Annales Fennici Mathematici
Volumen 46, 2021, 739–769

HOW TO KEEP A SPOT COOL?

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Abstract. Let D be a planar domain, let a be a *reference* point fixed in D, and let b_k, $k = 1, \dots, n$, be n *controlling* points fixed in $D \setminus \{a\}$. Suppose further that each b_k is connected to the boundary ∂D by an arc l_k. In this paper, we propose the problem of finding a shape of arcs l_k, $k = 1, \dots, n$, which provides the minimum to the harmonic measure $\omega(a, \bigcup_{k=1}^n l_k, D \setminus \bigcup_{k=1}^n l_k)$. This problem can also be interpreted as a problem on the minimal temperature at a, in the steady-state regime, when the arcs l_k are kept at constant temperature T_1 while the boundary ∂D is kept at constant temperature $T_0 < T_1$.

In this paper, we mainly discuss the first non-trivial case of this problem when D is the unit disk $\mathbf{D} = \{z \colon |z| < 1\}$ with the reference point $a = 0$ and two controlling points $b_1 = ir$, $b_2 = -ir$, $0 < r < 1$. It appears, that even in this case our minimization problem is highly nontrivial and the arcs l_1 and l_2 providing minimum for the harmonic measure are not the straight line segments as it could be expected from symmetry properties of the configuration of points under consideration.

1. Introduction

Consider a devise consisting of an insulated planar plate D, a reference point a in D, and $n \geq 1$ distinct controlling points b_k in $D \setminus \{a\}$. Suppose that each controlling point b_k is connected to the boundary ∂D of D with a wire l_k. Suppose further that, to make our devise operational, we keep all wires l_k at constant temperature T_1 and that the temperature of the surrounding media along ∂D is $T_0 < T_1$.

We wish to know what shape of the wires l_k, $1 \leq k \leq n$, guarantees the minimal temperature at the reference point a after long enough period of time. Thus, we deal with a variational problem for the *steady-state heat distribution* in planar domains.

As is well known, the heat distribution is governed by the heat equation $\omega_t = k\Delta\omega$, which in the steady-state case (when $\omega_t = 0$) reduces to the Laplace equation

$$\Delta\omega = \omega_{xx} + \omega_{yy} = 0. \tag{1.1}$$

Solutions to the Laplace equations are harmonic functions. If ω is a solution to (1.1) with boundary values T_1 on $L = \cup_{k=1}^n l_k$ and T_0 on ∂D, then $\widetilde{\omega} = (\omega - T_0)/(T_1 - T_0)$ is a solution to (1.1) with boundary value 1 on L and 0 on ∂D. Therefore, every solution of our variational problem for the steady-state distribution of heat coincides up to scaling with the *harmonic measure* of the set $L = \bigcup_{k=1}^n l_k$, which we call a *configuration of wires*, with respect to a given domain D.

To get formal definitions, we assume that D is a domain on the complex plane $\mathbf{C}$ and that E is a Borel set on ∂D. Then $\omega(z, E, D)$ will denote the harmonic measure of E with respect to D. More precisely, $\omega(z, E, D)$ is a Perron solution to the Dirichlet problem in D with boundary values 1 on E and 0 on $E' = \partial D \setminus E$. If E is a closed set on $\overline{\mathbf{C}}$ and $z \in D \setminus E$, then $\omega(z, E, D)$ will denote the harmonic measure of the set $E \cap \partial(D_z)$ with respect to the connected component D_z of $D \setminus E$,

https://doi.org/10.5186/aasfm.2021.4648
2020 Mathematics Subject Classification: Primary 31A15, 30C85.
Key words: Heat distribution, harmonic measure, quadratic differential, symmetrization.

which contains z. In case when the domain D in the problem under consideration is fixed, we often use a shorter notation $\omega_E(z) = \omega(z, E, D)$. For more information on the harmonic measure, the reader may consult a monograph [9] by Garnett and Marshall, which contains many useful properties and applications.

Now, the problem on the shape which provides the minimum temperature at a given point can be stated as follows.

Problem P1. *Given a domain $D \subset \mathbf{C}$, a point $a \in D$, and a set $B = \{b_1, \dots, b_n\}$ of points $b_k \in D \setminus \{a\}$, let $\mathcal{L} = \mathcal{L}(a, B, D)$ denote the family of the all configurations of wires of the form $L = \bigcup_{k=1}^n l_k$, where l_k is a continuum in $D \setminus \{a\}$ connecting the point b_k to the boundary ∂D. The problem is to identify all possible sets $L^* = \bigcup_{k=1}^n l_k^*$ in $\mathcal{L}(a, B, D)$ such that*

$$\omega_{L^*}(a) = \inf \omega_L(a), \tag{1.2}$$

where the infimum is taken over all L in $\mathcal{L}(a, B, D)$.

If a domain in Problem $P1$ is simply connected then we can use the Riemann mapping function to transplant the problem into the unit disk $\mathbf{D} = \{z\colon |z| < 1\}$. Thus, in the case of simply connected domains we may assume that $D = \mathbf{D}$ and $a = 0$; see Figure 1, which presents an example of an admissible set of wires in this case.

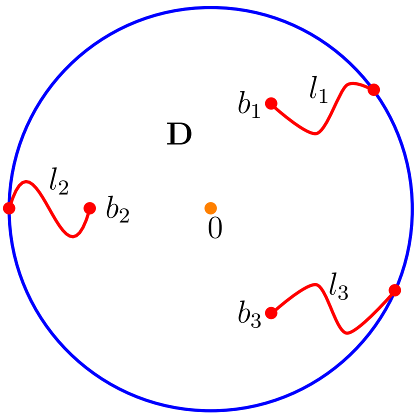

Figure 1. Round plate with three wires.

One particularly interesting case of Problem $P1$ in $\mathbf{D}$ is when the set B consists of n points equally distributed along the circle $C_r = \{z\colon |z| = r\}$, $0 < r < 1$. Assuming that $B = B_n(r)$, where $B_n(r) = \{b_{r,k} = re^{\pi i(2k-1)/n}, 1 \le k \le n\}$, we obtain the following special case of Problem P1, which we will call Problem P_n^* or Problem $P_n^*(r)$, when we want to emphasize dependence on r.

Problem P_n^*. *Given $0 < r < 1$, and $n \ge 1$, identify all possible configurations of wires $L^* = \bigcup_{k=1}^n l_k^*$ in $\mathcal{L}(0, B_n(r), \mathbf{D})$ such that $\omega(0, L^*, \mathbf{D}) = \omega_n^*(r)$, where*

$$\omega_n^*(r) = \inf \omega(0, L, \mathbf{D}) \tag{1.3}$$

and the infimum is taken over all L in $\mathcal{L}(0, B_n(r), \mathbf{D})$. It is also desirable to find explicit expression for the infimum $\omega_n^*(r)$ (when possible) or at least to find its asymptotic for fixed $0 < r < 1$ as $n \to \infty$ and asymptotic for fixed n as $r \to 1^-$ and as $r \to 0^+$.

Problem P_n^* was suggested by this author as a counterpart of the Gonchar problem on the maximal harmonic measure of n radial segments of the form $l(r, \alpha) = \{z = te^{i\alpha}\colon r \le t \le 1\}$, where $0 < r < 1$, $0 \le \alpha < 2\pi$. Gonchar conjectured that, for fixed

$n \geq 1$ and $0 < r < 1$, the maximum of the harmonic measure $\omega(0, \bigcup_{k=1}^{n} l(r, \alpha_k), \mathbf{D})$ of n radial segments of equal length occurs for symmetric configurations only; i.e. for rotations of the set $L_n(r) = \bigcup_{k=1}^{n} l_{r,k}$, where $l_{r,k} = \{te^{\pi i(2k-1)/n} \colon r \leq t \leq 1\}$. Using conformal mapping from the domain $\mathbf{D} \setminus L_n(r)$ onto $\mathbf{D}$, one can easily verify the following formula for the harmonic measure of the symmetric configuration $L_n(r)$:

$$(1.4) \qquad \omega(0, L_n(r), \mathbf{D}) = \frac{2}{\pi} \arcsin \frac{1 - r^n}{1 + r^n} = \frac{2}{\pi} \arccos \frac{2r^{n/2}}{1 + r^n} = 1 - \frac{4}{\pi} r^{n/2} + o(r^{n/2}).$$

Gonchar's conjecture was confirmed by Dubinin in [4] who, for this purpose, invented a new geometric transformation called *dissymmetrization*. Some remaining open question related to Gonchar's problem will be discussed in Section 10.

Motivated by Dubinin's solution of Gonchar's problem, this author suggested a conjecture that the symmetric configuration $L_n(r)$, extremal for the Gonchar problem, is extremal for Problem P_n^* as well. A more general motivation for this conjecture was the Julio Frederick Curie heuristic principle, which suggests that in the physical world "*symmetric assumptions imply symmetric consequences*". As an additional supporting point to our conjecture, we recall that for $n = 1$, this conjecture is a direct consequence of the well known Beurling's projection theorem for harmonic measure of a compact subset of the unit disk; see [3]. In general, problems whose solutions do not inherit some of the symmetry properties present in the assumptions of these problems (although they may inherit some other of these symmetry properties) are rare and present significant interest.

After I communicated this conjecture to A. Fryntov, he soon surprised me with a counterexample, which shows that for large n the n-symmetric set of segments $L_n(r)$ is not a minimizing set of wires for Problem P_n^*. This counterexample, which was published by Fryntov in [8] and was also mentioned in [25], can be constructed as follows.

For fixed $0 < r < 1$ and $n \geq 1$, consider sets

$$F_1(r, n) = \left\{ z = re^{i\theta} \colon \frac{\pi}{n} \leq \theta \leq 2\pi - \frac{\pi}{n} \right\} \cup [-1, -r]$$

and

$$F_2(r, n) = \left\{ z = re^{i\theta} \colon \frac{\pi}{n} \leq \theta \leq 2\pi - \frac{\pi}{n} \right\} \cup [-\infty, -r].$$

Let $D_1(r, n) = \mathbf{D} \setminus F_1(r, n)$, $D_2(r, n) = \mathbf{C} \setminus F_2(r, n)$. Then by the classical Carleman extension principle for the harmonic measure (see Chapter VIII, §4 in [10])

$$(1.5) \qquad \omega(0, F_1(r, n), D_1(r, n)) < \omega(0, F_1(r, n), D_2(r, n)).$$

Harmonic measure $\omega(0, F_1(r, n), D_2(r, n))$ can be calculated explicitly via conformal mapping from $D_2(r, n)$ onto the upper half-plane $\mathbf{H} = \{z \colon \Im z > 0\}$, see Section 2(c). For any fixed $0 < r < 1$, the right-hand side of (1.5) satisfies the following asymptotic relation, which was obtained in [25]:

$$(1.6) \qquad \omega(0, F_1(r, n), D_2(r, n)) = 1 - \frac{\sqrt{r}}{4(1 + r)} \frac{\pi}{n^2} + o(n^{-2}) \quad \text{as } n \to \infty.$$

Comparing (1.4), (1.5), and (1.6), we conclude (after Fryntov) that, for a fixed $0 < r < 1$,

$$\omega(0, F_1(r, n), D_1(r, n)) < \omega(0, L_n(r), \mathbf{D}) \quad \text{for all sufficiently large } n.$$

Our discussion above shows that symmetric configuration $L_n(r)$ is extremal for Problem P_n^* when $n = 1$ and is not extremal for this problem when n is large enough.

This leaves a possibility that $L_n(r)$ remains extremal for small $n > 1$. One of the goals of this paper is to show that $L_n(r)$ is not extremal even for the simplest nontrivial case of Problem P_n^* when $n = 2$. This break of at least one of the symmetries makes the problem more difficult and therefore more interesting.

In what follows, we mostly discuss questions related to Problem P_2^*, except for the last section, where we discuss related open problems. Since $L_2(r) = [ir, i] \cup [-i, -ir]$ is an admissible configuration of wires for Problem P_2^*, we may use formula (1.4) to obtain the following inequalities:

$$(1.7) \qquad 0 < \frac{2}{\pi} \arcsin \frac{1-r}{1+r} < \omega_2^*(r) \le \omega_{L_2(r)}(0) = \frac{2}{\pi} \arcsin \frac{1-r^2}{1+r^2} < 1$$

for all $0 < r < 1$. The second inequality in (1.7) follows from the Beurling projection theorem for the harmonic measure, which was mentioned above.

The rest of the paper has the following structure. In Section 2, we discuss several examples where the harmonic measure can be calculated explicitly. Some of the formulas presented in this section will be needed for our proofs. In Section 3, we show that for every r, $0 < r < 1$, there exists at least one configuration of wires L^* in $\mathcal{L}(0, B_2(r), \mathbf{D})$ extremal for Problem P_2^*. In Section 4, we prove that every extremal configuration L^* possesses certain symmetry properties. For these purposes, some symmetrization transformations will be used.

Then we will distinguish two cases of Problem P_2^*, when an admissible configuration of wires $L = l_1 \cup l_2$ is *connected* and when it is *disconnected*. These two possibilities lead to two auxiliary problems. The first of these auxiliary problems, discussed in Section 5, deals with pairs of wires l_1 and l_2, which have a common *contact point* on $\mathbf{T}$. In this case, the set $L = l_1 \cup l_2$ is connected.

The second auxiliary problem, discussed in Sections 6 and 7, deals with disjoint wires l_1 and l_2, which are symmetric to each other with respect to the real axis. To introduce this problem, we first give necessary definitions.

We recall that a *quadrilateral* $Q = Q[z_1, z_2, z_3, z_4]$ is a configuration consisting of a simply connected domain Q with four distinct boundary points z_1, z_2, z_3, z_4, called *vertices*, marked on ∂Q. We assume here that the vertices z_k are enumerated in the positive direction on ∂Q and also assume cyclic agreement, i.e., that $z_5 = z_1$, $z_0 = z_4$. The boundary arcs $\alpha_k \subset \partial Q$ connecting the vertices z_k and z_{k+1} are *sides* of Q. The sides α_1 and α_3 will be called *horizontal* and the sides α_2 and α_4 will be called *vertical*. Then $\mathrm{mod}(Q)$ will denote the module of Q with respect to the family of curves $\gamma \subset Q$ connecting vertical sides of Q.

If the wires l_1 and l_2 are disjoint, then the domain $Q = \mathbf{D} \setminus (l_1 \cup l_2)$ can be considered as a quadrilateral, which has l_1 and l_2 as a pair of its horizontal sides. The complementary boundary set $\partial Q \setminus (\overline{l_1 \cup l_2})$ consists of two arcs, which we denote by l_1' and l_2', of the unit circle. These two arcs l_1' and l_2' are vertical sides of Q. Then our second auxiliary problem is the following. For given $m > 0$, find the minimum of the harmonic measures $\omega(0, l_1 \cup l_2, \mathbf{D})$ over all pairs of wires l_1, l_2, which are symmetric to each other with respect to the real axis and such that $\mathrm{mod}(\mathbf{D} \setminus (l_1 \cup l_2)) = m$.

An alternative way to treat the second auxiliary problem is to consider the triple $(l_1, l_2, \mathbf{D})$ as a condenser with plates $l_+ = l_1$ and $l_- = l_2$ carrying positive and negative charges, respectively. Then our second auxiliary problem can be restated as a problem on the minimal temperature at the origin among all symmetric condensers of the form $(l_+, l_-, \mathbf{D})$, which have a prescribed capacity. We assume here that a condenser at work supports temperature 1 on the plates l_+ and l_- and temperature 0 on $\partial\mathbf{D}$.

In Section 6, we consider an auxiliary problem on the extremal partitioning of the unit disk into three quadrilaterals, which we need for our proofs in Section 7. This problem is a particular case of Jenkins' module problem, which has found numerous applications in different areas of mathematics; see [14], [29], [27], and [28]. Then, in Section 7, we discuss and solve the problem on the minimal harmonic measure in quadrilaterals of a fixed module m.

In Section 8, we derive systems of equations, whose solutions can be used to calculate the minimal harmonic measure numerically. In Section 9, we discuss how the change in the harmonic measure is related to the change in the module of related quadrilaterals. Finally, in Section 10, we present some open problems and questions for future discussion.

2. Examples of harmonic measures and related moduli

In this section, we find explicit formulas for the harmonic measures and related quantities in a few simple cases. Some of these formulas are needed for our proofs and discussions in further sections.

(a) The upper half-plane H provides a basic example for calculation of the harmonic measures. If a, b are real such that $a < b$ and if $\Im z > 0$, then it follows directly from the definition of the harmonic measure that

$$\omega(z, [a,b], \mathbf{H}) = \frac{1}{\pi} \arg \frac{z-b}{z-a}. \tag{2.1}$$

Applying (2.1) to the set consisting of segments $[1, 1/k]$ and $[-1/k, -1]$, where $0 < k < 1$, we obtain

$$\omega(it, [-1/k, -1] \cup [1, 1/k], \mathbf{H}) = \frac{2}{\pi} \arctan \frac{(1-k)t}{1+kt^2}, \quad t > 0. \tag{2.2}$$

(b) Symmetric radial slits. For $0 < r < \rho \le 1$ and $n \ge 1$, let $L_n(r,\rho) = \cup_{k=1}^n l_k(r,\rho)$, where $l_k(r,\rho) = \{z = te^{\frac{\pi i(2k+1)}{n}} : r \le t \le \rho\}$, $1 \le k \le n$. To find the harmonic measure $\omega(0, L_n(r,\rho), \mathbf{D} \setminus L_n(r))$, we first note that the function $\varphi_1(z) = z^n$ is an n to 1 mapping from $\mathbf{D} \setminus L_n(r)$ onto $\mathbf{D} \setminus [-1, -r^n]$. This implies that

$$\omega(0, L_n(r,\rho), \mathbf{D} \setminus L_n(r)) = \omega(0, [-\rho^n, -r^n], \mathbf{D} \setminus [-1, -r^n]). \tag{2.3}$$

To find the harmonic measure in the right-hand side of this equation, we note that the function

$$\varphi_2(z) = i\sqrt{\frac{z}{(1-z)^2} + \frac{r^n}{(1+r^n)^2}}$$

maps $\mathbf{D} \setminus [-1, -r^n]$ conformally onto $\mathbf{H}$ such that $\varphi_2(0) = \frac{ir^{n/2}}{1+r^n}$ and $\varphi([-\rho^n, -r^n]) = [-a, a]$, where

$$a = \frac{\sqrt{(\rho^n - r^n)(1 - r^n\rho^n)}}{(1+r^n)(1+\rho^n)}.$$

Now, using (2.1), we find

$$\omega(0, L_n(r,\rho), \mathbf{D} \setminus L_n(r)) = \frac{2}{\pi} \arccos \left(\frac{r^{n/2}}{1+r^n} : \frac{\rho^{n/2}}{1+\rho^n} \right). \tag{2.4}$$

For $\rho = 1$, the latter equation implies (1.4).

In the special case when $n = 2$, the domain $Q = \mathbf{D} \setminus L_2(r)$ can be considered as a quadrilateral with segments $[-i, -ir]$ and $[ir, i]$ as its pair of horizontal sides. It is left to the interested reader to verify that the module of this quadrilateral can be calculated as

$$m_2(r) = \frac{2\mathcal{K}(r^2)}{\mathcal{K}'(r^2)} = \frac{\pi}{2}\frac{1}{\log(1/r)} - \frac{\pi \log 2}{2}\frac{1}{(\log(1/r))^2} + o(\frac{1}{(\log(1/r))^2}. \tag{2.5}$$

Here and below $\mathcal{K}(\tau) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\tau^2x^2)}}$ and $\mathcal{K}'(\tau) = \mathcal{K}(\sqrt{1-\tau^2})$ denote the complete elliptic integrals of the first kind. The asymptotic expansion in (2.5) follows from the well-known series expansions for the complete elliptic integrals given by formulas (8.113) in [11].

(c) Circular slit. For $0 < r < 1$ and $0 < \varepsilon < \pi$, let $E(r, \varepsilon) = \{z = re^{i\theta} : \varepsilon \le \theta \le 2\pi - \varepsilon\} \cup [-1, -r]$ and let $G(r, \varepsilon) = \mathbf{C} \setminus (E(r, \varepsilon) \cup (-\infty, -1])$. To find $\omega(0, E(r, \varepsilon), G(r, \varepsilon))$, we consider a function $\varphi_0 = \varphi_3 \circ \varphi_2 \circ \varphi_1$ with φ_1, φ_2, φ_3 defined by equations:

$$\varphi_1(z) = \frac{r-z}{r+z}, \quad \varphi_2(z) = \sqrt{z^2 + \cot^2(\varepsilon/2)}, \quad \varphi_3(z) = i\sqrt{\frac{z - \cot(\varepsilon/2)}{z + \csc(\varepsilon/2)}}. \tag{2.6}$$

Then it is a standard exercise in conformal mappings to check that $\varphi_0(z)$ maps $G(r, \varepsilon)$ conformally onto $\mathbf{H}$ such that $\varphi_0(0) = i \sin(\varepsilon/4)$. Using explicit expressions (2.6) and formula (2.1), after routine calculations one can find that

$$\begin{aligned} &\omega(0, E(r, \varepsilon), G(r, \varepsilon)) \\ &= 1 - \frac{2}{\pi}\arctan\left(\sin\frac{\varepsilon}{4}\sqrt{\frac{1 + r - \sqrt{1 + 2r\cos\varepsilon + r^2}}{(1+r)\cos\frac{\varepsilon}{2} + \sqrt{1 + 2r\cos\varepsilon + r^2}}}\right). \end{aligned} \tag{2.7}$$

Taking two term Taylor approximation of (2.7) with $\varepsilon = \pi/n$, we obtain the asymptotic expansion shown in (1.6).

(d) Harmonic measure in quadrilaterals and rectangles. Let $Q = Q[z_1, z_2, z_3, z_4]$ be a quadrilateral with horizontal sides α_1 and α_2. It is well known that there is a univalent function $f(z)$ which maps Q conformally onto a rectangle of the form $R(m) = \{z : |\Re z| < 1, |\Im z| < m\}$ with some $m > 0$ such that the vertices z_1, z_2, z_3, z_4 are mapped to the geometric vertices $-1 - im$, $1 - im$, $1 + im$, $-1 + im$, respectively. It is also well known that the parameter m is defined uniquely by the condition $m = \mathrm{mod}(Q)$. Since the harmonic measure is conformally invariant we have

$$\omega(z, \alpha_1 \cup \alpha_2, Q) = \omega(f(z), [-1 - im, 1 - im] \cup [-1 + im, 1 + im], R(m)).$$

Thus, to study the harmonic measure in quadrilaterals, we may work with the harmonic measure in appropriate rectangles. In particular, we are interested in the formula for the harmonic measure $\omega(t, h_- \cup h_+, R(m))$, $-1 < t < 1$, where $h_- = \{z = \tau - im : -1 < \tau < 1\}$ and $h_+ = \{z = \tau + im : -1 < \tau < 1\}$ are horizontal sides of $R(m)$.

Let $f_0(z)$ denote the elliptic function

$$f_0(z) = \mathrm{sn}((i/2)\mathcal{K}'(k)z + (i/2)\mathcal{K}'(k), k)$$

with $0 < k < 1$ defined by the equation

$$\frac{2\mathcal{K}(k)}{\mathcal{K}'(k)} = m.$$

It follows from the well-know properties of elliptic functions that $f_0(z)$ maps $R(m)$ conformally onto the upper half-plane such that $f_0(h_-) = [1, 1/k]$ and $f_0(h_+) = [-1/k, -1]$. Since the harmonic measure is conformally invariant, we may use (2.2) to obtain the following formula for the harmonic measure of horizontal sides of $R(m)$:

$$\omega(t, h_- \cup h_+, R(m)) = \frac{2}{\pi} \arctan \frac{i(k-1)\mathrm{sn}((i/2)\mathcal{K}'(k)t + (i/2)\mathcal{K}'(k), k)), k)}{1 - k\,\mathrm{sn}^2((i/2)\mathcal{K}'(k)t + (i/2)\mathcal{K}'(k), k)}. \tag{2.8}$$

In particular, for $t = 0$, this implies the following formula for the harmonic measure of the rectangle $R(m)$ at its center $z = 0$:

$$\omega = \omega(0, h_- \cup h_+, R(m)) = \frac{2}{\pi}\mathrm{arccot}\frac{2\sqrt{k}}{1-k} = \frac{2}{\pi}\arccos\frac{2\sqrt{k}}{1+k}. \tag{2.9}$$

Solving (2.9) for k, we obtain

$$k = \frac{1 - \sin\frac{\pi\omega}{2}}{1 + \sin\frac{\pi\omega}{2}}.$$

Substituting this into (2.8) and then simplifying, we obtain the following more direct relation between the module m of the rectangle $R(m)$ and the harmonic measure ω of its horizontal sides evaluated at the center of $R(m)$:

$$\frac{\mathcal{K}'(\sin\frac{\pi\omega}{2})}{\mathcal{K}(\sin\frac{\pi\omega}{2})} = m. \tag{2.10}$$

(e) Harmonic measure and the module of a triad. According to Jenkins, a triad (D, α, z_0) is a configuration consisting of a simply connected domain D, an open boundary arc α on ∂D, and a point $z_0 \in D$. Then the module $\mathrm{mod}(D, \alpha, z_0)$ of the triad (D, α, z_0) is defined to be the maximal module among all quadrilaterals $Q \subset D\setminus\{z_0\}$, which have its horizontal pair of sides on the complementary boundary arc $\partial D \setminus \alpha$ and separate α from z_0 inside D. The harmonic measure $\omega_t = \omega(z_0, \alpha, D)$ and the module $m_t = \mathrm{mod}(D, \alpha, z_0)$ of a triad are related via the following formula due to Hersch [13]:

$$\frac{\mathcal{K}'(\sin\frac{\pi\omega_t}{2})}{\mathcal{K}(\sin\frac{\pi\omega_t}{2})} = 2m_t. \tag{2.11}$$

The similarity of formulas (2.10) and (2.11) is not coincidental. Indeed, let Q^* be the quadrilateral of the maximal module for the triad (D, α, z_0). Then there is a function $w = f(z)$ which maps Q^* conformally onto a rectangle $R^+(m_t) = \{w\colon -1 < \Re w < 1,\, 0 < \Im w < m_t\}$ such that the horizontal sides of Q^* corresponds to the horizontal sides of $R^+(m_t)$ and $f(z_0) = 0$. It follows from the conformal invariance of the harmonic measure and from the Schwarz reflection principle for harmonic functions that

$$\omega(z_0, \alpha, D) = \omega(0, [-1 - im_t, 1 - im_t] \cup [-1 + im_t, 1 + im_t], R(m_t)). \tag{2.12}$$

Since $\mathrm{mod}(R(m_t)) = 2\,\mathrm{mod}(Q^*)$, (2.11) follows from (2.10) and (2.12).

Equation (2.11) implies that the harmonic measure $\omega(z_0, \alpha, D)$ is a strictly decreasing function of the module of the triad (D, α, z_0). An advantage of relation (2.11) is that the module of a triad usually admits easier estimates in terms of the geometric characteristics of D and α than the harmonic measure itself.

3. Existence of extremal configurations of wires

Lemma 1. *For every $0 < r < 1$, there exists at least one configuration of wires $L(r) = l_+(r) \cup l_-(r)$ in $\mathcal{L}(0, B_2(r), \mathbf{D})$ that is extremal for Problem P_2^*.*

Proof. Let $L_k = l_1^k \bigcup l_2^k$, $k = 1, 2, \ldots$, be a sequence of sets such that $\omega_{L_k}(0) \to \omega_2^*(r)$ as $k \to \infty$. Let D_k be a connected component of $\mathbf{D} \setminus L_k$ containing 0 and let f_k be a conformal mapping from $\mathbf{D}$ onto D_k such that $f_k(0) = 0$. The set $\{f_k\}_{k=1}^\infty$ is a normal family of analytic functions. Therefore, taking a subsequence if necessary, we may assume that $f_k \to f_0$ uniformly on compact subsets of $\mathbf{D}$, where f_0 is analytic on $\mathbf{D}$ such that $f_0(0) = 0$. Since $\omega_2^*(r) < 1$, it follows from the Nevanlinna–Beurling projection lemma (see Lemma 10 in [1]) that there is r_0, $0 < r_0 < 1$ such that $\mathbf{D}_{r_0} \subset D_k = f_k(\mathbf{D})$ for all k. This implies that f_0 is not constant and therefore f_0 is univalent by Hurwitz theorem.

Taking a subsequence, if necessary, we may assume that either each L_k is disconnected or each L_k is connected.

Assume first that the sets L_k are disconnected for all k. Then $\mathbf{T} \setminus L_k$ contains exactly two open arcs, say Γ_1^k and Γ_2^k, lying on ∂D_k. Precomposing f_k with rotation if necessary, we may assume that for $j = 1, 2$, $\Gamma_j^k = f_k(\gamma_j^k)$, where $\gamma_1^k = \{\zeta = e^{i\theta} \colon 0 < \theta < \varphi_1^k\}$, $\gamma_2^k = \{z = e^{i\theta} \colon \alpha_2^k < \theta < \alpha_3^k\}$ with some α_1^k, α_2^k, α_3^k such that $0 < \alpha_1^k < \alpha_2^k < \alpha_3^k < 2\pi$ for $k = 1, 2, \ldots$. Taking a subsequence if necessary, we may assume that for $j = 1, 2, 3$, $\alpha_j^k \to \alpha_j^0$ as $k \to \infty$. Then $0 \le \alpha_1^0 \le \alpha_2^0 \le \alpha_3^0 \le 2\pi$ and

$$2\pi \lim_{k\to\infty} \omega_{L_k}(0) = 2\pi - \alpha_3^0 + \alpha_2^0 - \alpha_1^0 = 2\pi\omega_2^*(r).$$

Since $\omega_2^*(r) < 1$ by (1.7), the latter implies that at least one of the limit arcs $\gamma_1^0 = \{z = e^{i\theta} \colon 0 < \theta < \alpha_1^0\}$ and $\gamma_2^0 = \{z = e^{i\theta} \colon \alpha_2^0 < \theta < \alpha_3^0\}$ is not empty. Since $|f_k(e^{i\theta})| = 1$ for $e^{i\theta} \in \gamma_j^k$ and all k, the Schwarz reflection principle implies that f_k can be analytically continued across γ_j^k. If $\gamma_j^0 \ne \emptyset$, the latter implies that f_0 can be analytically continued across γ_j^0 and $|f_0(e^{i\theta})| = 1$ for all $e^{i\theta} \in \gamma_j^0$. Therefore,

$$2\pi\omega(0, f_0(\gamma_1^0 \cup \gamma_2^0), f(\mathbf{D})) = \alpha_1^0 + \alpha_3^0 - \alpha_2^0 = 2\pi(1 - \omega_2^*(r)). \tag{3.1}$$

Since $f_0(\mathbf{D})$ is simply connected and f_0 omits the points $\pm ir$ in $\mathbf{D}$, it follows that there are continua l_1^0 and l_2^0 in $\overline{\mathbf{D}} \setminus (f_0(\mathbf{D} \cup \gamma_1^0 \cup_2^0))$ such that l_1^0 joins ir and $\mathbf{T}$ and l_2^0 joins $-ir$ and $\mathbf{T}$. It follows from (3.1) that

$$\omega(0, l_1^0 \cup l_2^0, \mathbf{D}) \le \omega(0, \partial f_0(\mathbf{D} \setminus f_0(\gamma_1^0 \cup \gamma_2^0), f_0(\mathbf{D}) \le \omega_2^*(r).$$

Since $l_1^0 \cup l_2^0$ is an admissible set for Problem P_2^*, we must have $\omega(0, l_1^0, l_2^0, \mathbf{D}) = \omega_2^*(r)$, which proves the lemma in the case under consideration.

In the case when all sets L_k are connected, the proof follows the same lines. □

Remark 1. The existence proof given above is standard. It can be easily modified to show that if D is finitely connected, then there is at least one configuration of wires extremal for Problem $P1$.

Remark 2. One of the consequences of Fryntov's construction of a counterexample mentioned in the introduction is that, for large n, an extremal configuration for Problem P_n^* is not unique. Indeed, if it is unique then it must possess the same group of symmetries as the set of controlling points $B_n(r)$ possesses. In the latter case, straightforward application of Carleman's extension principle for the harmonic measure (see Chapter VIII, §4 in [10]) implies that an extremal configuration must coincide with $L_n(r)$, but that is not the case for large n.

As we will see later, an extremal configuration of Problem P_n^* is not unique even in the simplest case when $n = 2$.

If we assume that the existence result stated in Remark 1 is established, then the following lemma is an immediate corollary of Carleman's extension principle.

Lemma 2. *For fixed $n \geq 1$, the minimal harmonic measure $\omega_n^*(r)$ strictly decreases from 1 to 0 as r runs from 0 to 1.*

Proof. Suppose that $0 < r_1 < r_2 < 1$ and suppose that $L = \bigcup_{k=1}^n l_k$ is an admissible configuration of wires for the problem $P_n^*(r_1)$. Let $\widetilde{l}_k$ be the shortest subarc of l_k connecting $b_{r,k}$ to the circle $C_s = \{z\colon |z| = s\}$. Then the set $\widetilde{L} = \{z : sz \in \cup_{k=1}^n \widetilde{l}_k\}$ is an admissible configuration of wires for the problem $P_n^*(r_2)$. It follows from the conformal invariance of the harmonic measure and Carleman's extension principle that

$$\omega(0, \widetilde{L}, \mathbf{D}) = \omega(0, \textstyle\bigcup_{k=1}^n \widetilde{l}_k, \mathbf{D}_s) < \omega(0, L, \mathbf{D}), \tag{3.2}$$

where $\mathbf{D}_s = \{z\colon |z| < s\}$. The latter inequality implies the non-strict inequality $\omega_n^*(r_2) \leq \omega_n^*(r_1)$. Thus, to prove non-strict monotonicity of $\omega_n^*(r)$, we do not need to know that an extremal configuration of wires does exist.

Assuming existence of an extremal configuration of wires for the problem under consideration and letting L to be one of those extremal configurations, we conclude from (3.2) that $\omega_n^*(r)$ strictly decreases in the interval $0 < r < 1$. □

4. Symmetries of extremal configurations of wires

To prove Lemma 3 below, we will use a geometric transformation called *polarization*. This transformation, for many useful properties of which the reader may consult a recent book [6] and papers [5], [23], [25], [28], can be defined as follows, see [5]. Let L be a directed straight line on $\mathbf{C}$ and let H^+ and H^- be the left half-plane and the right half-plane with respect to L, respectively. For $z \in \overline{\mathbf{C}}$, let z^* denote the reflection of z in L. The polarization E_L of a given set $E \subset \overline{\mathbf{C}}$ into H^+ (or with respect to L) is then defined by

$$E_L = ((E \cup E^*) \cap \overline{H}^+) \cup ((E \cap E^*) \cap H^-), \tag{4.1}$$

where $E^* = \{z\colon z^* \in E\}$.

Lemma 3. *Let $L(r)$ be an extremal configuration of wires of Problem P_2^*. Then $L(r)$ lies either in the closed half-disk $\overline{\mathbf{D}}_- = \{z \in \overline{\mathbf{D}}\colon \Re z \leq 0\}$ or in the closed half-disk $\overline{\mathbf{D}}_+ = \{z \in \overline{\mathbf{D}}\colon \Re z \geq 0\}$.*

Proof. Let $L^-(r)$ be the polarization of $L(r)$, defined as in (4.1), into the left half-plane $H_l = \{z\colon \Re z < 0\}$. Then there exists a configuration of wires $\hat{L}$, admissible for Problem P_2^*, such that $\hat{L} \subset L^-(r) \cap H_l$.

Applying the polarization comparison theorem for harmonic measure [23, Theorem 2] and Carleman's principle for harmonic measure, see [12], and using the fact that $\hat{L}$ is an admissible configuration, we obtain

$$\omega(0, L(r), \mathbf{D}) \geq \omega(0, L^-(r), \mathbf{D}) \geq \omega(0, \hat{L}, \mathbf{D}) \geq \omega(0, L(r), \mathbf{D}). \tag{4.2}$$

Thus, each relation in (4.2) must hold with the sign of equality. By Theorem 2 in [23], equality occurs in the first of the inequalities (4.2) if and only if $L^-(r) = L(r)$ up to reflection in the imaginary axis. It follows from the maximum principle for harmonic functions that equality occurs in the second of these inequalities if and only if $\hat{L} = L^{-}(r)$. This shows that $L(r)$ satisfies the desired properties. □

To prove our next lemma, we need some facts about hyperbolic ellipsis. The hyperbolic plane will be identified with the unit disk $\mathbf{D}$ supplied with the hyperbolic distance

$$\rho(z_1, z_2) = \frac{1}{2} \log \frac{1 + \left|\frac{z_1 - z_2}{1 - z_1 \bar{z}_2}\right|}{1 - \left|\frac{z_1 - z_2}{1 - z_1 \bar{z}_2}\right|}, \quad z_1, z_2 \in \mathbf{D}.$$

For $0 < r, t < 1$, let $\mathcal{E}(r,t)$ denote the *hyperbolic ellipse* with foci at the points $\pm ir$ defined by $\mathcal{E}(r,t) = \{z \in \mathbf{D} \colon \rho(z, ir) + \rho(z, -ir) = \rho(t, ir) + \rho(t, -ir)\}$. To define $\mathcal{E}(r,t)$ for $t = 0$ and $t = 1$, we put $\mathcal{E}(r,0) = [-ir, ir]$ and $\mathcal{E}(r,1) = \mathbf{T}$. Thus, $t \in \mathcal{E}(r,t)$ for $0 \le t \le 1$.

A closed set $E \subset \overline{\mathbf{D}}$ is said to be symmetric with respect to $\mathcal{E}(r,t)$ if the intersection $E \cap \mathcal{E}(r,t)$ is either empty, or the whole $\mathcal{E}(r,t)$, or consists of a single closed arc of $\mathcal{E}(r,t)$ (possibly degenerate), which is symmetric with respect to the negative real axis.

Let $L(r) \subset \overline{\mathbf{D}}_{-}$ be an extremal configuration of wires for Problem P_2^* and let $D(r) = \mathbf{D} \setminus L(r)$. It follows from the maximum principle for harmonic functions that $D(r)$ is a simply connected domain. Let $\Omega(r)$ denote a closed set on $\overline{\mathbf{D}}_{-}$ bounded by $L(r)$, $[-ir, ir]$, and γ, where γ is a closed arc of the semicircle $\mathbf{T}^{-} = \{z \in \mathbf{T} \colon \Re z \le 0\}$ joining the connected components of $L(r)$ if $L(r)$ is not connected and $\gamma = \emptyset$ otherwise. More precisely, $\Omega(r) = \mathbf{C} \setminus \widetilde{\Omega}(r)$, where $\widetilde{\Omega}(r)$ is the unbounded connected component of $\mathbf{C} \setminus (L(r) \cup \gamma \cup [-ir, ir])$.

Lemma 4. *If $L(r) \subset \overline{\mathbf{D}}_{-}$ is an extremal configuration of wires for Problem P_2^* and $\Omega(r)$ is the corresponding closed set defined above, then $L(r)$ is symmetric with respect to the real axis and for each $t \in [0,1]$, the set $\Omega(r)$ is symmetric with respect to $\mathcal{E}(r,t)$.*

Proof. Let $\mathcal{R}$ be a doubly-sheeted Riemann surface over the disk $\mathbf{D}$ branched at the points $\pm ir$. The surface $\mathcal{R}$ can be obtained by gluing two copies, let $G_1(r)$ and $G_2(r)$, of the slit disk $\mathbf{D} \setminus [-ir, ir]$ across their boundary segments $[-ir, ir]$. Let $\widehat{L}(r)$ be the lift of $L(r)$ onto $\mathcal{R}$. Then $\mathcal{R} \setminus \widehat{L}(r)$ consists of two copies, let D_1 and D_2, of the domain $D(r)$ lying on $\mathcal{R}$.

Let $\zeta = f(z)$ be a conformal mapping from $G_1(r)$ onto a suitable annulus $A(\rho, 1)$ with some $\rho = \rho(r) < 1$ such that $f(1) = 1$. Applying the Schwarz reflection principle, we find that f can be analytically continued to a function, still denoted by f, which maps $\mathcal{R}$ conformally and one-to-one onto the annulus $A(\rho^2, 1)$. Then f maps $G_2(r)$ conformally onto $A(\rho^2, \rho)$.

We do not need the explicit expression of f, although it is available, see [17]. Simple geometric properties of f, which we explore below, can be found in [17].

First we note that f maps each hyperbolic ellipse $\mathcal{E}(r,t)$, $0 < t < 1$, embedded in $G_1(r)$, onto a circle C_τ with some $\tau = \tau(r,t)$ such that $\rho < \tau < 1$. Similarly, f maps $\mathcal{E}(r,t)$, $0 < t < 1$, embedded in $G_2(r)$, onto a circle C_τ with some $\tau = \tau(r,t)$ such that $\rho^2 < \tau < \rho$.

Let $\Omega_1 = f(D_1)$ and $\Omega_2 = f(D_2)$. The symmetry properties of f imply that the pair of nonoverlapping domains $\{\Omega_1, \Omega_2\}$ is invariant under the mapping $\zeta \mapsto -\rho^2/\zeta$; see [17]. The relation between hyperbolic ellipsis and circles mentioned above implies that to prove the lemma, it suffices to show that Ω_1 is circularly symmetric with respect to the positive real axis.

Since harmonic measure is conformally invariant and since $f(0) = \rho$ if $0 \in D_1$, we have

$$\omega(0, \mathbf{T}, D(r)) = \omega(\rho, \mathbf{T} \cup \mathbf{T}_{\rho^2}, \Omega_1) = \omega(-\rho, \mathbf{T} \cup \mathbf{T}_{\rho^2}, \Omega_2). \tag{4.3}$$

Let Ω_1^* and Ω_2^* be the circular symmetrizations of Ω_1 and Ω_2 with respect to the positive and negative real semi-axes, respectively. It follows from a symmetrization result of Krzyz [16] or from a more general Theorem 7 in Baernstein's paper [1] that

$$\omega(\rho, \mathbf{T} \cup \mathbf{T}_{\rho^2}, \Omega_1^*) \geq \omega(\rho, \mathbf{T} \cup \mathbf{T}_{\rho^2}, \Omega_1). \tag{4.4}$$

Furthermore, it follows from Theorem 10 in [7] that equality occurs in (4.4) if and only if Ω_1^* coincides with Ω_1 up to rotation about the origin.

Since the pair of domains $\{\Omega_1, \Omega_2\}$ is invariant under the mapping $\zeta \mapsto -\rho^2/\zeta$, it is easy to see that the domains Ω_1^* and Ω_2^* do not overlap and the pair $\{\Omega_1^*, \Omega_2^*\}$ is also invariant under this mapping. The latter implies that the inverse function f^{-1} is univalent on Ω_1^* (as well as on Ω_2^*) and the set $D^* = f^{-1}(\Omega_1^*)$ omits the points $\pm ir$. Therefore the compact set $L^* = \overline{(\mathbf{D} \setminus D^*)}$ is an admissible configuration of wires for Problem P_2^*. Combining this with the conformal invariance property of harmonic measure and equations (4.3) and (4.4), we find that

$$\omega(0, L^*, \mathbf{D}) = 1 - \omega(\rho, \mathbf{T} \cup \mathbf{T}_{\rho^2}, \Omega_1^*) \leq \omega(0, L(r), \mathbf{D}) \leq \omega(0, L^*, \mathbf{D}).$$

Thus, we must have the sign of equality in each relation of this chain. Therefore, (4.4) must hold with the sign of equality. Then $\Omega_1 = e^{i\alpha}\Omega_1^*$ and $\Omega_2 = e^{i\alpha}\Omega_2^*$ for some $\alpha \in \mathbf{R}$. Hence Ω_1 and Ω_2 are circularly symmetric with respect to the rays R_α and $R_{\alpha+\pi}$, respectively. Here $R_\beta = \{z = te^{i\beta} \colon t \geq 0\}$. Since the pair of domains $\{\Omega_1, \Omega_2\}$ is invariant under the mapping $\zeta \mapsto -\rho^2/\zeta$, it follows that Ω_2 is also circularly symmetric with respect to the ray $R_{\pi-\alpha}$. Therefore we must have $\alpha + \pi = \pi - \alpha$, mod (2π). Or equivalently, $\alpha = 0, \mod (\pi)$. Since $\rho \in \Omega_1$, we conclude that $\alpha = 0$ and therefore $\Omega_1 = \Omega_1^*$. Now the desired conclusion follows. $\square$

Now that symmetry with respect to the real axis of any extremal configuration is established, we have a simpler way (at least theoretically) to evaluate related harmonic measures. To explain this, we consider a disconnected admissible configuration of wires $L = l_+ \cup l_-$, where $l_+ \subset \mathbf{D}^+ = \{z \colon \Im z > 0\}$ and $l_- = \{z \colon \bar{z} \in l_+\}$. Then the domain $D_L = \mathbf{D} \setminus L$ can be considered as a quadrilateral with l_+ and l_- as a pair of its opposite sides. It is well-known that any such quadrilateral can be mapped by an analytic function $w = f(z)$ conformally onto a rectangle $R(m)$ considered in Section 2 with some $m > 0$ in such a way that its boundary arcs l_+ and l_- correspond to the vertical sides of $R(m)$. Since L is symmetric with respect to the real axis, it follows that $f(0)$ is pure imaginary and $-m < \Im f(0) < m$. Furthermore, if L is also symmetric with respect to the imaginary axis then $f(0) = 0$. Thus, the question on evaluation of the harmonic measure in the domain D_L is reduced to the question on evaluation of the harmonic measure in the rectangle $R(m)$. The latter question was discussed in part **(d)** of Section 2.

5. Pairs of wires with a common contact point on T

The collection of wires in Fryntov's counterexample forms a connected set. Thus, there is a possibility that, at least for certain values of r, two wires extremal for Problem P_2^* will merge at some point and then will follow to the unit circle $\mathbf{T}$ along the same path. In this case, we say that the wires have a common contact point on $\mathbf{T}$. To discuss this case, we consider the family $\mathcal{L}^c_-(r)$ consisting of all pairs of wires in $\overline{\mathbf{D}}_-$, which have a common contact point on $\mathbf{T}$ and connect the points ir and $-ir$ with $\mathbf{T}$. Let

$$\omega^*_{con}(r) = \inf_{L \in \mathcal{L}^c_-(r)} \omega_L(0) \tag{5.1}$$

Since every $L \in \mathcal{L}^c_-(r)$ is an admissible configuration of wires for Problem P^*_2 it follows that $\omega^*_2(r) \le \omega^*_{con}(r)$ for all $0 < r < 1$.

The minimal harmonic measure problem over the class $\mathcal{L}^c_-(r)$ is a particular case of D. Gaier's problem considered in [27]. Other interesting extremal problems on the harmonic measure, related to Gaier's problem, were discussed in [18], [19], and [20]. It follows from Theorem 1 in [27] that every Gaier's problem has a unique extremal configuration formed by a subset of critical trajectories of an appropriate quadratic differential. Two quadratic differentials relevant to the Problem P^*_2 are presented in the equations (5.2) and (5.3) below. For necessary definitions and properties of quadratic differentials, we recommend the reader to consult books [15], [29], [17] and papers [25] and [28]. Here we remind only that a trajectory γ of a quadratic differential $Q(z)\,dz^2$ is a smooth open arc or Jordan curve such that $Q(z)\,dz^2 > 0$ along γ. A trajectory γ of $Q(z)\,dz^2$ is called *critical* if at least one of its endpoints is a zero or a simple pole of $Q(z)\,dz^2$. Furthermore, by a *closed critical trajectory* we will understand the closure of the corresponding critical trajectory.

For $0 < r < 1$, $0 \le p \le 1$, and $\pi/2 < \varphi \le \pi$, let $Q_1(z)\,dz^2 = Q_1(z, r, p)\,dz^2$ and $Q_2(z)\,dz^2 = Q_2(z, r, \varphi)\,dz^2$ denote quadratic differentials defined by the following equations:

$$Q_1(z)\,dz^2 = \frac{(z+p)(z+1/p)}{z(z^2+r^2)(z^2+1/r^2)}\,dz^2 \tag{5.2}$$

and

$$Q_2(z)\,dz^2 = \frac{(z-e^{i\varphi})(z-e^{-i\varphi})}{z(z^2+r^2)(z^2+1/r^2)}\,dz^2. \tag{5.3}$$

Examples of possible trajectory structures of the quadratic differentials $Q_1(z)\,dz^2$ and $Q_2(z)\,dz^2$ are shown in Figures 2 and 3, respectively.

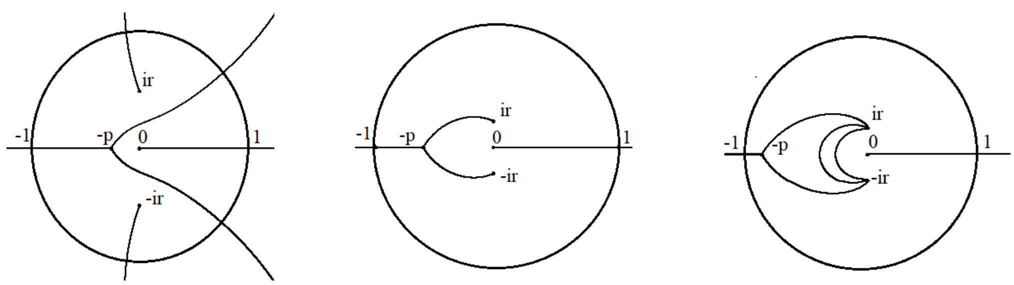

Figure 2. Critical trajectories of $Q_1(z)\,dz^2$.

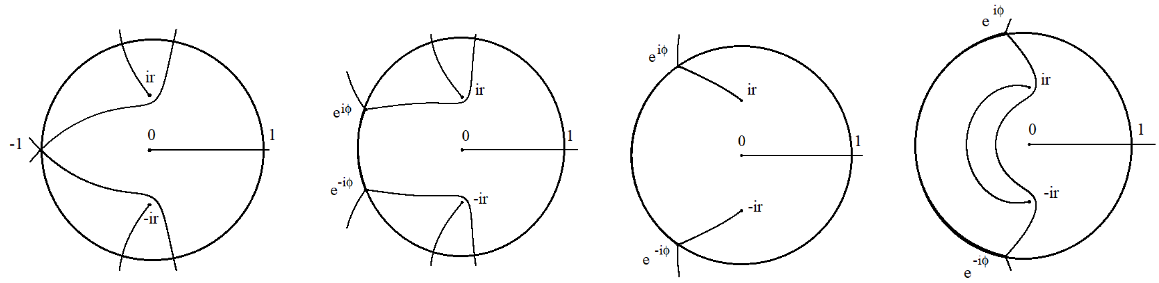

Figure 3. Critical trajectories of $Q_2(z)\,dz^2$.

Theorem 1. **(1)** *For every $0 < r < 1$, there is a unique continuum $E(r) \in \mathcal{L}^c_-(r)$ minimizing the harmonic measure $\omega_L(0)$ over the class $\mathcal{L}^c_-(r)$.*

(2) *Let $r_0 = 2 - \sqrt{3}$. Then $E(r_0)$ consists of two closed critical trajectories γ_+ and γ_- of a quadratic differential $Q(z, r_0)\,dz^2$ having the form (5.2) with $r = r_0$ and $p = 1$. Here γ_+ and γ_- denote the closed critical trajectories of $Q(z, r_0)\,dz^2$ joining $z = -1$ with the points ir_0 and $-ir_0$, respectively. The minimal harmonic measure in this case is $\omega^*_{con}(r_0) = \frac{2}{\pi}\arccos(1/3)$.*

(3) *If $0 < r < r_0$, then $E(r)$ consists of the interval $[-1, -p]$, where $p = p(r)$ is determined by the equation (5.4) below, and two closed critical trajectories γ_+ and γ_- of the quadratic differential $Q(z, r)\,dz^2$ of the form (5.2). Here γ_+ and γ_- are the closed critical trajectories of $Q(z, r)\,dz^2$ joining $z = -p$ with the points ir and $-ir$, respectively. For every r, $0 < r < r_0$, there is a unique $p = p(r)$, $0 < p < 1$, satisfying the equation*

$$\int_0^1 \sqrt{\frac{(1-x)(1-p^2x)}{x(x^2+(r/p)^2)(1+r^2p^2x^2)}}\,dx = \int_0^\pi \sqrt{\frac{1+2p\cos t+p^2}{1+2r^2\cos 2t+r^4}}\,dt. \tag{5.4}$$

The minimal harmonic measure $\omega^*_{con}(r)$ in this case can be found from equation (2.12) with the module $m = m_{con}(r)$ defined by equation

$$2m = \int_0^\pi \sqrt{\frac{1+2p\cos t+p^2}{1+2r^2\cos 2t+r^4}}\,dt \Bigg/ \int_0^1 \sqrt{\frac{(x+p)(1+px)}{x(x^2+r^2)(1+r^2x^2)}}\,dx. \tag{5.5}$$

(4) *For $r_0 < r < 1$, the extremal continuum $E(r)$ consists of the circular arc $\{e^{i\theta} : \varphi \le \theta \le 2\pi - \varphi\}$, where $\varphi = \varphi(r)$ is determined by the equation (5.6) below, and two closed critical trajectories γ_+ and γ_- of the quadratic differential $Q(z, r)\,dz^2$ of the form (5.3). Here γ_+ and γ_- are the closed critical trajectories of $Q(z, r)\,dz^2$ joining the points ir and $-ir$ with the points $e^{i\varphi}$ and $e^{-i\varphi}$, respectively. For every r, $r_0 < r < 1$, there is a unique $\varphi = \varphi(r)$, $\pi/2 < \varphi < \pi$, satisfying the equation*

$$\int_0^1 \sqrt{\frac{1+2x\cos\varphi+x^2}{x(x^2+r^2)(1+r^2x^2)}}\,dx = \int_0^\varphi \sqrt{\frac{2(\cos t-\cos\varphi)}{1+2r^2\cos 2t+r^4}}\,dt. \tag{5.6}$$

The minimal harmonic measure $\omega^*_{con}(r)$ in this case can be found from equation (2.12) with the module $m = m_{con}(r)$ defined by equation

$$2m = \int_0^1 \sqrt{\frac{1+2x\cos\varphi+x^2}{x(x^2+r^2)(1+r^2x^2)}}\,dx \Bigg/ \int_0^1 \sqrt{\frac{1-2x\cos\varphi+x^2}{x(x^2+r^2)(1+r^2x^2)}}\,dx. \tag{5.7}$$

(5) *The minimal harmonic measure $\omega^*_{con}(r)$ strictly decreases from* 1 *to* $\frac{1}{2}$ *as* r *varies from* 0 *to* 1.

Proof. First, we prove all statements concerning extremal configurations. After that we will explain our approach to calculate the corresponding harmonic measures.

Part (1) follows from Theorem 1 in [25], which also implies that the extremal continuum $E(r)$ consists of three or two closed critical trajectories of a quadratic differential $Q(z,r)\,dz^2$ having the form (5.2) or (5.3). In particular, $E(r)$ consists of two closed critical trajectories if and only if $Q(z,r)\,dz^2$ has a second order zero at $z=-1$. In addition, Theorem 1 in [25] implies that for every r, $0<r<1$, there exists precisely one quadratic differential of the form (5.2) or (5.3), which has critical trajectories as described in parts (2), (3), or (4) of this theorem.

Assume first, that the quadratic differential $Q(z,r)\,dz^2$ has the form (5.2) with $p=p(r)\in(0,1)$. Let $C=C_1\cup C_2\cup C_3\cup C_4\cup C_5$ be a positively oriented closed contour, where C_1 is the semicircle $\{e^{i\theta}\colon 0\le\theta\le\pi\}$, $C_2=[-1,-p(r)]$, C_3 is the trajectory γ_+ traversed twice, $C_4=[-p,0]$, and $C_5=[0,1]$. Since the branch of the radical $\sqrt{Q(z,r)}$ is analytic in the interior of C, $\int_C\sqrt{Q(z,r)}\,dz=0$ by Cauchy's Theorem. Since $\sqrt{Q(z,r)}\,dx$ is real on $C_2\cup C_3\cup C_5$ and it is pure imaginary on $C_1\cup C_4$, we must have $\int_{C_1\cup C_4}\sqrt{Q(z,r)}\,dz=0$. Parameterizing C_1 and C_4, it is easy to see that the latter equation is equivalent to (5.4).

If the quadratic differential $Q(z,r)\,dz^2$ has the form (5.3) with $\varphi=\varphi(r)\in(\pi/2,\pi)$, then a similar argument leads to the equation (5.6).

Assume now that for some $r\in(0,1)$, $Q(z,r)\,dz^2$ has a second order pole at $z=-1$; i.e. $Q(z,r)\,dz^2$ has the form (5.2) with $p=1$. Changing variables via conformal mapping $\zeta=k(z)$, where $k(z)=z/(1-z)^2$ is the Koebe function, and transplanting $Q(z,r)\,dz^2$ into the plane $\mathbf{C}_\zeta$, we obtain the quadratic differential

$$Q_1(\zeta)\,d\zeta^2=C\frac{d\zeta^2}{\zeta(\zeta-\zeta_0)(\zeta-\bar\zeta_0)}$$

with $\zeta_0=ir/(1-ir)^2$ and some $C>0$. This change of variables defines $Q_1(\zeta)\,d\zeta^2$ only in the slit plane $\mathbf{C}_\zeta\setminus(-\infty,-1/4]$. But since $Q_1(\zeta)\,d\zeta^2<0$ for all real $\zeta<0$, the quadratic differential $Q_1(\zeta)\,d\zeta^2$ can be extended to a quadratic differential (still denoted by $Q_1(\zeta)\,d\zeta^2$) defined on $\overline{\mathbf{C}}_\zeta$.

Now one can easily verify that the circular arc $\gamma_0=\{\zeta=|\zeta_0|e^{i\psi}\colon \operatorname{Arg}\zeta_0\le\psi\le 2\pi-\operatorname{Arg}\zeta_0\}$ is a closed critical trajectory of $Q_1(\zeta)\,d\zeta^2$, which corresponds under the mapping $\zeta=k(z)$ to the union $\gamma_+\cup\gamma_-$. Then we must have $k(-1)=-|\zeta_0|$. This is equivalent to the equation $r/(1+r^2)=1/4$, the only solution of which in the interval $(0,1)$ is $r_0=2-\sqrt{3}$.

Thus, we conclude that $Q(z,r)\,dz^2$ has a second order zero at $z=-1$ if and only if $r=r_0$. In addition, a standard continuity argument based on the Carathèodory convergence theorem (see, for example, the proof of Theorem 1 in [27]) shows that the quadratic differential $Q(z,r)\,dz^2$ depends continuously on the parameter r. Since the semicircle $\{e^{i\theta}\colon \pi/2\le\theta\le 3\pi/2\}$ is the only arc, which minimizes the harmonic measure $\omega_L(0)$ over all arcs $L\subset\overline{\mathbf{D}}_-$ joining the points $z=i$ and $z=-i$, the continuity argument implies also that $Q(z,r)\,dz^2$ converges to the quadratic differential $Q(z,1)\,dz^2=[z(z^2+1)]^{-1}\,dz^2$ as $r\to 1$. Therefore, $Q(z,r)\,dz^2$ must have the form (5.3) for all $r\in(r_0,1)$.

Let us show that for all $r\in(0,r_0)$ the quadratic differential $Q(z,r)\,dz^2$ has the form (5.2). If not, then $Q(z,r)\,dz^2$ has the form (5.3). Notice that the trajectory structure of $Q(z,r)\,dz^2$ coincides with the trajectory structure of the quadratic differential

$$Q_1(z,r)\,dz^2 = \frac{(z-e^{i\varphi})(z-e^{-i\varphi})}{z(z^2+r^2)(1+r^2z^2)}\,dz^2.$$

We can choose a sequence $r_k \to 0$ such that $\varphi(r_k) \to \varphi^*$, $\pi/2 \le \varphi^* \le \pi$, as $k \to \infty$. Then $Q_1(z,r_k)$ converges to $Q^*(z) = z^{-3}(z-e^{i\varphi^*})(z-e^{-i\varphi^*})$ uniformly in $\mathbf{C}\setminus\{0\}$. We note that the limit quadratic differential $Q^*(z)\,dz^2$ has two critical trajectories, let γ_+^* and γ_-^*, which join $z=0$ with the points $e^{i\varphi^*}$ and $e^{-i\varphi^*}$, respectively. We assume here that $\Im z \ge 0$ for all $z \in \gamma_+^*$. Since $z=0$ is the third order pole of $Q^*(z)\,dz^2$ and $Q^*(z)\,dz^2 > 0$ for $z \in (0,1)$ it follows that the trajectories γ_+^* and γ_-^* each forms a zero angle with the segment $[0,1]$ at $z=0$. In particular, γ_+^* contains a point z^* such that $\Re z^* > 0$.

Since $Q_1(z,r_k)$ converges to $Q^*(z)$ uniformly in $\mathbf{C}\setminus\{0\}$, it is not difficult to see that for all sufficiently large k, the critical trajectory γ_+ of the quadratic differential $Q(z,r_k)\,dz^2$ must intersect the disk $\{z\colon |z-z^*| \le (1/2)\Re z^*\}$. The latter contradicts to part (1) of this theorem, which says that $E(r) \in \mathcal{L}_-^c(r)$ and therefore $\gamma_+ \cup \gamma_-$ can not have common points with the half plane $\{z\colon \Re z > 0\}$. This contradiction shows that $Q(z,r)\,dz^2$ has the form (5.2) for all $r \in (0,r_0)$. Now all statements concerning extremal configurations are proved.

To find $m_{com}(2-\sqrt{3})$, we use the fact, mentioned above in this proof, that the Koebe function $k(z)$ maps $\mathbf{D}\setminus(\gamma_+\cup\gamma_-)$ onto a domain G that is the plane slit along the ray $(-\infty,-\frac{1}{4}]$ and along a circular arc centered at $-\frac{1}{4}$. Then it is a standard exercise in Complex Analysis to find a function mapping G conformally onto the upper half-plane. Then the interested reader may use formula (2.1) to verify that $\omega_{con}^*(2-\sqrt{3}) = \frac{2}{\pi}\arccos(1/3)$.

To prove equation (5.4) in the part **(3)** of this theorem, we note that the critical trajectories γ_+ and γ_- have one of their endpoints at $z=-p$ if and only if the Q_1-length of the interval $[-p,0]$ is equal to the Q_1-length of the upper half-circle $\mathbf{T}^+ = \{z \in \mathbf{T}\colon \Im z \ge 0\}$; i.e. if and only if $\int_{-p}^0 |Q_1(z)|^{1/2}\,|dz| = \int_{\mathbf{T}} |Q_1(z)|^{1/2}\,|dz|$. The latter equation after some routine calculations yields (5.4).

To obtain (5.5), we note that the triple $(\mathbf{D}\setminus(\gamma_+\cup\gamma_-\cup[-1,-p]), \gamma_+\cup\gamma_-\cup[-1,-p], 0)$ can be considered as a triad, whose module m can be calculated as in (5.5). Then the minimal harmonic measure $\omega_{con}^*(r)$ can be obtained as a solution to the equation (2.12).

Equations (5.6) and (5.7) in the part (4) of this theorem can be justified using same arguments as in our discussion of equations (5.4) and (5.5) above.

To prove (5), we assume that $0 < r_1 < r_2 < 1$. For $0 < s < 1$, let $f_s(z)$ denote the Riemann mapping function from the domain $\Omega(s) = \mathbf{D}\setminus([-i,-is]\cup[is,i])$ onto $\mathbf{D}$ such that $f_s(0)=0$, $f_s'(0)>0$. There is s_0, $r_1 < s_0 < 1$, such that $f_{s_0}(ir_1) = ir_2$, $f_{s_0}(-ir_1) = -ir_2$. Then the continuum $f_{s_0}(E(r_1))$ is in $\mathcal{L}_-^c(r_2)$. Now, using the conformal invariance property of the harmonic measure and the Carleman's principle for the harmonic measure, we obtain

$$\omega(0,E(r_2),\mathbf{D}) \le \omega(0,f_{s_0}(E(r_1)),\mathbf{D}) = \omega(0,E(r_1),\Omega(s_0)) < \omega(0,E(r_1),\mathbf{D}),$$

which proves that $\omega_{con}^*(r)$ is strictly decreasing on $0 < r < 1$. □

Remark 3. It follows from general results of the theory of Jenkins' module problem (see Theorem 4.1 in [27]) that the quadratic differentials $Q(z,r)\,dz^2$ defined in parts (3) and (4) of Theorem 1, depend continuously on the parameter r, which means that the solution $p=p(r)$ of equation (5.4) and solution $\varphi=\varphi(r)$ of equation

(5.6) each depends continuously on r. This continuity property can be also obtained directly by considering $p(r)$ and $\varphi(r)$ as implicit functions defined by the relevant equation. This approach implies also that $p(r)$ and $\varphi(r)$ are differentiable functions of the parameter r.

The graphs of functions $p(r)$ and $\varphi(r)$ shown in Figure 4[1] suggest that $p(r)$ strictly increases from 0 to 1 a r varies from 0 to $2-\sqrt{3}$ and $\varphi(r)$ strictly decreases from π to $\pi/2$ as r varies from $2-\sqrt{3}$ to 1, but we were not able to prove these monotonicity properties rigorously. Some partial results concerning properties of $p(r)$ are stated in the following corollary.

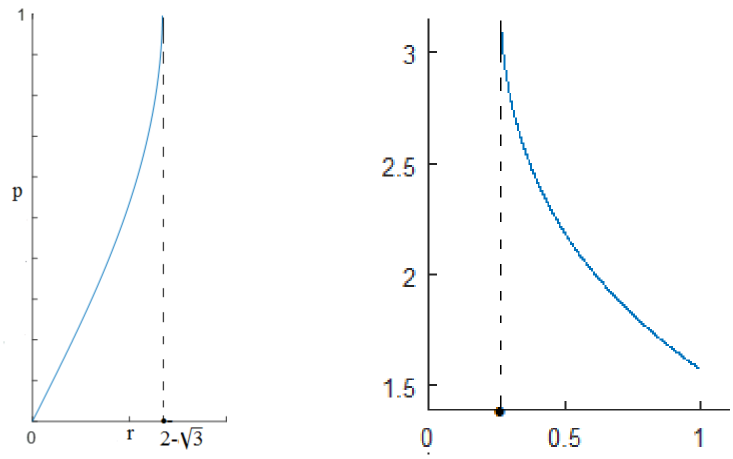

Figure 4. Functions $p(r)$ (left graph) and $\varphi(r)$ (right graph).

Corollary 1. *For $0 < r < 2-\sqrt{3}$, let $p = p(r)$ be solution to equation (5.4). Then:*

(a) $p(r) \to 1$ *as* $r \to 2-\sqrt{3}$;

(b) $p(r)/r \to 1/\delta$ *as* $r \to 0$, *where* $\delta = 0.423543\ldots$ *is the unique solution to the equation*

$$\int_0^1 \sqrt{\frac{1-x}{x(x^2+\delta^2)}}\,dx = \pi. \tag{5.8}$$

Proof. Part **(a)** follows from the fact that the extremal configuration in the problem under consideration depends continuously on the parameter r and that the corresponding quadratic differentials $Q(z,r)\,dz^2$ have the form (5.2) for $0 < r < 2-\sqrt{3}$ and the form (5.3) for $2-\sqrt{3} < r < 1$.

To prove part **(b)**, we note first that there are constants $c_1 > 0$ and $c_2 > 0$ such that $c_1 < p(r)/r < c_2$ for all $r > 0$ small enough. Indeed, if $p(r_n)/r_n \to \infty$ for some sequence $r_n \to 0$, then the value of the integral in the left-hand side of (5.4) will diverge to ∞ while the integral in the right-hand side of (5.4) remains bounded. Similar argument shows that $p(r_n)/r_n$ cannot converge to 0 if $r_n \to 0$. Now, assuming that $p(r_n)/r_n \to 1/\delta$, with $0 < \delta < \infty$, and taking the limit in (5.4), we obtain (5.8), which defines δ uniquely. $\square$

[1]Numerical computations presented in this paper were performed by Dr. J. Padgett using "Mathematica" package. In particular, he provided us with the graphs of the functions $p(r)$ and $\varphi(r)$ shown in Figure 4 as well as with the graphs of the module and harmonic measure functions shown in Figure 5.

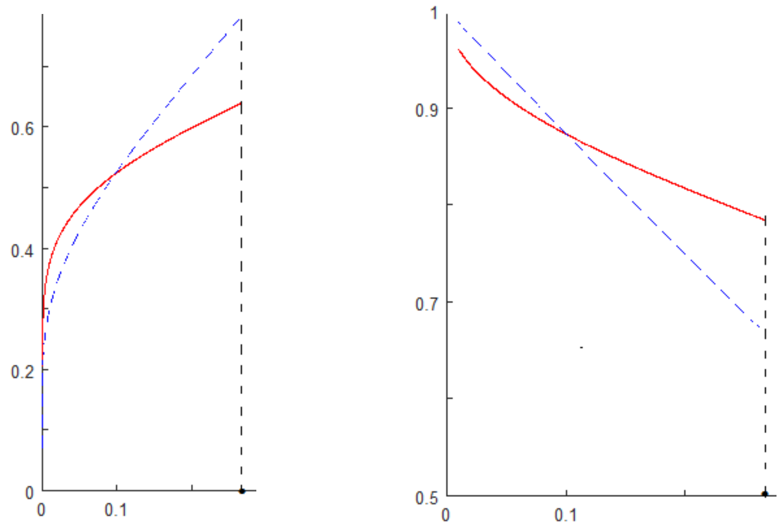

Figure 5. Moduli (left graph) and harmonic measures (right graph). Solid lines represent functions $m_{con}(r)$ and $\omega^*_{con}(r)$ while dashed lines represent functions $(1/2)m_2(r)$ and $\omega_{L_2(r)}(0)$.

Next, we want to compare the minimal harmonic measure $\omega^*_{con}(r)$ for wires with a common contact point defined by equation (5.1) with the harmonic measure $\omega(0, L_2(r), \mathbf{D})$ of two radial slits. The graphs of these two harmonic measures displayed in Figure 5 (right graph) show that $\omega^*_{con}(r) < \omega(0, l_2(r), \mathbf{D})$ for all $r < r_*$, where $r_* = 0.1005\ldots$ This gives a numerical evidence that the configuration $L_2(r)$ consisting of two radial slits is not an extremal configuration of wires for Problem $P^*_2(r)$ for small r. To support this with rigorous argument, we find the asymptotic expansions for the the moduli of triads relative to each of these configurations of wires.

For $0 < r < 2-\sqrt{3}$, consider the triad $(\mathbf{D} \setminus E(r), E(r), 0)$, where $E(r)$ is the extremal set of wires from Theorem 1. Then the module $m = \mathrm{mod}(\mathbf{D} \setminus E(r), E(r), 0)$ of this triad is given by formula (5.5) with $p = p(r)$ defined by equation (5.4). Using the relation $p(r)/r \to 1/\delta$ of Corollary 1, we find asymptotic of the integrals in the right-hand side of (5.5):

$$\int_0^\pi \sqrt{\frac{1+2p\cos t + p^2}{1+2r^2\cos 2t + r^4}}\, dt = \pi + o(1) \quad \text{as } r \to 0 \tag{5.9}$$

and

$$\begin{aligned}
&\int_0^1 \sqrt{\frac{(x+p)(1+px)}{x(x^2+r^2)(1+r^2x^2)}}\, dx = \int_0^{1/r} \sqrt{\frac{(\tau+p/r)(1+pr\tau)}{\tau(\tau^2+1)(1+r^4\tau^2)}}\, d\tau \\
&\le \int_0^1 \sqrt{\frac{(\tau+p/r)(1+pr\tau)}{\tau(\tau^2+1)(1+r^4\tau^2)}}\, d\tau + \int_1^{1/r} \sqrt{\frac{(\tau+p/r)(1+p)}{\tau(\tau^2+1)(1+r^2)}}\, d\tau \\
&\le A\log(1/r) + o(\log(1/r)) \quad \text{as } r \to 0,
\end{aligned} \tag{5.10}$$

where

$$A = \sqrt{\frac{(1+p/r)(1+p)}{(1+r^2)(1+r^4)}} \to \sqrt{1+\delta^{-1}} = 1.83331\ldots \quad \text{as } r \to 0. \tag{5.11}$$

Combining (5.9)–(5.11) with (5.5), we obtain

$$m_{con}(r) \geq \frac{\pi}{2} \frac{1}{\sqrt{1+\delta^{-1}}} \frac{1}{\log(1/r)} + o\left(\frac{1}{\log(1/r)}\right) \quad \text{as } r \to 0. \tag{5.12}$$

We want to compare $m_{con}(r)$ with the module related to the configuration $L_2(r)$. By (2.3), we have $\omega(0, L_2(r), \mathbf{D} \setminus L_2(r) = \omega(0, [-1, -r^2], \mathbf{D} \setminus [-1, -r^2])$. This together with equation (2.12) imply that $\omega^*_{con}(r) < \omega(0, L_2(r), \mathbf{D} \setminus L_2(r))$ if and only if

$$m_{con}(r) > \frac{1}{2} m_2(r), \tag{5.13}$$

where $m_2(r)$ is given by (2.5). Using (5.12) and (2.12), we conclude that there is r_1, $0 < r_1 < r_0$, such that (5.13) holds for all r such that $0 < r < r_1$. Thus, we have established the following.

Corollary 2. *There is r_1, $0 < r_1 < r_0$, such that*

$$\omega^*_{con}(r) < \omega_{L_2(r)}(0)$$

for all r such that $0 < r < r_1$. In particular, the set $L_2(r)$ consisting of two radial wires is not extremal for Problem $P^*_2(r)$ if $r > 0$ is small enough.

6. Disconnected sets of wires: a related module problem

As we have seen in Section 5, the minimal harmonic measure problem for connected sets of wires is linked to a special module problem. So, it is reasonable to expect that the minimal harmonic measure problem for disconnected sets is also related to some module problem. This is indeed the case and the relevant module problem will be introduced in this section.

First, we recall some definitions. A simply connected domain G with four distinct boundary points marked on ∂G is called a *quadrilateral*. Each quadrilateral has two pairs of opposite sides, one of which is considered as "distinguished". Then the module of G, denoted by $\mathrm{mod}(G)$, is defined as the module of a family of all rectifiable arcs γ in G, which join the distinguished sides of G, see [15] or [17].

Next, we introduce configurations (G_0, G_1, G_2) consisting of three quadrilaterals in the unit disk $\mathbf{D}$. For a given $0 < r < 1$, let $\mathbf{D}'_r = \mathbf{D} \setminus \{0, \pm ir\}$. For $\varepsilon > 0$ small enough, let γ^ε_0, γ^ε_1, and γ^ε_2 be Jordan arcs on $\mathbf{D}'_r$ defined as follows: $\gamma^\varepsilon_0 = [e^{i\varepsilon}, i\sin\varepsilon] \cup \{\varepsilon e^{i\theta} \colon \pi/2 \leq \theta \leq 3\pi/2\} \cup [-i\sin\varepsilon, e^{-i\varepsilon}]$, $\gamma^\varepsilon_1 = [ie^{i\varepsilon}, -\sin\varepsilon + ir] \cup \{ir + \varepsilon e^{i\theta} \colon \pi \leq \theta \leq 2\pi\} \cup [\sin\varepsilon + ir, ie^{-i\varepsilon}]$, $\gamma^\varepsilon_2 = \{z \colon \bar{z} \in \gamma^\varepsilon_1\}$.

By Γ_0, Γ_1, and Γ_2 we will denote the homotopy classes of arcs on $\mathbf{D}'_r$ having the arcs γ^ε_1, γ^ε_1, and γ^ε_2 as their representatives, respectively. We will say that a quadrilateral $G \subset \mathbf{D}'_r$ is associated with the class Γ_k if the arcs γ in G joining the distinguished sides of G belong to Γ_k.

Problem *M3Q*. (Module problem for three quadrilaterals in $\mathbf{D}$) *For given $0 < r < 1$ and $\alpha > 0$, find the maximum*

$$M(r, \alpha) = \max\{\alpha^2 \,\mathrm{mod}(G_0) + \mathrm{mod}(G_1) + \mathrm{mod}(G_2)\} \tag{6.1}$$

and identify all extremal configurations $\{G^*_0, G^*_1, G^*_2\}$ over all triples $\{G_0, G_1, G_2\}$ of nonoverlapping quadrilaterals G_0, G_1, and G_2 on $\mathbf{D}'_r$ associated with the classes Γ_0, Γ_1, and Γ_2, respectively.

In the statement of Problem $M3Q$ we allow triples with one or more degenerate quadrilaterals. In this case we put $\mathrm{mod}(G_k) = 0$ if $G_k = \emptyset$.

Problem $M3Q$ is a particular case of the Jenkins' module problem discussed in many texts, see [14], [17], [27]. Its solution is given by Theorem 2 below, whose statement requires some terminology of the theory of quadratic differentials $Q(z)\,dz^2$ defined on the unit disk $\mathbf{D}$ and real-valued on its boundary $\mathbf{T}$. The critical trajectories of every such quadratic differential divide $\mathbf{D}$ into a finite system of domains called *the domain configuration* of $Q(z)\,dz^2$. In this paper, we deal with quadratic differentials, the domain configuration of which consists of at most three domains, say G_0, G_1, and G_2, each of which is a quadrilateral with respect to $Q(z)\,dz^2$. The latter means that each G_k can be considered as a quadrilateral as defined above, which has a pair of distinguished sides on $\mathbf{T}$ such that G_k is swept out by the regular trajectories of $Q(z)\,dz^2$ having their end points on the opposite distinguished sides of G_k.

Theorem 2. *For every $0 < r < 1$ and $\alpha > 0$, there is a unique triple, denoted by $\{G_0(r,\alpha), G_1(r,\alpha), G_2(r,\alpha)\}$, which realizes the maximum in (6.1).*

(1) *If $0 < \alpha \le 1$, then $G_0(r,\alpha) = \emptyset$, $G_1(r,\alpha) = \mathbf{D}^+ \setminus [ir, i]$, $G_2(r,\alpha) = \mathbf{D}^- \setminus [-i, -ir]$.*

(2) *Let $0 < r \le r_0 = 2 - \sqrt{3}$ and let $\alpha_0 = \alpha_0(r)$ be defined by the equation*

$$\alpha_0 = \frac{2\int_0^1 \sqrt{\frac{(x+p)(x+1/p)}{x(x^2+r^2)(x^2+1/r^2)}}\,dx}{\int_0^1 \sqrt{\frac{(x+p)(x+1/p)}{x(x^2+r^2)(x^2+1/r^2)}}\,dx + \int_0^p \sqrt{\frac{(x-p)(x-1/p)}{x(x^2+r^2)(x^2+1/r^2)}}\,dx}, \tag{6.2}$$

where $p = p(r)$ is determined from the equation (5.4), and let $\alpha_0(r_0) = 2$. If $1 < \alpha < \alpha_0$, then the extremal triple coincides with the domain configuration $\{G_0, G_1, G_2\}$ of the quadratic differential having the form (5.2) with $p = p(r,\alpha)$ uniquely determined from the equation

$$\alpha \int_0^p \sqrt{\frac{(x-p)(x-1/p)}{x(x^2+r^2)(x^2+1/r^2)}}\,dx = (2-\alpha)\int_0^1 \sqrt{\frac{(x+p)(x+1/p)}{x(x^2+r^2)(x^2+1/r^2)}}\,dx. \tag{6.3}$$

If $\alpha \ge \alpha_0$, then $G_1(r,\alpha) = G_2(r,\alpha) = \emptyset$ and $G_0(r,\alpha)$ is a single quadrilateral, which constitutes the domain configuration of the quadratic differential having the form (5.2) with p (independent of α) determined from the equation (6.3) with $\alpha = \alpha_0(r)$.

(3) *Let $r_0 < r < 1$ and let $\alpha_1 = \alpha_1(r) > 2$ be defined by the equation*

$$\alpha_1 = \frac{2\int_0^1 \sqrt{\frac{x^2+2x\cos\varphi+1}{x(x^2+r^2)(x^2+1/r^2)}}\,dx}{\int_0^1 \sqrt{\frac{x^2+2x\cos\varphi+1}{x(x^2+r^2)(x^2+1/r^2)}}\,dx + \int_0^\varphi \sqrt{\frac{\cos t-\cos\varphi}{\cos 2t-(1+r^4)/(2r^2)}}\,dt}, \tag{6.4}$$

where $\varphi = \varphi(r)$ is determined from the equation (5.6). If $1 < \alpha \le 2$, then the extremal triple coincides with the domain configuration of the quadratic differential having the form (5.2), where $p = p(r,\alpha) \in (0,1)$ is uniquely determined from the equation (6.3) if $1 < \alpha < 2$ and $p(r,2) = 1$. If $2 < \alpha < \alpha_1$, then the extremal triple coincides with the domain configuration of the quadratic differential having the form (5.3) with $\varphi = \varphi(r,\alpha)$, $\pi < \varphi < \pi/2$, uniquely determined from the equation

$$(\alpha-2)\int_0^1 \sqrt{\frac{x^2+2x\cos\varphi+1}{x(x^2+r^2)(x^2+1/r^2)}}\,dx = \alpha\int_0^\varphi \sqrt{\frac{\cos t-\cos\varphi}{\cos 2t-(1+r^4)/(2r^2)}}\,dt. \tag{6.5}$$

If $\alpha \ge \alpha_1(r)$, then $G_1(r,\alpha) = G_2(r,\alpha) = \emptyset$ and $G_0(r,\alpha)$ is the single quadrilateral, which constitutes the domain configuration of the quadratic differential *having the form (5.3) with φ (independent of α) uniquely determined from the equation (6.5) with $\alpha = \alpha_1(r)$.*

Proof. The main part of this theorem follows from Jenkins' Theorem 1 in [14]. A particular case of Jenkins' theorem concerning the module problem on the unit disk was discussed in details by Kuz'mina in [17]. It follows from Theorem 0.2 in [17] that for every $0 < r < 1$ and $\alpha > 0$ the extremal triple is unique and coincides with the domain configuration of a quadratic differential $Q(z, r, \alpha)\, dz^2$ having the form (5.2) or (5.3). It is also well known that the quadratic differential $Q(z, r, \alpha)\, dz^2$ and therefore its domain configuration $\{G_0, G_1, G_2\}$ depend continuously on the parameters r and α; see Theorem 4.1 in [27]. Figure 3 shows three possible domain configurations extremal for Problem $M3Q$.

If neither of the quadrilaterals G_0, G_1, and G_2 is degenerate, then the Q-lengths of the trajectories γ and γ' of $Q(z, r, \alpha)\, dz^2$ lying in the quadrilaterals G_0 and G_1, respectively, satisfy the following equation:

$$\alpha \int_\gamma |Q^{1/2}(z, r\alpha)|\, |dz| = \int_{\gamma'} |Q^{1/2}(z, r, \alpha)|\, |dz|. \tag{6.6}$$

If $Q(z, r, \alpha)\, dz^2$ has the form (5.2), then integrating along corresponding critical trajectories one can easily see that (6.6) is equivalent to the equation (6.3). If $Q(z, r, \alpha)\, dz^2$ has the form (5.3), then, integrating along corresponding critical trajectories once more, we find that (6.6) is equivalent to the equation (6.3).

Next we will discuss possible degenerate configurations. The following facts are special cases of well-known results concerning degeneration of some domains of the extremal partitions related to the Jenkins' module problem; see [27, Theorem 4.3]:

(a) There are real $\alpha' = \alpha'(r)$ and $\alpha'' = \alpha''(r)$, $0 < \alpha' < \alpha''$, such that $G_0(r, \alpha) = \emptyset$ for all $\alpha \le \alpha'$ and $G_0(r, \alpha) \ne \emptyset$ for all $\alpha > \alpha'$ and $G_1(r, \alpha) = G_2(r, \alpha) = \emptyset$ for all $\alpha \ge \alpha''$ and $G_1(r, \alpha) \ne \emptyset$ (and therefore $G_2(r, \alpha) \ne \emptyset$) for all $\alpha < \alpha''$.

(b) To find $\alpha'(r)$, we consider the module problem (6.1) with $\alpha = 0$. Then the extremal triple $\{G_0, G_1, G_2\}$ consists of the degenerate quadrilateral $G_0 = \emptyset$ and quadrilaterals $G_1 = \mathbf{D}^+ \setminus [ir, i]$ and $G_2 = \mathbf{D}^- \setminus [-i, -ir]$. Let $Q_0(z)\, dz^2$ denote the quadratic differential associated with this problem, which has the form

$$Q_0(z)\, dz^2 = C(r) \left[(z^2 + r^2)(z^2 + 1/r^2) \right]^{-1} dz^2,$$

where the constant $C(r)$ is chosen in such a way that the Q_0-length of every trajectory γ of $Q_0(z)\, dz^2$ lying in G_1 equals 1; i.e. $\int_\gamma |Q_0^{1/2}(z)|\, |dz| = 1$. Then it follows from Theorem 4.3 in [27] that

$$\alpha'(r) = \inf_{\gamma' \in \Gamma_0} Q_0\text{-length}(\gamma') = 2 \int_0^1 Q_0^{1/2}(x)\, dx = 1.$$

(c) To find $\alpha''(r)$, we consider the problem on the maximum $\max \operatorname{mod}(G_0)$ over all quadrilaterals G_0 associated with the class Γ_0. Let $Q_1(z)\, dz^2$ be the quadratic differential associated with this problem, which has the form

$$Q_1(z)\, dz^2 = C_1(r) Q(z, r)\, dz^2,$$

where $Q(z, r)\, dz^2$ is the quadratic differential described in Theorem 1 and the constant $C_1(r) > 0$ is chosen in such a way that the minimal Q_1-length of arcs $\gamma \in \Gamma_1$ equals 1; i.e.

$$\inf_{\gamma \in \Gamma_1} \int_\gamma |Q_1(z)|\, |dz| = 1.$$

Then, using Theorem 4.3 in [27] once more, we conclude that $\alpha''(r) = \alpha_0(r)$, where $\alpha_0(r)$ is defined by (6.2), if the quadratic differential $Q(z,r)\,dz^2$ has the form (5.2) and $\alpha''(r) = \alpha_1(r)$, where $\alpha_1(r)$, is defined by (6.4) if $Q(z,r)\,dz^2$ has the form (5.3). □

7. Minimal harmonic measure in quadrilaterals of a fixed module

Let $0 < r < 1$ and let $\alpha \geq 1$ be such that $G_1(r,\alpha) \neq \emptyset$. Then $G_2(r,\alpha) = \{z\colon \bar{z} \in G_1(r,\alpha)\}$ is not empty as well. Let $\gamma^+(r,\alpha)$ denote the side of $G_1(r,\alpha)$ joining the point ir with the unit circle $\mathbf{T}$. Similarly, let $\gamma^-(r,\alpha)$ denote the side of $G_2(r,\alpha)$ joining $-ir$ with $\mathbf{T}$. Then the domain $\Omega(r,\alpha) = \mathbf{D} \setminus (\gamma^+(r,\alpha) \cup \gamma^-(r,\alpha))$ can be considered as a quadrilateral having $\gamma^+(r,\alpha)$ and $\gamma^-(r,\alpha)$ as a pair of its non-distinguished sides.

It follows from part (1) of Theorem 2 that $\Omega(r,1) = \mathbf{D} \setminus ([ir,i] \cup [-i. - ir])$. Mapping $\Omega(r,1)$ conformally onto a suitable rectangle, it is not difficult to find that $\operatorname{mod}(\Omega(r,1)) = m(r)$, where

$$m(r) = 2\mathcal{K}(r^2)/\mathcal{K}'(r^2). \tag{7.1}$$

Lemma 5. **(1)** *For a fixed r such that $0 < r < r_0$, the module* $\operatorname{mod}(\Omega(r,\alpha))$ *is a continuous function of α, which strictly decreases from $m(r)$ to 0 as α runs from 1 to $\alpha_0(r)$.*

(2) *For a fixed r such that $r_0 < r < 1$, the module* $\operatorname{mod}(\Omega(r,\alpha))$ *is a continuous function of α, which strictly decreases from $m(r)$ to* $\operatorname{mod}(\Omega(r,\alpha_1(r))) > 0$ *as α runs from 1 to $\alpha_1(r)$.*

Proof. It is well know (see, for instance, Theorems 4.1 and Theorem 4.2 in [27]) that the extremal triple $\{G_0(r,\alpha), G_1(r,\alpha), G_2(r,\alpha)\}$ depends continuously on the parameters r and α. This implies continuity of $\operatorname{mod}(\Omega(r,\alpha))$ as a function of r and α. It follows from Theorem 1 in [14] (or from part (1) of Theorem 2) that for all $\alpha \geq 1$, $\operatorname{mod}(\Omega(r,\alpha)) \leq \operatorname{mod}(\Omega(r,1))$.

Thus, to prove strict monotonicity of $\operatorname{mod}(\Omega(r,\alpha))$ with respect to α, it remains to show that $\operatorname{mod}(\Omega(r,\alpha_1)) \neq \operatorname{mod}(\Omega(r,\alpha_2))$ whenever $\alpha_1 \neq \alpha_2$. Proceeding by contradiction, let us assume that $\operatorname{mod}(\Omega(r,\alpha_1)) = \operatorname{mod}(\Omega(r,\alpha_2))$ for some r and α_1, α_2 such that $1 \leq \alpha_1 < \alpha_2$. Then there is a conformal mapping $f_1\colon \Omega(r,\alpha_1) \to \Omega(r,\alpha_2)$, which maps the sides $\gamma^+(r,\alpha_1)$ and $\gamma^-(r,\alpha_1)$ onto the sides $\gamma^+(r,\alpha_2)$ and $\gamma^-(r,\alpha_2)$, respectively. Since $\Omega(r,\alpha_1)$ and $\Omega(r,\alpha_2)$ are symmetric with respect to the real axis it follows that $f_1([-1,1]) = [-1,1]$. Hence, $-1 < f_1(0) < 1$.

Assume first that $a = f_1(0) < 0$. Then the triple $\{f_1(G_0(r,\alpha_1)), f_1(G_1(r,\alpha_1)), f_1(G_2(r,\alpha_1))\}$ is admissible for the module problem on $M(r,\alpha_1)$. Let $\widetilde{G}_0 = f_1(G_0(r,\alpha_1)) \cup [a,0)$. Then $\widetilde{G}_0$ can be considered as a quadrilateral, the distinguished sides of which coincide with the distinguished sides of $f_1(G_0(r,\alpha_1))$. Now the comparison theorem for quadrilaterals implies the inequality $\operatorname{mod}(\widetilde{G}_0) > \operatorname{mod}(f_1(G_0(r,\alpha_1)))$. Since the triple $\{\widetilde{G}_0, f_1(G_1(r,\alpha_1)), f_1(G_2(r,\alpha_1))\}$ is admissible for the module problem on $M(r,\alpha_1)$, the latter inequality leads to the following contradiction:

$$\alpha_1^2 \operatorname{mod}(\widetilde{G}_0) + \operatorname{mod}(f_1(G_1(r,\alpha_1))) + \operatorname{mod}(f_1(G_2(r,\alpha_1))) > M(r,\alpha_1),$$

which proves strict monotonicity in the case under consideration.

If $f_1(0) > 0$, then we have $f_2(0) < 0$, where $f_2 = f_1^{-1}$ maps $\Omega(r,\alpha_2)$ conformally onto $\Omega(r,\alpha_1)$. In this case, repeating our previous proof with f_1 replaced by f_2 and with $\Omega(r,\alpha_1)$ replaced by $\Omega(r,\alpha_2)$, we again will get a contradiction.

If $f_1(0) = 0$, then arguing as in the previous cases we conclude that the triple $\{f_1(G_0(r,\alpha_1)), f_1(G_1(r,\alpha_1)), f_1(G_2(r,\alpha_1))\}$ is extremal for the module problem on $M(r,\alpha_1)$. Since the extremal triple is unique we must have $f_1(G_k(r,\alpha_1)) = G_k(r,\alpha_1)$ for $k = 0,1,2$. Thus, $\gamma^+(r,\alpha_1) = \gamma^+(r,\alpha_2)$. Since $Q(z,r,\alpha_k)\,dz^2 > 0$ along $\gamma^+(r,\alpha_k)$ we must have

$$Q(z,r,\alpha_1)/Q(z,r,\alpha_2) > 0 \quad \text{for } z \in \gamma^+(r,\alpha_1). \tag{7.2}$$

Assume that the quadratic differentials $Q(z,r,\alpha_1)\,dz^2$ and $Q(z,r,\alpha_2)\,dz^2$ each has the form (5.2) with $p = p_1$ and $p = p_2$, respectively. Taking the limit in (7.2) as $z \to ir$ along $\gamma^+(r,\alpha_1)$, we obtain the inequality

$$\frac{1 - r^2 + ir(p_1 + 1/p_1)}{1 - r^2 + ir(p_2 + 1/p_2)} > 0.$$

Therefore we must have $p_1 = p_2$. Hence, $\alpha_1 = \alpha_2$ by (6.3) contradicting the assumption $\alpha_1 < \alpha_2$. Thus, $\operatorname{mod}(\Omega(r,\alpha_1)) \neq \operatorname{mod}(\Omega(r,\alpha_2))$ in the case under consideration.

In the cases when $Q(z,r,\alpha_1)\,dz^2$ has the form (5.2) and $Q(z,r,\alpha_2)\,dz^2$ has the form (5.3) or when both quadratic differentials have the form (5.3) the proof is similar. So, we leave it to the reader. Thus, strict monotonicity of $\operatorname{mod}(\Omega(r,\alpha))$ as a function of α is proved. If $0 < r \le r_0$, then one of the distinguished sides of $\Omega(r,\alpha)$ shrinks to a point as α approaches $\alpha_0(r)$ from the left. This implies that $\operatorname{mod}(\Omega(r,\alpha)) \to 0$ as $\alpha \to \alpha_0(r)$. If $r_0 < r < 1$, then the family of the quadrilaterals $\Omega(r,\alpha)$ converges to the non-degenerate quadrilateral $\Omega(r,\alpha_1(r))$ as α approaches $\alpha_1(r)$ from the left. This implies that $\operatorname{mod}(\Omega(r,\alpha)) \to \operatorname{mod}(\Omega(r,\alpha_1(r)))$ as $\alpha \to \alpha_1(r)$. The proof of the lemma is complete. □

The monotonicity property established in Lemma 5 allows us introducing, for each r, $0 < r < 1$, a parametric family of quadrilaterals $\Omega_r(m)$ depending continuously on the parameter m, $0 < m \le m(r)$, as follows.

If $0 < r \le r_0$, then for $0 < m \le m(r)$, we put $\Omega_r(m) = \Omega(r,\alpha(m))$, where $\alpha(m)$ is chosen such that $\operatorname{mod}(\Omega(r,\alpha(m))) = m$.

If $r_0 < r < 1$, then for $m_1(r) \le m \le m(r)$, where $m_1(r) = \operatorname{mod}(\Omega(r,\alpha_1(r)))$, the quadrilateral $\Omega_r(m)$ is defined as above; i.e. $\Omega_r(m) = \Omega(r,\alpha(m))$. For $0 < m < m_1(r)$, we define $\Omega_r(m)$ to be the domain $\mathbf{D} \setminus (\gamma^+(r,\alpha_1(r)) \cup \gamma^-(r,\alpha_1(r)))$ considered as a quadrilateral, the distinguished sides of which are the circular arcs $\{e^{i\theta}\colon |\theta| < \varphi(r)\}$ and $\{e^{i\theta}\colon |\theta - \pi| < \pi - \varphi_r(m)\}$, where $\varphi_r(m)$, $\varphi(r) < \varphi_r(m) < \pi$, is chosen such that $\operatorname{mod}(\Omega_r(m)) = m$.

For $0 < r < 1$ and $0 < m \le m(r)$, let $L_r(m)$ be the union of the closed non-distinguished sides of the quadrilateral $\Omega_r(m)$. For $m = 0$, we put $L_r(0) = L(r)$, where $L(r)$ is the extremal connected set of wires defined in Theorem 1. Then, for each fixed $0 < r < 1$, $\{L_r(m)\colon 0 \le m \le m(r)\}$ is a family of sets of wires $L_r(m) \in \overline{\mathbf{D}}_-$ depending continuously on the parameter m, which are symmetric with respect to $\mathbf{R}$ and admissible for the Problem $P_2^*(r)$.

It follows from Lemmas 3 and 4 that finding the minimal harmonic measure of Problem $P_2^*(r)$, we may restrict ourselves with sets of wires L, which are symmetric with respect to $\mathbf{R}$ and lay on $\overline{\mathbf{D}}_-$. The connected extremal sets of wires are completely described by Theorem 1. Below in this section we consider only disconnected sets of wires $L \subset \overline{\mathbf{D}}_-$, which are symmetric with respect to $\mathbf{R}$. Let $L = l_+ \cup l_-$, where $l_+ = L \cap \mathbf{H}_+$, $l_- = L \cap \mathbf{H}_-$. The domain $\Omega(L) = \mathbf{D} \setminus L$ can be considered as a quadrilateral having l_+ and l_- as its pair of non-distinguished sides. Since $\pm ir \in L$, the module of $\Omega(L)$ satisfies the inequality $0 < \operatorname{mod}(\Omega(L)) \le m(r)$. Theorem 3 below solves the problem 1.3 over the subclass of symmetric with respect to $\mathbf{R}$ disconnected sets of wires L such that $\mathrm{mod}(\Omega(L)) = m$, where m, $0 < m \le m(r)$, is fixed.

Theorem 3. *Let r and m be fixed such that $0 < r < 1$, $0 < m \le m(r)$. Let $L \subset \overline{\mathbf{D}}_-$ be a symmetric disconnected set of wires admissible for the Problem $P_2^*(r)$ such that* $\mathrm{mod}(\Omega(L)) = m$. *Then*

$$\omega_L(0) \ge \omega_{L_r(m)}(0). \tag{7.3}$$

Equality occurs in (7.3) if and only if $L = L_r(m)$.

Proof. Arguing by contradiction, suppose that there is a set of wires $L \ne L_r(m)$ satisfying the assumptions of the theorem such that

$$\omega_L(0) \le \omega_{L_r(m)}(0). \tag{7.4}$$

Since $\mathrm{mod}(\Omega(L)) = \mathrm{mod}(\Omega_r(m))$, there is a function f, which maps $\Omega_r(m)$ conformally onto $\Omega(L)$ such that $f(L_r(m)) = L$. Since $\Omega(L)$ and $\Omega_r(m)$ are symmetric with respect to $\mathbf{R}$ we have the inequality $-1 < f(0) < 1$. It follows from the polarization comparison theorem for harmonic measure [23, Theorem 2] that $\omega(x, L, \Omega(L))$ strictly decreases on the interval $0 \le x \le 1$. Therefore, (7.4) implies that $f(0) \le 0$. Let $\{G_0^*, G_1^*, G_2^*\}$ be the triple extremal for the module Problem $M3Q$ with $\alpha = \alpha(m)$. Since $f(0) \le 0$, one can easily see that the triple $\{f(G_0^*), f(G_1^*), f(G_2^*)\}$ is admissible for this problem. By Theorem 2, the extremal triple of the module Problem $M3Q$ is unique. Since $L \ne L_r(m)$, the uniqueness and conformal invariance of moduli imply the following inequality

$$\begin{aligned} M(r, \alpha(m)) &= (\alpha(m))^2 \,\mathrm{mod}(G_0^*) + \mathrm{mod}(G_1^*) + \mathrm{mod}(G_2^*) \\ &= (\alpha(m))^2 \,\mathrm{mod}(f(G_0^*)) + \mathrm{mod}(f(G_1^*)) + \mathrm{mod}(f(G_2^*)) < M(r, \alpha(m)), \end{aligned}$$

which, of course, is absurd. Therefore, our assumption (7.4) is not satisfied. The proof is complete. □

8. Computation of the harmonic measure and module

As Theorems 1 and 3 show, for every r and m such that $0 < r < 1$, $0 \le m \le m(r)$, there is a unique (up to reflection in the imaginary axis) set of wires $L_r(m)$, which minimizes the harmonic measure at $z = 0$. These theorems also provide qualitative solutions, in terms of related quadratic differentials, to the corresponding problems on the minimal harmonic measure. In this section, we discuss how to compute the minimal harmonic measure $\omega_{L_r(m)}(0)$ numerically.

For computational purposes, it is convenient to transplant the quadratic differentials (5.2) and (5.3) and the corresponding module problems into two auxiliary complex planes $\mathbf{C}_\zeta$ and $\mathbf{C}_w$.

Let $\zeta = g_1(z)$, where $g_1(z) = 2z/(1+z^2)$. In terms of ζ, the quadratic differential differentials (5.2) and (5.3) both take the same form

$$Q_1(\zeta, t, q)\, d\zeta^2 = -\frac{\zeta + q}{\zeta(\zeta^2 - 1)(\zeta^2 + t^2)}\, d\zeta^2. \tag{8.1}$$

Here $t = 2r/(1 - r^2)$ and $q = 2p/(1 + p^2)$ if $Q(z, r, \alpha)\, dz^2$ has the form (5.2) or $q = -\csc\varphi$ if it has the form (5.3). We note also that the range $0 < q \le 1$ corresponds to the form (5.2) while the range $1 < q \le q_0$, where $q_0 = -\csc(\varphi(r))$, corresponds to the form (5.3).

The module Problem $M3Q$ is equivalent to the following module problem on the extremal partition of $\mathbf{C}_\zeta$. For given $t > 0$ and $\alpha \geq 1$, find the maximum

$$M_1(t, \alpha) = \max\left\{\alpha^2 \operatorname{mod}(D_0) + 4 \operatorname{mod}(D_1)\right\} \tag{8.2}$$

over all pairs $\{D_0, D_1\}$ of doubly-connected domains D_0 and D_1 on $\mathbf{C}'_\zeta = \overline{\mathbf{C}} \setminus \{0, \pm 1, \pm it\}$ such that the curves γ separating the boundary components of D_0 are homotopic on $\mathbf{C}'_\zeta$ to narrow ellipses having foci at 0 and 1 and the curves γ separating the boundary components of D_1 are homotopic on $\mathbf{C}'_\zeta$ to narrow ellipses having foci at ± 1. Figures 6 and 7 show domain configurations of the quadratic differential $Q_1(\zeta, t, q)\, d\zeta^2$ for some typical values of t and q.

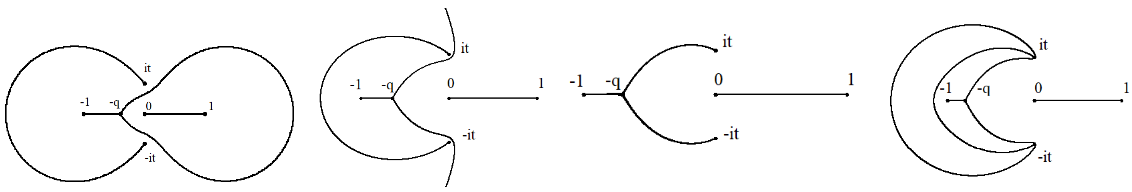

Figure 6. Geometry of critical trajectories of $Q_1(\zeta, t, q)\, d\zeta^2$ for $t > 0$ and $0 < q < 1$.

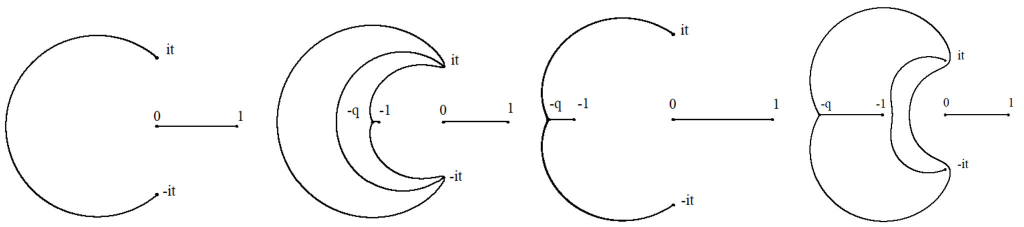

Figure 7. Geometry of critical trajectories of $Q_1(\zeta, t, q)\, d\zeta^2$ for $t > 0$ and $q \geq 1$.

Let $\{D_{\zeta,0}(t, \alpha), D_{\zeta,1}(t, \alpha)\}$ be the extremal pair of domains of the problem (8.2). The problems (6.1) and (8.2) correspond each other in the following sense: $g_1(G_0(r, \alpha)) = D_{\zeta,0}(t, \alpha)$, $g_1(G_1(r, \alpha)) = D_{\zeta,1}(t, \alpha) \cap \mathbf{H}_+$, and $g_1(G_2(r, \alpha)) = D_{\zeta,1}(t, \alpha) \cap \mathbf{H}_-$. Thus, the pair of the quadrilaterals $g_1(G_1(r, \alpha))$ and $g_1(G_2(r, \alpha))$, being glued along the corresponding intervals of the real axis, form the doubly-connected domain $D_{\zeta,1}(t, \alpha)$.

Below we assume that $D_{\zeta,1}(t, \alpha) \neq \emptyset$. Let $\gamma_{\zeta,1} = \gamma_{\zeta,1}(t, \alpha)$ denote the boundary component of $D_{\zeta,1}(t, \alpha)$ joining the points it and $-it$. Let $w = g_2(\zeta)$ be a conformal mapping from the domain $D_\zeta(t, \alpha) = \overline{\mathbf{C}} \setminus \gamma_{\zeta,1}(t, \alpha)$ onto the domain $D_w^\tau = \mathbf{C}_w \setminus ((-\infty, -1/\tau] \cup [1/\tau, \infty))$ with some $0 < \tau < 1$ such that $g_2(-1) = -1$, $g_2(1) = 1$. Existence of such mapping g_2 follows from the Riemann mapping theorem but the parameter τ can not be arbitrarily prescribed. Its value $\tau = \tau(t, \alpha)$ is uniquely determined by the following equation for the moduli of the corresponding doubly-connected domains:

$$\operatorname{mod}(\overline{\mathbf{C}}_\zeta \setminus (\gamma_{\zeta,1}(t, \alpha) \cup [-1, 1])) = \operatorname{mod}(\mathbf{C}_w \setminus ((-\infty, -1/\tau] \cup [-1, 1] \cup [1/\tau, \infty))). \tag{8.3}$$

Using the reflection principle for the module of a quadrilateral and equation (8.3), we obtain the following relation between the parameters m and τ:

$$m = 4 \operatorname{mod}(\mathbf{C}_w \setminus ((-\infty, -1/\tau] \cup [-1, 1] \cup [1/\tau, \infty))) = \frac{\mathcal{K}'(\tau)}{\mathcal{K}(\tau)}. \tag{8.4}$$

Changing variables in (8.1) via conformal mapping $w = g_2(\zeta)$, we obtain an equivalent quadratic differential $Q_2(w,\tau,a,b)\,dw^2$ defined in the w-plane:

$$Q_2(w,\tau,a,b)\,dw^2 = \frac{w-b}{(w-a)(w^2-1)(w^2-1/\tau^2)}\,dw^2, \tag{8.5}$$

where $a = g_2(0)$ and $b = g_2(-q)$ are such that $-\frac{1}{\tau} < b < a < 1$.

Moreover, we claim that $a > 0$. Indeed, since $\gamma_{\zeta,1}(t,\alpha)$ is a subset of the closed left half-plane, it follows from the polarization comparison theorem for harmonic measure [23, Theorem 2] that

$$\mathrm{mod}(\overline{\mathbf{C}}_\zeta \setminus (\gamma_{\zeta,1}(t,\alpha) \cup [-1,0])) < \mathrm{mod}(\overline{\mathbf{C}}_\zeta \setminus (\gamma_{\zeta,1}(t,\alpha) \cup [0,1])).$$

Since the module of a doubly connected domain is conformally invariant, the latter inequality implies that

$$\mathrm{mod}(\mathbf{C}_w \setminus ((-\infty,-\tfrac{1}{\tau}] \cup [-1,a] \cup [\tfrac{1}{\tau},\infty))) < \mathrm{mod}(\mathbf{C}_w \setminus ((-\infty,-\tfrac{1}{\tau}] \cup [a,1] \cup [\tfrac{1}{\tau},\infty))),$$

which, in turn, implies that $a > 0$.

The module problem corresponding to the quadratic differential (8.5) is the following. For given τ, a, and α such that $0 < \tau < 1$, $0 \le a < 1$, $\alpha \ge 1$, find the maximum

$$M_2(\tau,a,\alpha) = \max\left\{\alpha^2\,\mathrm{mod}(D_{w,0}) + 4\,\mathrm{mod}(D_{w,1})\right\} \tag{8.6}$$

over all pairs $\{D_{w,0}, D_{w,1}\}$ of doubly-connected domains $D_{w,0}$ and $D_{w,1}$ on the punctured plane $\mathbf{C}'_w = \mathbf{C} \setminus \{a, \pm 1, \pm 1/\tau\}$ such that the curves γ separating the boundary components of $D_{w,0}$ are homotopic on $\mathbf{C}'_w$ to narrow ellipses having foci at a and 1 and the curves γ separating the boundary components of $D_{w,1}$ are homotopic on $\mathbf{C}'_w$ to narrow ellipses having foci at ± 1.

Let $\{D_{w,0}(\tau,a,\alpha), D_{w,1}(\tau,a,\alpha)\}$ be the extremal pair of doubly-connected domains of the problem (8.6). Since the module of a doubly-connected domain is invariant under conformal mapping it follows that $D_{w,k}(\tau,a,\alpha) = g_2(D_{\zeta,k}(t,\alpha))$, $k = 0,1$. Figure 8 shows domain configurations of the quadratic differential $Q_2(w,\tau,a,b)\,dw^2$ for some typical values of τ, a, and b.

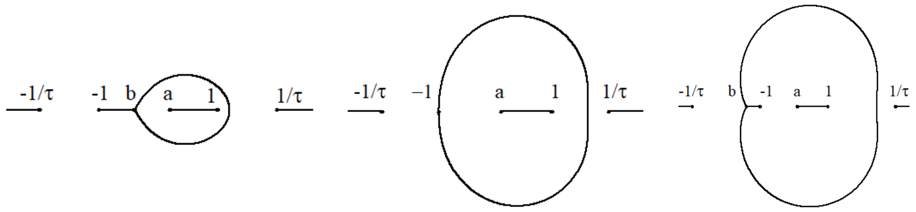

Figure 8. Geometry of critical trajectories of $Q_2(w,\tau,a,b)\,dw^2$ for $0 < \tau < 1$, $0 < a < 1$, $-1/\tau < b < a$.

Let $g = g_2 \circ g_1$. Then g maps $\Omega(r,\alpha)$ conformally onto $D^1_w = \mathbf{C}_w \setminus ((-\infty,-1] \cup [1,\infty))$ such that $g(0) = a$ and $g(\gamma^{\pm}(r,\alpha)) = \overline{\mathbf{R}} \setminus (-1/\tau, 1/\tau)$. Since the harmonic measure is conformally invariant, we have

$$\omega_r(m) = \omega(a, \overline{\mathbf{R}} \setminus (-1/\tau, 1/\tau), D^1_w). \tag{8.7}$$

To compute the harmonic measure in the right-hand side of (8.7), we consider the function $w_1 = g_3(w)$, where $g_3(w) = \sqrt{\frac{w-1}{w+1}}$ with the branch of the radical defined by $g_3(0) = i$. The function g_3 maps D^1_w conformally onto the upper half plane $\mathbf{H}_+$.

Let $g_3(a) = iA$, $g_3(1/\tau \pm i0) = \pm\kappa$. Then $A = \sqrt{\frac{1-a}{1+a}} > 0$, $0 < \kappa = \sqrt{\frac{1-\tau}{1+\tau}} < 1$ and $g_3(-1/\tau \pm i0) = \pm 1/\kappa$.

Using symmetry and conformal invariance of the harmonic measure we obtain

$$\omega(a, \overline{\mathbf{R}} \setminus (-1/\tau, 1/\tau), D_w^1) = 2\omega(iA, [\kappa, 1/\kappa], \mathbf{H}_+) = \frac{2}{\pi}\left(\arctan\frac{1}{\kappa A} - \arctan\frac{\kappa}{A}\right).$$

This after some algebra gives the desired formula for the harmonic measure $\omega_r(m)$:

$$\omega_r(m) = \frac{2}{\pi}\left(\arctan\sqrt{\frac{(1+\tau)(1+a)}{(1-\tau)(1-a)}} - \arctan\sqrt{\frac{(1-\tau)(1+a)}{(1+\tau)(1-a)}}\right). \tag{8.8}$$

We recall that τ is related to m via formula (8.4). Next, we discuss how the parameters τ, a, and b defined above can be calculated for given values of t and q. Since t and q are related to r and m, the latter will also provide a way to express a and b in terms of r and m.

First, we assume that $0 < r \le 2 - \sqrt{3}$. In this case, r and m are in a one-to-one correspondence with the parameters t, $0 < t \le \sqrt{3}/3$, and q, $0 < q \le \frac{2p(r)}{1+p^2(r)}$, of the quadratic differential (8.5). Thus, our goal is to express τ, a, and b as functions of t and q. Using the fact that Q_1-lengths of the critical trajectories (orthogonal trajectories) of $Q_1(\zeta)\,d\zeta^2$ are proportional to Q_2-lengths of the corresponding critical trajectories (orthogonal trajectories) of $Q_2(w)\,dw^2$ and integrating over appropriate intervals, we obtain the following system of equations:

$$\frac{\int_{[-1,-q]}\sqrt{Q_1(\zeta)}\,d\zeta}{\int_{[0,1]}\sqrt{Q_1(\zeta)}\,d\zeta} = \frac{\int_{[-1,b]}\sqrt{Q_2(w)}\,dw}{\int_{[a,1]}\sqrt{Q_2(w)}\,dw}, \tag{8.9}$$

$$\frac{\int_{[-q,0]}\sqrt{Q_1(\zeta)}\,d\zeta}{\int_{[0,1]}\sqrt{Q_1(\zeta)}\,d\zeta} = \frac{\int_{[b,a]}\sqrt{Q_2(w)}\,dw}{\int_{[a,1]}\sqrt{Q_2(w)}\,dw}, \tag{8.10}$$

$$\frac{\Im\int_{[0,it]}\sqrt{Q_1(\zeta)}\,d\zeta}{\int_{[0,1]}\sqrt{Q_1(\zeta)}\,d\zeta} = \frac{\Im\int_{[1,1/\tau]}\sqrt{Q_2(w)}\,dw}{\int_{[a,1]}\sqrt{Q_2(w)}\,dw}. \tag{8.11}$$

Using appropriate parameterizations of the intervals of integration, we rewrite equations (8.9)–(8.11) in the following form, which includes only real-valued integrals:

$$\frac{\int_q^1\sqrt{\frac{x-q}{x(1-x^2)(x^2+t^2)}}\,dx}{\int_0^1\sqrt{\frac{x+q}{x(1-x^2)(x^2+t^2)}}\,dx} = \frac{\int_{-b}^1\sqrt{\frac{x+b}{(x+a)(1-x^2)(1-\tau^2x^2)}}\,dx}{\int_a^1\sqrt{\frac{x-b}{(x-a)(1-x^2)(1-\tau^2x^2)}}\,dx}, \tag{8.12}$$

$$\frac{\int_0^q\sqrt{\frac{q-x}{x(1-x^2)(x^2+t^2)}}\,dx}{\int_0^1\sqrt{\frac{x+q}{x(1-x^2)(x^2+t^2)}}\,dx} = \frac{\int_b^a\sqrt{\frac{x-b}{(a-x)(1-x^2)(1-\tau^2x^2)}}\,dx}{\int_a^1\sqrt{\frac{x-b}{(x-a)(1-x^2)(1-\tau^2x^2)}}\,dx}, \tag{8.13}$$

$$\frac{\int_0^t\sqrt{\frac{x+\sqrt{x^2+q^2}}{2x(1+x^2)(t^2-x^2)}}\,dx}{\int_0^1\sqrt{\frac{x+q}{x(1-x^2)(x^2+t^2)}}\,dx} = \frac{\int_1^{\frac{1}{\tau}}\sqrt{\frac{x-b}{(x-a)(x^2-1)(1-\tau^2x^2)}}\,dx}{\int_a^1\sqrt{\frac{x-b}{(x-a)(1-x^2)(1-\tau^2x^2)}}\,dx}. \tag{8.14}$$

It follows from our previous results, for fixed t and q, such that $0 < t \le t(r_0) = \sqrt{3}/3$, $0 < q \le \frac{2p(r)}{1+p^2(r)}$ with $p(r)$ defined by equation (5.4), there is a unique solution

$a = a(t,q)$, $b = b(t,q)$, $\tau = \tau(t,q)$ of the system of equations (8.12)–(8.14) such that

$$0 < a < 1, \quad -1 < b < a, \quad 0 < \tau < 1. \tag{8.15}$$

Suppose now that $2-\sqrt{3} < r < 1$. In this case, r and m are in a one-to-one correspondence with the parameters t, $\sqrt{3}/3 < t < \infty$, and q, $0 < q \le -\csc\varphi(r)$, where $\varphi(r)$ is defined by equation (5.6). In this case, if $0 < q \le 1$, then $a(t,q)$, $b(t,q)$, and $\tau(t,q)$ can be found from equations (8.12)–(8.14). In the case $1 < q \le -\csc\varphi(r)$, we use the proportionality of the Q_1-lengths and Q_2-lengths of appropriate intervals to find the following counterpart of system of equations (8.12)-(8.14):

$$\frac{\int_1^q \sqrt{\frac{q-x}{x(x^2-1)(x^2+t^2)}}\,dx}{\int_0^1 \sqrt{\frac{x+q}{x(1-x^2)(x^2+t^2)}}\,dx} = \frac{\int_1^{-b} \sqrt{\frac{x+b}{(x+a)(x^2-1)(1-\tau^2x^2)}}\,dx}{\int_a^1 \sqrt{\frac{x-b}{(x-a)(1-x^2)(1-\tau^2x^2)}}\,dx}, \tag{8.16}$$

$$\frac{\int_0^1 \sqrt{\frac{q-x}{x(1-x^2)(x^2+t^2)}}\,dx}{\int_0^1 \sqrt{\frac{x+q}{x(1-x^2)(x^2+t^2)}}\,dx} = \frac{\int_{-1}^{a} \sqrt{\frac{x-b}{(a-x)(1-x^2)(1-\tau^2x^2)}}\,dx}{\int_a^1 \sqrt{\frac{x-b}{(x-a)(1-x^2)(1-\tau^2x^2)}}\,dx}, \tag{8.17}$$

$$\frac{\int_0^t \sqrt{\frac{x+\sqrt{x^2+q^2}}{2x(1+x^2)(t^2-x^2)}}\,dx}{\int_0^1 \sqrt{\frac{x+q}{x(1-x^2)(x^2+t^2)}}\,dx} = \frac{\int_1^{\frac{1}{\tau}} \sqrt{\frac{x-b}{(x-a)(x^2-1)(1-\tau^2x^2)}}\,dx}{\int_a^1 \sqrt{\frac{x-b}{(x-a)(1-x^2)(1-\tau^2x^2)}}\,dx}. \tag{8.18}$$

Once more, it follows from our previous results that for fixed t and q, such that $\sqrt{3}/3 < t < \infty$, $1 < q \le -\csc\varphi(r)$ with $\varphi(r)$ defined by equation (5.6), there is a unique solution $a = a(t,q)$, $b = b(t,q)$, $\tau = \tau(t,q)$ of the system of equations (8.16)-(8.18) such that

$$0 < a < 1, \quad -1/\tau < b < -1, \quad 0 < \tau < 1. \tag{8.19}$$

Computational Problem. Systems of equations, similar to systems (8.12)–(8.14) and (8.16)–(8.18), arise quite often in extremal problems related to quadratic differentials. It appears that numerical solution of such problems is rather challenging, especially when the number of unknown parameters is relatively large (say greater than 5). The reason here is, of course, the presence of singularities inside of the integrals. It will be very useful to have a program build in "Mathematica" or "MATLAB", which can be used to solve systems of equations as above with reasonably large number of integral equations with singularities.

Remark 4. Numerical computation of the harmonic measure usually requires numerical computation of relevant conformal mappings, which is rather difficult problem. An interesting approach to compute the harmonic measure, capacity of a condenser and some other conformally invariant characteristics of a planar domain was recently suggested by Nasser and Vuorinen [21], [22].

9. Relative growth of the harmonic measure and module

Summarizing our findings in the previous sections, we obtain the following qualitative solution of Problem $P_2^*(r)$ for every fixed r, $0 < r < 1$. If L is an extremal set of wires for (1.3) with $n = 2$, then either $L \subset \overline{\mathbf{D}}_-$ or $L \subset \overline{\mathbf{D}}_+$. In the latter case, the set of wires $-L = \{z\colon\ -z \in L\}$ is also extremal and $-L \subset \overline{\mathbf{D}}_-$. If $L \subset \overline{\mathbf{D}}_-$, then $L = L_r(m')$ for some $m' \in [0, m(r)]$. Thus, every extremal set of wires up to a symmetry with respect to the imaginary axis belongs to the parametric family of sets of wires $\{L_r(m)\colon 0 \le m \le m(r)\}$ related to the module Problem $M3Q$. Since the harmonic measure $\omega(z, E, \Omega)$ is an increasing function of E, we conclude in addition that for $r \in (r_0, 1)$ each extremal set of wires is disconnected.

This discussion shows that in order to find the minimal harmonic measure $\omega_2^*(r)$ and identify the extremal sets of wires, we have to find the absolute minimum of the function $\omega_r(m) := \omega_{L_r(m)}(0)$ on the interval $0 \le m \le m(r)$ if $0 < r \le r_0$ or on the interval $m_1(r) \le m \le m(r)$ if $r_0 < r < 1$. Thus, on this stage we have to deal with a standard calculus problem for a complicated transcendental function of the parameter m defined implicitly.

If the harmonic measure $\omega_r(m)$ defined by equation (8.8) achieves its minimum for some values of τ and m then we must have $d\omega_r(m) = 0$ for these values. Differentiating (8.8) and then simplifying, we find that the latter equation is equivalent to the equation

$$a\tau(1-\tau^2)\, da = (1-a^2)\, d\tau. \tag{9.1}$$

As we have shown in Section 8, the quadratic differentials (8.1) and (8.5) are related to the extremal problems (8.2) and (8.6). Therefore, for fixed t, the zeros and some poles of these quadratic differentials, as well as the module m of the correspondent quadrilateral $\Omega_r(m)$ are actually functions of the weight $\alpha \ge 1$. To emphasize this dependence, we will write $\tau = \tau(\alpha)$, $a = a(\alpha)$, $m = m(\alpha)$, etc. Thus, if $\omega_r(m(\alpha))$ achieves its minimum at $\alpha = \alpha_0$, then the functions $a(\alpha)$ and $\tau(\alpha)$ satisfy the equation (9.1) at $\alpha = \alpha_0$.

Next, we will discuss how the weighted sum of moduli $M_2(\tau, a, \alpha)$ depend on α. Since

$$M_2(\tau(t,\alpha), a(t,\alpha), \alpha) = M_1(t,\alpha),$$

differentiating we obtain

$$\frac{\partial M_2}{\partial \tau}\frac{\partial \tau}{\partial \alpha} + \frac{\partial M_2}{\partial a}\frac{\partial a}{\partial \alpha} + \frac{\partial M_2}{\partial \alpha} = \frac{\partial M_1}{\partial \alpha}. \tag{9.2}$$

To find $\partial M_2/\partial \alpha$ and $\partial M_1/\partial \alpha$, we apply the differentiation formula of Theorem 5.1 in [27]. Then we obtain

$$\frac{\partial M_1}{\partial \alpha} = 2\alpha \operatorname{mod}(D_{\zeta,0}), \quad \frac{\partial M_2}{\partial \alpha} = 2\alpha \operatorname{mod}(D_{w,0}). \tag{9.3}$$

Since $D_{\zeta,0}$ and $D_{w,0}$ are conformally equivalent, we have $\operatorname{mod}(D_{\zeta,0}) = \operatorname{mod}(D_{w,0})$. Using this and (9.3), we simplify (9.2) as follows

$$\frac{\partial M_2}{\partial a}\, da = -\frac{\partial M_2}{\partial \tau}\, d\tau. \tag{9.4}$$

To find $\partial M_2/\partial a$ and $\partial M_2/\partial \tau$, we consider M_2 as a function of simple poles a, $v_1 = 1/\tau$, and $v_2 = -1/\tau$ of the quadratic differential (8.5). Then applying the differentiation formula of Theorem 5.2 in [27], we find:

$$\frac{\partial M_2}{\partial a} = \pi C \lim_{w\to a}(Q_2(w,\tau,a,b)(w-a)) = \pi C \frac{\tau^2(a-b)}{(1-a^2)(1-a^2\tau^2)}, \tag{9.5}$$

$$\frac{\partial M_2}{\partial v_1} = \pi C \lim_{w\to 1/\tau}(Q_2(w,\tau,a,b)(w-1/\tau)) = \pi C \frac{\tau^3(1-b\tau)}{2(1-\tau^2)(1-a\tau)}, \tag{9.6}$$

$$\frac{\partial M_2}{\partial v_2} = \pi C \lim_{w\to -1/\tau}(Q_2(w,\tau,a,b)(w+1/\tau)) = -\pi C \frac{\tau^3(1+b\tau)}{2(1-\tau^2)(1+a\tau)}. \tag{9.7}$$

Here $C = C(\tau, a, b) > 0$ is a normalizing constant defined by the following condition:

$$C^{1/2} = \alpha / \left(2 \int_a^1 \sqrt{Q_2(w, \tau, a, b)}\, dw \right).$$

Using (9.6) and (9.7) after some algebra, we obtain

$$\frac{\partial M_2}{\partial \tau} = \frac{\partial M_2}{\partial v_1}\frac{dv_1}{d\tau} + \frac{\partial M_2}{\partial v_2}\frac{dv_2}{d\tau} = -\pi C \frac{\tau(1 - ab\tau^2)}{(1-\tau^2)(1-a^2\tau^2)}. \tag{9.8}$$

Finally, substituting (9.5) and (9.8) into (9.4) and simplifying, we obtain the following equation for the functions $\tau = \tau(t, \alpha)$ and $a = a(t, \alpha)$:

$$\tau(1-\tau^2)(a-b)\, da = (1-a^2)(1-ab\tau^2)\, d\tau. \tag{9.9}$$

Comparing equations (9.1) and (9.9), we obtain the following necessary condition for the quadratic differential (8.5) if it corresponds to the minimum of the harmonic measure in the problem P_2^* for a fixed r, $0 < r < 1$.

Lemma 6. *For a given $0 < r < 1$, let $Q_2(w, a, b, \tau)\, dw^2$ be a transplantation into the w-plane of the quadratic differential $Q(z, r, \alpha)\, dz^2$. If $Q(z, r, \alpha)\, dz^2$ corresponds to a disconnected set of wires extremal for the Problem $P_2^*(r)$, then $b = b(r, \alpha) = 0$.*

10. Further discussion and questions

It follows from our discussions in the previous sections that an extremal configuration of Problem P_2^* exists, is not unique, and every such configuration consists of arcs of critical trajectories of a related quadratic differential. In the case of a general domain D with n controlling points $b_k \in D$, we suggest the following steps toward a solution of Problem $P1$.

Problem *P*2.

(a) Under the assumptions of Problem $P1$, prove that there is at least one configuration of wires $L \in \mathcal{L}(a, B, D)$ minimizing the harmonic measure in (1.2).

(b) Prove that every configuration of wires, extremal for Problem $P1$, consists of arcs of trajectories of some Jenkins–Strebel quadratic differential.

Next, we want to recall a generalization of the Gonchar's problem mentioned in the Introduction, which was suggested by Baernstein II [2].

Problem *P*3. *Let $\Theta_n = (\theta_1, \theta_2, \ldots, \theta_n)$ denote the set of angles $0 \le \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi$ and let $\Theta_n^* = (\theta_1^*, \theta_2^*, \ldots, \theta_n^*)$ with $\theta_k^* = 2\pi k/n$, denote the set of equally distributed angles. Let E be a compact set on the interval $(0, 1]$ such that $\omega(0, E, \mathbf{D} \setminus E) > 0$, let $E_{\Theta_n} = \bigcup_{k=1}^n \left(e^{i\theta_k} E\right)$, and let $E_{\Theta_n^*} = \bigcup_{k=1}^n \left(e^{2\pi i(k-1)/n} E\right)$. Prove that*

$$\omega(0, E_{\Theta_n}, \mathbf{D} \setminus E_{\Theta_n}) \le \omega(0, E_{\Theta_n^*}, \mathbf{D} \setminus E_{\Theta_n^*}) \tag{10.1}$$

with equality sign if and only if E_{Θ_n} coincides with $E_{\Theta_n^*}$ up to rotation about the origin. Baernstein II suggested even a more general problem to prove the following inequality for integral means:

$$\int_0^{2\pi} \Phi(\omega(re^{i\theta}, E_{\Theta_n}, \mathbf{D} \setminus E_{\Theta_n}))\, d\theta \le \int_0^{2\pi} \Phi(\omega(re^{i\theta}, E_{\Theta_n^*}, \mathbf{D} \setminus E_{\Theta_n^*}))\, d\theta,$$

where $0 < r < 1$ and $\Phi(t)$ is a non-negative, non-decreasing, convex function.

Problem $P3$ is more challenging than the original Gonchar's problem because now we have to work with multiply connected domains $\mathbf{D} \setminus E_{\Theta_n}$ while in the original problem these domains were simply connected. Baernstein himself solved Problem $P3$ for $n = 2$ and $n = 3$. The only known case when the inequality (10.1) is proved for all n for multiply connected domains is when $E = [r_1, r_2]$ with $0 < r_1 < r_2 < 1$, see [26].

The following version of Gonchar's problem also presents significant interest.

Problem *P*4. For $0 < r_1 < r_2 < 1$, let $F = [r_1, 1]$, $E = [r_1, r_2]$. Prove that

$$\omega(0, E_{\Theta_n}, \mathbf{D} \setminus F_{\Theta_n}) \le \omega(0, E_{\Theta_n^*}, \mathbf{D} \setminus F_{\Theta_n^*}) \tag{10.2}$$

with equality sign if and only if E_{Θ_n} coincides with $E_{\Theta_n^*}$ up to rotation about the origin.

We note that, although Problem $P4$ deals with the harmonic measure $\omega(0, E_{\Theta_n}, \mathbf{D} \setminus F_{\Theta_n})$ considered with respect to a simply connected domain, Dubinin's dissymmetrization approach (in its original form) fails and therefore additional new ideas are needed.

We want to mention one more problem, about $n \ge 2$ wires growing in $\mathbf{D}$ starting at the points $z_k = e^{2\pi i(k-1)/n}$, which is in line with previous problems.

Problem *P*5. Find

$$M_n^*(s) = \max_L \omega(0, L, \mathbf{D} \setminus L), \tag{10.3}$$

where the maximum is taken over all sets $L = \cup_{k=1}^n l_k$ consisting of $n \ge 2$ Jordan arcs $l_k \subset \overline{\mathbf{D}}$ each of length s, $0 < s < \min\{1, 2\pi/n\}$, such that l_k has its initial point at $z = e^{2\pi i(k-1)/n}$. Describe all sets L, for which the maximum in (10.3) is attained. The problem to find exact values of $M_n^*(s)$ for all s sounds very difficult to this author. A more realistic problem is to find asymptotic of $M_n^*(s)$ as $s \to 0$ and as $s \to \min\{1, 2\pi/n\}$.

For the case of one arc, i.e. when $n = 1$, problem (10.3) was solved in [24]. In this case, the straight line segment $[1 - s, 1]$ is extremal for all s, $0 < s < 1$. It is clear that the union of straight line segments $[(1 - s)e^{2\pi i(k-1)/n}, e^{2\pi i(k-1)/n}]$, $k = 1, \ldots, n$, is not extremal when $n > 6$ and s is close to $\frac{2\pi}{n}$. Thus, an extremal configuration is not unique and is not trivial in this case. The latter makes Problem $P5$ rather challenging.

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Received 28 May 2020 • Accepted 5 September 2020