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POINTWISE CONVERGENCE ALONG A TANGENTIAL CURVE FOR THE FRACTIONAL SCHRÖDINGER EQUATION

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Abstract. In this paper we study the pointwise convergence problem along a tangential curve for the fractional Schrödinger equations in one spatial dimension and estimate the capacitary dimension of the divergence set. We extend a prior paper by Lee and the first author for the classical Schrödinger equation, which in itself contains a result due to Lee, Vargas and the first author, to the fractional Schrödinger equation. The proof is based on a decomposition argument without time localization, which has recently been introduced by the second author.

1. Introduction

Let $d \geq 1$. We consider the fractional Schrödinger equation on $\mathbf{R}^d \times \mathbf{R}$

$$\begin{cases} i\partial_t u + (-\Delta)^{\frac{m}{2}} u = 0, \\ u(\cdot, 0) = f, \end{cases}$$

with the initial data $f \in H^s(\mathbf{R}^d)$ and $m > 0$. Here, $H^s(\mathbf{R}^d)$ denotes the Sobolev space of order s whose norm is given by

$$\|f\|_{H^s(\mathbf{R}^d)} = \|(1-\Delta)^{\frac{s}{2}} f\|_{L^2(\mathbf{R}^d)}.$$

Then, the solution can be formally written as

$$u(x,t) = e^{it(-\Delta)^{\frac{m}{2}}} f(x) = (2\pi)^{-d} \int_{\mathbf{R}^d} e^{i(x\cdot\xi + t|\xi|^m)} \widehat{f}(\xi)\, \mathrm{d}\xi,$$

where $\widehat{\cdot}$ is the Fourier transform defined by $\widehat{f}(\xi) = \int_{\mathbf{R}^d} e^{-ix\cdot\xi} f(x)\, \mathrm{d}x$. While the classical and standard Schrödinger equation when $m = 2$ has drawn attention in numerous fields, the fractional Schödinger equation with general m has also been found to be influential in recent years. In fact, it is not only a model case of a general dispersive equation [10, 24] but also one of the fundamental equations in quantum mechanics [25, 26]. Since then, it has become an important subject studied by a number of authors from a variety of perspectives, see for example [7, 8, 18, 19, 20, 21, 22, 23, 33].

A fundamental question is the following pointwise convergence problem of determining the minimal exponent s for which

$$\lim_{t\to 0} e^{it(-\Delta)^{\frac{m}{2}}} f(\gamma(x,t)) = f(x) \quad \text{a.e.} \tag{1}$$

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for $f \in H^s(\mathbf{R}^d)$. Here, γ is a continuous function such that

$$\gamma\colon \mathbf{R}^d \times [-1,1] \to \mathbf{R}^d, \quad \gamma(x,0) = x.$$

The key quality of γ for (1) is the way of convergence whether *tangential* or *non-tangential* to the hyperplane $\mathbf{R}^d \times \{0\}$.

The simplest example of γ is $\gamma(x,t) = x$ and may be considered as the prototypical non-tangential case. For this γ, (1) reduces to the seminal pointwise convergence problem, so-called Carleson's problem, originating in [5]. It turns out for $d = 1$ and $m = 2$ that (1) holds for all $f \in H^s(\mathbf{R})$ if and only if $s \geq \frac{1}{4}$, due to [5] and Dahlberg and Kenig [12]. Later, Sjölin [37] generalized their results and proved $s \geq \frac{1}{4}$ is also necessary and sufficient even for $m > 1$. In higher dimensions, $d \geq 2$, the problem has attracted significant attention and been studied by many authors. When $m = 2$, it has recently been understood that $s \geq \frac{d}{2(d+1)}$ is necessary for (1) by Bourgain in [4] and $s > \frac{d}{2(d+1)}$ is sufficient for (1) by Du, Guth and Li [15] and Du and Zhang [16]. Ko and the first author [6] also proved $s > \frac{d}{2(d+1)}$ is sufficient for (1) when $m > 1$ as well. The reader may also refer to in particular the work of Vega [39], Lee [27], Bourgain [3], and Du, Guth, Li and Zhang [17] as papers which have played an important role in earlier developments.

In the study of pointwise convergence problem for the Schrödinger equation with harmonic oscillator potential, Lee and Rogers [28] showed that any $\gamma \in C^1(\mathbf{R}^d \times [-1,1] \to \mathbf{R}^d)$, such as $\gamma(x,t) = x - (t^\kappa, 0, \cdots, 0)$ with $\kappa \geq 1$, is essentially equivalent to the vertical line in the context of pointwise convergence problem of (1).

In contrast to the non-tangential case above, here we consider the tangential case. The curve γ is said to satisfy Hölder condition of order $\kappa \in (0,1]$ in t if

$$(2) \qquad |\gamma(x,t) - \gamma(x,t')| \leq C_1|t - t'|^\kappa, \quad x \in \mathbf{R}^d, \quad t, t' \in [-1,1]$$

and be bilipschitz in x if

$$(3) \qquad \frac{1}{C_2}|x - x'| \leq |\gamma(x,t) - \gamma(x',t)| \leq C_2|x - x'|, \quad t \in [-1,1], \quad x, x' \in \mathbf{R}^d$$

for some $C_1, C_2 > 0$. Then, we denote by $\Gamma(d,\kappa)$ the collection of such curves, namely,

$$\Gamma(d,\kappa) = \{\gamma\colon \mathbf{R}^d \times [-1,1] \to \mathbf{R}^d\colon \gamma \text{ satisfies } \gamma(x,0) = x, \text{ (2) and (3)}\}.$$

Note that $\Gamma(d,\kappa)$ contains $\gamma(x,t) = x - (t^\kappa, 0, \cdots, 0)$ with $0 \leq \kappa \leq 1$. As a tangential case for $\gamma \in \Gamma(1,\kappa)$, Lee, Vargas and the first author [9] observed the crucial difference in nature from the non-tangential case; $s > \max\{\frac{1}{4}, \frac{1-2\kappa}{2}\}$ is the sharp sufficient condition for (1).

We further consider a refinement of this question: quantifying the sets on which the convergence (1) fails in more precise way than Lebesgue measure. Let $0 < \alpha \leq d$. A positive Borel measure μ is said to be α-dimensional if there exists a constant c such that

$$(4) \qquad \mu(\mathbf{B}(x,r)) \leq cr^\alpha,$$

where $\mathbf{B}(x,r)$ is the ball centered at $x \in \mathbf{R}^d$ with radius $r > 0$. For $f \in H^s(\mathbf{R}^d)$ and $\gamma \in \Gamma(d,\kappa)$, let us define the *divergence set* by

$$\mathfrak{D}(\gamma, f) = \{x \in \mathbf{R}^d\colon e^{it(-\Delta)^{\frac{m}{2}}} f(\gamma(x,t)) \not\to f(x) \text{ as } t \to 0\},$$

and for a subset X in $\mathbf{R}^d$, we also define the *capacitary dimension* by

$$\dim_c(X) = \sup\{\alpha\colon \mathcal{M}^\alpha(X) \neq \emptyset\},$$

where $\mathcal{M}^\alpha(X) = \{\mu\colon \mu \text{ is } \alpha\text{-dimensional and } 0 < \mu(X) < \infty\}$. By the forthcoming Frostman's lemma, note that for a Borel set X there exists $\mu \in \mathcal{M}^\alpha(X)$ if and only if the Hausdorff dimension of X is greater than or equal to α. In this case, the capacitary dimension of X coincides with the Hausdorff dimension (see also [32]). Sjögren and Sjölin [36] considered this refined version of Carleson's problem for $\gamma(x,t) = x$ and $m \geq 2$. Later, Barceló, Bennett, Carbery and Rogers [1] extended their result to $m > 1$ and also obtained the sharp bound of the Hausdorff dimension of the divergence set by combining with work of Žubrinić [41]. In higher dimension, there are still some gaps remaining for which we refer the readers to contributions by Bennett and Rogers [2], Lucà and Rogers [29, 30, 31], and aforementioned papers [15, 16, 17]. For $\gamma \in \Gamma(d,\kappa)$ and $m = 2$, Lee and the first author obtained the capacitary dimension of the divergence set in [8] as a refinement of aforementioned result by them with Vargas[1].

Let $d = 1$ and write $\Gamma(\kappa) = \Gamma(1,\kappa)$ in the rest of the paper. Our goal of the present note is to estimate the capacitary dimension of divergence sets for the pointwise convergence to the fractional Schrödinger equation along the curve $\gamma \in \Gamma(\kappa)$. Let us define evolution operator S_t on appropriate input functions by

$$S_t f(x) = \frac{1}{2\pi} \int_{\mathbf{R}} e^{i(x\xi + t|\xi|^m)} \widehat{f}(\xi)\, \mathrm{d}\xi.$$

Our main results are the following.

Theorem 1. *Let $m > 1$, $0 < \kappa \leq 1$, μ be an α-dimensional measure and $\gamma \in \Gamma(\kappa)$. If $s > \max\left\{\frac{1}{4}, \frac{1-\alpha}{2}, \frac{1-m\alpha\kappa}{2}\right\}$, then*

$$\lim_{t\to 0} S_t(f(\gamma(x,t))) = f(x), \quad \mu\text{-a.e. } x$$

for all $f \in H^s(\mathbf{R})$.

By a standard argument, this is reduced to the following local maximal estimate.

Theorem 2. *Let $m > 1$, $0 < \kappa \leq 1$, μ be an α-dimensional measure and $\gamma \in \Gamma(\kappa)$. If $s > \max\left\{\frac{1}{4}, \frac{1-\alpha}{2}, \frac{1-m\alpha\kappa}{2}\right\}$, then there exists a constant C such that*

$$\left(\int_{-1}^{1} \sup_{t\in[-1,1]} |S_t f(\gamma(x,t))|^2 \,\mathrm{d}\mu(x)\right)^{\frac{1}{2}} \leq C\|f\|_{H^s} \tag{5}$$

for all $f \in H^s(\mathbf{R})$.

It is straightforward to obtain some results of the above type for $m \neq 2$ (more specifically $m \geq 2$) by appropriately modifying the argument in [9] or [14], however, as the second author observed in his study of a related problem in a different setting [35], there are some barriers with such an approach to treat m near 1. In particular, building on the ideas in [35] in which we completely avoid time localization techniques, we are able to handle the full range of $m > 1$ and give us the sharp sufficient conditions. Here, saying sharp is meant by that: Suppose $s < \max\left\{\frac{1}{4}, \frac{1-\alpha}{2}, \frac{1-m\alpha\kappa}{2}\right\}$, then there exist $\gamma \in \Gamma(\kappa)$, α-dimensional μ and $f \in H^s(\mathbf{R})$ such that (5) fails. Since the counterexamples can be provided by adjusting the corresponding well-known constructions (for instance, [9] and [37]) without any major difficulty, we rather focus on the sufficient conditions. As corollaries of Theorem 1, we have the following.

[1]The result due to [9] coincides with the case $\alpha = 1$ in [8]. Consequently,

$$|\mathfrak{D}(\gamma, f)| = 0$$

for all $f \in H^s(\mathbf{R})$ with $s > \max\left\{\frac{1}{4}, \frac{1-2\kappa}{2}\right\}$, where $|\cdot|$ is the Lebesgue measure defined in $\mathbf{R}$.

Corollary 3. *Let $m > 1$, $0 < \kappa \leq 1$, $\gamma \in \Gamma(\kappa)$. If $s > \frac{1}{4}$, then*

$$\dim_c(\mathfrak{D}(\gamma, f)) \leq \max\left\{1 - 2s, \frac{1-2s}{m\kappa}\right\}.$$

The special case when μ is the (1-dimensional) Lebesgue measure extends the result in [9] from $m = 2$ to $m > 1$ as follows. Here, note that the required regularity on s for (1) depends not only on κ but m as well.

Corollary 4. *Let $m > 1$, $0 < \kappa \leq 1$ and $\gamma \in \Gamma(\kappa)$. If $s > \max\{\frac{1}{4}, \frac{1-m\kappa}{2}\}$, then*

$$\left(\int_{-1}^{1} \sup_{t\in[-1,1]} |S_t f(\gamma(x,t))|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \leq C\|f\|_{H^s} \tag{6}$$

holds for all $f \in H^s(\mathbf{R})$.

Ding and Niu have also considered (6) for $m \geq 2$ in [14] and claim that the sharp sufficient condition is $s > \max\{\frac{1}{4}, \frac{1-2\kappa}{2}\}$ (in particular, their condition is independent of m). Unfortunately, it appears that the arguments in [14] are not complete and the sharp regularity threshold is $\max\{\frac{1}{4}, \frac{1-m\kappa}{2}\}$; we note that the necessity of $s \geq \max\{\frac{1}{4}, \frac{1-m\kappa}{2}\}$ for (6) follows from Section 5 in the present paper by taking $\alpha = 1$.

Combining Corollary 4 with the result from [35], the results in [9] have been completely extended from the standard Schrödinger equation to the fractional Schrödinger equation with[2] $m > 1$.

Remark. Although Theorem 2 is stated for the fractional Schrödinger evolution operator, by simply following the provided proof in Section 4 the same conclusion is valid for a wider class of evolution operators such as

$$S_t^{\Phi} f(x) = \frac{1}{2\pi}\int_{\mathbf{R}} e^{i(x\xi + t\Phi(\xi))}\widehat{f}(\xi)\,\mathrm{d}\xi,$$

where $\Phi\colon \mathbf{R} \to \mathbf{R}$ is a C^2-function for which there exist constants $C_3, C_4 > 0$ such that

$$|\xi|^{2-m}\left|\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\Phi(\xi)\right| \geq C_3 \quad \text{and} \quad |\xi|\left|\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\Phi(\xi)\right| \geq C_4\left|\frac{\mathrm{d}}{\mathrm{d}\xi}\Phi(\xi)\right|$$

for all $|\xi| \geq 1$. This class trivially contains $|\xi|^m$ whenever $m > 1$.

Throughout the paper, we denote $I = [-1, 1]$, $A \gtrsim B$ if $A \geq CB$, $A \lesssim B$ if $A \leq CB$ and $A \sim B$ if $C^{-1}B \leq A \leq CB$ for some constant $C > 0$. The domain of certain norms is sometimes abbreviated, for its meaning is clear from the context. In the following Section 2, we present useful lemmas, then first prove Theorem 1 and Corollary 3 in Section 3. In Section 4, we prove Theorem 2 by employing the philosophy in [35] of a decomposition argument without time localization. Finally, in Section 5 we see the shaprness of Theorem 2.

2. Preliminaries

In this section, as we have informed, we introduce useful lemmas which we use multiple times in the rest of the paper.

Lemma 5. *Let $d \geq 1$ and X be a set in $\mathbf{R}^d$. If $\dim_c(X) > \alpha$ then there exists α-dimensional finite measure μ such that $\operatorname{supp}\mu \subset X$ and $0 < \mu(X) < \infty$.*

[2] One can also consider the fractional Schrödinger equation with $0 < m \leq 1$ but the nature appears to be certainly different. For instance, see [11, 34, 40] for $\gamma(x,t) = x$.

For a proof of Lemma 5, we refer the reader to [32]. We need the following lemmas in order to prove Theorem 2 in Section 4.

Lemma 6. (van der Corput's lemma) *Let $-\infty < a < b < \infty$, ϕ be a sufficiently smooth real-valued function and ψ be a bounded smooth complex-valued function. Suppose we have $|\frac{\mathrm{d}^k}{\mathrm{d}\xi^k}\phi(\xi)| \geq 1$ for all $\xi \in [a,b]$. If $k=1$ and $\frac{\mathrm{d}}{\mathrm{d}\xi}\phi(\xi)$ is monotonic on (a,b), or simply $k \geq 2$, then there exists a constant C_k such that*

$$\left|\int_a^b e^{i\lambda\phi(\xi)}\psi(\xi)\,\mathrm{d}\xi\right| \leq C_k\lambda^{-\frac{1}{k}}\left(\int_a^b\left|\frac{\mathrm{d}}{\mathrm{d}\xi}\psi(\xi)\right|\mathrm{d}\xi + \|\psi\|_{L^\infty}\right)$$

for all $\lambda > 0$.

For a proof of Lemma 6, we refer the reader to [38].

Lemma 7. *Let $0 < \alpha \leq 1$ and μ be an α-dimensional measure. There exists a constant C such that for any interval $[a,b]$ $(-\infty < a, b < \infty)$*

$$\begin{aligned}(7)\qquad &\left|\iint\iint g(x,t)h(x',t')\chi_{[a,b]}(x-x')\,\mathrm{d}\mu(x)\,\mathrm{d}t\,\mathrm{d}\mu(x')\,\mathrm{d}t'\right| \\ &\leq C(b-a)^\alpha\|g\|_{L^2_x(\mathrm{d}\mu)L^1_t}\|h\|_{L^2_x(\mathrm{d}\mu)L^1_t}.\end{aligned}$$

Moreover, for $0 < \rho < \alpha$ there exists a constant C such that

$$\begin{aligned}(8)\qquad &\left|\iint\iint g(x,t)h(x',t')|x-x'|^{-\rho}\,\mathrm{d}\mu(x)\,\mathrm{d}t\,\mathrm{d}\mu(x')\,\mathrm{d}t'\right| \\ &\leq C\|g\|_{L^2_x(\mathrm{d}\mu)L^1_t}\|h\|_{L^2_x(\mathrm{d}\mu)L^1_t}.\end{aligned}$$

Here, the both integrals are taken over (x,t), $(x',t') \in I \times I$.

Proof of Lemma 7. Denoting $G(x) = \|g(x,\cdot)\|_{L^1}$ and $H(x') = \|h(x',\cdot)\|_{L^1}$,

$$\begin{aligned}&\left|\iint\iint g(x,t)h(x',t')\chi_{[a,b]}(x-x')\,\mathrm{d}\mu(x)\,\mathrm{d}t\,\mathrm{d}\mu(x')\,\mathrm{d}t'\right| \\ &\leq \int_{-1}^1\int_{-1}^1 G(x)H(x')\chi_{[a,b]}(x-x')\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(x').\end{aligned}$$

By invoking the Cauchy–Schwarz inequality on $L^2(I\times I, \mathrm{d}\mu\,\mathrm{d}\mu)$ and (4),

$$\begin{aligned}&\int_{-1}^1\int_{-1}^1 G(x)H(x')\chi_{[a,b]}(x-x')\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(x') \\ &\lesssim \left(\iint G(x)^2\chi_{[a,b]}(x-x')\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(x')\right)^{\frac{1}{2}}\left(\iint H(x')^2\chi_{[a,b]}(x-x')\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(x')\right)^{\frac{1}{2}} \\ &\lesssim (b-a)^\alpha\|G\|_{L^2_x(\mathrm{d}\mu)}\|H\|_{L^2_x(\mathrm{d}\mu)}.\end{aligned}$$

Now, (8) follows from (7), immediately. In fact, by applying a dyadic decomposition,

$$\begin{aligned}&\left|\iint\iint g(x,t)h(x',t')|x-x'|^{-\rho}\,\mathrm{d}\mu(x)\,\mathrm{d}t\,\mathrm{d}\mu(x')\,\mathrm{d}t'\right| \\ &\lesssim \sum_{j=0}^\infty 2^{\rho j}\iint G(x)H(x')\chi_{[2^{-j},2^{-j+1}]}(x-x')\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(x') \\ &\lesssim \sum_{j=0}^\infty 2^{(\rho-\alpha)j}\|G\|_{L^2_x(\mathrm{d}\mu)}\|H\|_{L^2_x(\mathrm{d}\mu)} \lesssim \|G\|_{L^2_x(\mathrm{d}\mu)}\|H\|_{L^2_x(\mathrm{d}\mu)}\end{aligned}$$

whenever $\rho - \alpha < 0$. □

3. Some reduction arguments

3.1. Proof of (Theorem 2 $\Longrightarrow$ Theorem 1). Fix an arbitrary $f \in H^s(\mathbf{R})$. Then, it is enough to show that $\mu(\mathfrak{D}(\gamma, f)) = 0$. Now, choose a sequence $\{f_n\}_{n\in\mathbf{N}} \subset C_0^\infty(\mathbf{R})$ which converges in H^s-norm to $f \in H^s(\mathbf{R})$. Then, we divide the divergence set into localized pieces as follows and show that all terms turn out to be 0.

$$\mu(\mathfrak{D}(\gamma, f)) \leq \sum_{j\in\mathbf{Z}} \sum_{\ell=1}^{\infty} \mu(\{x \in I + j\colon \lim_{t\to 0} |S_t f(\gamma(x,t)) - f(x)| > \ell^{-1}\}).$$

Now, for each $n \geq 1$, $j = 0$ and $\lambda \geq 1$ observe that

$$\begin{aligned}
&\mu(\{x \in I\colon \lim_{t\to 0} |S_t f(\gamma(x,t)) - f(x)| > \lambda^{-1}\}) \\
&\leq \mu(\{x \in I\colon \limsup_{t\to 0} |S_t f(\gamma(x,t)) - S_t f_n(\gamma(x,t))| > (3\lambda)^{-1}\}) \\
&\quad + \mu(\{x \in I\colon \limsup_{t\to 0} |S_t f_n(\gamma(x,t)) - f_n(x)| > (3\lambda)^{-1}\}) \\
&\quad + \mu(\{x \in I\colon |f_n(x) - f(x)| > (3\lambda)^{-1}\}) \\
&\leq \mu(\{x \in I\colon \sup_{t\in I} |S_t\big(f(\gamma(x,t)) - f_n(\gamma(x,t))\big)| > (3\lambda)^{-1}\}) \\
&\quad + 0 + \mu(\{x \in I\colon |f_n(x) - f(x)| > (3\lambda)^{-1}\}).
\end{aligned}$$

By invoking the Chebyshev's inequality and Theorem 2 we obtain

$$\mu(\{x \in I\colon \lim_{t\to 0} |S_t f(\gamma(x,t)) - f(x)| > \lambda^{-1}\}) \lesssim \lambda^2 \|f - f_n\|^2_{H^s(\mathbf{R})}, \tag{9}$$

which tends to 0 as $n \to \infty$. For other j, make translation $x \mapsto x + j$ and we define a measure μ_j by $\mu_j(x) = \mu(x + j)$ and a curve γ_j by $\gamma_j(x,t) = \gamma(x + j, t)$, both of which satisfy the required conditions[3] for Theorem 2 so that (9) holds with I replaced by $I + j$. Therefore, for all $j \in \mathbf{Z}$ and $\ell \geq 1$

$$\mu(\{x \in I + j\colon \lim_{t\to 0} |S_t f(\gamma(x,t)) - f(x)| > \ell^{-1}\}) = 0$$

holds as desired. □

3.2. Proof of (Theorem 1 $\Longrightarrow$ Corollary 3). Let $s > \frac{1}{4}$ and $f \in H^s(\mathbf{R})$. If we suppose $\dim_c(\mathfrak{D}(f,\gamma)) > \max\{1 - 2s, \frac{1-2s}{m\kappa}\} \geq 0$, then one would find $0 < \alpha < 1$ satisfying $\dim_c(\mathfrak{D}(f,\gamma)) > \alpha > \max\{1 - 2s, \frac{1-2s}{m\kappa}\} \geq 0$. Here, note that the second inequality is equivalent to $s > \max\{\frac{1-\alpha}{2}, \frac{1-m\alpha\kappa}{2}\}$. Hence, by Lemma 5 there would exist an α-dimensional measure μ such that $\mu(\mathfrak{D}(f,\gamma)) > 0$, which contradicts Theorem 1, and we must have $\dim_c(\mathfrak{D}(f,\gamma)) \leq \max\{1 - 2s, \frac{1-2s}{m\kappa}\}$. □

4. Proof of Theorem 2

Let

$$s_* = \min\left\{\frac{1}{4}, \frac{\alpha}{2}, \frac{m\alpha\kappa}{2}\right\}.$$

By following the standard steps via Littlewood–Paley decomposition, it is enough to show the following proposition. (For the details, for instance, see [35].)

[3]Strictly speaking, $\gamma_j \notin \Gamma(\kappa)$ because $\gamma_j(x,0) = x + j$, however this translation effect is negligible for Theorem 2 since $|\gamma_j(x,t) - \gamma_j(x',t')|$ is essentially equivalent to $|\gamma(x,t) - \gamma(x',t')|$ for any x, x', t, t'.

Proposition 8. *Let $\varepsilon > 0$. Then, there exists a constant C_ε such that*

$$\Big\| \sup_{t\in I} |S_t f(\gamma(\cdot,t))| \Big\|_{L^2(I,\mathrm{d}\mu)} \le C_\varepsilon \lambda^{\frac12 - s_* + \varepsilon} \|f\|_{L^2} \tag{10}$$

holds for all $\lambda \ge 1$ and $f \in L^2(\mathbf{R})$ whose Fourier support is contained in $\{\xi \in \mathbf{R} \colon \frac{\lambda}{2} \le |\xi| \le 2\lambda\}$.

Proof of Proposition 8. Let

$$Tf(x,t) = \chi(x,t) \int_{\mathbf{R}} e^{i(\gamma(x,t)\xi + t|\xi|^m)} f(\xi) \psi(\tfrac{\xi}{\lambda})\,\mathrm{d}\xi,$$

where $\chi = \chi_{I\times I}$ and $\psi \in C_0^\infty((-2,-\frac12)\cup(\frac12,2))$. Then, by the Plancherel theorem, (10) follows from

$$\|Tf\|^2_{L^2_x(\mathrm{d}\mu)L^\infty_t} \lesssim \lambda^{1-2s_*+\varepsilon} \|f\|^2_{L^2}. \tag{11}$$

By duality, (11) is equivalent to

$$\|T^*F\|^2_{L^2} \lesssim \lambda^{1-2s_*+\varepsilon} \|F\|^2_{L^2_x(\mathrm{d}\mu)L^1_t}, \tag{12}$$

where

$$T^*F(\xi) = \psi(\tfrac{\xi}{\lambda}) \iint \chi(x',t') e^{-i(\gamma(x',t')\xi + t'|\xi|^m)} F(x',t')\,\mathrm{d}\mu(x')\,\mathrm{d}t'.$$

Then,

$$\begin{aligned}
\|T^*F\|^2_{L^2} &= \int \psi(\tfrac{\xi}{\lambda})^2 \iint \iint \chi(x,t)\chi(x',t') \\
&\qquad \times e^{i((\gamma(x,t)-\gamma(x',t'))\xi + (t-t')|\xi|^m)} \bar F(x,t) F(x',t')\,\mathrm{d}\mu(x)\,\mathrm{d}t\,\mathrm{d}\mu(x')\,\mathrm{d}t'\,\mathrm{d}\xi \\
&= \int_W \int_{W'} \chi(w)\chi(w')\bar F(w) F(w') K_\lambda(w,w')\,\mathrm{d}_\mu w\,\mathrm{d}_\mu w' \\
&= \sum_{\ell=1}^3 \iint_{V_\ell} \chi(w)\chi(w')\bar F(w) F(w') K_\lambda(w,w')\,\mathrm{d}_\mu w\,\mathrm{d}_\mu w' \\
&= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.
\end{aligned}$$

Here, we denote $W = I \times I$, $w = (x,t) \in W$, $w' \in (x',t') \in W$ and $\mathrm{d}_\mu w = \mathrm{d}\mu(x)\,\mathrm{d}t$. Also,

$$\begin{aligned}
K_\lambda(w,w') &= \int_{\mathbf{R}} e^{i\phi(\xi,w,w')} \psi(\tfrac{\xi}{\lambda})^2\,\mathrm{d}\xi = \lambda \int_{\mathbf{R}} e^{i\phi(\lambda\xi,w,w')} \psi(\xi)^2\,\mathrm{d}\xi, \\
\phi(\xi,w,w') &= (\gamma(x,t) - \gamma(x',t'))\xi + (t-t')|\xi|^m
\end{aligned}$$

and

$$\begin{cases}
V_1 = \{(w,w') \in W\times W \colon |x-x'| \le 2\lambda^{-\frac{2s_*}{\alpha}}\}, \\
V_2 = \{(w,w') \in W\times W \colon |x-x'| > 2\lambda^{-\frac{2s_*}{\alpha}} \text{ and } \frac{1}{C_2}|x-x'| \le 2C_1|t-t'|^\kappa\}, \\
V_3 = \{(w,w') \in W\times W \colon |x-x'| > 2\lambda^{-\frac{2s_*}{\alpha}} \text{ and } \frac{1}{C_2}|x-x'| > 2C_1|t-t'|^\kappa\}.
\end{cases}$$

Then, (12) follows from

$$\mathcal{I}_\ell \lesssim \lambda^{1-2s_*+\varepsilon} \|F\|^2_{L^2_x(\mathrm{d}\mu)L^1_t}$$

for each $\ell = 1,2,3$.

The term $\mathcal{I}_1$. By using the trivial estimate

$$|K_\lambda(w,w')| \lesssim \lambda$$

and Lemma 7, we obtain

$$\mathcal{I}_1 \lesssim \lambda^{1-2s_*} \|F\|^2_{L^2_x(\mathrm{d}\mu)L^1_t}.$$

The term $\mathcal{I}_2$. Observe that

$$\left|\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\phi(\lambda\xi, w, w')\right| \gtrsim \lambda^m |t-t'||\xi|^{m-2} \gtrsim \lambda^m |x-x'|^{\frac{1}{\kappa}} \gtrsim \lambda^m \lambda^{-\frac{2s_*}{\alpha\kappa}} \ge 1$$

since $\frac{2s_*}{\alpha\kappa} = \min\{\frac{1}{2\alpha\kappa}, \frac{1}{\kappa}, m\} \le m$. Then, by Lemma 6 for arbitrary small $\varepsilon > 0$

$$\begin{aligned}|K_\lambda(w, w')| &\lesssim \lambda(\lambda^m |x-x'|^{\frac{1}{\kappa}})^{-\frac{1}{2}} \lesssim \lambda(\lambda^m |x-x'|^{\frac{1}{\kappa}})^{-\frac{2s_*}{m}} \\ &\sim \lambda^{1-2s_*} |x-x'|^{-\frac{2s_*}{m\kappa}} \lesssim \lambda^{1-2s_*+\varepsilon} |x-x'|^{-\frac{2s_*}{m\kappa}+\varepsilon}\end{aligned}$$

since $\frac{2s_*}{m} = \min\{\frac{1}{2m}, \frac{\alpha}{m}, \alpha\kappa\} < \frac{1}{2}$ and our separation assumption. Therefore, applying Lemma 7 with $\rho = \frac{2s_*}{m\kappa} - \varepsilon = \min\{\frac{1}{2m\kappa}, \frac{\alpha}{m\kappa}, \alpha\} - \varepsilon < \alpha$, it follows that

$$\mathcal{I}_2 \lesssim \lambda^{1-2s_*+\varepsilon} \|F\|^2_{L^2_x(\mathrm{d}\mu)L^1_t}.$$

The term $\mathcal{I}_3$. It remains to consider $\mathcal{I}_3$. First note that we have

$$|\gamma(w) - \gamma(w')| \ge \frac{1}{2C_2}|x-x'| \tag{13}$$

for $(w, w') \in V_3$ by using (2) and (3). Next, we split K_λ into $\mathcal{K}_1$ and $\mathcal{K}_2$ as follows.

$$\begin{aligned}K_\lambda(w, w') &= \lambda \int_{U_1} e^{i\phi(\lambda\xi, w, w')} \psi(\xi)^2 \,\mathrm{d}\xi + \lambda \int_{U_2} e^{i\phi(\lambda\xi, w, w')} \psi(\xi)^2 \,\mathrm{d}\xi \\ &=: \mathcal{K}_1 + \mathcal{K}_2,\end{aligned}$$

where

$$\begin{cases} U_1 = \{\xi \in \operatorname{supp}\psi \colon \frac{1}{C_2}|x-x'| > 4m\lambda^{m-1}|t-t'||\xi|^{m-1}\}, \\ U_2 = \{\xi \in \operatorname{supp}\psi \colon \frac{1}{C_2}|x-x'| \le 4m\lambda^{m-1}|t-t'||\xi|^{m-1}\}. \end{cases}$$

For $\mathcal{K}_1$, we use (13) in order to estimate the phase

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}\xi}\phi(\lambda\xi, w, w')\right| &\ge \lambda|\gamma(w) - \gamma(w')| - m\lambda^m |t-t'||\xi|^{m-1} \\ &\ge \frac{1}{2C_2}\lambda|x-x'| - m\lambda^m|t-t'||\xi|^{m-1} \\ &> \frac{1}{4C_2}\lambda|x-x'| \gtrsim \lambda^{1-\frac{2s_*}{\alpha}} \ge 1\end{aligned}$$

since $\frac{2s_*}{\alpha} = \min\{\frac{1}{2\alpha}, 1, m\kappa\} \le 1$. Here, note that the interval U_1 consists of at most two intervals since $\frac{\mathrm{d}}{\mathrm{d}\xi}\phi(\lambda\xi, w, w')$ is monotone on each interval $(-\infty, -1]$ and $[1, \infty)$. Thus, Lemma 6 gives that

$$\mathcal{K}_1 \lesssim \lambda(\lambda|x-x'|)^{-1} \lesssim \lambda(\lambda|x-x'|)^{-\min\{\frac{1}{2}, \alpha\}}.$$

On the other hand, for $\mathcal{K}_2$,

$$\left|\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\phi(\lambda\xi, w, w')\right| \gtrsim \lambda^m |t-t'||\xi|^{m-2} \gtrsim \lambda|x-x'|$$

so that we are allowed to apply Lemma 6 to obtain

$$\mathcal{K}_2 \lesssim \lambda(\lambda|x-x'|)^{-\frac{1}{2}} \lesssim \lambda(\lambda|x-x'|)^{-\min\{\frac{1}{2}, \alpha\}}.$$

Hence, for $(w,w')\in V_3$ we have the following kernel estimate

$$|K_\lambda(w,w')|\lesssim \lambda^{1-\min\{\frac12,\alpha\}}|x-x'|^{-\min\{\frac12,\alpha\}}\lesssim \lambda^{1-\min\{\frac12,\alpha\}+\varepsilon}|x-x'|^{-\min\{\frac12,\alpha\}+\varepsilon}.$$

Here, we used the separation assumption, $|x-x'|\gtrsim \lambda^{-\frac{2s_*}{\alpha}}$. By Lemma 7 with $\rho=\min\{\frac12,\alpha\}-\varepsilon<\alpha$ we conclude that

$$\mathcal{I}_3\lesssim \lambda^{1-\min\{\frac12,\alpha\}+\varepsilon}\|F\|^2_{L^2_x(\mathrm{d}\mu)L^1_t}\lesssim \lambda^{1-2s_*+\varepsilon}\|F\|^2_{L^2_x(\mathrm{d}\mu)L^1_t}.$$ □

5. The necessary conditions regarding Theorem 2

In this section, we present $s\ge\max\left\{\frac14,\frac{1-\alpha}{2},\frac{1-m\alpha\kappa}{2}\right\}$ is necessary for Theorem 2, otherwise there exist $\gamma\in\Gamma(\kappa)$, α-dimensional measure μ and an initial data $f\in H^s$ such that (5) fails. Throughout the section, we shall let $\lambda\ge1$, $\gamma(x,t)=x-t^\kappa$, $\mu(x)=|x|^{-1+\alpha}\,\mathrm{d}x$ and ψ_0 be a smooth radial bump function whose support is in a small neighborhood of the origin. Also, we fix $m>1$ and $0<\kappa\le1$, and we assume that the maximal estimate (5) holds.

The necessity of $s\ge\frac{1-\alpha}{2}$. In this case, we will follow the idea in [9]. Let

$$\widehat{f_1}(\xi)=\psi_0(\lambda^{-\frac1m}\xi).$$

With this initial data,

$$|S_tf_1(\gamma(x,t))|\sim\left|\int e^{i((x-t^\kappa)\xi+t|\xi|^m)}\widehat{f_1}(\xi)\,\mathrm{d}\xi\right|=\lambda^{\frac1m}\left|\int e^{i\phi_1(\eta,x,t)}\psi_0(\eta)\,\mathrm{d}\eta\right|,$$

where

$$\phi_1(\eta,x,t)=\lambda^{\frac1m}(x-t^\kappa)\eta+\lambda t|\eta|^m.$$

For $x\in(0,\frac{1}{100}\lambda^{-\frac1m})$ and $|t|<\frac{1}{100}\lambda^{-1}$, we have

$$|\phi_1(\eta,x,t)|\le\frac12$$

so that

$$\begin{aligned}|S_tf_1(\gamma(x,t))|&\gtrsim\lambda^{\frac1m}\left|\int(\cos\phi_1(\eta,x,t))\psi_0(\eta)\,\mathrm{d}\eta\right|\\&\gtrsim\lambda^{\frac1m}\chi_{(0,\frac{1}{100}\lambda^{-\frac1m})\times(0,\frac{1}{100}\lambda^{-1})}(x,t).\end{aligned}$$

Hence,

$$\begin{aligned}\left\|\sup_{t\in I}|S_tf_1(\gamma(\cdot,t))|\right\|_{L^2(I,\mathrm{d}\mu)}&\ge\left\|\sup_{t\in(0,\frac{1}{100}\lambda^{-1})}|S_tf_1(\gamma(\cdot,t))|\right\|_{L^2((0,\frac{1}{100}\lambda^{-\frac1m}),\mathrm{d}\mu)}\\&\gtrsim\lambda^{\frac1m}\lambda^{-\frac{\alpha}{2m}}.\end{aligned}$$

On the other hand,

$$\|f_1\|_{H^s}\sim\left(\int(1+|\xi|^2)^s|\psi_0(\lambda^{-\frac1m}\xi)|^2\,\mathrm{d}\xi\right)^{\frac12}\lesssim\lambda^{\frac sm}\lambda^{\frac{1}{2m}}.$$

Therefore, combining the above calculations, we obtain

$$\lambda^{\frac1m}\lambda^{-\frac{\alpha}{2m}}\lesssim\lambda^{\frac sm}\lambda^{\frac{1}{2m}}.$$

As letting $\lambda\to\infty$, it is necessary that

$$\frac1m-\frac{\alpha}{2m}\le\frac sm+\frac{1}{2m},$$

which is

$$s \geq \frac{1-\alpha}{2}.$$

The necessity of $s \geq \frac{1-m\alpha\kappa}{2}$. Here, choose the same initial data f_1 as above. For $x \in (0, \frac{1}{100}\lambda^{-\kappa})$ and $t = t(x) = x^{\frac{1}{\kappa}}$, one can estimate

$$|\phi_1(\eta, x, t)| \leq \frac{1}{2}.$$

Then, following a similar argument as above, we have

$$\lambda^{\frac{1}{m}}\lambda^{-\frac{\alpha\kappa}{2}} \lesssim \lambda^{\frac{s}{m}}\lambda^{\frac{1}{2m}}.$$

As letting $\lambda \to \infty$, it is necessary that

$$\frac{1}{m} - \frac{\alpha\kappa}{2} \leq \frac{s}{m} + \frac{1}{2m},$$

which clearly gives

$$s \geq \frac{1-m\alpha\kappa}{2}.$$

The necessity of $s \geq \frac{1}{4}$. In this case, we will refer to the idea in [37] (see page 712). Let

$$\widehat{f_2}(\xi) = \lambda^{-1}\psi_0(\lambda^{-1}\xi + \lambda).$$

Then, by the change of variables $-\eta = \lambda^{-1}\xi + \lambda$,

$$|S_t f_2(\gamma(x,t))| \sim \left| \int e^{i((x-t^\kappa)\xi + t|\xi|^m)}\lambda^{-1}\psi_0(\lambda^{-1}\xi + \lambda)\,\mathrm{d}\xi \right| = \left| \int e^{i\phi_2(\eta,x,t)}\psi_0(-\eta)\,\mathrm{d}\eta \right|,$$

where

$$\phi_2(\eta, x, t) = -(x - t^\kappa)\lambda\eta + \lambda^m t|\lambda + \eta|^m.$$

By a Taylor expansion,

$$\begin{aligned}(\lambda+\eta)^m = \lambda^m(1+\lambda^{-1}\eta)^m &= \lambda^m\left(1 + m\lambda^{-1}\eta + \frac{m(m-1)}{2}\lambda^{-2}\eta^2 + O(\lambda^{-3}|\eta|^3)\right)\\ &= \lambda^m + m\lambda^{-(1-m)}\eta + \frac{m(m-1)}{2}\lambda^{-(2-m)}\eta^2 + O(\lambda^{-(3-m)}|\eta|^3),\end{aligned}$$

and it follows that

$$\begin{aligned}\phi_2(\eta,x,t) &= -\lambda x\eta + \lambda t^\kappa\eta + \lambda^{2m}t + m\lambda^{-(1-2m)}t\eta\\ &\quad + \frac{m(m-1)}{2}\lambda^{-(2-2m)}t\eta^2 + O(\lambda^{-(3-2m)}t|\eta|^3)\\ &= \lambda^{2m}t + \lambda(-x + t^\kappa + m\lambda^{-(2-2m)}t)\eta\\ &\quad + \frac{m(m-1)}{2}\lambda^{-(2-2m)}t\eta^2 + O(\lambda^{-(3-2m)}t|\eta|^3).\end{aligned}$$

For $x \in (0, \frac{1}{100(m-1)})$, we can choose $t(x)$ such that $x = t(x)^\kappa + m\lambda^{-(2-2m)}t(x)$. In fact, if we consider the function $\tau(t) = t^\kappa + m\lambda^{-(2-2m)}t$, then $\tau\colon [0,\infty) \to [0,\infty)$ is a strictly increasing bijection and

$$0 = \tau^{-1}(0) < \tau^{-1}(x) = t(x) < \tau^{-1}(\tfrac{1}{100(m-1)}) < \frac{\lambda^{2-2m}}{100m(m-1)}.$$

Therefore, for such choice of $(x, t(x))$, it follows that

$$|\phi_2(\eta, x, t(x)) - \lambda^{2m}t(x)| \lesssim 0 + \frac{1}{100} + O(\lambda^{-1}) \leq \frac{1}{2},$$

which implies that

$$|S_t f_2(\gamma(x,t(x)))| \sim \left| \int \cos(\phi_2(\eta,x,t(x)) - \lambda^{2m} t(x)) \psi_0(-\eta)\, \mathrm{d}\eta \right| \gtrsim \chi_{(0,\frac{1}{100(m-1)})}(x).$$

Hence,

$$\left\| \sup_{t\in I} |S_t f_2(\gamma(\cdot,t))| \right\|_{L^2(I,\mathrm{d}\mu)} \gtrsim 1.$$

On the other hand,

$$\|f_2\|_{H^s} = \left(\int (1+|\xi|^2)^s |\lambda^{-1}\psi_0(\lambda^{-1}\xi+\lambda)|^2\, \mathrm{d}\xi \right)^{\frac{1}{2}} \lesssim \lambda^{2s}\lambda^{-\frac{1}{2}}.$$

Therefore, combining the calculations above implies that

$$1 \lesssim \lambda^{2s}\lambda^{-\frac{1}{2}}.$$

As $\lambda \to \infty$, it is necessary that

$$s \geq \frac{1}{4}.$$

This ends the proof that $s \geq \max\left\{\frac{1}{4}, \frac{1-\alpha}{2}, \frac{1-m\alpha\kappa}{2}\right\}$ is necessary for (5) to hold.

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