Annales Fennici Mathematici
Volumen 46, 2021, 795–818

CHARACTERIZATION OF COMPACT LINEAR INTEGRAL OPERATORS IN THE SPACE OF FUNCTIONS OF BOUNDED VARIATION

Piotr Kasprzak

Adam Mickiewicz University in Poznań, Faculty of Mathematics and Computer Science
ul. Uniwersytetu Poznańskiego 4, 61-614 Poznań, Poland; kasp@amu.edu.pl

Abstract. Although various operators in the space of functions of bounded variation have been studied by quite a few authors, no simple necessary and sufficient conditions guaranteeing compactness of linear integral operators acting in such spaces have been known. The aim of the paper is to fully characterize the class of kernels which generate compact linear integral operators in the BV-space. Using this characterization we show that certain weakly singular and convolution operators (such as the Abel and Volterra operators), when considered as transformations of $BV[a,b]$, are compact. We also provide a detailed comparison of those new necessary and sufficient conditions with various other conditions connected with compactness of linear (integral) operators in the space of functions of bounded variation which already exist in the literature.

1. Introduction

Compactness in its various forms and shapes, since its introduction more than a century ago, is one of the key concepts of analysis. It is especially evident in fixed point theory, where results involving compactness conditions (like, for example, the Schauder fixed point theorem and many of its generalizations, or the Leray–Schauder degree) constitute a main branch of the theory. For this sole reason it seems that the quest for characterizing compactness of subsets of certain spaces or operators is worth embarking on.

Until very recently the only one (not very useful from the application point of view) characterization of compactness in the space of functions of bounded variation was known (see [17, Exercise IV.13.48]). However, in [9] Bugajewski and Gulgowski gave another necessary and sufficient condition for compactness of subsets of $BV[a,b]$, which is simpler and easier to use.

The situation concerning compactness of linear operators in the space of functions of bounded variation is more or less similar—no full (and "applicable" at the same time) characterization has been known in the literature. Although linear operators acting to $BV[a,b]$ (or to some of its subspaces) or in $BV[a,b]$ were studied, respectively, in the late 1930' and very recently (for "older" results we refer the reader, for example, to [16, 19, 26, 27, 42]–see also [17, Table VI, pp. 543–551]—the "newer" ones can be found in [4, 5, 10, 11, 38, 39, 40]), there are only few sufficient conditions guaranteeing compactness of such operators, which, by no means, are necessary. There are also two results due to Gelfand which describe the general form of a continuous and compact linear operator from an arbitrary Banach space X into the subspace of $BV[a,b]$ consisting of those function which vanish at the point a (see [19, Section II.9]). However, because of their generality (properties of the given operator are

https://doi.org/10.5186/aasfm.2021.4652
2020 Mathematics Subject Classification: Primary 47G10, 47B07, 26A45.
Key words: Abel operator, bounded variation, compact linear operator, convolution kernel, integral operator, space of functions of bounded variation, Volterra operator, weakly singular kernel.

characterized by some abstract function of bounded variation with values in the dual space X^*), it seems that they cannot be used directly, for example, in applications to integral equations. (Another approach to characterize the general form of a linear operator on $BV[a, b]$—this time utilizing the correspondence between functions of bounded variation and additive set functions—is described in [24]). We have devoted the whole last section of the paper to presenting a detailed comparison of our conditions and various conditions connected with compactness of linear (integral) operators acting in $BV[a, b]$.

The aim of the paper is to fill in the gap and to provide necessary and sufficient conditions for a linear integral operator in the space of functions of bounded variation to be compact. It is worth pointing out that our conditions are imposed on the kernel of the operator, and thus they can be easily utilized in integral equations. Although, we consider integral operators defined by the Lebesgue integral, the proof of the main result relies heavily on the fact that it is possible (at least in certain situations) to look at the given transformation also from the "Riemann–Stieltjes perspective"; this approach allows us to "keep" the absolute value on the outside of the integral, which is important when computing variations. In the second part of the paper we show that certain weakly singular and convolution kernels satisfy our conditions, and thus generate compact integral operators on $BV[a, b]$.

On the one hand, the paper is a continuation of the article [10], where necessary and sufficient conditions for continuity of linear integral operators in $BV[a, b]$ were studied. On the other hand, it can be also regarded as a part of a larger whole, as various topological and algebraic properties of linear and nonlinear operators on $BV[a, b]$ have been studied extensively recently (acting conditions as well as continuity of nonlinear superposition operators—also known as Nemytskii operators—were studied in, respectively, [6, 32] and [22, 29], while properties of multiplication operators and sets of multipliers were studied in [4, 5, 7, 13]). The works of Maćkowiak and Gulgowski concerning continuity of Nemytskii operators in $BV[a, b]$ (see [22, 29]) are especially interesting here, since, together with the results from [10] and the compactness results from this paper, they enable more flexibility of assumptions while, for example, looking for solutions of integral equations in the space of functions of bounded variation (balancing the linear and nonlinear parts of the nonlinear operators associated with such problems is crucial here).

The paper is organized as follows. In Section 2 we gather definitions and basic facts from functional and classical analysis which will be needed throughout the article. In Section 3 we briefly recall the necessary and sufficient conditions under which the linear integral operator maps the space of functions of bounded variation into itself and is continuous. Section 4 is devoted to proving the main theorem of the paper characterizing compact linear integral operators in such a space. To illustrate our result we show, for example, that the kernels of the Volterra and Abel integral operators satisfy our conditions. The aim of the last fifth section is to compare the results established in the paper with those concerning compactness of integral operators on the BV-space already existing in the literature.

2. Preliminaries

The aim of this section is to recall some basic facts concerning functions of bounded variation, Riemann–Stieltjes integration and operator theory that will be needed in the sequel.

Let us begin with some notation and conventions. If X is a normed space, then by $\overline{B}_X(x,r)$ we will denote the closed ball in X with center at x and radius $r > 0$. Sometimes, for simplicity, we will, however, write $\overline{B}_{BV}(x,r)$ instead of $\overline{B}_{BV[a,b]}(x,r)$. Unless stated otherwise, we will always assume that all the intervals $[a,b]$ consist of at least two points, that is, will assume that $-\infty < a < b < +\infty$. If $A \subseteq \mathbf{R}$, then by χ_A we will denote the characteristic function of the set A. To avoid misunderstandings, throughout the paper the Lebesgue integral will be denoted by "$(L)\int$," the (original/ordinary) Riemann–Stieltjes integral will be denoted by "$(RS)\int$" and the Perron–Stieltjes integral will be denoted by "$(PS)\int$." While both the Lebesgue and Riemann–Stieltjes integrals are nowadays regarded as classics and do not need to be advertised, the Perron–Stieltjes (or, equivalently, Kurzweil–Stieltjes) integral is not so well-known, although—similarly to its famous relatives—it has found applications in various mathematical theories, for example, in generalized differential equations. In the sequel we will use the fact that if a function f is Riemann–Stieltjes integrable with respect to a function g, then f is also Perron–Stieltjes integrable with respect to g and both those integrals coincide. For more information on the Perron–Stieltjes (Kurzweil–Stieltjes) integral see [33, Chapter 6] or [40, Section I.4] and the references therein.

2.1. Compact operators. Let us recall that a linear operator $T\colon X \to Y$, acting between Banach spaces X and Y, is called *compact* if the image $T(A)$ of any bounded subset A of X (or, equivalently, the image $T(\overline{B}_X(0,1))$) is a relatively compact subset of Y. Since the notion of a compact operator is well-known, we will not dwell on this topic any longer (for more details see, for example, [31, Section 3.4] or [14, Section VI.3]).

2.2. Functions of bounded variation. Let f be a real-valued function defined on the interval $[a,b]$. The number

$$\operatorname*{var}_{[a,b]} f := \sup \sum_{i=1}^{n} \bigl|f(t_i) - f(t_{i-1})\bigr|,$$

where the supremum is taken over all finite partitions $a = t_0 < t_1 < \cdots < t_n = b$ of the interval $[a,b]$, is called the *(Jordan) variation* of the function f over the interval $[a,b]$. If $\operatorname{var}_{[a,b]} f < +\infty$, then the function f is said to be of *bounded variation* (or just a *BV-function*). Sometimes, when considering functions of two (or more) variables, we will write $\operatorname{var}_{t\in[a,b]} f(t,s)$ to denote the variation of the horizontal sections $t \mapsto f(t,s)$ with s fixed. To avoid unnecessary technicalities, we restrict our further considerations to the unit interval $[0,1]$. The linear space consisting of all functions $f\colon [0,1] \to \mathbf{R}$ of bounded variation, that is the space $BV[0,1] = \{f\colon [0,1] \to \mathbf{R} \mid \operatorname{var}_{[0,1]} f < +\infty\}$, endowed with the BV-norm $\|f\|_{BV} := |f(0)| + \operatorname{var}_{[0,1]} f$, is a Banach space. It is well-known that functions of bounded variation are bounded and that for every $f \in BV[0,1]$ we have $\|f\|_\infty \leq \|f\|_{BV}$, where the symbol $\|f\|_\infty$ stands for the supremum norm of the function f, that is, $\|f\|_\infty := \sup_{t\in[0,1]}|f(t)|$. Finally, let us recall that if the function $f\colon [0,1] \to \mathbf{R}$ is of bounded variation, then it can be written as a difference of two real-valued non-decreasing functions defined on $[0,1]$. In particular, functions of bounded variation are Borel measurable. For a thorough treatment of functions of bounded variation in the sense of Jordan we refer the reader to [3, Chapter 1] and [12, Section 13].

To prove our main theorem, we will need a result concerning compact subsets of the space $BV[0,1]$ established recently by Bugajewski and Gulgowski in [9]. However, before we will be able to state it, we need the following definition.

Definition 1. [9, Definition 2] A non-empty set $A \subseteq BV[0,1]$ is said to be *equivariated*, if for each $\varepsilon > 0$ there exists a finite partition $0 = t_0^\varepsilon < t_1^\varepsilon < \cdots < t_n^\varepsilon = 1$ of the interval $[0,1]$ such that for every $f \in A$ we have $\operatorname{var}_{[0,1]} f \leq \varepsilon + \sum_{i=1}^n |f(t_i^\varepsilon) - f(t_{i-1}^\varepsilon)|$.

Proposition 1. [9, Lemma 3] *If A is a non-empty and relatively compact subset of $BV[0,1]$, then A is equivariated.*

In the sequel we will also need the following two-dimensional generalization of the Jordan variation. Let f be a real-valued function defined on the rectangle $I := [a,b] \times [c,d]$. The number

$$\vartheta\text{-}\operatorname*{var}_I f := \sup \sum_{i=1}^m \sum_{j=1}^n \bigl| f(t_i, s_j) - f(t_{i-1}, s_j) - f(t_i, s_{j-1}) + f(t_{i-1}, s_{j-1}) \bigr|,$$

where the supremum is taken over all finite partitions $a = t_0 < \cdots < t_m = b$ and $c = s_0 < \cdots < s_n = d$ of the intervals $[a,b]$ and $[c,d]$, respectively, is called the *Vitali variation* of the function f over I. More information on the Vitali variation can be found in [3, Section 1.4] or [40, Section I.6].

2.3. Riemann–Stieltjes integrals. The proof of our main result relies heavily on the fact that in certain situations it is possible to rewrite the Lebesgue integral as the Riemann–Stieltjes integral.

Proposition 2. [28, Theorem 7.4.10] *Let $f\colon [0,1] \to \mathbf{R}$ be a Lebesgue integrable function and let $g\colon [0,1] \to \mathbf{R}$ be of bounded variation. Then,*

$$(L) \int_0^1 f(t) g(t) \,\mathrm{d}t = (RS) \int_0^1 g(t) \,\mathrm{d}F(t),$$

where F is the primitive of f, that is, $F(t) = (L) \int_0^t f(s) \,\mathrm{d}s$ for $t \in [0,1]$.

Although this result seems to be well-known, we have not found any book containing its simple proof. Hence, for readers' convenience, let us share an elementary argument, which we learnt from Jürgen Appell during our stay at Oberwolfach Research Institute for Mathematics (see [2]).

Proof. First, let us note that since F is absolutely continuous, the Riemann–Stieltjes integral $(RS) \int_0^1 g(t) \,\mathrm{d}F(t)$ exists (see [28, Theorem 1.5.6]). Now, let us fix $\varepsilon > 0$. Then, there clearly exists a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of the interval $[0,1]$ such that $\max_{1 \leq i \leq n} (L) \int_{t_{i-1}}^{t_i} |f(t)| \,\mathrm{d}t \leq \frac{1}{2}(1 + \operatorname{var}_{[0,1]} g)^{-1} \varepsilon$ and

$$\left| \sum_{i=1}^n g(t_i)[F(t_i) - F(t_{i-1})] - (RS) \int_0^1 g(t) \,\mathrm{d}F(t) \right| \leq \frac{1}{2} \varepsilon.$$

And thus,

$$\begin{aligned}
&\left|(L)\int_0^1 f(t)g(t)\,\mathrm{d}t - (RS)\int_0^1 g(t)\,\mathrm{d}F(t)\right| \\
&\leq \frac{1}{2}\varepsilon + \left|(L)\int_0^1 f(t)g(t)\,\mathrm{d}t - \sum_{i=1}^n g(t_i)[F(t_i)-F(t_{i-1})]\right| \\
&\leq \frac{1}{2}\varepsilon + \sum_{i=1}^n (L)\int_{t_{i-1}}^{t_i} |g(t)-g(t_i)||f(t)|\,\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2}\varepsilon + \sum_{i=1}^n \operatorname*{var}_{[t_{i-1},t_i]} g \cdot (L)\int_{t_{i-1}}^{t_i} |f(t)|\,\mathrm{d}t \\
&\leq \frac{1}{2}\varepsilon + \operatorname*{var}_{[0,1]} g \cdot \max_{1\leq i\leq n} (L)\int_{t_{i-1}}^{t_i} |f(t)|\,\mathrm{d}t \leq \varepsilon.
\end{aligned}$$

As the number ε was arbitrary, we obtain the desired equality. □

Another result which will come in handy is the integration-by-parts formula for the Riemann–Stieltjes integral.

Proposition 3. [28, Theorem 1.6.7] *If $f, g\colon [0,1] \to \mathbf{R}$ are two functions of bounded variation which do not have common points of discontinuity, then*

$$(RS)\int_0^1 f(t)\,\mathrm{d}g(t) + (RS)\int_0^1 g(t)\,\mathrm{d}f(t) = f(1)g(1) - f(0)g(0).$$

3. Integral operators in BV-spaces

In this short section we recall some results concerning linear integral operators in the space of functions of bounded variation generated by the function $k\colon [0,1]\times[0,1] \to \mathbf{R}$, that is, operators of the form

$$(Kx)(t) = (L)\int_0^1 k(t,s)x(s)\,\mathrm{d}s, \quad t\in[0,1], \tag{1}$$

where $x \in BV[0,1]$. Clearly, the formula (1) makes sense for any $x \in BV[0,1]$ only if the function k satisfies the following condition

(H1) for every $t \in [0,1]$ the function $s \mapsto k(t,s)$ is Lebesgue integrable on $[0,1]$.

(In the last section, we will also briefly discuss integral operators given by the formula (1) defined on a different space than $BV[0,1]$; for example, on $L^1[0,1]$. We will not, however, introduce a new symbol for such operators, as it will be always clear from which space the functions x are taken.)

In [10], the following theorem, which characterizes continuous linear integral operators in the space of functions of bounded variation, was established.

Theorem 1. [10, Theorem 4] *Let $k\colon [0,1]\times[0,1] \to \mathbf{R}$ be a kernel satisfying the condition* (H1) *and let K be the linear integral operator given by* (1). *The operator K maps the space $BV[0,1]$ into itself and is continuous if and only if the following condition is satisfied*:

(H2) *there exists a constant $M > 0$ such that $\operatorname{var}_{t\in[0,1]}\big((L)\int_0^\xi k(t,s)\,\mathrm{d}s\big) \leq M$ for every $\xi \in [0,1]$.*

Remark 1. It is known that if T is a continuous linear operator acting in a reflexive Banach space X, then for every point $x \in X$ and radius $r > 0$ the set $T(\overline{B}_X(x,r))$ is closed (cf. [14, Exercise 1, p. 181]). Interestingly, although the space of functions of bounded variation is not reflexive (see, for example, [36]), continuous integral operators acting in $BV[0,1]$ also have the above-mentioned property. Indeed, if $K\colon BV[0,1] \to BV[0,1]$ is an integral operator generated by a kernel $k\colon [0,1] \times [0,1] \to \mathbf{R}$ satisfying the conditions (H1) and (H2) and if $(y_n)_{n\in\mathbf{N}}$ is a sequence in $K(\overline{B}_{BV}(x,r))$ which converges to a function $y_0 \in BV[0,1]$ with respect to the BV-norm, then for every $n \in \mathbf{N}$ we have $y_n = Kx_n$ for some $x_n \in \overline{B}_{BV}(x,r)$ and, in view of Helly's selection theorem (see [12, Theorem 13.16]), the sequence $(x_n)_{n\in\mathbf{N}}$ admits a subsequence $(x_{n_m})_{m\in\mathbf{N}}$ converging pointwise on $[0,1]$ to a function $x_0 \in \overline{B}_{BV}(x,r)$. Therefore, by the Lebesgue dominated convergence theorem, $y_{n_m} = Kx_{n_m} \to Kx_0$ pointwise on $[0,1]$. As the BV-convergence is stronger than the pointwise one, this proves that $y_0 = Kx_0 \in K(\overline{B}_{BV}(x,r))$.

In [10], it was also shown that if a kernel satisfies the conditions (H1)–(H2), then the corresponding integral operator does not need to be compact (see [10, Example 3]). Therefore, a natural problem arises: give necessary and sufficient conditions on the kernel $k\colon [0,1] \times [0,1] \to \mathbf{R}$ under which the linear integral operator K maps the space $BV[0,1]$ into itself and is compact.

4. Compact integral operators in BV-spaces

In this section we are going to give a complete answer the question raised at the end of Section 3; namely, we are going to provide necessary and sufficient conditions on the kernel under which the linear integral operator, generated by this kernel, maps the space $BV[0,1]$ into itself and is compact.

Let us begin with the following abstract result.

Proposition 4. *Let $k\colon [0,1] \times [0,1] \to \mathbf{R}$ satisfy the conditions* (H1) *and* (H2) *and let $K\colon BV[0,1] \to BV[0,1]$ be the linear integral operator given by* (1). *Then, the following conditions are equivalent*:

(i) *the operator K is compact*,
(ii) *for every sequence $(x_n)_{n\in\mathbf{N}}$ of elements of $\overline{B}_{BV}(0,1)$ which is pointwise convergent on $[0,1]$ to a function $x\colon [0,1] \to \mathbf{R}$, we have $\lim_{n\to\infty}\|Kx_n - Kx\|_{BV} = 0$*,
(iii) *for every sequence $(x_n)_{n\in\mathbf{N}}$ of elements of $\overline{B}_{BV}(0,1)$ which is pointwise convergent on $[0,1]$ to the zero function, we have $\lim_{n\to\infty}\|Kx_n\|_{BV} = 0$.*

Proof. (i) $\Rightarrow$ (ii) First, let us note that if a sequence $(x_m)_{m\in\mathbf{N}}$ of elements of the closed unit ball in $BV[0,1]$ is pointwise convergent to a function $x\colon [0,1] \to \mathbf{R}$, then necessarily x is of bounded variation; actually, it can be shown that x belongs to $\overline{B}_{BV}(0,1)$—cf. [28, Theorem 1.3.5]. In particular, by Theorem 1, the function Kx is well-defined and is of bounded variation. Using the Lebesgue dominated convergence theorem, it is easy to show that $Kx_m \to Kx$ pointwise on $[0,1]$ as $m \to +\infty$. To show that $Kx_m \to Kx$ with respect to the BV-norm, let us consider an arbitrary subsequence $(Kx_{m_n})_{n\in\mathbf{N}}$ of the sequence $(Kx_m)_{m\in\mathbf{N}}$. The operator K is compact, and so there exists a subsequence $(Kx_{m_{n_l}})_{l\in\mathbf{N}}$ and a function $y \in BV[0,1]$ such that $\|Kx_{m_{n_l}} - y\|_{BV} \to 0$ as $l \to +\infty$. But then, since the BV-convergence is stronger than the uniform convergence (and thus, than the pointwise convergence), we obtain $y = Kx$ and $\|Kx_{m_{n_l}} - Kx\|_{BV} \to 0$ as $l \to +\infty$. This shows that $Kx_m \to Kx$ with respect to the BV-norm.

(ii) $\Rightarrow$ (iii) The proof is obvious.

(iii) $\Rightarrow$ (i) Let $(x_m)_{m\in\mathbf{N}}$ be an arbitrary sequence of elements of the closed unit ball in $BV[0,1]$. By Helly's selection theorem (see [12, Theorem 13.16]) there exists a subsequence $(x_{m_n})_{n\in\mathbf{N}}$ of $(x_m)_{m\in\mathbf{N}}$ and a function $x \in \overline{B}_{BV}(0,1)$ such that $x_{m_n} \to x$ pointwise on $[0,1]$. Then, clearly, the functions $y_{m_n} := \frac{1}{2}(x_{m_n} - x)$ form a sequence satisfying the predecessor of the implication in (iii), and thus $Ky_{m_n} \to 0$ in $BV[0,1]$ as $n \to +\infty$. Therefore, $Kx_{m_n} \to Kx$ with respect to the BV-norm, which implies that the operator K is compact. □

In the discussion of compact integral operators in $BV[0,1]$ the following condition imposed on the kernel $k\colon [0,1]\times[0,1] \to \mathbf{R}$ will play a key role:

(H3) for every $\varepsilon > 0$, there exists $\delta := \delta(\varepsilon) > 0$ such that

$$\operatorname*{var}_{t\in[0,1]}\left((L)\int_a^b k(t,s)\,\mathrm{d}s\right) \le \varepsilon$$

for any subinterval $[a,b]$ of $[0,1]$ of length not exceeding δ.

Remark 2. It turns out that if the kernel $k\colon [0,1]\times[0,1] \to \mathbf{R}$ satisfies (H1), then the condition (H3) is stronger than the condition (H2). To show this let $\delta := \delta(1)$ be a number chosen according to the condition (H3) for $\varepsilon = 1$, and let $M := [1/\delta] + 1$; here, $[\cdot]$ denotes the floor function, that is, $[a]$ is the greatest integer less than or equal to a. Of course, we may assume that $\delta < 1$. Then, for every $\xi \in [0,1]$ there exists some $l \in \{0,1,\ldots,M-1\}$ such that $l\delta \le \xi < (l+1)\delta$, and thus

$$\begin{aligned}\operatorname*{var}_{t\in[0,1]}\left((L)\int_0^\xi k(t,s)\,\mathrm{d}s\right) &\le \sum_{i=0}^{l-1}\operatorname*{var}_{t\in[0,1]}\left((L)\int_{i\delta}^{(i+1)\delta} k(t,s)\,\mathrm{d}s\right) + \operatorname*{var}_{t\in[0,1]}\left((L)\int_{l\delta}^{\xi} k(t,s)\,\mathrm{d}s\right)\\ &\le l+1 \le M.\end{aligned}$$

This shows that

$$\sup_{\xi\in[0,1]}\operatorname*{var}_{t\in[0,1]}\left((L)\int_0^\xi k(t,s)\,\mathrm{d}s\right) \le M,$$

and proves our claim.

Now, we are in position to prove the characterization of compact integral operators in BV-spaces. For completeness, let us add that the idea to use refinements of a given partition in the proof was inspired by a simple result for the Riemann–Stieltjes integral—cf. [28, Lemma 1.5.2].

Theorem 2. *Let $k\colon [0,1]\times[0,1] \to \mathbf{R}$ be a kernel satisfying the condition* (H1) *and let K be the linear integral operator given by* (1). *The operator K maps the space $BV[0,1]$ into itself and is compact if and only if k satisfies the condition* (H3).

Proof. First, let us assume that the kernel k satisfies the conditions (H1) and (H3). Note that in view of Remark 2 and Theorem 1, the operator K maps the space $BV[0,1]$ into itself.

Let us consider an arbitrary sequence $(x_v)_{v\in\mathbf{N}}$ of elements of the closed unit ball in $BV[0,1]$ which is pointwise convergent on $[0,1]$ to the zero function. We will show that $Kx_v \to 0$ with respect to the BV-norm.

Given $\varepsilon > 0$ let us consider an arbitrary (but fixed) partition $\Xi\colon 0 = \xi_0 < \xi_1 < \cdots < \xi_n = 1$ of the interval $[0,1]$ such that $\max_{1\le i\le n}|\xi_i - \xi_{i-1}| \le \delta$, where $\delta := \delta(\frac{1}{6}\varepsilon)$

is chosen according to (H3). Moreover, let $v_0 \in \mathbf{N}$ be such that

$$|x_v(1)| \le \frac{1}{3M}\varepsilon \quad \text{and} \quad \sum_{i=1}^{n}|x_v(\xi_i) - x_v(\xi_{i-1})| \le \frac{1}{3M}\varepsilon \quad \text{for every } v \ge v_0,$$

where $M := \sup_{\xi\in[0,1]} \operatorname{var}_{t\in[0,1]}\big((L)\int_0^\xi k(t,s)\,\mathrm{d}s\big) < +\infty$ (cf. Remark 2).

Now, let us fix $v \ge v_0$ and an arbitrary partition $0 = t_0 < t_1 < \cdots < t_m = 1$ of the interval $[0,1]$. As the function $\xi \mapsto (L)\int_0^\xi [k(t_j,s) - k(t_{j-1},s)]\,\mathrm{d}s$ is absolutely continuous, the Riemann–Stieltjes integral

$$(RS)\int_0^1 \bigg((L)\int_0^\xi [k(t_j,s) - k(t_{j-1},s)]\,\mathrm{d}s\bigg)\,\mathrm{d}x_v(\xi)$$

exists for every $j \in \{1,\ldots,m\}$ (see [28, Theorem 1.5.5]). In particular, there is a partition $\overline{\Xi}$: $0 = \overline{\xi}_0 < \overline{\xi}_1 < \cdots < \overline{\xi}_p = 1$ which is a refinement of Ξ such that

$$\sum_{j=1}^{m}\bigg|\overline{S}_j^v - (RS)\int_0^1 \bigg((L)\int_0^\xi [k(t_j,s) - k(t_{j-1},s)]\,\mathrm{d}s\bigg)\,\mathrm{d}x_v(\xi)\bigg| \le \frac{1}{6}\varepsilon, \tag{2}$$

where $\overline{S}_j^v$ is the approximating Riemann–Stieltjes sum, that is,

$$\overline{S}_j^v := \sum_{l=1}^{p}\bigg((L)\int_0^{\overline{\xi}_l} [k(t_j,s) - k(t_{j-1},s)]\,\mathrm{d}s\bigg)[x_v(\overline{\xi}_l) - x_v(\overline{\xi}_{l-1})].$$

(Clearly, the partition $\overline{\Xi}$ depends on v; however, to simplify the notation, we have decided not to use any superscript and not to write $\overline{\xi}_l^v$.) Let us set

$$S_j^v := \sum_{i=1}^{n}\bigg((L)\int_0^{\xi_i} [k(t_j,s) - k(t_{j-1},s)]\,\mathrm{d}s\bigg)[x_v(\xi_i) - x_v(\xi_{i-1})].$$

Since $\overline{\Xi}$ is a refinement of Ξ, we have $\xi_i = \overline{\xi}_{\alpha_i}$ where $0 = \alpha_0 < \cdots < \alpha_n = p$, and thus we can write

$$S_j^v = \sum_{i=1}^{n}\sum_{l=\alpha_{i-1}+1}^{\alpha_i}\bigg((L)\int_0^{\xi_i} [k(t_j,s) - k(t_{j-1},s)]\,\mathrm{d}s\bigg)[x_v(\overline{\xi}_l) - x_v(\overline{\xi}_{l-1})]$$

and

$$\overline{S}_j^v = \sum_{i=1}^{n}\sum_{l=\alpha_{i-1}+1}^{\alpha_i}\bigg((L)\int_0^{\overline{\xi}_l} [k(t_j,s) - k(t_{j-1},s)]\,\mathrm{d}s\bigg)[x_v(\overline{\xi}_l) - x_v(\overline{\xi}_{l-1})].$$

Then,

$$\begin{aligned}
\sum_{j=1}^{m}|S_j^v - \overline{S}_j^v| &= \sum_{j=1}^{m}\bigg|\sum_{i=1}^{n}\sum_{l=\alpha_{i-1}+1}^{\alpha_i}\bigg((L)\int_{\overline{\xi}_l}^{\xi_i} [k(t_j,s) - k(t_{j-1},s)]\,\mathrm{d}s\bigg)[x_v(\overline{\xi}_l) - x_v(\overline{\xi}_{l-1})]\bigg| \\
&\le \sum_{i=1}^{n}\sum_{l=\alpha_{i-1}+1}^{\alpha_i}\sum_{j=1}^{m}\bigg|(L)\int_{\overline{\xi}_l}^{\xi_i} [k(t_j,s) - k(t_{j-1},s)]\,\mathrm{d}s\bigg| \cdot |x_v(\overline{\xi}_l) - x_v(\overline{\xi}_{l-1})| \\
&\le \sum_{i=1}^{n}\sum_{l=\alpha_{i-1}+1}^{\alpha_i}\operatorname*{var}_{t\in[0,1]}\bigg((L)\int_{\overline{\xi}_l}^{\xi_i} k(t,s)\,\mathrm{d}s\bigg) \cdot |x_v(\overline{\xi}_l) - x_v(\overline{\xi}_{l-1})| \\
&\le \frac{1}{6}\varepsilon \cdot \sum_{i=1}^{n}\sum_{l=\alpha_{i-1}+1}^{\alpha_i}|x_v(\overline{\xi}_l) - x_v(\overline{\xi}_{l-1})|,
\end{aligned}$$

and so

$$\sum_{j=1}^{m}|S_j^v - \overline{S}_j^v| \leq \frac{1}{6}\varepsilon \cdot \operatorname*{var}_{[0,1]} x_v \leq \frac{1}{6}\varepsilon.$$

This, together with (2), implies that

$$\sum_{j=1}^{m}\left|S_j^v - (RS)\int_0^1 \left((L)\int_0^{\xi}[k(t_j,s)-k(t_{j-1},s)]\,\mathrm{d}s\right)\mathrm{d}x_v(\xi)\right| \leq \frac{1}{3}\varepsilon.$$

Moreover, we have

$$\begin{aligned}\sum_{j=1}^{m}|S_j^v| &= \sum_{j=1}^{m}\left|\sum_{i=1}^{n}\left((L)\int_0^{\xi_i}[k(t_j,s)-k(t_{j-1},s)]\,\mathrm{d}s\right)[x_v(\xi_i)-x_v(\xi_{i-1})]\right| \\ &\leq \sum_{i=1}^{n}\sum_{j=1}^{m}\left|(L)\int_0^{\xi_i}[k(t_j,s)-k(t_{j-1},s)]\,\mathrm{d}s\right| \cdot |x_v(\xi_i)-x_v(\xi_{i-1})| \\ &\leq M\cdot\sum_{i=1}^{n}|x_v(\xi_i)-x_v(\xi_{i-1})|,\end{aligned}$$

and thus

$$\begin{aligned}&\sum_{j=1}^{m}\left|(RS)\int_0^1\left((L)\int_0^{\xi}[k(t_j,s)-k(t_{j-1},s)]\,\mathrm{d}s\right)\mathrm{d}x_v(\xi)\right| \\ &\leq \sum_{j=1}^{m}\left|S_j^v-(RS)\int_0^1\left((L)\int_0^{\xi}[k(t_j,s)-k(t_{j-1},s)]\,\mathrm{d}s\right)\mathrm{d}x_v(\xi)\right| + \sum_{j=1}^{m}|S_j^v| \\ &\leq \frac{1}{3}\varepsilon + M\cdot\sum_{i=1}^{n}|x_v(\xi_i)-x_v(\xi_{i-1})|.\end{aligned}$$

This, since the partition $0 = t_0 < t_1 < \cdots < t_m = 1$ was arbitrary, gives

$$\operatorname*{var}_{t\in[0,1]}\left((RS)\int_0^1\left((L)\int_0^{\xi}k(t,s)\,\mathrm{d}s\right)\mathrm{d}x_v(\xi)\right) \leq \frac{1}{3}\varepsilon + M\cdot\sum_{i=1}^{n}|x_v(\xi_i)-x_v(\xi_{i-1})|.$$

Therefore, as $v \geq v_0$, we have

$$\operatorname*{var}_{t\in[0,1]}\left((RS)\int_0^1\left((L)\int_0^{\xi}k(t,s)\,\mathrm{d}s\right)\mathrm{d}x_v(\xi)\right) \leq \frac{2}{3}\varepsilon.$$

By Propositions 2 and 3, for every $t \in [0,1]$ we have

$$\begin{aligned}(Kx_v)(t) &= (L)\int_0^1 k(t,s)x_v(s)\,\mathrm{d}s \\ &= x_v(1)\cdot(L)\int_0^1 k(t,s)\,\mathrm{d}s - (RS)\int_0^1\left((L)\int_0^{\xi}k(t,s)\,\mathrm{d}s\right)\mathrm{d}x_v(\xi),\end{aligned}$$

and hence

$$\operatorname*{var}_{[0,1]} Kx_v \leq |x_v(1)|\cdot M + \frac{2}{3}\varepsilon \leq \varepsilon \qquad \text{for } v \geq v_0.$$

In other words, $\lim_{v\to\infty}\operatorname{var}_{[0,1]} Kx_v = 0$. To end the proof of the sufficiency part it is enough to observe that from the assumptions it follows easily that

$$\lim_{v\to\infty}(Kx_v)(0) = \lim_{v\to\infty}(L)\int_0^1 k(0,s)x_v(s)\,\mathrm{d}s = 0,$$

and thus $\|Kx_v\|_{BV} \to 0$ as $v \to +\infty$. This, by Proposition 4, shows that the linear integral operator K is compact.

Now, we proceed to the second part of the proof. Let us assume that the integral operator K maps the space $BV[0,1]$ into itself and is compact, and let us fix $\varepsilon > 0$. Since, by the assumption, $K(\overline{B}_{BV}(0,2))$ is a relatively compact subset of $BV[0,1]$, in view of Proposition 1, there exists a finite partition $0 = t_0^\varepsilon < t_1^\varepsilon < \cdots < t_n^\varepsilon = 1$ of the interval $[0,1]$ such that

$$\operatorname*{var}_{[0,1]} K\chi_{(a,b)} \leq \frac{1}{2}\varepsilon + \sum_{i=1}^{n} \bigl|(K\chi_{(a,b)})(t_i^\varepsilon) - (K\chi_{(a,b)})(t_{i-1}^\varepsilon)\bigr| \quad \text{for every } (a,b) \subseteq [0,1].$$

Choose δ to be a positive real number such that

$$(L)\int_A |k(t_i^\varepsilon, s)|\,\mathrm{d}s \leq \frac{\varepsilon}{4(n+1)} \quad \text{for } i = 0,\dots,n,$$

whenever A is a Lebesgue measurable subset of $[0,1]$ with measure not exceeding δ.

If $[a,b]$ is an arbitrary subinterval of $[0,1]$ of length at most δ, then

$$\begin{aligned}\operatorname*{var}_{t\in[0,1]}\biggl((L)\int_a^b k(t,s)\,\mathrm{d}s\biggr) &= \operatorname*{var}_{[0,1]} K\chi_{(a,b)} \leq \frac{1}{2}\varepsilon + \sum_{i=1}^{n}\bigl|(K\chi_{(a,b)})(t_i^\varepsilon) - (K\chi_{(a,b)})(t_{i-1}^\varepsilon)\bigr| \\ &\leq \frac{1}{2}\varepsilon + 2\sum_{i=0}^{n}(L)\int_a^b |k(t_i^\varepsilon, s)|\,\mathrm{d}s \leq \varepsilon.\end{aligned}$$

This shows that

$$\operatorname*{var}_{t\in[0,1]}\biggl((L)\int_a^b k(t,s)\,\mathrm{d}s\biggr) \leq \varepsilon$$

for any subinterval $[a,b]$ of $[0,1]$ such that $b-a \leq \delta$. The proof is complete. □

Remark 3. At this point it should be noted that there is also another approach which allows to prove the sufficiency part of the above theorem. It is based on the notion of an abstract Riemann–Stieltjes integral and a certain convergence result for such an integral (we refer a reader interested in this topic to, for example, [25, Section 3.3]). Since this concept is significantly less known than the classical Riemann–Stieltjes integral, we chose to provide in full detail a little bit longer but more elementary proof. Now, however, we will sketch the other approach.

Let us assume that the kernel $k\colon [0,1]\times[0,1] \to \mathbf{R}$ satisfies the conditions (H1) and (H3) (and, thus, also (H2)—see Remark 2) and let us define the mapping $F\colon [0,1] \to BV[0,1]$ which to each real number $\xi \in [0,1]$ assigns the BV-function $t \mapsto (L)\int_0^\xi k(t,s)\,\mathrm{d}s$. Note that thanks to (H1) and (H2) the map F is well-defined, and thanks to the absolute continuity of the integral as well as (H1) and (H3) it is continuous when $BV[0,1]$ is considered with the norm topology (in other words, F is *strongly continuous*). This, in particular, implies that for each $x \in BV[0,1]$ the abstract Riemann–Stieltjes integral $(RS)\int_0^1 F(\xi)\,\mathrm{d}x(\xi)$ exists (in the norm topology) and is an element of $BV[0,1]$ (see [25, Theorem 3.3.2]). Now, if we take an arbitrary sequence $(x_n)_{n\in\mathbf{N}}$ of elements of $\overline{B}_{BV}(0,1)$ which is pointwise convergent on $[0,1]$ to the zero function and apply convergence results for the Riemann–Stieltjes integral, both the classical and the abstract one, (see [28, Theorem 1.6.10] and [25, Corollary 2, p. 65]), we get $\lim_{n\to\infty}\bigl\|(RS)\int_0^1 F(\xi)\,\mathrm{d}x_n(\xi)\bigr\|_{BV} = 0$. Observe also that by

Propositions 2 and 3 for every $t \in [0,1]$ we have

$$(3) \qquad (Kx_n)(t) = x_n(1) \cdot (L)\int_0^1 k(t,s)\,\mathrm{d}s - (RS)\int_0^1 \Big((L)\int_0^\xi k(t,s)\,\mathrm{d}s \Big)\,\mathrm{d}x_n(\xi)$$

(cf. the proof of Theorem 2), and so

$$(4) \qquad \|Kx_n\|_{BV} \le |x_n(1)| \cdot \|F(1)\|_{BV} + \Big\|(RS)\int_0^1 F(\xi)\,\mathrm{d}x_n(\xi)\Big\|_{BV};$$

let us add that in (3) the Riemann–Stieltjes integral should be understood in the classical sense, while in (4) in the abstract one. Thus, $\|Kx_n\|_{BV} \to 0$ as $n \to +\infty$. To end the proof it suffices to apply Proposition 4.

Remark 4. Let us note that using a similar technique to that used in [10, Remark 7] we can show that *not* every linear compact operator acting in $BV[0,1]$ is an integral operator.

Indeed, it is easy to see that the linear operator $T\colon BV[0,1] \to BV[0,1]$ given by the formula $(Tx)(t) = x(0)$ for $t \in [0,1]$ is compact. However, if there existed a kernel $k\colon [0,1] \times [0,1] \to \mathbf{R}$ such that $Kx = Tx$, that is,

$$(5) \qquad (L)\int_0^1 k(t,s)x(s)\,\mathrm{d}s = x(0) \quad \text{for every } t \in [0,1] \text{ and } x \in BV[0,1],$$

then for every fixed $t \in [0,1]$ and every $a \in (0,1)$, taking $\chi_{(a-h,a+h)}$ in place of x in (5), we would have

$$\frac{1}{2h}(L)\int_{a-h}^{a+h} k(t,s)\,\mathrm{d}s = 0,$$

provided that $h > 0$ is sufficiently small. This, together with the Lebesgue differentiation theorem, would imply that for every $t \in [0,1]$ the function $s \mapsto k(t,s)$ is zero a.e. on $[0,1]$, and hence $Tx = Kx = 0$ for every $x \in BV[0,1]$—a contradiction.

From Theorem 2 and the results on the continuity of autonomous Nemytskii operators we immediately get the following corollary concerning compactness of nonlinear integral operators.

Corollary 1. *Let $k\colon [0,1]\times[0,1] \to \mathbf{R}$ be a kernel satisfying the conditions* (H1) *and* (H3), *and let $f\colon \mathbf{R} \to \mathbf{R}$ be locally Lipschitz continuous, that is, we assume that for each $r > 0$ there exists a constant $L_r \ge 0$ such that $|f(u) - f(w)| \le L_r|u - w|$ for all $u, w \in [-r, r]$. Then, the nonlinear integral operator F which to each $x \in BV[0,1]$ assigns the function given by the formula*

$$F(x)(t) = (L)\int_0^1 k(t,s)f(x(s))\,\mathrm{d}s, \quad t \in [0,1],$$

maps the space $BV[0,1]$ into itself and is completely continuous[1].

Proof. The result is a direct consequence of Theorem 2 and the result of Maćkowiak which says that the autonomous superposition operator $x \mapsto f \circ x$, generated by a function f satisfying a local Lipschitz condition, acts in $BV[0,1]$ and is continuous

[1]A word of caution is in order: for mathematicians working in nonlinear analysis and/or fixed point theory the notions of a compact and completely continuous operator may mean something slightly different than for those working in functional analysis. Here, writing that a *nonlinear* operator $F\colon X \to Y$ acting between metric spaces X and Y is *completely continuous*, we mean that it is continuous and for every bounded set A in X, the image $F(A)$ is contained in a compact subset of Y (cf. [15, Definition 1.1, p. 112]).

(see [29, Theorem 7]); the fact that such a superposition operator carries bounded subsets of $BV[0,1]$ into bounded ones is trivial. □

Remark 5. It is worth noting here that [29, Theorem 7] was first obtained by Morse in 1937 (see [34, Theorem 7.1]). However, the original proof was very long and tedious. In [29], Maćkowiak gave a new, much shorter, proof of this result.

As regards non-autonomous superposition operators in $BV[0,1]$ (that is, operators which to each $x \in BV[0,1]$ assign the function $t \mapsto f(t, x(t))$, where $t \in [0,1]$), no *simple* necessary and sufficient conditions guaranteeing continuity of such operators are known. (Although, in Section 5 of [29], Maćkowiak gave some such conditions, they cannot be labelled as "user-friendly".) For completeness, let us add that, for example, continuous differentiability of the non-autonomous generator $(t,u) \mapsto f(t,u)$ on a certain set is one of sufficient conditions guaranteeing that the corresponding non-autonomous Nemytskii operator maps the space $BV[0,1]$ into itself and is uniformly continuous on bounded subsets (for more details on this topic see [22, 29]).

After introducing a new condition, it is a good practise to provide some examples of objects (in this case, kernels) satisfying it.

Proposition 5. *If $f\colon [0,1] \to \mathbf{R}$ is a Lebesgue integrable function, then the kernel $k\colon [0,1]\times[0,1] \to \mathbf{R}$ given by the formula*

$$k(t,s) = \begin{cases} f(t-s), & \text{if } 0 \le s < t \le 1, \\ 0, & \text{if } 0 \le t \le s \le 1, \end{cases} \tag{6}$$

satisfies the conditions (H1) *and* (H3).

Proof. Since it is clear that the kernel satisfies the condition (H1), we will only focus on showing that it also satisfies (H3). It can be shown that given an interval $[a,b] \subseteq [0,1]$, we have

$$(L)\int_a^b k(t,s)\,\mathrm{d}s = \begin{cases} 0, & \text{if } 0 \le t \le a, \\ (L)\displaystyle\int_0^{t-a} f(s)\,\mathrm{d}s, & \text{if } a \le t \le b, \\ (L)\displaystyle\int_{t-b}^{t-a} f(s)\,\mathrm{d}s, & \text{if } b \le t \le 1. \end{cases}$$

For a given $\varepsilon > 0$ let $\delta > 0$ be chosen in such a way that $(L)\int_0^\delta |f(s)|\,\mathrm{d}s \le \frac{1}{2}\varepsilon$ and $(L)\int_{\mathbf{R}} |\varphi(s) - \varphi(s+\tau)|\,\mathrm{d}s \le \frac{1}{2}\varepsilon$ for all $0 \le \tau \le \delta$, where φ is defined by $\varphi(s) = 0$ for $s \notin [0,1]$ and $\varphi(s) = f(s)$ for $s \in [0,1]$. Note that the existence of such a number δ follows from the absolute continuity of the Lebesgue integral and the continuity of the translation of a given function in the L^1-norm (cf. [28, Theorem 6.2.8] and [23, Theorem 13.24]). Now, let us fix an interval $[a,b] \subseteq [0,1]$ such that $b - a \le \delta$, and let us observe that

$$\operatorname*{var}_{t\in[0,a]}\left((L)\int_a^b k(t,s)\,\mathrm{d}s\right) = 0.$$

Of course, only the case $a \neq 0$ is interesting here; if the number a happens to be equal to zero, then we simply skip this part and move on to the next one.

If $a = t_0 < t_1 < \cdots < t_n = b$ is an arbitrary finite partition of $[a, b]$, then

$$\sum_{i=1}^{n}\Big|(L)\int_a^b k(t_i, s)\,\mathrm{d}s - (L)\int_a^b k(t_{i-1}, s)\,\mathrm{d}s\Big| = \sum_{i=1}^{n}\Big|(L)\int_{t_{i-1}-a}^{t_i-a} f(s)\,\mathrm{d}s\Big|$$
$$\le (L)\int_0^{b-a} |f(s)|\,\mathrm{d}s,$$

which proves that

$$\operatorname*{var}_{t\in[a,b]}\Big((L)\int_a^b k(t, s)\,\mathrm{d}s\Big) \le (L)\int_0^{b-a} |f(s)|\,\mathrm{d}s \le \frac{1}{2}\varepsilon.$$

To estimate the variation of the function $t \mapsto (L)\int_a^b k(t, s)\,\mathrm{d}s$ over the interval $[b, 1]$ we will use the fact that this function is absolutely continuous on that interval. As before, this part is needed only if $b \neq 1$. For $t \in [b, 1]$ we have

$$(L)\int_a^b k(t, s)\,\mathrm{d}s = (L)\int_{t-b}^{t-a} f(s)\,\mathrm{d}s = (L)\int_a^t f(s-a)\,\mathrm{d}s - (L)\int_b^t f(s-b)\,\mathrm{d}s.$$

Therefore,

$$\operatorname*{var}_{t\in[b,1]}\Big((L)\int_a^b k(t, s)\,\mathrm{d}s\Big) = (L)\int_b^1 |f(s-a) - f(s-b)|\,\mathrm{d}s$$

(see [3, Theorem 3.19]). But

$$(L)\int_b^1 |f(s-a) - f(s-b)|\,\mathrm{d}s = (L)\int_0^{1-b} |f(s+b-a) - f(s)|\,\mathrm{d}s$$
$$= (L)\int_0^{1-b} |\varphi(s+b-a) - \varphi(s)|\,\mathrm{d}s$$
$$\le (L)\int_{\mathbf{R}} |\varphi(s+b-a) - \varphi(s)|\,\mathrm{d}s \le \frac{1}{2}\varepsilon.$$

Thus,

$$\operatorname*{var}_{t\in[b,1]}\Big((L)\int_a^b k(t, s)\,\mathrm{d}s\Big) \le \frac{1}{2}\varepsilon.$$

Summing the above cases and using the additivity of the variation as a function of the interval, we finally get

$$\operatorname*{var}_{t\in[0,1]}\Big((L)\int_a^b k(t, s)\,\mathrm{d}s\Big) \le \varepsilon.$$

This completes the proof. □

From Proposition 5 we immediately get the following corollary.

Corollary 2. *Let $\alpha \in (0, 1)$. The weakly singular kernel $k\colon [0, 1] \times [0, 1] \to \mathbf{R}$ given by the following formula*

$$k(t, s) = \begin{cases} (t-s)^{-\alpha} & \text{for } 0 \le s < t \le 1, \\ 0 & \text{for } 0 \le t \le s \le 1, \end{cases} \tag{7}$$

satisfies the conditions (H1) *and* (H3).

Another consequence of Theorem 2 and Proposition 5 is the following result concerning convolutions.

Corollary 3. *Let $f\colon [0,1] \to \mathbf{R}$ be a Lebesgue integrable function. Then, the convolution, that is, the operator given by $x \mapsto f * x$, where $(f*x)(0) = 0$ and*

$$(f * x)(t) = (L)\int_0^t f(t-s)x(s)\,\mathrm{d}s, \quad t \in (0,1],$$

is a compact linear operator on the space $BV[0,1]$.

Remark 6. The fact that the convolution operator acts in the space $BV[0,1]$ and is continuous has been already known in the literature (see, for example, [21, Theorem 2.2 (iii)]). It seems, however, that the fact that this operator is compact is new.

Two well-known examples of linear integral operators are the Volterra operator V and the Abel operator J_α (see, for example, [1, 14, 20]) defined by the formulas

$$(Vx)(t) = (L)\int_0^t x(s)\,\mathrm{d}s, \quad t \in [0,1], \tag{8}$$

and

$$(J_\alpha x)(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \cdot (L)\displaystyle\int_0^t (t-s)^{\alpha-1}x(s)\,\mathrm{d}s, & \text{if } t \in (0,1], \\ 0, & \text{if } t = 0, \end{cases}$$

respectively; here $\alpha \in (0,1)$ is fixed and Γ is the Gamma function, and the function x belongs to a suitable function space. Let us add that the function $J_\alpha x$ is sometimes also called the *Abel transform of x*.

Of course, the operators V and J_α can be regarded as convolution operators generated by the kernel (6) with the Lebesgue integrable function f given by $f(t) = 1$ and $f(t) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1}$ for $t \in (0,1]$, respectively. Compactness properties of the Volterra and Abel operators are well-known for the spaces of integrable or continuous functions (see, for example, [14, pp. 43–45], [20, Section 4.3] and [37]). Using Corollary 3 and the above characterization of those operators as convolutions we obtain a result of a similar vein for the space $BV[0,1]$.

Corollary 4. *The Volterra operator V and the Abel operator J_α with $\alpha \in (0,1)$ map the space $BV[0,1]$ into itself and are compact.*

Remark 7. Using a similar approach to that used in the proof of [20, Theorem 4.1.4] it can be shown that the Abel transform $J_\alpha x$ of a BV-function x, where $\alpha \in (0,1)$, is not only of bounded variation but is also Hölder continuous with exponent α. (This is especially interesting, since neither of the spaces $BV[0,1]$ and $Lip_\alpha[0,1]$, that is, the space of all Hölder continuous real-valued functions with exponent α defined on the interval $[0,1]$, is a linear subspace of the other one—cf. [3, Examples 1.23–1.25].) Indeed, if $x \in BV[0,1]$, then for any $t, \tau \in [0,1]$ such that $t > \tau > 0$ we have

$$\begin{aligned}
&\Gamma(\alpha) \cdot |(J_\alpha x)(t) - (J_\alpha x)(\tau)| \\
&\qquad = \left|(L)\int_0^\tau [(t-s)^{\alpha-1} - (\tau-s)^{\alpha-1}]x(s)\,\mathrm{d}s + (L)\int_\tau^t (t-s)^{\alpha-1}x(s)\,\mathrm{d}s\right| \\
&\leq (L)\int_0^\tau [(\tau-s)^{\alpha-1} - (t-s)^{\alpha-1}]|x(s)|\,\mathrm{d}s + (L)\int_\tau^t (t-s)^{\alpha-1}|x(s)|\,\mathrm{d}s \\
&\leq 2\alpha^{-1}\|x\|_\infty (t-\tau)^\alpha + \alpha^{-1}\|x\|_\infty(\tau^\alpha - t^\alpha) \leq 2\alpha^{-1}\|x\|_{BV}|t-\tau|^\alpha;
\end{aligned}$$

of course, if $\tau = 0$ the same reasoning applies. This, in particular, shows that J_α with $\alpha \in (0,1)$ maps continuously $BV[0,1]$ into the space $Lip_\alpha[0,1]$ endowed with the norm

$$\|x\|_\alpha = |x(0)| + \sup\left\{ \frac{|x(t) - x(\tau)|}{|t-\tau|^\alpha} \;\middle|\; t, \tau \in [0,1], t \neq \tau \right\}. \tag{9}$$

In connection with Corollary 4, a natural question arises whether the operator $J_\alpha \colon BV[0,1] \to Lip_\alpha[0,1]$ is also compact. The answer to this question is negative as evidenced by the following example. Let us consider the sequence $(x_n)_{n\in\mathbf{N}}$ of increasing functions $x_n \colon [0,1] \to \mathbf{R}$ given by the formulas

$$x_n(t) = \frac{\Gamma(1+\alpha+\frac{1}{n})}{\Gamma(1+\frac{1}{n})} t^{\frac{1}{n}} \quad \text{for } t \in [0,1].$$

Since $\frac{4}{5} \leq \Gamma(t) \leq 2$ for $t \in [1,3]$, we clearly have $x_n \in BV[0,1]$ and $\|x_n\|_{BV} \leq \frac{5}{2}$ for every $n \in \mathbf{N}$. It is also easy to see that $(J_\alpha x_n)(t) = t^{\alpha+\frac{1}{n}}$ for $t \in [0,1]$. This, in particular, means that the sequence $(J_\alpha x_n)_{n\in\mathbf{N}}$ is pointwise convergent on $[0,1]$ to the function $x \in Lip_\alpha[0,1]$ given by $x(t) = t^\alpha$. So, if the Abel operator J_α were compact, the sequence $(J_\alpha x_n)_{n\in\mathbf{N}}$ should admit a subsequence convergent in the norm $\|\cdot\|_\alpha$ to the function x (because the norm convergence in $Lip_\alpha[0,1]$ is stronger than the pointwise one). However, for every $n \in \mathbf{N}$, we have

$$\|J_\alpha x_n - x\|_\alpha \geq \frac{|(J_\alpha x_n - x)(\frac{1}{2^n}) - (J_\alpha x_n - x)(0)|}{|\frac{1}{2^n} - 0|^\alpha} = \frac{1}{2}.$$

This shows that the Abel operator J_α is not compact when considered as a map acting from $BV[0,1]$ into $Lip_\alpha[0,1]$.

Similarly, it can be shown that although the Volterra operator V maps continuously $BV[0,1]$ into the space of all Lipschitz continuous mappings defined on the interval $[0,1]$ and endowed with the norm $\|\cdot\|_\alpha$ with $\alpha = 1$, it is not compact; it suffices to consider the functions $x_n(t) = t^n$.

Remark 8. For a fixed $n \in \mathbf{N}$ let $x_n(t) := t^n$ for $t \in [0,1]$; we also set $x_0(t) := 1$ for $t \in [0,1]$. A trivial observation that the set of monomials $A = \{x_n \mid n \in \mathbf{N} \cup \{0\}\}$ is mapped by the Volterra operator to a linearly independent set $V(A) = \{\frac{1}{n} x_n \mid n \in \mathbf{N}\}$ shows that V as an operator acting in the space of functions of bounded variation has not finite rank[2], that is, $V(BV[0,1])$ is not finite dimensional. Since $V \colon BV[0,1] \to BV[0,1]$ is compact, it means that the range of the Volterra operator is not a closed subspace of $BV[0,1]$ (cf. [31, Proposition 3.4.6]). Thus, V is an example of an operator showing that in the classical result from functional analysis which states that if a continuous linear operator $T \colon X \to Y$ carries closed bounded subsets of a Banach space X to closed subsets of a Banach space Y, then it has closed range (see, for example, [1, Problem 2.1.10 (c)]), one cannot replace the assumption "*all closed bounded subsets*" with "*all closed balls*" (cf. Remark 1).

5. A short excursion through the literature

In this last section we are going to compare the condition (H3) with some conditions connected with compactness of linear integral operators in spaces of functions of bounded variation which are known in the literature.

[2]The fact that the Volterra operator has not finite rank as an operator acting in, for example, $L^2[0,1]$ is well-known. However, the only place I could find it stated explicitly (and in an elementary way) was at the Mathematics Stack Exchange forum (question 1887169).

5.1. Bugajewski's condition. When dealing with the existence of BV-solutions to certain non-linear integral equations, Bugajewski in [8] introduced the following condition imposed on the kernel $k\colon [0,1]\times[0,1]\to\mathbf{R}$:

(H4) $\operatorname{var}_{t\in[0,1]} k(t,s) \leq m(s)$ for a.e. $s\in[0,1]$, where $m\colon[0,1]\to[0,+\infty)$ is a Lebesgue integrable function.

In [11], using the Lebesgue dominated convergence theorem and Helly's selection theorem, it was proved that under the conditions (H1) and (H4) the linear integral operator K given by (1) maps the space $BV[0,1]$ into itself and is compact (see [11, Proposition 6]). This, in particular, means that if the kernel $k\colon[0,1]\times[0,1]\to\mathbf{R}$ satisfies (H1), then the condition (H3) is weaker than the condition (H4). To see that (H3) is strictly weaker than (H4) it suffices to consider, for example, the weakly singular kernel

$$k(t,s)=\begin{cases}\frac{1}{2}(t-s)^{-\frac{1}{2}} & \text{for } 0\leq s<t\leq 1,\\ 0 & \text{for } 0\leq t\leq s\leq 1\end{cases}\tag{10}$$

(cf. Corollary 2 and [10, Example 2]).

Let us also note that the direct proof of the fact that under the general hypothesis (H1) the condition (H4) implies (H3) is quite straightforward. Indeed, if the kernel k satisfies (H4), for a given $\varepsilon>0$ let $\delta>0$ be a real number such that $(L)\int_A m(s)\,\mathrm{d}s\leq\varepsilon$, whenever A is a Lebesgue measurable set with measure not exceeding δ. Then, for every interval $[a,b]\subseteq[0,1]$ of length at most δ and an arbitrary finite partition $0=t_0<\cdots<t_n=1$ of $[0,1]$, we have

$$\sum_{i=1}^{n}|k(t_i,s)-k(t_{i-1},s)|\leq \operatorname*{var}_{t\in[0,1]} k(t,s)\leq m(s)$$

for a.e. $s\in[0,1]$, and thus

$$\begin{aligned}\sum_{i=1}^{n}\left|(L)\int_a^b\big[k(t_i,s)-k(t_{i-1},s)\big]\,\mathrm{d}s\right| &\leq (L)\int_a^b\sum_{i=1}^{n}|k(t_i,s)-k(t_{i-1},s)|\,\mathrm{d}s\\ &\leq (L)\int_a^b m(s)\,\mathrm{d}s\leq\varepsilon.\end{aligned}$$

This implies that

$$\operatorname*{var}_{t\in[0,1]}\left((L)\int_a^b k(t,s)\,\mathrm{d}s\right)\leq\varepsilon,$$

and shows that k satisfies (H3).

5.2. Schwabik's condition. Compact linear integral operators were also studied by Schwabik in, for example, [38, 39] (see also [40, Chapter II]). In particular, he proved that the integral operator S defined for $x\in BV[0,1]$ by the formula

$$(Sx)(t)=(PS)\int_0^1 x(s)\,\mathrm{d}_s l(t,s),\qquad t\in[0,1],\tag{11}$$

maps the space $BV[0,1]$ into itself and is compact provided that the kernel $l\colon[0,1]\times[0,1]\to\mathbf{R}$ satisfies the condition $\operatorname{var}_{[0,1]} l(0,s)+\vartheta\text{-}\operatorname{var}_I l<+\infty$, where $I:=[0,1]\times[0,1]$ (see [38, Theorem 3.1] or [40, Theorem II.1.5]).

Remark 9. Let us note that thanks to Proposition 2, if the kernel $k\colon[0,1]\times[0,1]\to\mathbf{R}$ satisfies the general condition (H1) it is always possible to represent the linear integral operator K given by (1) in the Schwabik's form (11). Indeed, let the function $l\colon [0,1]\times[0,1]\to \mathbf{R}$ be given by $l(t,s)=(L)\int_0^s k(t,z)\,\mathrm{d}z$. Then, for every $x\in BV[0,1]$ and $t\in[0,1]$ we have

$$\begin{aligned}(Kx)(t)&=(L)\int_0^1 k(t,s)x(s)\,\mathrm{d}s=(RS)\int_0^1 x(s)\,\mathrm{d}\Big((L)\int_0^s k(t,z)\,\mathrm{d}z\Big)\\&=(PS)\int_0^1 x(s)\,\mathrm{d}\Big((L)\int_0^s k(t,z)\,\mathrm{d}z\Big)=(PS)\int_0^1 x(s)\,\mathrm{d}_s l(t,s)=(Sx)(t);\end{aligned}$$

in the above equality we used the fact that the Perron–Stieltjes integral is a generalization of the Riemann–Stieltjes integral (see [33, Chapter 6] or [40, Section I.4]).

However, it is worth noting that the sufficiency part of Theorem 2 does not follow from Schwabik's result as evidenced by the following example.

Example 1. Let $k\colon [0,1]\times[0,1]\to\mathbf{R}$ be given by (10). From Corollary 2 we know that the kernel k satisfies both conditions (H1) and (H3), and hence it generates the linear integral operator K which maps the space $BV[0,1]$ into itself and is compact. However, this compactness result cannot be obtained using Schwabik's representation of K described in Remark 9 and his theorem, as it turns out the function $l(t,s):=(L)\int_0^s k(t,z)\,\mathrm{d}z$ is not of bounded Vitali variation on the square $I:=[0,1]\times[0,1]$. Indeed, for any $t,s\in[0,1]$ we have $l(t,s)=\sqrt{t}-\sqrt{t-\min\{t,s\}}$. And so, for any $n\in\mathbf{N}$ we obtain

$$\vartheta\text{-}\operatorname*{var}_I l\geq\sum_{i=1}^n\big|l(\tfrac{i}{n},\tfrac{i}{n})-l(\tfrac{i-1}{n},\tfrac{i}{n})-l(\tfrac{i}{n},\tfrac{i-1}{n})+l(\tfrac{i-1}{n},\tfrac{i-1}{n})\big|=\sqrt{n},$$

which shows that $\vartheta\text{-}\mathrm{var}_I\, l=+\infty$.

For completeness, let us add that in the papers [38, 39] Schwabik also studied operators of the form

$$(\hat{S}x)(t)=(PS)\int_0^1 l(t,s)\,\mathrm{d}x(s),\qquad t\in[0,1],$$

and proved some compactness results (see also [40, Chapter II]); here, as before, one of the main assumptions imposed on the kernel $l\colon[0,1]\times[0,1]\to\mathbf{R}$ was that $\vartheta\text{-}\mathrm{var}_I\, l<+\infty$. Functions satisfying (H1) and (H3), clearly, do not need to be of bounded Vitali variation. To see this it suffices to consider the kernel $k\colon[0,1]\times[0,1]\to\mathbf{R}$ given by $k(t,s)=t\cdot\chi_{\mathbf{Q}\cap[0,1]}(s)$.

5.3. Vulich's conditions. In the late 1930' Vulich studied the general form of a continuous linear operator acting in certain Banach spaces. Among other things, in [42] he proved the following characterization of continuous linear operators between the space of Lebesgue integrable functions $L^1[0,1]$ and $BV[0,1]$.

Theorem 3. [42, Theorem 3] *Each continuous linear operator mapping $L^1[0,1]$ into $BV[0,1]$ is an integral operator K given by* (1) *with the kernel $k\colon[0,1]\times[0,1]\to\mathbf{R}$ satisfying the following conditions*:

(H5) *for every $t\in[0,1]$ the function $s\mapsto k(t,s)$ is Lebesgue measurable on $[0,1]$*,

(H6) *there exists a constant $M>0$ such that for every $a,b\in[0,1]$ we have*

$$\Big|(L)\int_a^b k(0,s)\,\mathrm{d}s\Big|+\operatorname*{var}_{t\in[0,1]}\Big((L)\int_a^b k(t,s)\,\mathrm{d}s\Big)\leq M|a-b|.$$

Conversely, if a kernel $k\colon [0,1]\times[0,1]\to\mathbf{R}$ *of the linear integral operator* K *given by* (1) *satisfies the above conditions, then* K *maps* $L^1[0,1]$ *into* $BV[0,1]$ *and is continuous.*

Let us add that the proof of Vulich's result is completely different than the proof of Theorem 2, and, in a sense, much simpler, as it utilizes the fact that the set of step functions is dense in the space $L^1[0,1]$.

Remark 10. Note that our general hypothesis (H1) is "hidden" in the conditions (H5)–(H6); otherwise the integrals in (H6) would not be finite (or, would not exist at all) and the condition itself would make no sense. It is a matter of taste which approach to take: either state (H1) explicitly or impose a weaker regularity condition (like (H5)) instead and assume that all the integrals appearing in all the other conditions are understood to exist and be finite.

Remark 11. Let $k\colon [0,1]\times[0,1]\to\mathbf{R}$ be a kernel satisfying the condition (H1). If, additionally, k satisfies (H6), then for each fixed $t\in[0,1]$ it is essentially bounded on $[0,1]$ with respect to s. Indeed, for any fixed $\tau\in[0,1]$ the condition (H6) with $a:=\sigma-h$ and $b:=\sigma+h$, where $\sigma\in(0,1)$ and $0<h<\min\{\sigma,1-\sigma\}$, yields

$$\left|(L)\int_{\sigma-h}^{\sigma+h} k(\tau,s)\,\mathrm{d}s\right|\le\left|(L)\int_{\sigma-h}^{\sigma+h} k(0,s)\,\mathrm{d}s\right|+\operatorname*{var}_{t\in[0,1]}\left((L)\int_{\sigma-h}^{\sigma+h} k(t,s)\,\mathrm{d}s\right)\le 2Mh.$$

And so,

$$\left|\frac{1}{2h}\cdot(L)\int_{\sigma-h}^{\sigma+h} k(\tau,s)\,\mathrm{d}s\right|\le M,$$

which, together with the Lebesgue differentiation theorem, implies that $|k(\tau,\sigma)|\le M$ for a.e. $\sigma\in[0,1]$.

Remark 12. In view of Corollary 2 and Remark 11 it is easy to see that, under our general hypothesis (H1), the Vulich condition (H6) is strictly stronger that the condition (H3).

At this point it is also worth noting that not every continuous linear operator from $L^1[0,1]$ into $BV[0,1]$ is compact. We are going to show even more interesting example illustrating that an operator which is compact as an operator from $BV[0,1]$ into $BV[0,1]$ may fail to be compact when considered as an operator from $L^1[0,1]$ into $BV[0,1]$.

Example 2. Let us consider the Volterra operator defined by formula (8). According to Corollary 4 V is a compact linear integral operator when considered as an operator acting in $BV[0,1]$.

However, as we will see, V is not compact as an operator from $L^1[0,1]$ into $BV[0,1]$. First, note that V is a (norm-to-norm) continuous operator between $L^1[0,1]$ and $BV[0,1]$ (in fact, it is not difficult to check directly that the kernel generating V satisfies the conditions (H5)–(H6) with $M=1$), and thus it is also weak-to-weak sequentially continuous. Now, let us consider a sequence $(x_n)_{n\in\mathbf{N}}$ of functions in $L^1[0,1]$ given by the formulas $x_n(t)=\sin(2\pi nt)$ for $n\in\mathbf{N}$ and $t\in[0,1]$. Clearly, this sequence is bounded in $L^1[0,1]$; actually, it can be shown that it is weakly convergent to zero (cf. [18, Exercise 2.57]). Therefore, $(Vx_n)_{n\in\mathbf{N}}$ is weakly convergent to zero. Since evaluation functionals e_t, defined for any $t\in[0,1]$ and $x\in BV[0,1]$ by the formulas $e_t(x)=x(t)$, are continuous on the space of functions of bounded variation, this means that the sequence $(Vx_n)_{n\in\mathbf{N}}$ is pointwise convergent on $[0,1]$ to the zero function. So, if the operator $V\colon L^1[0,1]\to BV[0,1]$ were compact, some subsequence of $(Vx_n)_{n\in\mathbf{N}}$ would converge in the BV-norm to zero. But this is impossible because for $n \in \mathbf{N}$ we have

$$\operatorname*{var}_{[0,1]} Vx_n = \int_0^1 \big|\sin(2\pi nt)\big| \,\mathrm{d}t = \sum_{i=1}^{2n} \int_{\frac{i-1}{2n}}^{\frac{i}{2n}} \big|\sin(2\pi nt)\big| \,\mathrm{d}t = 2n \int_0^{\frac{1}{2n}} \sin(2\pi nt) \,\mathrm{d}t = \frac{2}{\pi}.$$

Thus, V as an operator from $L^1[0,1]$ into $BV[0,1]$ cannot be compact.

In the same paper [42], Vulich also studied linear operators which are continuous when different notion of convergence (that is, the so-called K-convergence) is considered in the target space (for the notion of K-normed spaces and related topics see, for example, [41, 30]). In particular, in a similar fashion to Theorem 3, he characterized the class of all such operators acting between $L^1[0,1]$ and $BV[0,1]$ (see [42, Theorem 4]). However, instead of the condition (H6), this time Vulich used the following one:

(H7) there is a constant $M > 0$ such that for any finite partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of the interval $[0,1]$ and any number $\varepsilon > 0$ we have

$$\sup\left|(L)\int_a^b k(0,s)\,\mathrm{d}s\right| + \sum_{i=1}^{n} \sup\left|(L)\int_a^b [k(t_i,s) - k(t_{i-1},s)]\,\mathrm{d}s\right| \leq M\varepsilon,$$

where the suprema are taken over the collection of all the intervals $[a,b] \subseteq [0,1]$ with length not exceeding ε.

Now, we will compare (H6) and (H7); of course, to avoid any doubts whether the integrals appearing in those conditions exist and are finite, we will assume that all the kernels involved satisfy our general hypothesis (H1) – cf. Remark 10.

Remark 13. It may come as a surprise that although (H7) closely resembles (H3) (if we additionally take into account the definition of the Jordan variation), it is not weaker than (H6); in fact, quite the opposite is true. To see this let us assume that a kernel $k\colon [0,1] \times [0,1] \to \mathbf{R}$ satisfies (H1) and (H7). Take the constant $M > 0$ as in (H7). If we fix an interval $[c,d] \subseteq [0,1]$ and apply (H7) with $\varepsilon := d - c$ and an arbitrary (but given) finite partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of the interval $[0,1]$, then we get

$$\left|(L)\int_c^d k(0,s)\,\mathrm{d}s\right| + \sum_{i=1}^{n} \left|(L)\int_c^d [k(t_i,s) - k(t_{i-1},s)]\,\mathrm{d}s\right| \leq M|c-d|.$$

This, as the partition $0 = t_0 < t_1 < \cdots < t_n = 1$ was arbitrary, implies that

$$\left|(L)\int_c^d k(0,s)\,\mathrm{d}s\right| + \operatorname*{var}_{t\in[0,1]}\left((L)\int_c^d k(t,s)\,\mathrm{d}s\right) \leq M|c-d|.$$

In other words, k satisfies (H6). In particular, (H7) is strictly stronger than (H3) (cf. Remark 12).

The fact that (H6) is weaker than (H7) may be less surprising, when we realize that the class of linear operators which are continuous when the K-convergence is considered in the target space is contained in the class of (norm-to-norm) continuous linear operators (see, for example, [42, p. 277]). To see that (H6) is strictly weaker than (H7) it suffices to consider the following example.

Example 3. It is easy to check that the weakly singular kernel $k\colon [0,1] \times [0,1] \to \mathbf{R}$ given by the formula (6) with $f(t) = 1$ satisfies (H6). (Of course, k satisfies also (H1).) However, if for an arbitrary number $M > 0$ we choose $n \in \mathbf{N}$ so that $n > M$ and set $t_i := \frac{i}{n}$ for $i = 0, \dots, n$, then we have

$$\sup_{|a-b|\leq\frac{1}{n}}\left|(L)\int_a^b k(0,s)\,\mathrm{d}s\right| + \sum_{i=1}^{n}\sup_{|a-b|\leq\frac{1}{n}}\left|(L)\int_a^b [k(t_i,s)-k(t_{i-1},s)]\,\mathrm{d}s\right|$$
$$\geq \sum_{i=1}^{n}\left|(L)\int_{t_{i-1}}^{t_i}[k(t_i,s)-k(t_{i-1},s)]\,\mathrm{d}s\right| = \sum_{i=1}^{n}|t_i - t_{i-1}| = 1 > \frac{1}{n}M.$$

This shows that k does not satisfy (H7).

5.4. Gelfand's conditions. Commenting his theorem, Vulich observed that a similar result to Theorem 3 was obtained earlier by Gelfand (see [42, the footnote 7 on p. 279]). Indeed, on page 278 in [19], inside a reasoning concerning other things, Gelfand wrote that any continuous linear operator acting between $L^1[0,1]$ and the space of functions of bounded variation is an integral operator K given by (1) with k satisfying the following condition

(H8) $\operatorname{ess\,sup}_{s\in[0,1]} \operatorname{var}_{t\in[0,1]} k(t,s) < +\infty$.

As regards the regularity of k, Gelfand states only that "$k(t,s)$ is measurable" without indicating whether he thinks about joint measurability or measurability with respect to a certain variable. Note that for a given bounded kernel $k\colon [0,1]\times[0,1] \to \mathbf{R}$ which is Lebesgue measurable on the square $[0,1]\times[0,1]$, the function $s \mapsto \operatorname{var}_{t\in[0,1]} k(t,s)$ (although well-defined and finite at each point) may fail to be Lebesgue measurable on $[0,1]$. To see this it suffices to consider a kernel given by $k(t,s) = \chi_{\{0\}\times A}(t,s)$, where A is a subset of $[0,1]$ which is not Lebesgue measurable, since then for each $s \in [0,1]$ we have $\operatorname{var}_{t\in[0,1]} k(t,s) = \chi_A(s)$.

Another problem with the condition (H8), putting aside the measurability issue, is that it does not guarantee the Lebesgue integrability (not to mention the essential boundedness) of the function $s \mapsto k(t,s)$ on the interval $[0,1]$; here, $t \in [0,1]$ is fixed. Thus, the integral appearing in the definition of the operator K may make no sense. Indeed, if $k\colon [0,1]\times[0,1] \to \mathbf{R}$ is given by the formula

$$k(t,s) = \begin{cases} s^{-1}, & \text{if } (t,s) \in [0,1]\times(0,1], \\ 0, & \text{if } (t,s) \in [0,1]\times\{0\}, \end{cases}$$

then for each $s \in [0,1]$ we have $\operatorname{var}_{t\in[0,1]} k(t,s) = 0$, and so k satisfies (H8). But, if we fix $t \in [0,1]$, we obtain the function $s \mapsto k(t,s)$ which is not Lebesgue integrable on $[0,1]$.

Moreover, Gelfand refers to the space of functions of bounded variation as the "conjugate" (dual) to the space of continuous functions, whereas it is known that the dual to $(C[0,1], \|\cdot\|_\infty)$ may be identified not with the whole space $BV[0,1]$, but rather with its proper subspace consisting of those functions which are right-continuous on $[0,1)$ and which at $t = 0$ take the value 0 (see, for example, [3, Theorem 4.31]). This raises some doubts, and we have to admit that, unfortunately, we were not able to fully understand Gelfand's exposition.

Now, let us try to compare (H8) with our condition (H3) and other conditions mentioned in this section, adding, whenever it is needed, additional regularity requirements on the kernel k.

Remark 14. Assuming that the function $s \mapsto \operatorname{var}_{t\in[0,1]} k(t,s)$ is Lebesgue measurable on $[0,1]$, it is easy to see that any kernel $k\colon [0,1]\times[0,1] \to \mathbf{R}$ satisfying (H8) also satisfies (H4). Moreover, if k, additionally, satisfies the general hypothesis (H1), then it is straightforward to check that from (H8) it follows that for any $a, b \in [0,1]$ we have

$$\operatorname*{var}_{t\in[0,1]}\left((L)\int_a^b k(t,s)\,\mathrm{d}s\right) \leq M|a-b|$$

with $M := \operatorname{ess\,sup}_{s\in[0,1]} \operatorname{var}_{t\in[0,1]} k(t,s)$. In particular, under the above regularity assumptions (H8) implies (H3).

This also shows that if we assume that all the vertical sections $s \mapsto k(t,s)$, where $t \in [0,1]$, and the function $s \mapsto \operatorname{var}_{t\in[0,1]} k(t,s)$ are Lebesgue measurable on $[0,1]$, and that $k(0,s) = 0$ for a.e. $s \in [0,1]$, then (H8) implies (H1) and (H6) (cf. [11, Remark 7]).

As evidenced by the following example, there are some kernels k which satisfy (H6) but do not satisfy (H8), even if we assume that $k(0,s) = 0$ for each $s \in [0,1]$ and that the functions $s \mapsto k(t,s)$, where $t \in [0,1]$, and $s \mapsto \operatorname{var}_{t\in[0,1]} k(t,s)$ are Lebesgue measurable on $[0,1]$.

Example 4. For any fixed $n \in \mathbf{N}$ and $l \in \{0,\dots,2^n-1\}$ let us set

$$B_l^n := \big\{\big(t, t-2^{-(2n+1)}l\big) \in \mathbf{R}^2 \;\big|\; t \in \big(2^{-(n+1)} + 2^{-(2n+1)}l, 2^{-n}\big)\big\},$$
$$C_l^n := \big\{\big(t, t+2^{-(2n+1)}l\big) \in \mathbf{R}^2 \;\big|\; t \in \big(2^{-(n+1)}, 2^{-n} - 2^{-(2n+1)}l\big)\big\},$$

and $A := \bigcup_{n=1}^{\infty}\bigcup_{l=0}^{2^n-1}(B_l^n \cup C_l^n)$. Moreover, let the kernel $k\colon [0,1]\times[0,1] \to \mathbf{R}$ be given by the formula $k(t,s) = \chi_A(t,s)$.

Since for a fixed $t \in [0,1]$ the function $s \mapsto k(t,s)$ is a.e. zero on $[0,1]$, it is easily seen that k satisfies the conditions (H6). It is also not difficult to check that the function $s \mapsto \operatorname{var}_{t\in[0,1]} k(t,s)$ is Lebesgue measurable on $[0,1]$, but k does not satisfy (H8). Indeed, from the definition of the set A, it follows that for a.e. $s \in (2^{-(n+1)}, 2^{-n})$ we have $\operatorname{var}_{t\in[0,1]} k(t,s) = 2^{n+1}$ for every $n \in \mathbf{N}$. And so, $\operatorname{ess\,sup}_{s\in[0,1]} \operatorname{var}_{t\in[0,1]} k(t,s) = +\infty$.

In [19], Gelfand proved also two results characterizing the general form of a continuous and compact linear operator T acting from an arbitrary Banach space X into the subspace of $BV[0,1]$ consisting of all functions which vanish at $t = 0$ (see [19, Section II.9]). Namely, he showed that such operators can be described by certain abstract functions of bounded variation; from the proofs of the results it follows that if T is the given operator, then as the corresponding function one can take the function from $[0,1]$ into the dual space of X, that is X^*, given by $t \mapsto e_t \circ T$; here, e_t are the evaluation functionals on $BV[0,1]$. Although interesting, these results seem of limited applicability due to their generality.

It is worth pointing out here that we have already used an idea similar to that used by Gelfand. In Remark 3 we translated the assumptions (H1) and (H3) into the (strong) continuity of the mapping $F\colon [0,1] \to BV[0,1]$ which to each $\xi \in [0,1]$ assigns the BV-function $t \mapsto (L)\int_0^{\xi} k(t,s)\,\mathrm{d}s$, and used it to prove the compactness of the linear integral operator K. Likewise, we can translate the conditions (H1) and (H2) as well as (H5) and (H6) into, respectively, boundedness and Lipschitz continuity of the map F. For instance, the map associated to the Volterra operator is given by $F(\xi)(t) = \min\{\xi, t\}$. Note also that, in contrast to Gelfand's approach, we do not need to deal with functions taking values in the dual.

5.5. Diagram. Finally, for readers' convenience, we summarize the relationships between various conditions appearing in this paper. We will write "(A) $\Longrightarrow$ (B)" to indicate that the condition (A) is strictly stronger than the condition (B). Talking about the condition (H8) we also need some regularity assumptions on the functions $s \mapsto k(t,s)$ and $s \mapsto \operatorname{var}_{t\in[0,1]} k(t,s)$; we will denote briefly this fact by writing (R). Furthermore, we silently assume that the condition (H1) is always satisfied, that is, instead of writing e.g. (H1)+(H3), we simply write (H3).

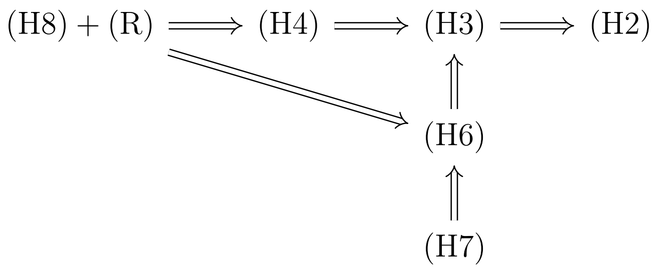

Figure 1. The diagram illustrating the relationships between various conditions imposed on the kernel k of the integral operator K.

Acknowledgements. I would like to thank Dariusz Bugajewski from Adam Mickiewicz University in Poznań and Jacek Gulgowski from University of Gdańsk for inspiring discussions concerning compactness in spaces of functions of bounded variation and for sharing with me the manuscript of their paper [9].

I also thank the anonymous referee for a careful reading and suggestions that improved the exposition of the paper.

References

[1] Abramovich, Y. A., and C. D. Aliprantis: Problems in operator theory. - Grad. Stud. Math. 51, Amer. Math. Soc., Providence, RI, 2002.

[2] Appell, J.: Personal communication, 2018.

[3] Appell, J., J. Banaś, and N. Merentes: Bounded variation and around. - De Gruyter Series in Nonlinear Analysis and Applications 17, De Gruyter, Berlin, 2014.

[4] Astudillo-Villaba, F. R., R. E. Castillo, and J. C. Ramos-Fernández: Multiplication operators on the spaces of functions of bounded p-variation in Wiener's sense. - Real Anal. Exchange 42:2, 2017, 329–344.

[5] Astudillo-Villalba, F. R., and J. C. Ramos-Fernández: Multiplication operators on the space of functions of bounded variation. - Demonstr. Math. 50:1, 2017, 105–115.

[6] Bugajewska, D., D. Bugajewski, P. Kasprzak, and P. Maćkowiak: Nonautonomous superposition operators in the spaces of functions of bounded variation. - Topol. Methods Nonlinear Anal. 48:2, 2016, 637–660.

[7] Bugajewska, D., and S. Reinwand: Some remarks on multiplier spaces II: BV-type spaces. - Z. Anal. Anwend. 38:3, 2019, 309–327.

[8] Bugajewski, D.: On BV-solutions of some nonlinear integral equations. - Integral Equations Operator Theory 46, 2003, 387–398.

[9] Bugajewski, D., and J. Gulgowski: On the characterization of compactness in the space of functions of bounded variation in the sense of Jordan. - J. Math. Anal. Appl. 484:2, 2020, 123752, 17 pages.

[10] Bugajewski, D., J. Gulgowski, and P. Kasprzak: On integral operators and nonlinear integral equations in the spaces of functions of bounded variation. - J. Math. Anal. Appl. 444:1, 2016, 230–250.

[11] Bugajewski, D., J. Gulgowski, J., and P. Kasprzak: On continuity and compactness of some nonlinear operators in the spaces of functions of bounded variation. - Ann. Mat. Pura Appl. (4) 195:5, 2016, 1513–1530.

[12] Carothers, N. L.: Real analysis. - Cambridge Univ. Press, Cambridge, 2000.

[13] Chaparro, H. C.: On multipliers between bounded variation spaces. - Ann. Funct. Anal. 9:3, 2018, 376–383.

[14] Conway, J. B.: A course in functional analysis. Second edition. - Grad. Texts in Math. 96, Springer-Verlag, New York, 1990.

[15] Dugundji, J.: Granas, A.: Fixed point theory. - Springer-Verlag, New York, 2003.

[16] Dunford, N., and B. J. Pettis: Linear operations on summable functions. - Trans. Amer. Math. Soc. 47:3, 1940, 323–392.

[17] Dunford, N., and J. T. Schwartz: Linear Operators. I. General Theory. - Pure Appl. Math. 7, Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London, 1958.

[18] Fonseca, I., and G. Leoni: Modern methods in the calculus of variations: L^p spaces. - Springer Monogr. Math., Springer, New York, 2007.

[19] Gelfand, I.: Abstrakte Funktionen und lineare Operatoren. - Rec. Math. [Mat. Sbornik] N.S. 4 (46):2, 1938, 235–286.

[20] Gorenflo, R., and S. Vessella: Abel integral equations. Analysis and applications. - Lecture Notes in Math. 1461, Springer-Verlag, Berlin, 1991.

[21] Gripenberg, G., S.-O. Londen, and Staffans, O.: Volterra integral and functional equations. - Encyclopedia Math. Appl. 34, Cambridge Univ. Press, Cambridge, 1990.

[22] Gulgowski, J.: Uniform continuity of nonautonomous superposition operators in ΛBV-spaces. - Forum Math. 31:3, 2019, 713–726.

[23] Hewitt, E., and K. Stromberg: Real and abstract analysis. A modern treatment of the theory of functions of a real variable. - Grad. Texts in Math. 25, Springer-Verlag, Berlin–Heidelberg–New York, 1975.

[24] Hildebrandt, T. H.: Linear operations on functions of bounded variation. - Bull. Amer. Math. Soc. 44:2, 1938, 75.

[25] Hille, E., and R. S. Phillips: Functional analysis and semi-groups. - Amer. Math. Soc. Colloq. Publ. 31, Amer. Math. Soc., Providence, R.I., 1996.

[26] Kantorovitch, L.: Linear operations in semi-ordered spaces I. - Rec. Math. [Mat. Sbornik] N.S. 7 (49):2, 1940, 209–284.

[27] Kantorovitch, L.: Sur la représentation des opérations linéaires. - Compos. Math. 5, 1938, 119–165.

[28] Łojasiewicz, S.: An introduction to the theory of real functions. Third edition. - A Wiley-Interscience Publication, John Wiley & Sons, Ltd., Chichester, 1988.

[29] Maćkowiak, P.: On the continuity of superposition operators in the space of functions of bounded variation. - Aequationes Math. 91:4, 2017, 759–777.

[30] Maddaus, I., Jr.: On types of "weak" convergence in linear normed spaces. - Ann. of Math. (2) 42, 1941, 229–246.

[31] Megginson, R. E.: An introduction to Banach space theory. - Grad. Texts in Math. 183, Springer-Verlag, New York, 1998.

[32] Michalak, A.: On superposition operators in spaces of regular and of bounded variation functions. - Z. Anal. Anwend. 35:3, 2016, 285–308.

[33] Monteiro, G. A., A. Slavík, and M. Tvrdý: Kurzweil–Stieltjes integral: theory and applications. - Series in Real Analysis 15, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2019.

[34] Morse, A. P.: Convergence in variation and related topics. - Trans. Amer. Math. Soc. 41:1, 1937, 48–83.

[35] Phillips, R. S.: On linear transformations. - Trans. Amer. Math. Soc. 48:3, 1940, 516–541.

[36] Prus-Wiśniowski, F., and W. H. Ruckle: The Banach spaces ΛBV are non-reflexive. - J. Math. Anal. Appl. 389:2, 2012, 1394–1396.

[37] Riemenschneider, S. D.: Compactness of a class of Volterra operators. - Tohoku Math. J. (2) 26, 1974, 385–387.

[38] Schwabik, Š.: On an integral operator in the space of functions with bounded variation. - Časopis Pěst. Mat. 97:3, 1972, 297–329.

[39] Schwabik, Š.: On an integral operator in the space of functions with bounded variation. II. - Časopis Pěst. Mat. 102:2, 1977, 189–202.

[40] Schwabik, Š., M. Tvrdý, and O. Vejvoda: Differential and integral equations: boundary value problems and adjoints. - D. Reidel Publishing Co., Dordrecht-Boston, Mass.-London, 1979.

[41] Vulich, B., On a generalized notion of convergence in a Banach space. - Ann. of Math. (2) 38:1, 1937, 156–174.

[42] Vulich, B.: Sur les formes générales de certaines opérations linéaires. - Rec. Math. [Mat. Sbornik] N.S. 2 (44):2, 1937, 275–305.

Received 13 June 2020 • Accepted 30 October 2020