilde{S}\backslash\widetilde{E} \to \widetilde{S}\backslash\widetilde{f(E)}$ that maps geodesics to geodesics.

Suppose $A \subset \widetilde{S}\backslash\widetilde{E}$ is a geodesic triangle. Then $\widetilde{f}\colon A \to \widetilde{f}(A)$ is a homeomorphism between two geodesic triangles that maps geodesics to geodesics, and hence $\widetilde{f}|_A \in \mathrm{PGL}(3, \mathbf{R})$ by Lemma 15. Further, since $\widetilde{S}\backslash\widetilde{E}$ is path connected, we see that $\widetilde{f}\colon \widetilde{S}\backslash\widetilde{E} \to \widetilde{S}\backslash\widetilde{f(E)}$ belongs to $\mathrm{PGL}(3, \mathbf{R})$. Now $\partial\Delta$ is invariant under $\widetilde{f}$, hence $\widetilde{f}$ is an isometry. Therefore $f\colon S\backslash E \to S\backslash f(E)$ is an isometry.

In fact, by a similar argument we obtain:

Proposition 16. *A homeomorphism between the interiors of two hyperbolic surfaces of finite type is an isometry if it maps geodesics to geodesics.*

Since E and $f(E)$ are quadrilaterals, we can extend the isometry $f\colon S\backslash E \to S\backslash f(E)$ to an isometry $F\colon S \to S$. Suppose that $p \in S$. Then there is a family of geodesics $\{\tau_i\}_{i\in I}$ whose unique intersection is p. Since a geodesic in S is uniquely determined by its intersection with E, we see that $f(\tau_i) = F(\tau_i)$ for $i \in I$. Hence $f(p) = F(p)$. It follows that $f\colon S \to S$ is an isometry.

Finally, Theorem 3 follows from the fact that our proof works for bi-geodesic mapping $f\colon S_1 \to S_2$.

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