

A NOTE ON HAUSDORFF MEASURES OF SELF-SIMILAR SETS IN \mathbf{R}^d

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Abstract. We prove that for all $s \in (0, d)$ and $c \in (0, 1)$ there exists a self-similar set $E \subset \mathbf{R}^d$ with Hausdorff dimension s such that $\mathcal{H}^s(E) = c|E|^s$. This answers a question raised by Zhiying Wen [16].

1. Introduction

In this paper, we investigate Hausdorff measures of self-similar sets in \mathbf{R}^d . Recall that given a finite family of contracting similitudes $\Phi = \{\phi_i\}_{i=1}^\ell$ on \mathbf{R}^d , there is a unique non-empty compact set $K \subset \mathbf{R}^d$ satisfying $K = \bigcup_{i=1}^\ell \phi_i(K)$. We call Φ an *iterated function system* (IFS) of similitudes and K the *self-similar set* generated by Φ . Moreover, given a probability vector $\mathbf{p} = (p_1, \dots, p_\ell)$, i.e. all $p_i > 0$ and $\sum_{i=1}^\ell p_i = 1$, there is a unique Borel probability measure ν supported on K satisfying

$$\nu = \sum_{i=1}^{\ell} p_i \nu \circ \phi_i^{-1}.$$

We call ν the *self-similar measure* generated by Φ and \mathbf{p} . We refer the reader to [9, 7] for the examples and detailed properties of self-similar sets and self-similar measures.

We say that Φ satisfies the *strong separation condition* (SSC) if $\phi_i(K) \cap \phi_j(K) = \emptyset$ for all distinct $i, j \in \{1, \dots, \ell\}$. Similarly, we say that Φ satisfies the *open set condition* (OSC) if there exists a non-empty bounded open set $V \subset \mathbf{R}^d$ such that $\phi_1(V), \dots, \phi_\ell(V)$ are disjoint subsets of V . Under the OSC, it is well-known that the Hausdorff dimension of K , denoted by $\dim_{\mathbf{H}} K$, equals the similarity dimension of Φ , i.e. the positive number s satisfying $\sum_{i=1}^\ell r_i^s = 1$, where $r_i \in (0, 1)$ is the contraction ratio of ϕ_i , $i = 1, \dots, \ell$. Also, the s -dimensional Hausdorff measure of K satisfies that $0 < \mathcal{H}^s(K) \leq |K|^s$, here and afterwards for $A \subset \mathbf{R}^d$, $|A|$ stands for the diameter of A . Moreover, it is known that

$$(1.1) \quad \mathcal{H}^s|_K = \mathcal{H}^s(K)\mu,$$

where $\mathcal{H}^s|_K$ stands for the restriction of the s -dimensional Hausdorff measure on K , μ is the self-similar measure generated by Φ and the probability vector (r_1^s, \dots, r_ℓ^s) . We call μ the *natural self-similar measure* on K . See [9] for the proofs of the above facts.

Nevertheless, given a self-similar set $K \subset \mathbf{R}^d$ satisfying the OSC with $\dim_{\mathbf{H}} K = s$, it is often challenging to determine the exact value of $\mathcal{H}^s(K)$. When $d = 1$, this problem was first studied independently by Marion [11, 12] and Ayer and Strichartz

[1]. They computed the exact value of $\mathcal{H}^s(K)$ under certain additional hypothesis. Their method is based on the relation (1.1) and the convex density theorem for Hausdorff measures (see [6, Theorem 2.3]). When $d > 1$ and $s > 1$ is not an integer, despite numerous studies, not a single example is known for which the exact value of $\mathcal{H}^s(K)$ is computed; see [2, 3, 4, 17] for related works. In this case, it is known that $\mathcal{H}^s(K) < |K|^s$ if the convex hull of K is a polytope (see [8, Corollary 1.6]). It remains an open question whether $\mathcal{H}^s(K) < |K|^s$ for any self-similar set $K \subset \mathbf{R}^d$ with similarity dimension $s > 1$. Regarding of this problem, Zhiying Wen raised the following question in [16].

Question 1.1. Let $s > 1$ and $\epsilon \in (0, 1)$. Does there exist a self-similar set K with similarity dimension s such that $\mathcal{H}^s(K) > (1 - \epsilon)|K|^s$?¹

In this paper, we give an affirmative answer to the above question by proving the following.

Theorem 1.2. For every $s \in (0, d)$ and $c \in (0, 1)$, there exists a self-similar set $K \subset \mathbf{R}^d$ so that its generating IFS satisfies the SSC, $\dim_{\mathbb{H}} K = s$ and $\mathcal{H}^s(K) = c|K|^s$.

Our strategy of the proof is as follows: given $s \in (0, d)$ and $\epsilon \in (0, 1)$, we construct a family of IFSs $\{\Phi_t\}_{t \in [0, 1]}$ with the corresponding self-similar sets $\{K_t\}_{t \in [0, 1]}$ such that: (1) for each $t \in [0, 1]$, Φ_t satisfies the SSC, $\dim_{\mathbb{H}} K_t = s$ and $|K_t| = 1$; (2) the mapping $t \mapsto \mathcal{H}^s(K_t)$ is continuous; (3) $\mathcal{H}^s(K_0) > 1 - \epsilon$; (4) $\mathcal{H}^s(K_1) < \epsilon$. See Proposition 2.2 for the details. A key part is the proof of (3), in which we apply the isodiametric inequality (see Lemma 2.4). Then Theorem 1.2 follows from the above result and a result in [13] about the continuity of Hausdorff measures of self-similar sets satisfying the SSC with respect to the defining data of the IFSs.

2. Proof of Theorem 1.2

Our proof of Theorem 1.2 is based on the following result.

Lemma 2.1. [14] Let $K \subset \mathbf{R}^d$ be a self-similar set generated by an IFS $\Phi = \{\phi_i\}_{i=1}^{\ell}$ which satisfies the SSC with $\dim_{\mathbb{H}} K = s$. Let μ be the natural self-similar measure on K . Set $\Delta = \min_{i \neq j} \text{dist}(\phi_i(K), \phi_j(K))$. Then we have

$$(2.1) \quad \mathcal{H}^s(K)^{-1} = \max \left\{ \frac{\mu(U)}{|U|^s} : U \subset \mathbf{R}^d \text{ is compact and convex} \right\}$$

$$(2.2) \quad = \max \left\{ \frac{\mu(U)}{|U|^s} : U \subset \mathbf{R}^d \text{ is compact and convex with } |U| \geq \Delta \right\}.$$

In particular, the above maximums are both attained.

We remark that when $d = 1$, Lemma 2.1 was proved earlier in [11, 15, 1], where it was used to compute Hausdorff measures of self-similar sets in \mathbf{R} under certain additional hypothesis. Moreover, Lemma 2.1 is not explicitly stated in [14], but it can be easily deduced from the results of [14]. Indeed, it was proved in [14,

¹It is known that for each $s \in (0, 1]$, there exists a self-similar set $K \subset \mathbf{R}$ with similarity dimension s such that $\mathcal{H}^s(K) = |K|^s$ (see e.g. [11, 1]).

Corollaries 1.5–1.6] that

$$\begin{aligned} \mathcal{H}^s(K)^{-1} &= \sup \left\{ \frac{\mu(U)}{|U|^s} : U \subset \mathbf{R}^d \text{ is open and convex} \right\} \\ &= \sup \left\{ \frac{\mu(U)}{|U|^s} : U \subset \mathbf{R}^d \text{ is open and convex with } |U| \geq \Delta \right\}. \end{aligned}$$

Then Lemma 2.1 follows from the above equalities together with a standard compactness argument.

Based on Lemma 2.1, we establish the following result, which is an essential part in our proof of Theorem 1.2.

Proposition 2.2. *For every $s \in (0, d)$ and $\epsilon \in (0, 1)$, there exist $r \in (0, 1)$, $\ell \in \mathbf{N}$ and a family of IFSs $\Phi_t = \{\phi_{t,i}(x) = rx + a_i(t)\}_{i=1}^\ell$ ($t \in [0, 1]$) on \mathbf{R}^d with $a_1, \dots, a_\ell: [0, 1] \rightarrow \mathbf{R}^d$ being continuous functions such that the following statements hold:*

- (i) *For each $t \in [0, 1]$, Φ_t satisfies the SSC and its attractor, denoted as K_t , has dimension s and diameter 1.*
- (ii) *$\mathcal{H}^s(K_0) > 1 - \epsilon$ and $\mathcal{H}^s(K_1) < \epsilon$.*

To prove Proposition 2.2, we first give an elementary lemma.

Lemma 2.3. *Let $0 < s < d$ and $\epsilon > 0$. Then there exists $N > 0$ such that for all $n \geq N$ and all $x \in [\frac{1}{4n}, 1]$,*

$$\frac{\left(x + \frac{\sqrt{d}}{n}\right)^d}{\left(1 - \frac{\sqrt{d}}{2n}\right)^d x^s} < 1 + \epsilon.$$

Proof. Let $y_0 > 0$ be large enough such that for all $y \geq y_0$,

$$(2.3) \quad \frac{\left(1 + \frac{\sqrt{d}}{y}\right)^d}{\left(1 - \frac{\sqrt{d}}{y}\right)^d} < 1 + \epsilon.$$

Then take $N > 0$ large enough such that for all $n \geq N$,

$$(2.4) \quad \frac{\left(1 + 4\sqrt{d}\right)^d}{\left(1 - \frac{\sqrt{d}}{2n}\right)^d} \left(\frac{y_0}{n}\right)^{d-s} < 1 + \epsilon.$$

Notice that such N exists since $s < d$. Let $n \geq N$ and $x \in [\frac{1}{4n}, 1]$. If $nx \geq y_0$, then $2n > nx \geq y_0$ and so by (2.3),

$$\frac{\left(x + \frac{\sqrt{d}}{n}\right)^d}{\left(1 - \frac{\sqrt{d}}{2n}\right)^d x^s} = \frac{\left(1 + \frac{\sqrt{d}}{nx}\right)^d}{\left(1 - \frac{\sqrt{d}}{2n}\right)^d} x^{d-s} \leq \frac{\left(1 + \frac{\sqrt{d}}{nx}\right)^d}{\left(1 - \frac{\sqrt{d}}{nx}\right)^d} x^{d-s} < 1 + \epsilon.$$

If $nx < y_0$, then

$$\begin{aligned} \frac{\left(x + \frac{\sqrt{d}}{n}\right)^d}{\left(1 - \frac{\sqrt{d}}{2n}\right)^d x^s} &\leq \frac{\left(x + 4\sqrt{d}x\right)^d}{\left(1 - \frac{\sqrt{d}}{2n}\right)^d x^s} && \text{(since } \frac{1}{n} \leq 4x\text{)} \\ &= \frac{\left(1 + 4\sqrt{d}\right)^d}{\left(1 - \frac{\sqrt{d}}{2n}\right)^d} x^{d-s} \\ &\leq \frac{\left(1 + 4\sqrt{d}\right)^d}{\left(1 - \frac{\sqrt{d}}{2n}\right)^d} \left(\frac{y_0}{n}\right)^{d-s} && \text{(since } nx < y_0\text{)} \\ &< 1 + \epsilon && \text{(by (2.4)).} \end{aligned}$$

This completes the proof of the lemma. □

Let \mathcal{L}^d denote the d -dimensional Lebesgue measure on \mathbf{R}^d . The following standard isodiametric inequality plays a key role in our proof of Proposition 2.2.

Lemma 2.4. [5, Theorem 2.4] *For every Lebesgue measurable set $A \subset \mathbf{R}^d$,*

$$\mathcal{L}^d(A) \leq \omega_d 2^{-d} |A|^d,$$

where ω_d denotes the Lebesgue measure of a unit ball in \mathbf{R}^d .

For $x = (x_1, \dots, x_d) \in \mathbf{R}^d$ and $\delta > 0$, let $B(x, \delta)$ be the closed ball in \mathbf{R}^d centered at x of radius δ , and let $Q(x, \delta)$ denote the cube $\prod_{i=1}^d [x_i - \delta, x_i + \delta]$. For $A \subset \mathbf{R}^d$, let $\#A$ be the cardinality of A .

Proof of Proposition 2.2. Fix $s \in (0, d)$ and $\epsilon \in (0, 1)$. We are going to construct the self-similar sets K_t ($t \in [0, 1]$) in the ball $B := B(0, 1/2)$. For this purpose, let N be as in Lemma 2.3. Pick a positive integer $n \geq N$ so that

$$(2.5) \quad \frac{\omega_d \left(n - \frac{\sqrt{d}}{2}\right)^d - 2}{(8n + 4)^s} > \epsilon^{-1}.$$

Set $F = B \cap (\mathbf{Z}^d / (2n))$. Let $\ell = \#F$ and b_1, \dots, b_ℓ be the different elements of F with

$$b_1 = (-1/2, 0, \dots, 0), \quad b_2 = (1/2, 0, \dots, 0).$$

Then by a simple volume argument (see e.g. [10, p. 17]),

$$(2.6) \quad \ell \geq \omega_d \left(n - \frac{\sqrt{d}}{2}\right)^d.$$

Let $r = \ell^{-1/s}$. Then by (2.5)–(2.6),

$$(2.7) \quad (8n + 4)r = (8n + 4)\ell^{-1/s} \leq (8n + 4)\omega_d^{-1/s} \left(n - \frac{\sqrt{d}}{2}\right)^{-d/s} < \epsilon^{1/s} < 1.$$

In particular, $8nr < 1$.

For $t \in [0, 1]$, let $\Phi_t = \{\phi_{t,i}(x) = rx + a_i(t)\}_{i=1}^\ell$, where

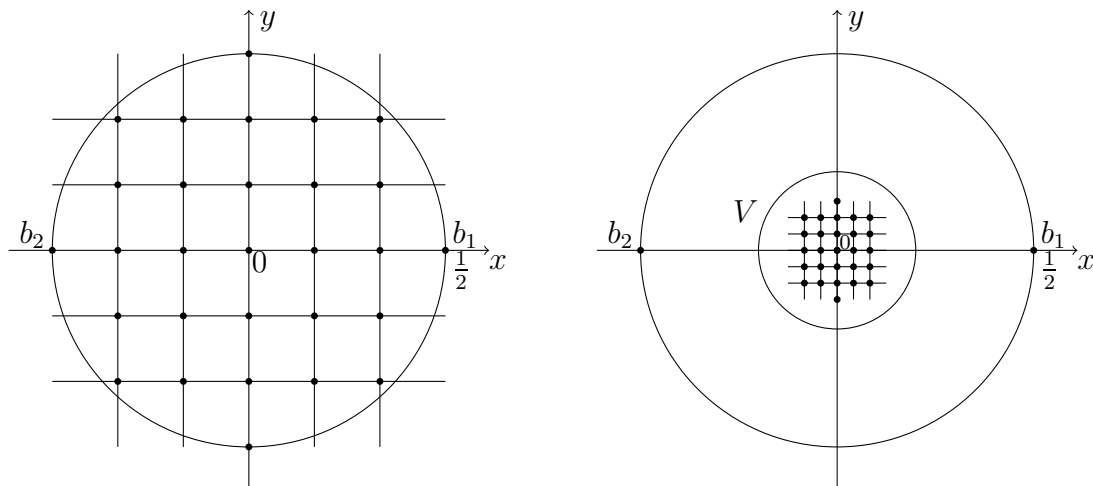
$$a_1(t) = (1 - r)b_1, \quad a_2(t) = (1 - r)b_2, \quad a_i(t) = (1 - r)(8nr)^t b_i \quad \text{for } i \in \{3, \dots, \ell\}.$$

Let K_t be the attractor of Φ_t . Notice that $a_1, \dots, a_\ell: [0, 1] \rightarrow \mathbf{R}^d$ are continuous functions. Clearly, for each $1 \leq i \leq \ell$, the fixed point of $\phi_{t,i}$ is $a_i(t)/(1-r)$. In particular, b_1, b_2 are the fixed points of $\phi_{t,1}$ and $\phi_{t,2}$, respectively. Hence $b_1, b_2 \in K_t$. Since $8nr < 1$, it is not difficult to check that for each $t \in [0, 1]$, $\phi_{t,i}(B)$ ($i = 1, \dots, \ell$) are pairwise disjoint and contained in B . This implies that $K_t \subset B$ (see e.g. [7]) and so $|K_t| \leq |B| = 1$, and Φ_t satisfies the SSC with $\dim_{\mathbf{H}} K_t = s$. Since $b_1, b_2 \in K_t$, $|K_t| \geq \|b_1 - b_2\| = 1$. Hence $|K_t| = 1$. This proves the part (i) of the proposition. In the following we prove that $\mathcal{H}^s(K_0) > 1 - \epsilon$ and $\mathcal{H}^s(K_1) < \epsilon$.

For $t \in \{0, 1\}$, let μ_t be the natural self-similar measure on K_t . That is, μ_t is the unique Borel probability measure supported on K_t satisfying

$$(2.8) \quad \mu_t = \sum_{i=1}^{\ell} r^s \mu_t \circ (\phi_{t,i})^{-1}.$$

We first show that $\mathcal{H}^s(K_0) > 1 - \epsilon$. Recall that $\Phi_0 = \{\phi_{0,i}(x) = rx + (1-r)b_i\}_{i=1}^{\ell}$. See Figure 1(a) for an illustration of the locations of b_1, \dots, b_ℓ , which are the fixed points of the elements of Φ_0 .



(a) The bold dots are the elements of F and also the fixed points of the similitudes in Φ_0 . (b) The bold dots are the fixed points of the similitudes in Φ_1 .

Figure 1. Φ_0 and Φ_1 , $n = 3$.

Since $|K_0| = 1$ and $8nr < 1$, we see that for $i \in \{1, \dots, \ell\}$, $\phi_{0,i}(K_0)$ is contained in the interior of $Q(b_i, 1/(4n))$. In particular, $Q(b_i, 1/(4n)) \cap \phi_{0,j}(K_0) = \emptyset$ for any $i, j \in \{1, \dots, \ell\}$ with $i \neq j$. Hence by (2.8),

$$(2.9) \quad \mu_0(Q(b_i, 1/(4n))) = r^s = \ell^{-1}, \quad i = 1, \dots, \ell.$$

By (2.2) there exists a compact convex set $U \subset \mathbf{R}^d$ such that

$$(2.10) \quad \frac{\mu_0(U)}{|U|^s} = \mathcal{H}^s(K_0)^{-1}$$

with $|U| \geq \min_{i \neq j} \text{dist}(\phi_{0,i}(K_0), \phi_{0,j}(K_0))$. Notice that for $i \in \{1, \dots, \ell\}$, b_i is the fixed point of $\phi_{0,i}$ and so $b_i \in \phi_{0,i}(K_0)$. Hence for all $i \neq j$, since $|\phi_{0,i}(K_0)| = |\phi_{0,j}(K_0)| = r$ and $8nr < 1$, we have by the triangle inequality,

$$\text{dist}(\phi_{0,i}(K_0), \phi_{0,j}(K_0)) \geq \|b_i - b_j\| - 2r \geq \frac{1}{2n} - 2r > \frac{1}{4n}.$$

It follows that $|U| \geq \frac{1}{4n}$. On the other hand, since $K_0 \subset B$, we have $\mu_0(U) = \mu_0(U \cap B)$ and so

$$(2.11) \quad \frac{\mu_0(U)}{|U|^s} \leq \frac{\mu_0(U \cap B)}{|U \cap B|^s}.$$

However, since $U \cap B$ is compact and convex, by (2.1) and (2.10) we see that the equality holds in (2.11). Therefore $|U| = |U \cap B|$. Hence replacing U by $U \cap B$ if necessary, we can assume that $U \subset B$ and thus $|U| \leq 1$. So we have

$$(2.12) \quad \frac{1}{4n} \leq |U| \leq 1.$$

Let $m = \#\mathcal{F}$, where $\mathcal{F} := \{Q(b, 1/(4n)) : b \in F, Q(b, 1/(4n)) \cap U \neq \emptyset\}$. Then by (2.9) and (2.6),

$$(2.13) \quad \mu_0(U) \leq m\ell^{-1} \leq m\omega_d^{-1} \left(n - \frac{\sqrt{d}}{2} \right)^{-d}.$$

On the other hand, notice that each cube in \mathcal{F} is of diameter $\sqrt{d}/(2n)$ and so it is contained in the closed $\sqrt{d}/(2n)$ -neighborhood of U , which we denote by $\overline{V}_{\sqrt{d}/(2n)}(U)$, and the intersection of any two different cubes in \mathcal{F} has zero Lebesgue measure. Hence by a simple volume argument and the isodiametric inequality (see Lemma 2.4),

$$(2.14) \quad m \left(\frac{1}{2n} \right)^d \leq \mathcal{L}^d \left(\overline{V}_{\sqrt{d}/(2n)}(U) \right) \leq \omega_d 2^{-d} \left(|U| + \frac{\sqrt{d}}{n} \right)^d.$$

Now by (2.13)–(2.14) and Lemma 2.3 (in which we take $|U| = x$ and recall (2.12)),

$$\frac{\mu_0(U)}{|U|^s} \leq \frac{\left(|U| + \frac{\sqrt{d}}{n} \right)^d}{\left(1 - \frac{\sqrt{d}}{2n} \right)^d |U|^s} < 1 + \epsilon.$$

This combining with (2.10) yields that $\mathcal{H}^s(K_0) > 1/(1 + \epsilon) > 1 - \epsilon$.

Finally, we show that $\mathcal{H}^s(K_1) < \epsilon$. Recall that Φ_1 consists of the similitudes $\phi_{1,1}(x) = rx + (1 - r)b_1$, $\phi_{1,2}(x) = rx + (1 - r)b_2$ and $\phi_{1,k}(x) = rx + (1 - r)8nr^s b_k$ for $k \in \{3, \dots, \ell\}$. Let $V = B(0, (4n + 1)r)$. Since $(8n + 4)r < 1$ (see (2.7)), it is easily checked that $\phi_{1,1}(K_1)$, $\phi_{1,2}(K_1)$ are both disjoint from V , and $\phi_{1,k}(K_1) \subset V$ for $k \in \{3, \dots, \ell\}$. See Figure 1(b) for an illustration of V and the locations of the fixed points of the elements of Φ_1 . Then by (2.5)–(2.6),

$$\frac{\mu_1(V)}{|V|^s} = \frac{1 - 2r^s}{(8n + 2)^s r^s} = \frac{\ell - 2}{(8n + 2)^s} \geq \frac{\omega_d \left(n - \frac{\sqrt{d}}{2} \right)^d - 2}{(8n + 2)^s} > \epsilon^{-1}.$$

Hence $\mathcal{H}^s(K_1) < \epsilon$ by Lemma 2.1. This completes the proof of the proposition. \square

With Proposition 2.2 in hand, we are ready to prove Theorem 1.2. The proof is a direct consequence of Proposition 2.2 combined with a continuity result in [13] about Hausdorff measures of self-similar sets generated by IFSs satisfying the SSC.

Proof of Theorem 1.2. Fix $s \in (0, d)$ and $c \in (0, 1)$. Let $\epsilon > 0$ be so small that $\epsilon < c < 1 - \epsilon$. Let Φ_t, K_t ($t \in [0, 1]$) be constructed as in Proposition 2.2. Since $a_1, \dots, a_\ell: [0, 1] \rightarrow \mathbf{R}^d$ are continuous functions, we easily see from [13, Theorem 1.2] that the mapping $t \mapsto \mathcal{H}^s(K_t)$ is continuous on $[0, 1]$. Since $\mathcal{H}^s(K_0) > 1 - \epsilon$, $\mathcal{H}^s(K_1) < \epsilon$ and $c \in (\epsilon, 1 - \epsilon)$, the continuity of $t \mapsto \mathcal{H}^s(K_t)$ implies that $\mathcal{H}^s(K_{t_0}) = c$ for some

$t_0 \in [0, 1]$. Therefore, $\mathcal{H}^s(K_{t_0}) = c|K_{t_0}|^s$ as $|K_{t_0}| = 1$. Letting $K = K_{t_0}$ we complete the proof of Theorem 1.2. \square

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