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BILIPSCHITZ MAPPINGS AND QUASIHYPERBOLIC MAPPINGS IN REAL BANACH SPACES

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Abstract. Suppose that $G \subsetneq E$ and $G' \subsetneq E'$ are domains, where E and E' denote real Banach spaces with dimension at least 2, and $f\colon G \to G'$ is a homeomorphism. The aim of this paper is to prove the validity of the implications: f is M-bilipschitz $\Rightarrow$ f is locally M-bilipschitz $\Rightarrow$ f is M-QH $\Rightarrow$ f is locally M-QH, and the invalidity of their opposite implications, i.e., f is locally M-QH $\not\Rightarrow$ f is M-QH $\not\Rightarrow$ f is locally M-bilipschitz $\not\Rightarrow$ f is M-bilipschitz. Among these results, the relationship that f is locally M-QH $\not\Rightarrow$ f is M-QH gives a negative answer to one of the open problems raised by Väisälä in 1999.

1. Introduction

Throughout the paper, we always assume that E and E' denote real Banach spaces with dimension at least 2, $G \subsetneq E$ and $G' \subsetneq E'$ are domains (open and connected sets). The norm of a vector z in E is written as $|z|$, and for every pair of points z_1, z_2 in E, the distance between them is denoted by $|z_1 - z_2|$.

The *quasihyperbolic length* of a rectifiable arc or path α in the norm metric in G is the number

$$\ell_k(\alpha) = \int_\alpha \frac{|dz|}{d_G(z)},$$

where $d_G(z)$ denotes the distance from z to the boundary ∂G of G in E. For each pair of points z_1, z_2 in G, the *quasihyperbolic distance* $k_G(z_1, z_2)$ between z_1 and z_2 is defined in the usual way:

$$k_G(z_1, z_2) = \inf \ell_k(\alpha),$$

where the infimum is taken over all rectifiable arcs α joining z_1 to z_2 in G.

Suppose that $f\colon G \to G'$ is a homeomorphism and $M \geq 1$ denotes a constant. The mapping f is said to be

(1) *M-bilipschitz* if

$$\frac{1}{M}|z_1 - z_2| \leq |z_1' - z_2'| \leq M|z_1 - z_2|$$

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for all $z_1, z_2 \in G$. Here and hereafter, the primes always denote the images in G' of the points in G under the mapping f;

(2) *locally M-bilipschitz* if each point $z \in G$ has a neighborhood D in G such that the restriction $f|_D$ of f in D is M-bilipschitz;

(3) *M-QH* if it is M-bilipschitz in the quasihyperbolic metric, i.e.,

$$\frac{1}{M} k_G(z_1, z_2) \leq k_{G'}(z_1', z_2') \leq M k_G(z_1, z_2)$$

for all $z_1, z_2 \in G$;

(4) *locally M-QH* if f is locally M-bilipschitz in the quasihyperbolic metric.

A mapping f is said to have *fully a given property* if for every subdomain Ω of G, the restriction $f|_\Omega$ of f has this property in Ω. For example, we obtain *fully M-QH* mappings, *fully φ-solid* (briefly, *φ-FQC*) mappings etc (see [6], for example, for the definitions of solid mappings and FQC mappings).

The class of QH mappings is useful in the quasiconformal (briefly, QC) theory of $\mathbf{R}^n$. For example, the Beurling–Ahlfors extension of a quasisymmetric (briefly, QS) mapping of $\mathbf{R}^1$ to a half plane is QH [2]. This property was used by Ahlfors to obtain a bilipschitz reflection across a quasicircle through ∞ [1]. When $n \neq 4$, the classes of QH mappings, QC mappings and solid mappings are rather close to each other (cf. [5, Theorem 7.4]). The class of QH mappings also plays an important role in the freely quasiconformal (briefly, FQC) theory of Banach spaces. For example, in [6], Väisälä proved that the concepts M-QH and fully M-QH are quantitatively equivalent ([6, Theorem 4.7]). This leads to the quantitative implications: M-QH $\Rightarrow$ φ-FQC $\Rightarrow$ φ-solid.

In [3], Heinonen and Koskela considered the problem whether the global QS structure of a space can be recaptured from a local or infinitesimal QC structure. Note that every QS mapping is FQC (QC in $\mathbf{R}^n$) and every QH mapping is FQC (QC in $\mathbf{R}^n$). Obviously, every M-QH mapping is locally M-QH. By analogy, a natural problem is whether the opposite implication is true or not. In fact, this problem was raised as an open problem by Väisälä in [7].

Open Problem 1.1. [7, Problem 13.2.13] Suppose that $f\colon G \to G'$ is a homeomorphism and each point has a neighborhood $D \subset G$ such that $f|_D\colon D \to f(D)$ is M-QH. Is f M'-QH with $M' = M'(M)$?

In [6], Väisälä obtained the following relationship between locally bilipschitz mappings and fully QH mappings.

Theorem A. [6, Theorem 4.8] *If a homeomorphism $f\colon G \to G'$ is locally M-bilipschitz in the norm metric, then f is fully M^2-QH.*

In this paper, we discuss the relationships among (locally) bilipschitz mappings and (locally) QH mappings. Our results are as follows.

Theorem 1.1. *Suppose that $f\colon G \to G'$ is a homeomorphism. Then*

(1) *the following implications are quantitatively true: f is M-bilipschitz $\Rightarrow$ f is locally M-bilipschitz $\Rightarrow$ f is M-QH $\Rightarrow$ f is locally M-QH.*

(2) *the opposite implications are invalid, i.e., f is locally M-QH $\nRightarrow$ f is M-QH $\nRightarrow$ f is locally M-bilipschitz $\nRightarrow$ f is M-bilipschitz.*

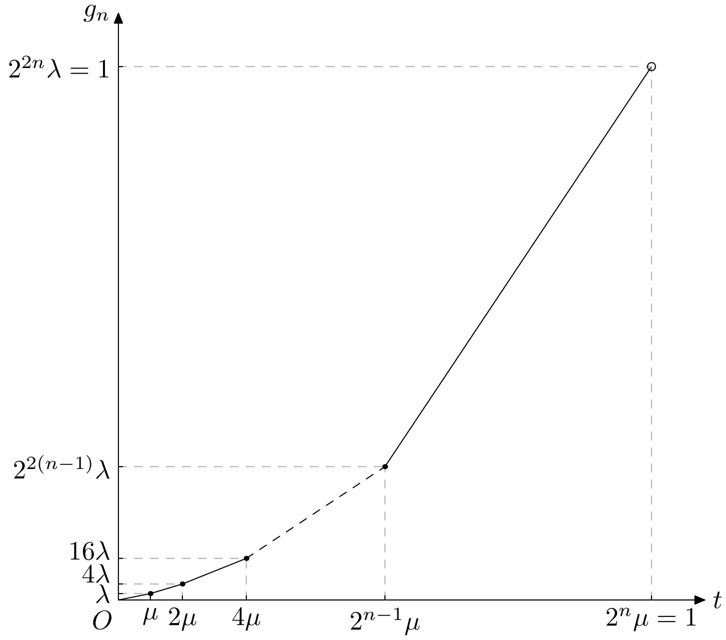
Here, for two conditions, we say that *Condition A implies Condition B quantitatively* if Condition A implies Condition B and the data $\gamma(B)$ of Codition B depends only on the data $\gamma(A)$ of Condition A.

By the statement (2) of Theorem 1.1, we see that the local quasihyperbolicity fails to imply the global quasihyperbolicity, quantitatively. This shows that the answer to Open Problem 1.1 is negative.

Theorem 1.1 is proved in the next section.

2. Proof of Theorem 1.1

We start this section with three examples.



$$(\mu = \tfrac{1}{2^n},\ \lambda = \tfrac{1}{2^{2n}})$$

Figure 1. The graph of g_n.

Example 2.1. Let $n \geq 3$ be an integer,

$$g_n(t) = \begin{cases} g_{n_j}(t) = \frac{3}{2^{j+1}}t - \frac{1}{2^{2j+1}}, & \text{if } t \in [\frac{1}{2^{j+1}}, \frac{1}{2^j}), \text{ where } j \in \{0, 1, \dots, n-1\}, \\ g_{n_n}(t) = \frac{1}{2^n}t, & \text{if } t \in [0, \frac{1}{2^n}) \end{cases}$$

(see Figure 1), and let

$$f_n(z) = \begin{cases} g_n(|z|)\frac{z}{|z|}, & \text{if } z \in \mathbf{D}\backslash\{0\}, \\ 0, & \text{if } z = 0, \end{cases}$$

where $\mathbf{D} = \{z \in \mathbf{C} \colon |z| < 1\}$, the unit disk in the complex plane $\mathbf{C}$, and $0 = (0, 0)$, the origin. Then we have the following:

(1) $g_n \colon [0, 1) \to [0, 1)$ is a homeomorphism with $g_n(0) = 0$ for each n.
(2) $f_n \colon \mathbf{D} \to \mathbf{D}$ is a radial homeomorphism for each n.
(3) For each n and every point $z_0 \in \mathbf{D}$, there is a neighborhood $\Omega \subset \mathbf{D}$ of z_0 such that the restriction $f_n|_\Omega$ is $\frac{256}{9}$-QH.
(4) For any constant $M \geq 1$, there exists a large integer $N \geq 3$ such that for all $n > N$, f_n are not M-QH.

Proof. For notational convenience, let

$$\mathbf{D}(z_0, r) = \{z \in \mathbf{C} \colon |z - z_0| < r\}$$

denote the open disk with center z_0 and radius $r > 0$. In particular, let $\mathbf{D}(r) = \mathbf{D}(0, r)$. Then $\mathbf{D} = \mathbf{D}(1)$. Also, we use the notation:

$$A(r_1, r_2) = \mathbf{D}(r_1) \backslash \overline{\mathbf{D}}(r_2)$$

for $0 < r_2 < r_1 \leq 1$, where $\overline{\mathbf{D}}(r_2)$ denotes the closure of $\mathbf{D}(r_2)$.

The first two assertions of the example directly follow from the definitions of g_n and f_n. To prove the last two assertions, we need some preparation which consists of the following three claims.

Claim 2.1.1. For each $n \geq 3$, the restriction $f_n|_{A(1,\frac{1}{4})}$ is $\frac{16}{3}$-bilipschitz.

Proof. Let z_1, $z_2 \in A(1, \frac{1}{4})$ with $z_1 \neq z_2$. We divide the proof into the following three cases.

Case 1. Suppose that $z_1, z_2 \in A(1, \frac{1}{2}) \cup \mathbf{S}(\frac{1}{2})$, where $\mathbf{S}(\frac{1}{2}) = \{z \in \mathbf{C} \colon |z| = \frac{1}{2}\}$.

By definition, we have that for $k \in \{1, 2\}$,

$$z_k' = f_n(z_k) = g_{n_0}(|z_k|)\frac{z_k}{|z_k|} = \frac{1}{2}(3|z_k| - 1)\frac{z_k}{|z_k|},$$

and so,

$$|z_1' - z_2'| = \frac{1}{2}\Big|3(z_1 - z_2) - \big(\frac{z_1}{|z_1|} - \frac{z_2}{|z_2|}\big)\Big|. \tag{2.1}$$

If $|z_1| = |z_2|$, then (2.1) implies

$$\frac{1}{2}|z_1 - z_2| \leq |z_1' - z_2'| \leq |z_1 - z_2|, \tag{2.2}$$

since $\frac{1}{2} \leq |z_1| < 1$.

If $\arg z_1 = \arg z_2$, then $\frac{z_1}{|z_1|} - \frac{z_2}{|z_2|} = 0$, and thus, by (2.1), we have

$$|z_1' - z_2'| = \frac{3}{2}|z_1 - z_2|. \tag{2.3}$$

Next, we consider the remaining case, that is, $|z_1| \neq |z_2|$ and $\arg z_1 \neq \arg z_2$. Without loss of generalization, we assume that

$$|z_1| < |z_2|.$$

Let $\xi \in [0, z_2]$ be such that $|\xi| = |z_1|$, where $[0, z_2]$ denotes the segment in $\mathbf{D}$ with the endpoints 0 and z_2. Obviously, for $k \in \{1, 2\}$,

$$|z_k - \xi| \leq |z_k - z_{k+1}| \quad \text{and} \quad |z_k' - \xi'| \leq |z_k' - z_{k+1}'|,$$

where $z_3 = z_1$. Then we infer from (2.2) and (2.3) that

$$|z_1' - z_2'| \leq |z_1' - \xi'| + |\xi' - z_2'| \leq \frac{5}{2}|z_1 - z_2|, \tag{2.4}$$

and

$$|z_1 - z_2| \leq |z_1 - \xi| + |\xi - z_2| \leq \frac{8}{3}|z_1' - z_2'|. \tag{2.5}$$

We conclude from (2.2)–(2.5) that

$$\frac{3}{8}|z_1 - z_2| \leq |z_1' - z_2'| \leq \frac{8}{3}|z_1 - z_2|. \tag{2.6}$$

Case 2. Suppose that $z_1, z_2 \in A(\frac{1}{2}, \frac{1}{4})$.

Under this assumption, we have that for $k \in \{1,2\}$,

$$z_k' = f_n(z_k) = g_{n_1}(|z_k|)\frac{z_1}{|z_k|} = \frac{1}{8}(6|z_k|-1)\frac{z_k}{|z_k|}.$$

This leads to

$$|z_1'-z_2'| = \frac{1}{8}\left|6(z_1-z_2) - \left(\frac{z_1}{|z_1|} - \frac{z_2}{|z_2|}\right)\right|.$$

Similar arguments as in Case 1 ensure the following: If $|z_1| = |z_2|$, then

$$\frac{1}{4}|z_1-z_2| \le |z_1'-z_2'| \le \frac{1}{2}|z_1-z_2|. \tag{2.7}$$

If $\arg z_1 = \arg z_2$, then

$$|z_1'-z_2'| = \frac{3}{4}|z_1-z_2|,$$

and if $|z_1| \ne |z_2|$ and $\arg z_1 \ne \arg z_2$, then

$$\frac{3}{16}|z_1-z_2| \le |z_1'-z_2'| \le \frac{5}{4}|z_1-z_2|.$$

Hence, in this case, we have

$$\frac{3}{16}|z_1-z_2| \le |z_1'-z_2'| \le \frac{16}{3}|z_1-z_2|. \tag{2.8}$$

Case 3. Suppose that $z_1 \in A(\frac{1}{2}, \frac{1}{4})$ and $z_2 \in A(1, \frac{1}{2}) \cup \mathbf{S}(\frac{1}{2})$.

In this case, we know that

$$z_1' = g_{n_1}(|z_1|)\frac{z_1}{|z_1|} = \frac{1}{8}(6|z_1|-1)\frac{z_1}{|z_1|} \quad \text{and} \quad z_2' = g_{n_0}(|z_2|)\frac{z_2}{|z_2|} = \frac{1}{2}(3|z_2|-1)\frac{z_2}{|z_2|}.$$

If $\arg z_1 = \arg z_2$, then

$$|z_1'-z_2'| = \big|g_{n_1}(|z_1|) - g_{n_0}(|z_2|)\big|. \tag{2.9}$$

Since

$$\frac{3}{4} \le \frac{\big|g_{n_1}(|z_1|) - g_{n_0}(|z_2|)\big|}{|z_1-z_2|} \le \frac{3}{2},$$

we know from (2.9) that

$$\frac{3}{4} \le \frac{|z_1'-z_2'|}{|z_1-z_2|} \le \frac{3}{2}. \tag{2.10}$$

In the following, we assume that $\arg z_1 \ne \arg z_2$. Let $\zeta \in [0, z_2]$ be such that

$$|\zeta| = |z_1|.$$

Obviously, for $k \in \{1,2\}$,

$$|z_k - \zeta| \le |z_k - z_{k+1}| \quad \text{and} \quad |z_k' - \zeta'| \le |z_k' - z_{k+1}'|,$$

where $z_3 = z_1$. Then we deduce from (2.7) and (2.10) that

$$|z_1'-z_2'| \le |z_1'-\zeta'| + |\zeta'-z_2'| \le 2|z_1-z_2|, \tag{2.11}$$

and

$$|z_1-z_2| \le |z_1-\zeta| + |\zeta-z_2| \le \frac{16}{3}|z_1'-z_2'|. \tag{2.12}$$

We conclude from (2.10)–(2.12) that

$$\frac{3}{16}|z_1-z_2| \le |z_1'-z_2'| \le \frac{16}{3}|z_1-z_2|. \tag{2.13}$$

Now, we know from (2.6), (2.8) and (2.13) that the restriction $f_n|_{A(1,\frac{1}{4})}$ is $\frac{16}{3}$-bilipschitz. $\square$

Let $j \in \mathbf{Z}$, where $\mathbf{Z}$ is the set of all integers, and let

$$T_j(z) = 2^j z$$

in $\mathbf{C}$. Define

$$f_{n,j} = \begin{cases} T_{2j} \circ f_n|_{A(\frac{1}{2^j},\frac{1}{2^{j+2}})} \circ T_{-j}, & \text{if } j \in \{0, 1, \ldots, n-2\}, \\ T_{n-1} \circ f_n|_{\mathbf{D}\left(\frac{1}{2^{n-1}}\right)}, & \text{if } j = n-1. \end{cases}$$

Note that $f_{n,n-1}$ is a self-homeomorphism of $\mathbf{D}\big(\frac{1}{2^{n-1}}\big)$.

By a similar reasoning as in the proof of Claim 2.1.1, we have the following claim.

Claim 2.1.2. $f_{n,n-1}$ is 4-bilipschitz.

Claim 2.1.3. For each $j \in \{0, 1, \ldots, n-2\}$, the following statement holds:

$$f_n|_{A(1,\frac{1}{4})} = f_{n,j}.$$

Proof. Let $z \in A(1, \frac{1}{4})$. For the proof, we consider two possibilities: $z \in A(1, \frac{1}{2}) \cup \mathbf{S}(\frac{1}{2})$ and $z \in A(\frac{1}{2}, \frac{1}{4})$. For the first possibility, we get

$$f_n|_{A(1,\frac{1}{4})}(z) = g_{n_0}(|z|)\frac{z}{|z|} = \frac{1}{2}(3|z|-1)\frac{z}{|z|}.$$

Moreover, since

$$T_{-j}(z) = 2^{-j}z \in A\left(\frac{1}{2^j}, \frac{1}{2^{j+1}}\right) \cup \mathbf{S}\left(\frac{1}{2^{j+1}}\right),$$

it follows that

$$\begin{aligned} f_{n,j}(z) &= T_{2j} \circ f_n|_{A(\frac{1}{2^j},\frac{1}{2^{j+2}})}\big(T_{-j}(z)\big) \\ &= T_{2j}\left(g_{n_j}(|2^{-j}z|)\frac{2^{-j}z}{|2^{-j}z|}\right) = \frac{1}{2}(3|z|-1)\frac{z}{|z|}, \end{aligned}$$

and so,

$$f_n|_{A(1,\frac{1}{2})\cup\mathbf{S}(\frac{1}{2})} = f_{n,j}.$$

For the remaining possibility, that is, $z \in A(\frac{1}{2}, \frac{1}{4})$, we obtain

$$f_n|_{A(1,\frac{1}{4})}(z) = g_{n_1}(|z|)\frac{z}{|z|} = \frac{1}{8}(6|z|-1)\frac{z}{|z|}.$$

On the other hand, since

$$T_{-j}(z) = 2^{-j}z \in A\left(\frac{1}{2^{j+1}}, \frac{1}{2^{j+2}}\right),$$

we deduce that

$$\begin{aligned} f_{n,j}(z) &= T_{2j} \circ f_n|_{A(\frac{1}{2^j},\frac{1}{2^{j+2}})}\big(T_{-j}(z)\big) \\ &= T_{2j}\left(g_{n_{j+1}}(|2^{-j}z|)\frac{2^{-j}z}{|2^{-j}z|}\right) = \frac{1}{8}(6|z|-1)\frac{z}{|z|}, \end{aligned}$$

from which it follows that

$$f_n|_{A(\frac{1}{2},\frac{1}{4})} = f_{n,j}.$$

Hence the claim is proved. $\square$

Now, we are ready to prove the third assertion of the example. Let $z_0 \in \mathbf{D}$. Then there is an integer $j \in \{0, 1, \dots, n-2\}$ such that $z_0 \in A(\frac{1}{2^j}, \frac{1}{2^{j+2}})$ or $z_0 \in \mathbf{D}\big(\frac{1}{2^{n-1}}\big)$.

If $z_0 \in A(\frac{1}{2^j}, \frac{1}{2^{j+2}})$, then by Claims 2.1.1 and 2.1.3, we know that $f_{n,j}$ is $\frac{16}{3}$-bilipschitz, and so, Theorem A ensures that $f_{n,j}$ is fully $\frac{256}{9}$-QH. Since

$$f_{n,j} = T_{2j} \circ f_n|_{A(\frac{1}{2^j}, \frac{1}{2^{j+2}})} \circ T_{-j},$$

and since both T_{2j} and T_{-j} are stretchings, we know that $f_n|_{A(\frac{1}{2^j}, \frac{1}{2^{j+2}})}$ is also fully $\frac{256}{9}$-QH. Therefore, there is a neighborhood $\Omega \subset A(\frac{1}{2^j}, \frac{1}{2^{j+2}})$ of z_0 such that the restriction $f_n|_\Omega$ is $\frac{256}{9}$-QH.

If $z_0 \in \mathbf{D}\big(\frac{1}{2^{n-1}}\big)$, then it follows from Claim 2.1.2 and Theorem A that $f_{n,n-1}$ is fully 16-QH. Thus, it follows that there is a neighborhood $\Omega \subset \mathbf{D}\big(\frac{1}{2^{n-1}}\big)$ of z_0 such that $f_n|_\Omega$ is 16-QH. These prove the third assertion of the example.

To prove the fourth assertion of the example, let $z_1 = 0$ and $z_2 = \frac{1}{2^n}$. Then

$$z_1' = f_n(z_1) = 0 \quad \text{and} \quad z_2' = f_n(z_2) = \frac{1}{2^{2n}}.$$

Since

$$k_{\mathbf{D}}(z_1, z_2) = \Big| \log \frac{d_{\mathbf{D}}(z_1)}{d_{\mathbf{D}}(z_2)} \Big| = \log \frac{2^n}{2^n - 1}$$

and

$$k_{\mathbf{D}}(z_1', z_2') = \Big| \log \frac{d_{\mathbf{D}}(z_1')}{d_{\mathbf{D}}(z_2')} \Big| = \log \frac{2^{2n}}{2^{2n} - 1},$$

we get

$$\frac{k_{\mathbf{D}}(z_1', z_2')}{k_{\mathbf{D}}(z_1, z_2)} \to 0$$

as $n \to +\infty$. This shows that the fourth assertion of the example is true, and hence, the example is proved. □

Example 2.2. Define the mapping $f\colon G \to G'$ by

$$f(z) = e^z,$$

where $G = \{z \in \mathbf{C}\colon -\frac{\pi}{2} < \operatorname{Im} z < \frac{\pi}{2}\}$ and $G' = \{z \in \mathbf{C}\colon \operatorname{Re} z > 0\}$. Then we have the following assertions.

(1) f is fully π^2-QH.

(2) For any constant $M \ge 1$, f is not locally M-bilipschitz.

Proof. It follows from [6, Example 4.11] that the mapping f is $\frac{\pi}{2}$-QH, and so, [6, Theorem 4.7] implies f is fully π^2-QH. This shows that the first assertion of the example is true.

Suppose on the contrary that there is an $M \ge 1$ such that f is locally M-bilipschitz. For each positive integer n, let $z_n = n$. Then there is a neighborhood $\Omega_n \subset G$ of z_n such that the restriction $f|_{\Omega_n}$ is M-bilipschitz. Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence such that (i) $\varepsilon_n > 0$ for each n, (ii) $\varepsilon_n \to 0$ as $n \to \infty$, and (iii) $\widetilde{z}_n = n + \varepsilon_n \in \Omega_n$. Then

$$\frac{|z_n' - \widetilde{z}_n'|}{|z_n - \widetilde{z}_n|} \le M \tag{2.14}$$

holds for any n, where $z_n' = f(z_n)$ and $\widetilde{z}_n' = f(\widetilde{z}_n)$. But

$$\frac{|z_n' - \widetilde{z}_n'|}{|z_n - \widetilde{z}_n|} = \frac{|e^n - e^{n+\varepsilon_n}|}{|n - (n + \varepsilon_n)|} \to +\infty \tag{2.15}$$

as $n \to +\infty$, which is the desired contradiction. Hence the second assertion in the example is also true. □

We borrow the following example from [6, Example 4.12].

Example 2.3. Let E be an infinite dimensional separable Hilbert space, and let $\{e_j\}_{j\in\mathbf{Z}}$ be an orthogonal basis of E indexed by $\mathbf{Z}$. Set

$$\gamma = \bigcup_{j\in\mathbf{Z}} \gamma_j \quad \text{and} \quad \gamma' = \bigcup_{j\in\mathbf{Z}} \gamma'_j,$$

where $\gamma_j = [a_j, a_{j+1}]$ with $a_j = j\sqrt{2}e_1$, which denotes the segment with the endpoints a_j and a_{j+1}, and $\gamma'_j = [e_j, e_{j+1}]$. Let $G = \bigcup_{z\in\gamma} \mathbf{B}(z,r)$, where $\mathbf{B}(z,r) = \{w \in E\colon |w-z| < r\}$ and $0 < r < \frac{1}{10}$. Then there exist a constant $M \geq 1$, a neighbourhood G' of γ' and a homeomorphism f such that

(1) $f\colon G \to G'$ is locally M-bilipschitz.
(2) f maps each γ_j isometrically onto γ'_j.
(3) f is not M'-bilipschitz for any constant $M' \geq 1$.

Proof. The first two statements in the example are from [6, Example 4.12]. From the first two statements, the third one follows because γ is unbounded, but γ' is bounded. □

Proof of Theorem 1.1. The first implication in Theorem 1.1(1) is obvious. By Theorem A, we know that every locally M-bilipschitz mapping is M^2-QH. It follows from [6, Theorem 4.7] that each M-QH mapping is fully $4M^2$-QH. Of course, it is locally $4M^2$-QH. These show that the rest two implications in Theorem 1.1(1) are also true. Theorem 1.1(2) easily follows from Examples 2.1−2.3. Hence the theorem is proved. □

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Added information. After the acceptance of this paper, we found three related references, see [4, 8, 9]. The main aim of these papers is to investiage Väisälä's open problem discussed in this paper. The authors answered the problem under certain additional conditions. Our result, Theorem 1.1, in this paper shows that the related conditions in [4, 8, 9] cannot be removed.

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