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THREE-TERM ARITHMETIC PROGRESSIONS IN SUBSETS OF $\mathbf{F}_q^\infty$ OF LARGE FOURIER DIMENSION

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Abstract. We show that subsets of $\mathbf{F}_q^\infty$ of large Fourier dimension must contain three-term arithmetic progressions. This contrasts with a construction of Shmerkin of a subset of $\mathbf{R}$ of Fourier dimension 1 with no three-term arithmetic progressions.

1. Introduction

In a recent paper, Ellenberg and Gijswijt [2] have shown that, for any odd prime q, there exists $\alpha_q < 1$ such that a subset of $\mathbf{F}_q^d$ with at least $q^{\alpha_q d}$ elements must contain a three-term arithmetic progression. This contrasts with the case for finite cyclic groups, where Behrend [1] constructed a counter example—a subset of $\mathbf{Z}/N\mathbf{Z}$ with $C_\epsilon N^{1-\epsilon}$ elements that does not contain a three-term arithmetic progression.

In this note, we will consider what happens in the vector space $\mathbf{F}_q^\infty$—a vector space of infinite dimension over $\mathbf{F}_q$. By $\mathbf{F}_q^\infty$, we mean the vector space consisting of infinite sequences of elements of $\mathbf{F}_q$ with the product topology. This is a compact abelian group that is isomorphic to the additive group of $\mathbf{F}_q[[t]]$, the ring of formal power series over $\mathbf{F}_q$.

In light of the result of Ellenberg and Gijswijt, one may be tempted to guess that a subset of $\mathbf{F}_q^\infty$ of full Hausdorff dimension must contain a three-term arithmetic progression; however, this has been shown not to be the case [3]. The construction was inspired by a similar construction of Keleti [5] of a subset of $\mathbf{R}$ of full Hausdorff dimension that does not contain any solutions to $x_4 - x_3 = x_2 - x_1$ with $x_1 \neq x_2$ and $x_3 \neq x_4$. We will present a more elementary counterexample in this paper for completeness. Because size in the sense of Hausdorff dimension is not enough to guarantee the existence of a three-term arithmetic progression, some additional condition, such as a Fourier decay condition, is needed.

In the real-variable setting, Łaba and Pramanik [6] have shown that a subset of $\mathbf{R}$ supporting a measure satisfying a Fourier decay condition as well as a ball condition depending on the rate of Fourier decay must contain a three-term arithmetic progression. However, Shmerkin [11] has constructed a subset of $\mathbf{R}$ of Fourier dimension 1 not containing any three-term arithmetic progressions. Shmerkin's construction relies on the Behrend example [1] of a large subset of $\{1, 2, \ldots, N\}$ that does not contain a three-term arithmetic progression. Because the result [2] of Ellenberg and Gijswijt implies that no such example can exist for finite vector spaces, it seems sensible to guess that a subset of $\mathbf{F}_q^\infty$ with large Fourier dimension must contain a three-term arithmetic progression. This is exactly what we will show:

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Theorem 1.1. *Let q be an odd prime. Let E be a compact subset of $\mathbf{F}_q^\infty$ supporting a probability measure μ such that for some $2/3 < \beta < 1$ and some $0 < \alpha < 1$, and some positive constants C_1 and C_2:*

1. *There exists $E' \subset E$ such that $\mu(E') > 0$ and for all balls $B \subset \mathbf{F}_q^\infty$,*
$$\mu'(B) \leq C_1 \operatorname{rad}(B)^\alpha.$$
Here, μ' is the measure $\mu \mathbf{1}_{E'}$ whose support is restricted to E'.
2. $\widehat{\mu}(\xi) \leq C_2|\xi|^{-\beta/2}$ *for all nontrivial characters ξ.*

We will see in Lemma 2.6 that condition 2 above implies condition 1 for all $\alpha < \beta$. Let α_q be such that, for sufficiently large d, any subset of $\mathbf{F}_q^d$ consisting of at least $q^{\alpha_q d}$ elements must contain a three-term arithmetic progression. Suppose that α and β satisfy the following condition:

$$1 - \alpha < \left(\frac{3\beta}{2} - 1\right)\left(\frac{1-\alpha_q}{3-\alpha_q}\right) \tag{1}$$

Then the set E must contain a three-term arithmetic progression. In particular, Lemma 2.6 implies that the condition (1) *is met provided that*

$$\beta > \frac{8 - 4\alpha_q}{9 - 5\alpha_q}. \tag{2}$$

This differs from the Łaba–Pramanik result [6] because the value α does not depend on the constants C_1 and C_2. This allows us to drop the first assumption provided that β is sufficiently close to 1. The counterexample of Shmerkin [11] shows that this assumption cannot be dropped in the Euclidean setting.

In order to properly interpret this theorem, we need to discuss some of the basic properties of the Fourier transform on $\mathbf{F}_q^\infty$.

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2. Fourier Analysis on $\mathbf{F}_q^\infty$

2.1. The abelian groups $\mathbf{F}_q^\infty$ and $\widehat{\mathbf{F}_q^\infty}$. Much of the material in this section can be found in Taibleson's book [12]. Let q be an odd prime, and let $\mathbf{F}_q^\infty$ be the group

$$\prod_{j=0}^\infty \mathbf{F}_q$$

equipped with the product topology. With respect to this topology, $\mathbf{F}_q^\infty$ is a compact abelian group. The topology on $\mathbf{F}_q^\infty$ is induced by an absolute value: given an element $x = (x_0, x_1, x_2, \ldots)$ of $\mathbf{F}_q^\infty$, we define $|x| = q^{-j}$, where j is the index of the first nonzero component of x. If $x = 0$, then we take $|x| = 0$. There is a natural projection $\pi_d \colon \mathbf{F}_q^\infty \to \mathbf{F}_q^d$ given by $\pi_d(x) = (x_0, \ldots, x_{d-1})$. Note that for any $d^* > d$, there is a natural projection from $\mathbf{F}_q^{d^*} \to \mathbf{F}_q^d$; we will abuse notation and also use π_d for this projection. As for $\mathbf{F}_q^d$, we define an absolute value on $\mathbf{F}_q^d$

by $|(x_0,\dots,x_{d-1})| = q^{-j}$, where j is the index of the first nonzero component of $(x_0,\dots,x_{d-1})$, and $|(0,0,\dots,0)| = 0$. Notice that if $x \in \mathbf{F}_q^\infty$ is such that $|\pi_d(x)| > 0$, then $|\pi_d(x)| = |x|$.

The compact abelian group $\mathbf{F}_q^\infty$ is equipped with a Haar probability measure dx. This measure assigns a measure of q^{-j} to any closed ball of radius q^{-j}. The pushforward of this measure under π_d yields the uniform probability measure on $\mathbf{F}_q^d$.

The Fourier character group $\widehat{\mathbf{F}_q^d}$ of $\mathbf{F}_q^d$ is isomorphic to $\mathbf{F}_q^d$ as an abelian group. We will write $(\xi_1,\dots,\xi_d)$ for a typical character on $\mathbf{F}_q^d$ (notice that the indexing will start from 1 instead of 0). We define an absolute value on $\widehat{\mathbf{F}_q^d}$ by $|\xi| = q^j$, where j is the maximum index of a nonzero component of $(\xi_1,\dots,\xi_d)$. The product $(\xi_1,\dots,\xi_d)\cdot(x_0,\dots,x_{d-1})$ is defined by $\xi_1 x_0 + \cdots + \xi_d x_{d-1}$, which is defined as an element of $\mathbf{F}_q$. We can therefore make sense of $\exp(\frac{2\pi i}{q}\xi\cdot x)$, which will be written as

$$e_q(\xi\cdot x) = \exp\left(\frac{2\pi i}{q}\xi\cdot x\right). \tag{3}$$

This describes the action of $\widehat{\mathbf{F}_q^d}$ on $\mathbf{F}_q^d$.

The Fourier character group $\widehat{\mathbf{F}_q^\infty}$ consists of sequences of the form $\xi = (\xi_1,\xi_2,\xi_3,\dots)$ where only finitely many ξ_j are nonzero. The absolute value $|\xi|$ of ξ is given by q^j, where j is the largest index of a nonzero component of ξ, with $|0|$ taken to be 0. Because all of the components of ξ after the jth component are zero, we can define a product $\xi\cdot x$ for $\xi \in \widehat{\mathbf{F}_q^\infty}$ and $x \in \mathbf{F}_q^\infty$ as the finite sum

$$\sum_{k=1}^{j} \xi_k x_{k-1}$$

which makes sense as an element of $\mathbf{F}_q$. We can thus define $e_q(\xi\cdot x)$ as before, giving the action of $\widehat{\mathbf{F}_q^\infty}$ on $\mathbf{F}_q^\infty$. Notice that each element of $\widehat{\mathbf{F}_q^\infty}$ can be viewed as an element of $\widehat{\mathbf{F}_q^d}$ where $d \ge j$ and $q^j = |\xi|$. In this sense, every element of $\widehat{\mathbf{F}_q^\infty}$ can be viewed as an element of $\widehat{\mathbf{F}_q^j}$ for some finite j. In fact, if $|\xi| \le q^d$, then $\xi\cdot x = \xi\cdot\pi_d(x)$. In other words, the function $x \mapsto \xi\cdot x$ is constant on closed balls of radius $|\xi|^{-1}$ for $\xi \ne 0$.

Given $d^* > d$, and $x \in \mathbf{F}_q^{d^*}$, we can write $x = (x_0,x_1,\dots,x_{d^*-1})$ as a sum $x = x' + x''$, where

$$x' = (x_0,\dots,x_{d-1},0,\dots,0),$$
$$x'' = (0,\dots,0,x_d,\dots,x_{d^*-1}).$$

We call this the *order d decomposition* on $\mathbf{F}_q^{d^*}$. Similarly, given $\xi \in \widehat{\mathbf{F}_q^{d^*}}$, we can write $\xi = \xi' + \xi''$, where

$$\xi' = (\xi_1,\dots,\xi_d,0,\dots,0),$$
$$\xi'' = (0,\dots,0,\xi_{d+1},\dots,\xi_{d^*}).$$

We will call this the *order d decomposition* of ξ. We note some trivial facts about these order d decompositions. First, we observe that $|x''| \le q^{-d}$ and $|\xi'| \le q^d$. We have $|\xi''| \ge q^{d+1}$ unless $\xi'' = 0$. We also have that $(x'+x'')\cdot(\xi'+\xi'') = x'\cdot\xi' + x''\cdot\xi''$.

The Haar measure on $\widehat{\mathbf{F}_q^\infty}$ will be denoted $d\xi$. This Haar measure is simply the counting measure on $\widehat{\mathbf{F}_q^\infty}$. Because the abelian group $\widehat{\mathbf{F}_q^\infty}$ is not compact, the Haar measure on $\widehat{\mathbf{F}_q^\infty}$ is not finite.

2.2. The Fourier transform on $\mathbf{F}_q^\infty$. The Fourier transform of an L^1 function $f\colon \mathbf{F}_q^\infty \to \mathbf{C}$ is given by

$$\widehat{f}(\xi) = \int f(x) e_q(-x\cdot\xi)\,dx$$

where dx is the Haar measure on $\mathbf{F}_q^\infty$ and e_q is as defined in (3). The Fourier transform of a finite measure μ on $\mathbf{F}_q^\infty$ is

$$\widehat{\mu}(\xi) = \int e_q(-x\cdot\xi)\,d\mu(x)$$

The Fourier transform of a function $f\colon \mathbf{F}_q^d \to \mathbf{C}$ is

$$\widehat{f}(\xi) = \sum_{x\in\mathbf{F}_q^d} f(x) e_q(-x\cdot\xi)$$

Notice that, if μ is a measure on $\mathbf{F}_q^\infty$ and μ_d is the pushforward of μ under π_d (which can be interpreted as a function on $\mathbf{F}_q^d$), and $|\xi| \le q^d$, then we have (by conflating $\xi \in \mathbf{F}_q^\infty$ with $\xi \in \mathbf{F}_q^d$ as above)

$$\begin{aligned}\widehat{\mu_d}(\xi) &= \sum_{x_0\in\mathbf{F}_q^d} \mu_d(x_0) e_q(-x_0\cdot\xi) = \sum_{x_0\in\mathbf{F}_q^d} \int_{\pi_d(x)=x_0} e_q(-x_0\cdot\xi)\,d\mu(x)\\ &= \sum_{x_0\in\mathbf{F}_q^d} \int_{\pi_d(x)=x_0} e_q(-x\cdot\xi)\,d\mu(x) = \widehat{\mu}(\xi)\end{aligned}$$

This means that the Fourier coefficients of a measure $\mu(\xi)$ where $|\xi| \le q^d$ can be computed directly in the finite vector space $\mathbf{F}_q^d$ without passing to the limit.

One important algebraic fact about Fourier analysis on $\mathbf{F}_q^d$ is the *convolution rule.* This rule states that for complex-valued functions f and g on $\mathbf{F}_q^d$:

$$\widehat{fg}(\xi_1) = q^{-d} \sum_{\xi_2\in\widehat{\mathbf{F}_q^d}} \widehat{f}(\xi_2)\widehat{g}(\xi_1-\xi_2). \tag{4}$$

The factor q^{-d} will be important in the following arguments.

2.3. Hausdorff and Fourier dimension of subsets of $\mathbf{F}_q^\infty$. A good general reference for Hausdorff and Fourier dimensions in Euclidean spaces is [8]. The notion of Fourier dimension occurring in this section is the $\mathbf{F}_q^\infty$ equivalent of the Euclidean Fourier dimension. Most of the material in this section appears in the thesis of Christos Papadimitropoulos [9].

Because $\mathbf{F}_q^\infty$ is a metric space, we can define the Hausdorff dimension of compact subsets of $\mathbf{F}_q^\infty$ in the usual manner. We will briefly review this definition now.

For a compact set $E \subset \mathbf{F}_q^\infty$, $t > 0$, define a t-covering of E to be a covering of E by closed balls of radius at most t. Define the s-dimensional t-Hausdorff content of E as follows:

$$\mathcal{H}_t^s(E) := \inf_{\mathcal{B}\ t\text{-covering of } E} \sum_{B\in\mathcal{B}} \operatorname{rad}(B)^s.$$

Here, the notation $\operatorname{rad}(B)$ denotes the radius of the ball B. The value of $\mathcal{H}_t^s(E)$ increases as $t \to 0$ because the infimum is taken over a smaller family of coverings. We define

$$\mathcal{H}^s(E) := \sup_{t>0} \mathcal{H}_t^s(E),$$

with the understanding that this supremum may be infinite.

$\mathcal{H}^s(E)$ is a non-increasing function of s. In fact, $\mathcal{H}^s(E)$ will be equal to either 0 or ∞ except for at most one value of s. Let $s_0 = \sup\{0 \le s \le 1 \colon \mathcal{H}^s(E) = \infty\}$, taking the supremum to be zero if no such s exist. Then s_0 is called the Hausdorff dimension of the set E. Note that $\mathcal{H}^{s_0}(E)$ may be equal to 0, ∞, or a finite non-zero value.

Frostman's Lemma relates the Hausdorff dimension of a compact subset E of $\mathbf{F}_q^\infty$ to the ball condition of measures supported on the set E. In fact, this statement holds without the assumption that E is compact, but that is all we will need.

The following version of Frostman's lemma can be found in Mattila [7] as Theorem 8.17.

Lemma 2.1. (Frostman's Lemma on compact metric spaces) *Let X be a compact metric space such that $\mathcal{H}^s(X) > 0$. Then there exists a Radon probability measure μ and a constant C such that*

$$\mu(B) \le Cr^s \text{ for all closed balls } B \text{ of radius } r \text{ and for all } r > 0. \tag{5}$$

Conversely, if X is a compact metric space supporting a Radon probability measure μ satisfying the condition (5)*, then we have $\mathcal{H}^s(X) > 0$.* □

Technically, the converse statement does not appear in Theorem 8.17. We will briefly present a proof of the converse below. This proof is a minor adaptation of the Euclidean version appearing in Theorem 2.7 from Mattila [8].

Proof of converse statement in Lemma 2.1. Suppose that X is a compact metric space supporting a probability measure μ satisfying the condition (5). Cover X by a collection of closed balls $\{B_j\}$. Then by (5), $\operatorname{rad}(B_j)^s > C^{-1}\mu(B_j)$ for each B_j. By subadditivity, we also have $\sum_j \mu(B_j) \ge \mu(X)$. Thus

$$\sum_j \operatorname{rad}(B_j)^s \ge C^{-1} \sum_j \mu(B_j) \ge C^{-1}\mu(X) = C^{-1}.$$

Because this holds for all coverings of X by closed balls, this shows that $\mathcal{H}^s(X) \ge C^{-1}$. □

The inequality (5) is called the s-dimensional ball condition. On $\mathbf{F}_q^\infty$, the s-dimensional ball condition is related to the finiteness of the s-energy of μ. The following lemma appears in [9] as Lemma 4.3.1 and Lemma 4.3.2:

Lemma 2.2. *Let μ be a Borel probability measure on $\mathbf{F}_q^\infty$ satisfying the s-dimensional ball condition* (5)*. If $t < s$, then the t-energy*

$$\iint |x-y|^{-t}\, d\mu(x)\, d\mu(y) \tag{6}$$

is finite. Here, we use the convention that the t-energy is infinite if $\mu \times \mu(\Delta)$ is nonzero, where Δ is the diagonal $\{(x,x) \colon x \in \mathbf{F}_q^\infty\}$. Conversely, if the t-energy (6) *is finite, then there exists a set $A \subset \mathbf{F}_q^\infty$ such that $\mu(A) > 0$ and such that the measure $\mathbf{1}_A\mu$ satisfies the t-dimensional ball condition.* □

There is also a Fourier-analytic expression for the t-energy. This lemma is a small modification of Lemma 4.3.4 (ii) in [9].

Lemma 2.3. *If μ is a probability measure on $\mathbf{F}_q^\infty$,*

$$\iint |x-y|^{-t} d\mu(x)\, d\mu(y) = \frac{1-q^{-1}}{1-q^{t-1}}|\widehat{\mu}(0)|^2 + \sum_{\xi \ne 0} \frac{1-q^{-t}}{1-q^{t-1}}|\widehat{\mu}(\xi)|^2|\xi|^{t-1}.$$

Therefore, the t-energy of μ is finite if and only if $\sum_{\xi\neq 0} |\widehat{\mu}(\xi)|^2 |\xi|^{t-1}$ *is finite.* $\square$

The proof of this result is based on a formula for the Fourier transform of the Riesz potential. We have included a proof of this result in the appendix for the interested reader.

We are now ready to define the Fourier dimension of a compact subset $E \subset \mathbf{F}_q^\infty$.

Definition 2.4. Let $E \subset \mathbf{F}_q^\infty$ be a compact set. The Fourier dimension of E is the supremum over all real numbers s such that there exists a probability measure μ_s supported on E such that

$$|\widehat{\mu}_s(\xi)| \leq C_s |\xi|^{-s/2} \quad \text{for all } \xi \neq 0. \tag{7}$$

We will now quickly verify that any measure satisfying (7) will have finite t-energy for any $t < s$- thus a set of Fourier dimension s_0 will support a measure with finite s-energy for any $s < s_0$.

Lemma 2.5. *Suppose that μ satisfies the condition* (7). *Then the t-energy of μ is finite for any $t < s$.*

Proof. By Lemma 2.3, it is sufficient to verify that the sum

$$\sum_{\xi\neq 0} |\widehat{\mu}(\xi)|^2 |\xi|^{t-1}$$

is finite. In order to estimate this sum, we split the summation region $\{\xi \colon \xi \neq 0\}$ into disjoint annuli $\{\xi \colon |\xi| = q^j\}$.

$$\begin{aligned}\sum_{\xi\neq 0} |\widehat{\mu}(\xi)|^2 |\xi|^{t-1} &= \sum_{j=1}^{\infty} \sum_{|\xi|=q^j} |\widehat{\mu}(\xi)|^2 |\xi|^{t-1} \leq C_s^2 \sum_{j=1}^{\infty} \sum_{|\xi|=q^j} |\xi|^{-s} |\xi|^{t-1} \\ &= C_s^2 \sum_{j=1}^{\infty} \sum_{|\xi|=q^j} q^{j(t-s-1)} \leq C_s^2 \sum_{j=1}^{\infty} q^{j(t-s-1)} q^j = C_s^2 \sum_{j=1}^{\infty} q^{j(t-s)},\end{aligned}$$

which converges because $s > t$. $\square$

Combining all of these facts gives the following simple statement:

Lemma 2.6. *Suppose μ is a Borel probability measure supported on a compact set $E \subset \mathbf{F}_q^\infty$ such that $|\widehat{\mu}(\xi)| \leq C|\xi|^{-\beta/2}$ for some constant C and all nonzero $\xi \in \widehat{\mathbf{F}_q^\infty}$. Then there exists a set A such that $\mu(A) > 0$ and such that $\mathbf{1}_A\mu$ satisfies the α-dimensional ball condition for any $\alpha < \beta$.*

Proof. Let $\alpha < \beta$. By Lemma 2.5, the α-energy of μ is finite. Therefore, by Lemma 2.2, there exists $A \subset \mathbf{F}_q^\infty$ such that $\mu(A) > 0$ and such that $\mathbf{1}_A\mu$ satisfies the α-dimensional ball condition. $\square$

3. An AP-free subset of $\mathbf{F}_q^\infty$ with Hausdorff dimension 1

We will present an example of a subset E of $\mathbf{F}_q^\infty$ of Hausdorff dimension 1 that does not contain a three-term arithmetic progression. This example has the advantage that E is easily seen to have Fourier dimension 0. The author would like to thank the anonymous referee for suggesting this example.

We define the set E as follows:

$$E := \{x \in \mathbf{F}_q^\infty \colon x_0 = x_1 = 0 \text{ and } x_{n^2} = (x_n)^2 \text{ for all integers } n \geq 2.\}$$

Because no conditions are imposed on x_j for non-square j, it is not difficult to show that E has Hausdorff dimension 1.

To see that E has no three-term arithmetic progressions, we use the following algebraic fact about $\mathbf{F}_q$.

Lemma 3.1. *Let q be an odd prime power. Then the only solutions in $\mathbf{F}_q$ to the system*

$$a - 2b + c = 0$$
$$a^2 - 2b^2 + c^2 = 0$$

are the trivial solutions for which $a = b = c$.

Proof. From the first equation and the fact that q is odd, we have that

$$b = \frac{a+c}{2}.$$

Thus

$$b^2 = \frac{a^2}{4} + \frac{ac}{2} + \frac{c^2}{4}$$

and

$$a^2 - 2b^2 + c^2 = \frac{a^2}{2} - ac + \frac{c^2}{2} = \frac{1}{2}(a^2 - 2ac + c^2) = \frac{1}{2}(a-c)^2.$$

So if $a^2 - 2b^2 + c^2 = 0$, then we have $a = c$. Thus $a = b = c$. □

Lemma 3.1 is easily seen to imply that E does not contain any nontrivial three-term arithmetic progressions. For if $(x, y, z) \in E^3$ solves the equation $x - 2y + z = 0$, then for each $j \geq 2$, we have the equations $x_j - 2y_j + z_j = 0$ and $x_j^2 - 2y_j^2 + z_j^2 = 0$, which implies $x_j = y_j = z_j$ for all $j \geq 0$ (by Lemma 3.1 and the fact that $x_0 = x_1 = y_0 = y_1 = z_0 = z_1 = 0$), so (x, y, z) must be trivial.

It only remains to be seen that the Fourier dimension of E is zero. A stronger statement is shown in the following lemma.

Lemma 3.2. *Let μ be a Borel probability measure supported on E, and let j be any non-square integer. Then there exists a vector ξ of the form $\xi = (0, \ldots, 0, \xi_{j+1}, 0, \ldots, 0, \xi_{j^2+1}, 0, 0, \ldots)$, with $(\xi_{j+1}, \xi_{j^2+1}) \neq (0, 0)$, such that $|\widehat{\mu}(\xi)| \geq C_q$, where $C_q > 0$ is a constant depending only on q.*

Proof. For $j \geq 2$, we define the auxiliary function $f_j \colon \mathbf{F}_q^2 \to \mathbf{R}_{\geq 0}$ as follows:

$$f_j(x_1, x_2) := \mu(\{y \in E \colon y_j = x_1, y_{j^2} = x_2\}).$$

because μ is supported on the set E, it follows that f_j is supported on the set

$$S := \{(x, x^2) \colon x \in \mathbf{F}_q\},$$

and furthermore, for $j \geq 2$, we have $\sum_{x \in \mathbf{F}_q} f_j(x, x^2) = 1$.

We define C_q to be the constant

$$C_q = \inf_{\substack{\mathrm{supp} g \subset S \\ g \geq 0 \\ \sum_{x \in \mathbf{F}_q} g(x,x^2) = 1}} \max_{\substack{(\xi_1,\xi_2) \in \widehat{\mathbf{F}_q^2} \\ (\xi_1,\xi_2) \neq (0,0)}} |\widehat{g}(\xi_1, \xi_2)|.$$

This constant is strictly greater than zero, since the only functions g on $\mathbf{F}_q^2$ for which $\widehat{g}(\xi_1, \xi_2) = 0$ for all nonzero (ξ_1, ξ_2) are the constant functions, none of which is close to a nonnegative function supported on S that sums to 1.

For each $j \geq 2$, we choose a 2-tuple $(\xi_{j+1}, \xi_{j^2+1}) \neq (0,0)$ such that

$$|\widehat{f}_j(\xi_{j+1}, \xi_{j^2+1})| \geq C_q.$$

Choose

$$\xi = (0, 0, \ldots, \xi_{j+1}, 0, 0, \ldots, \xi_{j^2+1}, 0, 0, \ldots),$$

where ξ_{j+1}, ξ_{j^2+1} are as above. We have that

$$\begin{aligned}
\widehat{\mu}(\xi) = \widehat{\mu}_{j^2+1}(\xi) &= \sum_{x \in \mathbf{F}_q^{(j^2+1)}} \mu_{j^2+1}(x) e_q(-x \cdot \xi) \\
&= \sum_{x_j} \sum_{x_{j^2}} e_q(-x_j \xi_{j+1} - x_{j^2} \xi_{j^2+1}) \sum_{\substack{y \in \mathbf{F}_q^{(j^2+1)} \\ y_j = x_j \\ y_{j^2} = x_{j^2}}} \mu_{j^2+1}(y) \\
&= \sum_{x_j} \sum_{x_{j^2}} e_q(-x_j \xi_{j+1} - x_{j^2} \xi_{j^2+1}) f_j(x_j, x_{j^2}) = \widehat{f}_j(\xi_{j+1}, \xi_{j^2+1})
\end{aligned}$$

which is at least C_q in absolute value, independent of j. □

Therefore, E is a set of Hausdorff dimension 1 with no three-term arithmetic progressions. Furthermore, any measure μ supported on E has the property that there exist ξ with arbitrarily large absolute value such that $|\widehat{\mu}(\xi)| \geq C_q$. This implies that E has Fourier dimension 0.

4. An example of a measure satisfying the conditions of Theorem 1.1

We will describe a procedure for constructing a random Cantor set in $\mathbf{F}_q^\infty$. Let $d \in \mathbf{N}$ be a large integer, and let $0 < \alpha < 1$ be a number such that αd is also an integer.

Let $\mathcal{R}$ be the set of all subsets of $\mathbf{F}_q^d$ containing exactly $q^{\alpha d}$ elements equipped with the uniform probability distribution. Let $\tilde{E}_1$ be a set chosen from $\mathcal{R}$ at random. Let E_1 denote the set

$$\{y \in \mathbf{F}_q^\infty \colon (y_0, \ldots, y_{d-1}) \in \tilde{E}_1\}.$$

We define the measure $d\mu_1$ to be the normalized Haar probability measure with support restricted to E_1.

$$d\mu_1 = q^{(1-\alpha)d} \mathbf{1}_{E_1} dx.$$

We will now describe an inductive procedure for constructing a set E_{j+1} from E_j. We will assume that we have constructed a subset $\tilde{E}_j$ of $\mathbf{F}_q^{dj}$ containing $q^{\alpha dj}$ elements, and a subset $E_j \subset \mathbf{F}_q^\infty$ given by

$$E_j = \{y \in \mathbf{F}_q^\infty \colon (y_0, \ldots, y_{jd-1}) \in \tilde{E}_j\}$$

This implies, in particular, that E_j is a union of closed balls of radius q^{-dj}.

For each element $x^{(j)}$ in $\tilde{E}_j$, choose a random element from $\mathcal{R}$, called $T_{x^{(j)}}$. Define $S_{x^{(j)}}$ to be the set

$$\{z \in \mathbf{F}_q^{(j+1)d} \colon (z_0, \ldots, z_{jd-1}) = x^{(j)}; (z_{jd}, \ldots, z_{(j+1)d-1}) \in T_{x^{(j)}}\}$$

We now define the set $\tilde{E}_{j+1}$ to be

$$\bigcup_{x^{(j)} \in \tilde{E}_j} S_{x^{(j)}}$$

and let E_{j+1} denote the set

$$\{y \in \mathbf{F}_q^\infty \colon (y_0, \ldots, y_{(j+1)d-1}) \in \tilde{E}_{j+1}\}.$$

Observe that $E_{j+1} \subset E_j$ for every j. We define the compact set $E \subset \mathbf{F}_q^\infty$ by

$$E = \bigcap_{j=1}^{\infty} E_j.$$

We will define the measure $d\mu_{j+1}$ to be the normalized Haar probability measure with support restricted to E_{j+1}:

$$d\mu_{j+1} = q^{(1-\alpha)d(j+1)} \mathbf{1}_{E_{j+1}}\, dx.$$

Observe that, for any $k < j+1$, and any ball B of radius q^{-kd} intersecting E, we have that

$$\mu_{j+1}(B) = \mu_k(B) = q^{-\alpha kd}.$$

It is clear, then, that the probability measures μ_j have a weak limit probability measure supported on E, which we will denote by μ. This measure μ is easily seen to satisfy the ball condition (1) of Theorem 1.1 for the exponent α.

We will obtain a probabilistic estimate on $\widehat{\mu}(\xi)$, with the goal of showing that (2) of Theorem 1.1 holds almost surely. First, we will need a straightforward lemma.

Lemma 4.1. *Let $\xi \in \widehat{\mathbf{F}_q^d} \backslash \{0\}$, and let Z_ξ be the complex-valued random variable*

$$Z_\xi := \sum_{x \in \mathbf{F}_q^d} e(-x \cdot \xi) \mathbf{1}_R(x),$$

where R is chosen uniformly at random from $\mathcal{R}$, the family of $q^{\alpha d}$-element subsets of $\mathbf{F}_q^d$. Then Z_ξ is a mean-zero random variable such that $|Z_\xi| \le q^{\alpha d}$.

Proof. The only statement in the lemma that is not obvious is the statement that Z_ξ has mean zero. To see this, choose $a \in \mathbf{F}_q^d$ such that $a \cdot \xi \neq 0$. This is possible because of the assumption that $\xi \neq 0$. Partition $\mathcal{R}$ into orbits under translation by a. Since $x + q \cdot a = x$ for any $x \in \mathbf{F}_q^d$, it follows that $R + q \cdot a = R$ for any $R \in \mathcal{R}$. Hence, by primality of q, each orbit contains either 1 or q elements. Let $\mathcal{O}$ be the collection of all orbits of $\mathcal{R}$ under translation by a. We have

$$\mathbf{E}[Z_\xi] = |\mathcal{R}|^{-1} \sum_{O \in \mathcal{O}} \sum_{R \in O} \sum_{x \in R} \mathbf{1}_R(x) e(-x \cdot \xi)$$

We will show that

$$\sum_{R \in O} \sum_{x \in R} \mathbf{1}_R(x) e(-x \cdot \xi) = 0 \quad \text{for each } O \in \mathcal{O}$$

First, suppose O is an orbit containing only one set R_0. Then $R_0 = R_0 + ja$ for each $0 \le j \le q-1$. Thus

$$\begin{aligned}
\sum_{x \in R_0} e(-x \cdot \xi) &= \frac{1}{q} \sum_{j=0}^{q-1} \sum_{x \in R_0 + ja} e(-x \cdot \xi) = \frac{1}{q} \sum_{x \in R_0} \sum_{j=0}^{q-1} e(-(x + ja) \cdot \xi) \\
&= \frac{1}{q} \sum_{x \in R_0} e(-x \cdot \xi) \sum_{j=0}^{q-1} e(-ja \cdot \xi) = 0
\end{aligned}$$

In the opposite case, in which the orbit O contains q elements, the calculation is even simpler. The orbit O consists of q different sets R_0, R_0+a, …, $R_0+(q-1)a$. Thus

$$\sum_{R\in O}\sum_{x\in R} e(-x\cdot\xi) = \sum_{j=0}^{q-1}\sum_{x\in R_0} e(-(x+ja)\cdot\xi) = \sum_{x\in R_0} e(-x\cdot\xi)\sum_{j=0}^{q-1} e(-ja\cdot\xi) = 0.$$

This shows that Z_ξ has mean zero, as desired. □

The importance of Lemma 4.1 is that it will allow us to apply Hoeffding's inequality [4]:

Lemma 4.2. (Hoeffding's inequality) *Let $X_1, \dots, X_n$ be a collection of independent real-valued random variables with mean zero such that $|X_j| \le C$ almost surely for all $1 \le j \le n$. Then*

$$\mathbf{P}\left(\left|\sum_{j=1}^{n} X_j\right| \ge t\right) \le \exp\left(-\frac{t^2}{2nC^2}\right).$$

□

We are now ready to estimate $\widehat{\mu}(\xi)$ for nonzero characters ξ. If $\xi \neq 0$, then there exists some j such that

$$q^{(j-1)d} < |\xi| \le q^{jd}.$$

Observe that $\widehat{\mu}_k(\xi) = \widehat{\mu}_j(\xi)$ for any $k > j$. So it is enough to compute $\widehat{\mu}_j(\xi)$.

Let $\xi = \xi' + \xi''$ be the order $(j-1)d$ decomposition of ξ. Now,

$$\widehat{\mu}_j(\xi) = q^{-\alpha dj}\sum_{x\in\tilde{E}_j} e(-x\cdot\xi) = q^{-\alpha dj}\sum_{x'\in\tilde{E}_{j-1}} e(-x'\cdot\xi')\sum_{x''\in T_{x'}} e(-x''\cdot\xi'').$$

Here, $T_{x'}$ is the randomly selected set described in the construction of E. Observe that

$$\sum_{x''\in T_{x'}} e(-x''\cdot\xi'')$$

has the same distribution as the random variable $Z_{\xi''}$ described in Lemma 4.1. Therefore, each of the sums

$$e(-x'\cdot\xi')\sum_{x''\in T_{x'}} e(-x''\cdot\xi'')$$

is a complex-valued random variable with mean zero and absolute value bounded above by $q^{\alpha d}$. We can apply Hoeffding's inequality separately to the real and imaginary parts of $\widehat{\mu}_j(\xi)$. Observe that the number of elements of $\tilde{E}_{j-1}$ is exactly equal to $q^{\alpha(j-1)d}$. So Lemma 4.2 applies with $n = q^{\alpha(j-1)d}$ and $C = q^{\alpha d}$. This suggests the choice $t = q^{(1+\epsilon)\alpha dj/2}$.

With this choice, we have

$$\frac{t^2}{2nC^2} = \frac{q^{\alpha dj(1+\epsilon)}}{2q^{\alpha d(j-1)}q^{2\alpha d}} = \frac{q^{\alpha d(j\epsilon-1)}}{2} \ge q^{\alpha dj\epsilon/2}$$

provided that $j \ge j_0(q,d,\epsilon)$. Therefore, Hoeffding's inequality shows that, for $j \ge j_0$,

$$\mathbf{P}\left(\left|\mathrm{Re}\sum_{x'\in\tilde{E}_{j-1}} e(-x'\cdot\xi')\sum_{x''\in T_{x'}} e(-x''\cdot\xi'')\right| \ge q^{(1+\epsilon)\alpha dj/2}\right) \le \exp\left(-q^{\alpha dj\epsilon/2}\right).$$

With a similar inequality for the imaginary part. Thus we have, for $j \ge j_0$,

$$\mathbf{P}\left(|\widehat{\mu}(\xi)| \ge 2q^{(-1+\epsilon)\alpha dj/2}\right) \le 2\exp\left(-q^{\alpha dj\epsilon/2}\right).$$

Because we have $|\xi| \le q^{jd}$,

$$\sum_{|\xi| \ge q^{j_0 d}} \mathbf{P}\left(|\widehat{\mu}(\xi)| \ge 2q^{(-1+\epsilon)\alpha dj/2}\right) \le \sum_{|\xi| \ge q^{j_0 d}} 2\exp\left(-|\xi|^{\alpha\epsilon}\right) < \infty.$$

So by the Borel–Cantelli lemma, it follows that, with probability 1, there exists some K such that

$$|\widehat{\mu}(\xi)| < 2q^{(-1+\epsilon)\alpha dj/2} \quad \text{whenever } K \le q^{(j-1)d} < |\xi| \le q^{jd}.$$

This inequality implies

$$|\widehat{\mu}(\xi)| < 2|\xi|^{(-1+\epsilon)\alpha/2} \quad \text{whenever } K \le |\xi|.$$

This shows that, for any $\epsilon > 0$, the measure μ satisfies an inequality of the same form as (2) of Theorem 1.1 for the value $\beta = (1-\epsilon)\alpha$ with probability 1. By continuity of probability, this shows that E almost surely has Fourier dimension equal to α.

Thus Theorem 1.1 implies that if $\alpha > \frac{8-4\alpha_q}{9-5\alpha_q}$, then the random set E constructed above will contain a three-term arithmetic progression almost surely.

5. A Varnavides-type theorem for thin subsets of $\mathbf{F}_q^d$

Varnavides's theorem [13, Theorem 10.9] gives a quantitative statement about the number of three-term arithmetic progressions in large subsets of $\{1, \dots, N\}$. We will prove a similar result for the finite group $\mathbf{F}_q^d$. This version of Varnavides's theorem was established by Pohoata and Roche-Newton [10] using the result of Ellenberg and Gijswijt [2] and the triangle removal lemma. We present a different proof using a simple counting argument instead of the triangle removal lemma. The proof is similar to the standard proof of Varnavides's theorem, and in particular uses the strategy of intersecting with random planes described by Tao and Vu [13, Exercise 10.1.9] to arrive at a quantitative statement for thin sets.

Proposition 5.1. (Varnavides's theorem for $\mathbf{F}_q^d$) *Let q be an odd prime, and let α_q be as in Theorem 1.1. Suppose $\alpha > \alpha_q$ and $\epsilon > 0$. Let $A \subset \mathbf{F}_q^d$ be such that $|A| \ge q^{\alpha d}$. Define the value c_q by*

$$c_q = \frac{3-\alpha_q}{1-\alpha_q}.$$

Then, provided that d is sufficiently large, A contains at least $q^{2d-(c_q+\epsilon)(1-\alpha)d}$ three-term arithmetic progressions.

Proof. Let η be a positive real number satisfying $1+\eta < \frac{1-\alpha_q}{1-\alpha}$. We will choose a parameter d' depending on η as follows.

$$d' = \left\lfloor d(1+\eta) \cdot \frac{1-\alpha}{1-\alpha_q} \right\rfloor. \tag{8}$$

Observe that the condition on η guarantees $d' < d$.

Let $\mathrm{Graff}(\mathbf{F}_q, d, d')$ denote the set of planes of dimension d' in $\mathbf{F}_q^d$. Then we define M to be the fraction of elements of $\mathrm{Graff}(\mathbf{F}_q, d, d')$ that contain at least one nontrivial arithmetic progression in the set A. We will need the following estimate on M.

Lemma 5.2. *Let q, A, and α be defined as in Lemma 5.1, and let M and d' be as defined above. If d is large enough depending on α, then we have the estimate*

$$M \ge \frac{1}{2} q^{(\alpha-1)d}.$$

Fix a nontrivial three-term arithmetic progression $P = \{x, x+a, x+2a\}$. We let L denote the fraction of elements of $\mathrm{Graff}(\mathbf{F}_q, d, d')$ that contain P. An easy symmetry argument shows that L does not depend on P. We will need the following estimate on L.

Lemma 5.3. *Let q and α be as defined in Lemma 5.1, and let L and d' be defined as above. Then L satisfies the estimate*

$$L = (1 + o_q(1))q^{2(d'-d)}.$$

Here, the $o_q(1)$ term approaches zero as $d \to \infty$ for a fixed q.

We will now complete the proof of Proposition 5.1 assuming Lemmas 5.2 and 5.3. Let $p(A)$ denote the number of nontrivial three-term arithmetic progressions in the set A. Let I be the number of incidences between d'-dimensional planes in $\mathbf{F}_q^d$ and nontrivial 3-term arithmetic progressions in A; that is, I is the number of pairs (P, W) where P is a three-term arithmetic progression contained in both the set A and the plane W. Then we have

$$I = \sum_{\substack{P \subset A \\ P \text{ 3-term AP}}} I_P,$$

where I_P is the number of planes in $\mathrm{Graff}(\mathbf{F}_q, d, d')$ containing the progression P. By definition of L, we have

$$I_P = |\mathrm{Graff}(\mathbf{F}_q, d, d')| \cdot L$$

for every P, and thus

$$I = |\mathrm{Graff}(\mathbf{F}_q, d, d')| \cdot L \cdot p(A). \tag{9}$$

On the other hand, the definition of M guarantees that we have

$$I \geq |\mathrm{Graff}(\mathbf{F}_q, d, d')| \cdot M. \tag{10}$$

Combining equation (9) and inequality (10) gives

$$p(A) \geq \frac{M}{L}. \tag{11}$$

Now, we combine equation (11) with Lemmas 5.2 and 5.3 to give, for sufficiently large d,

$$p(A) \geq \frac{1}{3}q^{(\alpha-1)d-2(d'-d)}.$$

Since $d' \leq d(1+\eta)(1-\alpha)(1-\alpha_q)^{-1}$, we get

$$p(A) \geq \frac{1}{3}q^{(\alpha-1)d-2(d'-d)} \geq \frac{1}{3}q^{2d-((1-\alpha)+2(1+\eta)(1-\alpha)(1-\alpha_q)^{-1})d}.$$

Now, if η is chosen to satisfy $\eta < (1-\alpha_q)\epsilon/2$, we have

$$p(A) \geq q^{2d-(1-\alpha)(1+\epsilon+2(1-\alpha_q)^{-1})d} = q^{2d-(c_q+\epsilon)(1-\alpha)d}. \qquad \square$$

It remains to prove Lemmas 5.2 and 5.3.

Proof of Lemma 5.2. The result of Ellenberg and Gijswijt [2] implies that if d' is sufficiently large, any subset of a d'-dimensional plane consisting of at least $q^{\alpha_q d'}$ elements will contain a three-term arithmetic progression, where $\alpha_q < 1$ is a real number depending only on q. In order to obtain a lower bound on M, it suffices to obtain a lower bound on the fraction M' of d'-dimensional planes that contain at least $q^{\alpha_q d'}$ elements of A. We will apply a pigeonhole-principle argument in order to obtain a lower bound for M'.

The average number of elements of A contained in a plane of dimension d' is at least $q^{\alpha d}q^{d'-d}$. On the other hand, the average number of elements of A in such a plane is bounded above by $(1-M')q^{\alpha_q d'} + M'q^{d'}$. This gives the inequality

$$q^{\alpha d+d'-d} \leq (1-M')q^{\alpha_q d'} + (M')q^{d'}.$$

When we isolate M' in this inequality, we arrive at the inequality

$$M \geq M' \geq 1 - \frac{1-q^{(\alpha-1)d}}{1-q^{(\alpha_q-1)d'}}. \tag{12}$$

From (12) and (8), we get

$$M \geq 1 - \frac{1-q^{(\alpha-1)d}}{1-q^{(1+\eta)(\alpha-1)d+1}}$$

If d is large enough depending on α, then M will be larger than $\frac{1}{2}q^{(\alpha-1)d}$ as can be seen by using e.g. the linearization of the function $\frac{1-x}{1-y}$ near $(0,0)$. This gives the desired lower bound on M. $\square$

Proof of Lemma 5.3. Let $\mathrm{Aff}(d,\mathbf{F}_q)$ denote the group of affine-linear maps of full rank on $\mathbf{F}_q^d$. Fix a d'-dimensional plane $V \subset \mathbf{F}_q^d$. We will compute L by counting in two different ways the number $\mathrm{Aff}_{P,V}$ of elements of $\mathrm{Aff}(d,\mathbf{F}_q)$ that map P into the plane V.

First, we will count the number $N_{d,q}$ of nontrivial three-term arithmetic progressions in $\mathbf{F}_q^d$. Note that, while y and b are sufficient to determine the three-term arithmetic progression $\{y, y+b, y+2b\}$, the three-term arithmetic progression $\{y, y+b, y+2b\}$ corresponds to multiple choices of y and b. For $q \neq 3$, this arithmetic progression can be described in exactly two ways:

$$\begin{aligned}
&\{y, y+b, y+2b\},\\
&\{(y+2b), (y+2b)-b, (y+2b)-2b\}.
\end{aligned}$$

When $q=3$, we have $2\cdot 2 = 1$, so the arithmetic progression $\{y, y+b, y+2b\}$ can be described in exactly six ways:

$$\begin{aligned}
&\{y, y+b, y+2b\},\\
&\{y, y+2b, y+2\cdot 2b\},\\
&\{(y+b), (y+b)+b, (y+b)+2b\},\\
&\{(y+b), (y+b)+2b, (y+b)+2\cdot 2b\},\\
&\{(y+2b), (y+2b)+b, (y+2b)+2b\},\\
&\{(y+2b), (y+2b)+2b, (y+2b)+2\cdot 2b\}.
\end{aligned}$$

So the number $N_{d,q}$ of nontrivial three-term arithmetic progressions in $\mathbf{F}_q^d$ is given by

$$N_{d,q} = \begin{cases} \frac{1}{2}q^d(q^d-1) & \text{if } q \neq 3,\\ \frac{1}{6}3^d(3^d-1) & \text{if } q = 3. \end{cases}$$

Now, we observe that the group $\mathrm{Aff}(d,\mathbf{F}_q)$ acts transitively on the set of nontrivial three-term arithmetic progressions in $\mathbf{F}_q^d$. Therefore, by the orbit-stabilizer lemma, it follows that for any nontrivial arithmetic progression $P' = \{y, y+b, y+2b\}$ the number of elements $\mathrm{Aff}_{P,P'}$ of $\mathrm{Aff}(d,\mathbf{F}_q)$ mapping P onto P' is exactly

$$\mathrm{Aff}_{P,P'} = \frac{1}{N_{d,q}}|\mathrm{Aff}(d,\mathbf{F}_q)|.$$

Summing this over all nontrivial three-term arithmetic progressions P' lying in the plane V gives

$$\mathrm{Aff}_{P,V} = \frac{N_{d',q}}{N_{d,q}} |\mathrm{Aff}(d, \mathbf{F}_q)|. \tag{13}$$

We will now count $\mathrm{Aff}_{P,V}$ in a different way. Let U be an arbitrary d'-dimensional plane in $\mathbf{F}_q^d$. We will first count $\mathrm{Aff}_{U,V}$, the number of elements of $\mathrm{Aff}(d, \mathbf{F}_q)$ that map U onto V. Let $\mathrm{Graff}(\mathbf{F}_q, d, d')$ denote the collection of all d'-dimensional planes in $\mathbf{F}_q^d$. We observe that $\mathrm{Aff}(d, \mathbf{F}_q)$ acts transitively on the set $\mathrm{Graff}(\mathbf{F}_q, d, d')$. Therefore, by the orbit-stabilizer lemma, we have

$$\mathrm{Aff}_{U,V} = \frac{1}{|\mathrm{Graff}(\mathbf{F}_q, d, d')|} |\mathrm{Aff}(d, \mathbf{F}_q)|.$$

We sum this over all d'-dimensional planes U containing the progression P. Observe that the number of such planes is $L \cdot |\mathrm{Graff}(\mathbf{F}_q, d, d')|$. Therefore, we get

$$\mathrm{Aff}_{P,V} = L |\mathrm{Aff}(d, \mathbf{F}_q)|. \tag{14}$$

Combining (13) and (14) gives

$$L = \frac{N_{d',q}}{N_{d,q}} = \frac{q^{d'}(q^{d'} - 1)}{q^d(q^d - 1)} = (1 + o(1)) q^{2(d'-d)}. \qquad \square$$

6. Proof of Theorem 1.1

6.1. Two trilinear forms. For any d, define μ_d to be the pushforward of μ under the projection π_d. In order to prove Theorem 1.1, we will introduce the trilinear forms

$$g_d(x) = \sum_{a \in \mathbf{F}_q^d \setminus \{0\}} \mu_d(x)\mu_d(x + a)\mu_d(x + 2a)$$

and

$$h_{d,d*}(x) = \sum_{\substack{a \in \mathbf{F}_q^{d^*} \\ |a| > q^{-d}}} \mu_{d^*}(x)\mu_{d^*}(x + a)\mu_{d^*}(x + 2a).$$

Note that the sum

$$S_g := \sum_{x \in \mathbf{F}_q^d} g_d(x)$$

gives the $\mu \times \mu \times \mu$-measure of the set of points (x, y, z) that lie within q^{-d} of a three-term arithmetic progression $(x, x + a, x + 2a)$ whose common difference a satisfies $|a| > q^{-d}$, and the sum

$$S_h := \sum_{x \in \mathbf{F}_q^d} h_{d,d^*}(x)$$

gives the $\mu \times \mu \times \mu$-measure of the set of points (x, y, z) that lie within q^{-d^*} of a three-term arithmetic progression $(x, x + a, x + 2a)$ whose common difference a satisfies $|a| > q^{-d}$.

The main result we will need to prove Theorem 1.1 is the following:

Lemma 6.1. *Suppose that μ is a measure satisfying the conditions of Theorem 1.1. Then there exists a $d \in \mathbf{N}$ such that for any $d^* > d$, we have the inequality*

$$\sum_{x \in \mathbf{F}_q^{d^*}} h_{d,d^*}(x) > 0. \tag{15}$$

Proof of Theorem 1.1 assuming Lemma 6.1. Let $d > 0$ be such that (15) holds whenever $d^* > d$. For any $d^* > d$, select an $(x_{d^*}, y_{d^*}, z_{d^*}) \in E^3$ and an $|a_{d^*}| \geq q^{-d}$ such that

$$|y_{d^*} - (x_{d^*} + a_{d^*})| \leq q^{-d^*}; \quad |z_{d^*} - (x_{d^*} + 2a_{d^*})| \leq q^{-d^*}. \tag{16}$$

Because $E \times E \times E$ is compact, the sequence $\{(x_{d^*}, y_{d^*}, z_{d^*})\}_{d^*>d}$ has at least one limit point. Choose such a limit point, and call this limit point (x, y, z). We claim that (x, y, z) is a nontrivial three-term arithmetic progression.

First, we check that (x, y, z) is a three-term arithmetic progression. The inequalities (16) imply that

$$\lim_{d^* \to \infty} x_{d^*} - 2y_{d^*} + z_{d^*} = 0.$$

Hence (x, y, z) is an arithmetic progression. On the other hand, equation (16), the bound $|a_{d^*}| > q^{-d}$, and the ultrametric inequality guarantee that $|y_{d^*} - x_{d^*}| > q^{-d}$ for each of the points $(x_{d^*}, y_{d^*}, z_{d^*})$, and therefore $|y - x| > q^{-d}$ as well. Thus E contains a nontrivial three-term arithmetic progression. □

It remains to prove Lemma 6.1. In the next subsection, we will obtain an estimate on $\sum_{x \in \mathbf{F}_q^d} g_d(x)$ that will be helpful in proving the lemma.

6.2. Locating approximate arithmetic progressions in *E*. We will prove the following estimate.

Lemma 6.2. *Let μ be a measure satisfying the first assumption of Theorem 1.1, and let $\epsilon > 0$ be a positive real number. If ϵ is sufficiently small depending on g, and if d is sufficiently large depending on μ, q, and ϵ, then*

$$S_g \geq q^{(-1 - c_q(1-\alpha) - \epsilon)d} \tag{17}$$

Proof. The first assumption of Theorem 1.1 implies that there exists a finite Borel measure μ' obtained by restricting the support of the measure μ to a subset E' of E such that μ' satisfies the ball condition of dimension α.

Let μ'_d be the pushforward of μ' under the projection π_d. Suppose $\alpha > \alpha_q$, where α_q is defined as in Lemma 5.1. The ball condition implies that $0 \leq \mu'_d(x) \leq C_1 q^{-\alpha d}$ for all $x \in \mathbf{F}_q^d$. Let $K = \mu'(\mathbf{F}_q^\infty)$. If we then define

$$\mu''_d(x) = \begin{cases} \mu'_d(x) & \text{if } \mu'_d(x) > Kq^{-d}/2, \\ 0 & \text{if } \mu'_d(x) \leq Kq^{-d}/2, \end{cases}$$

a simple pigeonholing argument shows $\sum_{x \in \mathbf{F}_q^d} \mu''_d(x) \geq K/2$.

Let $0 < \epsilon < \frac{1}{100}$ be so small that $\alpha - \epsilon > \alpha_q$. We will accumulate small losses in exponents in our argument that will, ultimately, be controlled by this ϵ.

Let A be the support of μ''_d in $\mathbf{F}_q^d$. Because $\mu''_d(x) \leq C_1 q^{-\alpha d}$ for $x \in A$ and $\sum_{x \in A} \mu''_d(x) \geq K/2$, we have the lower bound $|A| \geq \frac{Kq^{\alpha d}}{2C_1}$. We can absorb the constants by replacing α by the slightly smaller number $\alpha - \frac{\epsilon}{30c_q}$: we have for sufficiently large d (depending on α, ϵ, q, C_1, and K) that $|A| \geq q^{(\alpha - \frac{\epsilon}{30c_q})d}$.

We apply Lemma 5.1 to the set A. This lemma guarantees that, if d is sufficiently large, there are at least $q^{(2-(c_q+\frac{\epsilon}{3})(1-\alpha+\frac{\epsilon}{30c_q}))d} \geq q^{(2-c_q(1-\alpha)-\frac{2\epsilon}{3})d}$ three-term arithmetic progressions contained in A. Because there are at least $q^{(2-c_q(1-\alpha)-\frac{2\epsilon}{3})d}$ pairs (x, a) such that $\{x, x+a, x+2a\}$ is contained in A, and $\mu_d(x) \geq \mu'_d(x) \geq \frac{K}{2} q^{-d}$ on A, we have, by absorbing the constant $K^3/8$ into a $q^{-\frac{\epsilon}{3}d}$ term, that

$$\sum_{x \in \mathbf{F}_q^d} g(x) \geq q^{(-1-c_q(1-\alpha)-\epsilon)d},$$

as desired. □

6.3. Refining the approximate arithmetic progressions. The core remaining task is to relate the sum S_g to the sum S_h.

Lemma 6.3. *The sums S_g and S_h are related by the equation*

$$S_h = g^{d-d^*} S_g + S_{\neq 0}, \tag{18}$$

where

$$S_{\neq 0} := q^{-d^*-d} \sum_{\substack{\xi_1 \in \widehat{\mathbf{F}_q^{d^*}} \\ \xi_1{}'' \neq 0}} \sum_{\xi_2' \in \widehat{\mathbf{F}_q^d}} \widehat{\mu_{d^*}}(-\xi_1' - \xi_2' + \xi_1'')\widehat{\mu_{d^*}}(\xi_1' + \xi_1'')\widehat{\mu_{d^*}}(\xi_2' - 2\xi_1'')$$
$$\cdot \left(\sum_{\substack{a' \in \mathbf{F}_q^d \\ a' \neq 0}} e_q(-(2\xi_1' + \xi_2') \cdot a') \right).$$

Here, $\xi_1 = \xi_1' + \xi_1''$ is the order d decomposition of the character ξ_1.

Proof. In order to relate S_h and S_g, we observe that S_h is equal to $\widehat{h_{d^*,d}}(0)$. Therefore, by two applications of the convolution rule (4):

$$\begin{aligned}
\widehat{h_{d^*,d}}(0) &= \sum_{\substack{a \in \mathbf{F}_q^{d^*} \\ |a| > q^{-d}}} (\mu_{d^*}(\cdot)\mu_{d^*}(\cdot + a)\mu_{d^*}(\cdot + 2a))^\wedge(0) \\
&= \frac{1}{q^{d^*}} \sum_{\substack{a \in \mathbf{F}_q^{d^*} \\ |a| > q^{-d}}} \sum_{\xi_1 \in \widehat{\mathbf{F}_q^{d^*}}} (\mu_{d^*}(\cdot)\mu_{d^*}(\cdot + a))^\wedge(-\xi_1)\widehat{\mu_{d^*}}(\xi_1)e_q(-\xi_1 \cdot 2a) \\
&= \frac{1}{q^{2d^*}} \sum_{\substack{a \in \mathbf{F}_q^{d^*} \\ |a| > q^{-d}}} \sum_{\xi_1 \in \widehat{\mathbf{F}_q^{d^*}}} \sum_{\xi_2 \in \widehat{\mathbf{F}_q^{d^*}}} \widehat{\mu_{d^*}}(-\xi_1 - \xi_2)\widehat{\mu_{d^*}}(\xi_1)\widehat{\mu_{d^*}}(\xi_2)e_q(-(2\xi_1 + \xi_2) \cdot a).
\end{aligned}$$

Here, ξ_1 and ξ_2 are elements of $\mathbf{F}_q^{d^*}$. We will write

$$\begin{aligned}
\xi_1 &= (\xi_1^{(1)}, \ldots, \xi_1^{(d^*)}), \\
\xi_2 &= (\xi_2^{(1)}, \ldots, \xi_2^{(d^*)}).
\end{aligned}$$

We will apply the order d decomposition to $a = a' + a''$, $\xi_1 = \xi_1' + \xi_1''$, $\xi_2 = \xi_2' + \xi_2''$, and observe that the condition $|a| > q^{-d}$ is equivalent to the statement that $a' \neq 0$. So we can rewrite this sum as

$$\widehat{h_{d^*,d}}(0) = \frac{1}{q^{2d^*}} \sum_{a' \neq 0} \sum_{a''} \sum_{\xi_1', \xi_2'} \sum_{\xi_1'', \xi_2''} \big(\widehat{\mu_{d^*}}(-\xi_1' - \xi_1'' - \xi_2' - \xi_2'')\widehat{\mu_{d^*}}(\xi_1' + \xi_1'')\widehat{\mu_{d^*}}(\xi_2' + \xi_2'')$$
$$\cdot e_q(-(2\xi_1' + \xi_2') \cdot a')e_q(-(2\xi_1'' + \xi_2'') \cdot a'')\big).$$

We rearrange this sum so that the sums in a' and a'' are inside:

$$\widehat{h_{d^*,d}}(0) = \frac{1}{q^{2d^*}} \sum_{\xi_1',\xi_2'} \sum_{\xi_1'',\xi_2''} \widehat{\mu_{d^*}}(-\xi_1' - \xi_1'' - \xi_2' - \xi_2'')\widehat{\mu_{d^*}}(\xi_1' + \xi_1'')\widehat{\mu_{d^*}}(\xi_2' + \xi_2'')$$
$$\cdot \left(\sum_{a' \neq 0} e_q(-(2\xi_1' + \xi_2') \cdot a') \right) \left(\sum_{a''} e_q(-(2\xi_1'' + \xi_2'') \cdot a'') \right).$$

We will first consider the sum

$$\sum_{a''} e_q(-(2\xi_1'' + \xi_2'') \cdot a'').$$

This sum vanishes if $2\xi_1'' + \xi_2''$ is nonzero. If $2\xi_1'' + \xi_2''$ is equal to zero, then each summand is equal to 1, so the sum is equal to q^{d^*-d}, the number of summands. Therefore, we have

$$\widehat{h_{d^*,d}}(0) = q^{-d^*-d} \sum_{\xi_1',\xi_2'} \sum_{\xi_1''} \widehat{\mu_{d^*}}(-\xi_1' - \xi_2' + \xi_1'')\widehat{\mu_{d^*}}(\xi_1' + \xi_1'')\widehat{\mu_{d^*}}(\xi_2' - 2\xi_1'')$$
$$\cdot \left(\sum_{a' \neq 0} e_q(-(2\xi_1' + \xi_2') \cdot a') \right)$$

We can therefore write

$$\widehat{h_{d^*,d}}(0) = S_0 + S_{\neq 0},$$

where

$$S_0 := q^{-d^*-d} \sum_{\xi_1',\xi_2'} \widehat{\mu_{d^*}}(-\xi_1' - \xi_2')\widehat{\mu_{d^*}}(\xi_1')\widehat{\mu_{d^*}}(\xi_2') \left(\sum_{a' \neq 0} e_q(-(2\xi_1' + \xi_2') \cdot a') \right) \tag{19}$$

and

$$S_{\neq 0} := q^{-d^*-d} \sum_{\xi_1',\xi_2'} \sum_{\xi_1'' \neq 0} \widehat{\mu_{d^*}}(-\xi_1' - \xi_2' + \xi_1'')\widehat{\mu_{d^*}}(\xi_1' + \xi_1'')\widehat{\mu_{d^*}}(\xi_2' - 2\xi_1'')$$
$$\cdot \left(\sum_{a' \neq 0} e_q(-(2\xi_1' + \xi_2') \cdot a') \right).$$

The $S_{\neq 0}$ term is as promised in the statement of the lemma. We need only relate S_0 to S_g. By a calculation similar to the one above, we can compute $S_g = \widehat{g}_d(0)$.

$$\widehat{g}_d(0) = q^{-2d} \sum_{\eta_1,\eta_2 \in \widehat{\mathbf{F}_q^d}} \widehat{\mu}_d(-\eta_1 - \eta_2)\widehat{\mu}_d(\eta_1)\widehat{\mu}_d(\eta_2) \sum_{\substack{b \in \mathbf{F}_q^d \\ b \neq 0}} e_q(-(2\eta_1 + \eta_2) \cdot b).$$

Because ξ_1' and ξ_2' in (19) have absolute value at most q^d, it follows that $\xi_1' \cdot x = (\xi_1^{(1)}, \ldots, \xi_1^{(d)}) \cdot \pi_d(x)$ for any $x \in \mathbf{F}_q^{d^*}$. Thus, if η_1 is the vector $(\xi_1^{(1)}, \ldots, \xi_1^{(d)})$ and η_2 is the vector $(\xi_2^{(1)}, \ldots, \xi_2^{(d)})$, then $\widehat{\mu}_{d^*}(\xi_1')$ is equal to $\widehat{\mu}_d(\eta_1)$, and similarly for ξ_2' and $-\xi_1' - \xi_2'$. Re-indexing the sum in ξ_1' and ξ_2' in S_0 by η_1 and η_2, and $b = \pi_d(a)$, we observe

$$S_0 = q^{-d^*-d} \sum_{\eta_1,\eta_2} \widehat{\mu}_d(-\eta_1 - \eta_2)\widehat{\mu}_d(\eta_1)\widehat{\mu}_d(\eta_2) \sum_{\substack{b \in \mathbf{F}_q^d \\ b \neq 0}} e_q(-(2\eta_1 + \eta_2) \cdot b) = q^{d-d^*}\widehat{g}_d(0).$$

This establishes the equation (18). $\square$

The only remaining task is to bound the error term $S_{\neq 0}$. This will be accomplished in the next subsection.

6.4. Estimating the error term. In this section, we will establish the following estimate on $S_{\neq 0}$.

Lemma 6.4. *For some constant $C_{q,\beta}$ depending only on q and β (and in particular, not on d or d^*), the following bound holds for the sum $S_{\neq 0}$ defined in Lemma 6.3,*

$$|S_{\neq 0}| \leq C_{q,\beta} q^{-d^*+(1-3\beta/2)d}. \tag{20}$$

Proof. Because ξ_1'' is nonzero in the sum $S_{\neq 0}$, we have in fact that $|\xi_1''| \geq q^{d+1}$. In particular, this means that $|\xi_1''| > \max(|\xi_1'|, |\xi_2'|)$, and thus $|\xi_1' + \xi_1''|$, $|\xi_2' - 2\xi_1''|$, and $|-\xi_1' - \xi_2' + \xi_1''|$ are all equal to $|\xi_1''|$ by the ultrametric inequality. Applying the second assumption of Theorem 1.1 and the triangle inequality, we therefore have the estimate

$$\begin{aligned}
|S_{\neq 0}| &\leq q^{-d^*-d} \sum_{\xi_1',\xi_2'} \sum_{\xi_1''\neq 0} |\xi_1''|^{-3\beta/2} \left| \sum_{a'\neq 0} e_q(-(2\xi_1' + \xi_2')\cdot a') \right| \\
&= q^{-d^*-d} \left(\sum_{\xi_1''\neq 0} |\xi_1''|^{-3\beta/2} \right) \left(\sum_{\xi_1',\xi_2'} \left| \sum_{a'\neq 0} e_q(-(2\xi_1' + \xi_2')\cdot a') \right| \right).
\end{aligned}$$

First, we will estimate

$$\sum_{\xi_1',\xi_2'} \left| \sum_{a'\neq 0} e_q(-(2\xi_1' + \xi_2')\cdot a') \right|.$$

The sum

$$\sum_{a'\neq 0} e_q(-(2\xi_1' + \xi_2')\cdot a')$$

will take the value -1 if $2\xi_1' + \xi_2'$ is nonzero, and will take the value $q^d - 1$ otherwise. For a fixed ξ_1', there is exactly one choice of ξ_2' (namely, $-2\xi_1'$) such that $2\xi_1' + \xi_2' = 0$. Thus for each ξ_1', we have

$$\sum_{\xi_2'} \left| \sum_{a'\neq 0} e_q(-(2\xi_1' + \xi_2')\cdot a') \right| = 2(q^d - 1),$$

and thus

$$\sum_{\xi_1',\xi_2'} \left| \sum_{a'\neq 0} e_q(-(2\xi_1' + \xi_2')\cdot a') \right| = 2q^d(q^d - 1) \leq 2q^{2d}. \tag{21}$$

We will now estimate

$$\sum_{\xi_1''\neq 0} |\xi_1''|^{-3\beta/2}.$$

This sum can be rewritten

$$\sum_{j=d+1}^{d^*} (\#\{\xi_1'' \colon |\xi_1''| = q^j\}) q^{-3\beta j/2}.$$

Note that $|\xi_1''| = q^j$ whenever ξ_1'' has the form

$$\xi_1'' = (0, \ldots, 0, \xi_1^{(d+1)}, \xi_1^{(d+2)}, \ldots, \xi_1^{(j)}, 0, \ldots, 0)$$

where $\xi_1^{(j)} \neq 0$. There are q choices for each of $\xi_1^{(d+1)}, \dots, \xi_1^{(j-1)}$ and $q-1$ choices for $\xi_1^{(j)}$ and thus there are $(q-1)q^{j-d-1} \leq q^{j-d}$ values of ξ_1'' such that $|\xi_1''| = q^j$. Thus

$$\sum_{\xi_1'' \neq 0} |\xi_1''|^{-3\beta/2} \leq \sum_{j=d+1}^{d^*} q^{j-d} q^{-3\beta j/2} = q^{-d} \sum_{j=d+1}^{d^*} q^{j(1-3\beta/2)} \leq q^{-d} \sum_{j=d+1}^{\infty} q^{j(1-3\beta/2)}.$$

The sum $\sum_{j=d+1}^{\infty} q^{j(1-3\beta/2)}$ is convergent because of the assumption that $\beta > 2/3$. The geometric series formula gives the estimate $\sum_{j=d+1}^{\infty} q^{j(1-3\beta/2)} \leq C_{q,\beta} q^{d(1-3\beta/2)}$. Thus we get

$$\sum_{\xi_1'' \neq 0} |\xi_1''|^{-3\beta/2} \leq C_{q,\beta} q^{-3d\beta/2}. \tag{22}$$

Combining (21) and (22) and absorbing the constant 2 into $C_{q,\beta}$, we get

$$|S_{\neq 0}| \leq C_{q,\beta} q^{-d^*-d-3d\beta/2+2d} = C_{q,\beta} q^{-d^*+(1-3\beta/2)d}.$$

□

We will now use Lemmas 6.2, 6.3, and 6.4 to prove Lemma 6.1, thereby completing the proof of Theorem 1.1.

Proof of Lemma 6.1. By Lemma 6.3, we can write

$$S_h = q^{d-d^*} S_g + S_{\neq 0}.$$

Let $\epsilon > 0$. We can apply Lemma 6.2 and 6.4 to obtain an estimate on S_h.

$$\begin{aligned} S_h &\geq q^{d-d^*} S_g - |S_{\neq 0}| \geq q^{d-d^*} q^{-d+(-c_q(1-\alpha)-\epsilon)d} - C_{q,b} q^{-d^*+(1-3\beta/2)d} \\ &= q^{-d^*} \left(q^{(-c_q(1-\alpha)-\epsilon)d} - C_{q,b} q^{(1-3\beta/2)d} \right). \end{aligned}$$

In order to guarantee that this expression is positive for sufficiently large d, we need only verify the inequality

$$-c_q(1-\alpha) - \epsilon > 1 - 3\beta/2.$$

Rearranging this inequality and solving for $1-\alpha$ gives the inequality

$$1 - \alpha < \frac{1-\alpha_q}{3-\alpha_q} \left(\frac{3\beta}{2} - 1 \right) - c_q^{-1} \epsilon.$$

Under the assumption (1) of Theorem 1.1, this inequality holds provided that ϵ is sufficiently small depending on q, α, and β. Therefore, if d is sufficiently large, S_h is positive, as claimed. □

7. Concluding remarks

We crucially used the result of Ellenberg–Gijswijt [2] in the proof of Lemma 5.1. For this reason, the proof described here does not apply to the Euclidean setting. As stated before, Shmerkin [11] has provided a counterexample to Theorem 1.1 in $\mathbf{R}$.

The only use of the Fourier decay condition occurred in the estimate of the term $S_{\neq 0}$.

8. Appendix: Fourier analysis of the Riesz potential

The goal for this section is to prove Lemma 2.3. For this purpose, we define, for $0 < t < 1$, the Riesz potential $r_t(x)$ to be the L^1 function on $\mathbf{F}_q^\infty$ defined by the formula

$$r_t(x) = |x|^{-t} \quad \text{for } x \neq 0. \tag{23}$$

In terms of the Riesz potential, we observe that the t-energy of a measure μ is given by

$$\iint_{\mathbf{F}_q^\infty \times \mathbf{F}_q^\infty} r_t(x-y)\, d\mu(x)\, d\mu(y).$$

Again, we use the convention that this energy is infinite if $\mu \times \mu(\Delta) > 0$, where Δ is the diagonal.

We now restate Lemma 2.3 using this notation.

Lemma 8.1. *The t-energy of a measure μ is given by the formula*

$$\iint_{\mathbf{F}_q^\infty \times \mathbf{F}_q^\infty} r_t(x-y)\, d\mu(x)\, d\mu(y) = \frac{1-q^{-1}}{1-q^{t-1}}|\widehat{\mu}(0)|^2 + \sum_{\xi \neq 0} \frac{1-q^{-t}}{1-q^{t-1}}|\widehat{\mu}(\xi)|^2 |\xi|^{t-1}. \tag{24}$$

In order to establish this formula, we will need to compute the Fourier transform of the Riesz potential. The following formula appears as Lemma 4.3.4 (i) in [9].

Lemma 8.2. *Let $r_t(x)$ denote the Riesz potential on $\mathbf{F}_q^\infty$:*

$$r_t(x) = |x|^{-t} \quad \textit{for } x \neq 0.$$

Then

$$\begin{aligned} \widehat{r_t}(\xi) &= \frac{1-q^{-t}}{1-q^{t-1}}|\xi|^{t-1} \quad \textit{if } \xi \neq 0, \\ \widehat{r_t}(0) &= \frac{1-q^{-1}}{1-q^{t-1}}. \end{aligned}$$

Proof of Lemma 8.2. We will evaluate the integral defining $\widehat{r_t}(\xi)$ by partitioning the region of integration according to the absolute value of x.

$$\begin{aligned} \widehat{r_t}(\xi) &= \int_{\mathbf{F}_q^\infty} |x|^{-t} e_q(x\cdot\xi)\, dx \\ &= \sum_{j=0}^\infty \int_{|x|=q^{-j}} |x|^{-t} e_q(x\cdot\xi)\, dx = \sum_{j=0}^\infty q^{jt} \int_{|x|=q^{-j}} e_q(x\cdot\xi)\, dx \\ &= \sum_{j=0}^\infty q^{jt} \left(\int_{B_{q^{-j}}(0)} e_q(x\cdot\xi)\, dx - \int_{B_{q^{-j-1}}(0)} e_q(x\cdot\xi)\, dx \right) \\ &= \sum_{j=0}^\infty q^{jt} \int_{B_{q^{-j}}(0)} e_q(x\cdot\xi)\, dx - \sum_{j=0}^\infty q^{jt} \int_{B_{q^{-j-1}}(0)} e_q(x\cdot\xi)\, dx. \end{aligned} \tag{25}$$

We now consider the integral

$$\int_{B_{q^{-j}}(0)} e_q(x\cdot\xi)\, dx.$$

Suppose first $\xi \neq 0$. Then $|\xi| = q^d$ for some d. If $d > j$, then we have

$$x \cdot \xi = x_{d-1}\xi_d + \sum_{1 \leq k \leq d-1} x_{k-1}\xi_k,$$

so

$$\begin{aligned} \int_{B_{q^{-j}}(0)} e_q(x\cdot\xi)\,dx &= \int_{B_{q^{-j}}(0)} e_q(x_{d-1}\xi_d) e_q\left(\sum_{k=1}^{d-1} x_{k-1}\xi_k\right) dx \\ &= \sum_{a \in \mathbf{F}_q} e_q(a\xi_d) \int_{|x| \leq q^{-j}; x_{d-1}=a} e_q\left(\sum_{k=1}^{d-1} x_{k-1}\xi_k\right) dx \quad \text{if } j < d \end{aligned} \tag{26}$$

We implicitly use the fact that $d > j$, which implies $d - 1 \geq j$, and therefore the ball $B_{q^{-j}}(0)$ contains points for which $x_{d-1} = a$. Observe that the value of the inner integral does not depend on a. Therefore, we have

$$\int_{B_{q^{-j}}(0)} e_q(x \cdot \xi)\,dx = 0 \quad \text{if } d > j.$$

On the other hand, for $d \leq j$, we have that $x \cdot \xi = 0$ on $B_{q^{-j}}(0)$ since $x_0, \ldots, x_{j-1}$ are equal to zero for any x in this ball. Thus

$$\int_{B_{q^{-j}}(0)} e_q(x \cdot \xi)\,dx = q^{-j} \quad \text{if } j \geq d. \tag{27}$$

We plug (26) and (27) into (25):

$$\begin{aligned} \widehat{r}_t(\xi) &= \sum_{j=d}^{\infty} q^{jt}q^{-j} - \sum_{j=d-1}^{\infty} q^{jt}q^{-j-1} = \sum_{j=d}^{\infty} q^{jt}q^{-j} - \sum_{j=d}^{\infty} q^{(j-1)t}q^{-j} \\ &= (1-q^{-t})\sum_{j=d}^{\infty} q^{j(t-1)} = \frac{(1-q^{-t})q^{d(t-1)}}{1-q^{t-1}} = \frac{1-q^{-t}}{1-q^{t-1}}|\xi|^{t-1}. \end{aligned}$$

It remains to consider the $\xi = 0$ case. In this case, $x \cdot \xi = 0$ for all x, so (25) becomes

$$\widehat{r}_t(0) = \sum_{j=0}^{\infty} q^{jt}q^{-j} - \sum_{j=0}^{\infty} q^{jt}q^{-j-1} = (1-q^{-1})\sum_{j=0}^{\infty} q^{j(t-1)} = \frac{1-q^{-1}}{1-q^{t-1}},$$

as desired. ◻

Papadimitropoulos uses this expression for the Fourier transform of the Riesz potential in order to compute a Fourier-analytic formula for the t-energy. We will need the following formula for the partial sums of the Fourier series for the Riesz potential, which is given implicitly in the proof of Lemma 4.3.4 (ii) in [9].

Lemma 8.3. *Let $S_n r_t$ denote the partial sum of the Fourier series for the Riesz potential:*

$$S_n r_t(z) = \sum_{|\xi| \leq q^n} \widehat{r}_t(\xi) e(z \cdot \xi).$$

Then we have the following formula for $S_n r_t$:

$$S_n r_t(z) = \begin{cases} |z|^{-t} & \text{if } |z| > q^{-n}, \\ \frac{q^{tn}(1-q^{-1})}{1-q^{t-1}} & \text{if } |z| \leq q^{-n}. \end{cases}$$

In particular, for each n, we have that $S_n r_t$ is nonnegative and is bounded above by the integrable function

$$S_n r_t(z) \le \frac{1-q^{-1}}{1-q^{t-1}}|z|^{-t}.$$

Proof of Lemma 8.3. This proof is very similar to the proof of Lemma 8.2. We write out the sum

$$\begin{aligned}
\sum_{|\xi|\le q^n} \widehat{r}_t(\xi)e(z\cdot\xi) &= \widehat{r}_t(0) + \sum_{j=1}^{n}\sum_{|\xi|=q^j} \widehat{r}_t(\xi)e_q(z\cdot\xi) \\
&= \frac{1-q^{-1}}{1-q^{t-1}} + \sum_{j=1}^{n}\sum_{|\xi|=q^j}\left(\frac{1-q^{-t}}{1-q^{t-1}}\right) q^{(t-1)j} e_q(z\cdot\xi) \\
&= \frac{1-q^{-1}}{1-q^{t-1}} + \frac{1-q^{-t}}{1-q^{t-1}}\sum_{j=1}^{n} q^{(t-1)j}\sum_{|\xi|=q^j} e_q(z\cdot\xi) \qquad (28)\\
&= \frac{1-q^{-1}}{1-q^{t-1}} + \frac{1-q^{-t}}{1-q^{t-1}}\sum_{j=1}^{n} q^{(t-1)j}\left(\sum_{|\xi|\le q^j} e_q(z\cdot\xi) - \sum_{|\xi|\le q^{j-1}} e_q(z\cdot\xi)\right) \\
&= \frac{1-q^{-1}}{1-q^{t-1}} + \frac{1-q^{-t}}{1-q^{t-1}}\left(\sum_{j=1}^{n} q^{(t-1)j}\sum_{|\xi|\le q^j} e_q(z\cdot\xi) - \sum_{j=1}^{n} q^{(t-1)j}\sum_{|\xi|\le q^{j-1}} e_q(z\cdot\xi)\right).
\end{aligned}$$

But we know that

$$\sum_{|\xi|\le q^j} e_q(z\cdot\xi) = \begin{cases} q^j & \text{if } |z|\le q^{-j}, \\ 0 & \text{if } |z| > q^{-j}. \end{cases}$$

We will now split into cases depending on the value of $|z|$. We will first consider z such that $|z| > q^{-n}$. For such z, there exists some k such that $0\le k<n$ and such that $|z| = q^{-k}$. The sum then reduces to

$$\begin{aligned}
\sum_{|\xi|\le q^n} \widehat{r}_t(\xi)e(z\cdot\xi) &= \frac{1-q^{-1}}{1-q^{t-1}} + \frac{1-q^{-t}}{1-q^{t-1}}\left(\sum_{j=1}^{k} q^{(t-1)j}q^j - \sum_{j=1}^{k+1} q^{(t-1)j}q^{j-1}\right) \\
&= \frac{1-q^{-1}}{1-q^{t-1}} + \frac{1-q^{-t}}{1-q^{t-1}}\left(\sum_{j=1}^{k} q^{tj} - \sum_{j=1}^{k+1} q^{tj-1}\right) \\
&= \frac{1-q^{-1}}{1-q^{t-1}} + \left(\frac{1-q^{-t}}{1-q^{t-1}}\right)\left(q^t\left(\frac{1-q^{tk}}{1-q^t}\right) - q^{t-1}\left(\frac{1-q^{t(k+1)}}{1-q^t}\right)\right) \\
&= \frac{1-q^{-1}}{1-q^{t-1}} + \left(\frac{1-q^{-t}}{1-q^{t-1}}\right)\left(\frac{q^t - q^{t(k+1)} - q^{t-1} + q^{t(k+2)-1}}{1-q^t}\right) \\
&= \frac{1-q^{-1}-1+q^{tk}+q^{-1}-q^{t(k+1)-1}}{1-q^{t-1}} = q^{tk} = |z|^{-t}.
\end{aligned}$$

It remains to consider those z such that $|z|\le q^{-n}$. For such z, we have that for all $0\le j\le n$, we have

$$\sum_{|\xi|\le q^j} e_q(z\cdot\xi) = q^j.$$

Therefore, the sum reduces to

$$\begin{aligned}\sum_{|\xi|\le q^n} \widehat{r}_t(\xi)e(z\cdot\xi) &= \frac{1-q^{-1}}{1-q^{t-1}} + \frac{1-q^{-t}}{1-q^{t-1}}\left(\sum_{j=1}^{n} q^{tj} - \sum_{j=1}^{n} q^{tj-1}\right)\\ &= \frac{1-q^{-1}}{1-q^{t-1}} + \frac{1-q^{-t}}{1-q^{t-1}}\left(q^t(1-q^{-1})\frac{1-q^{tn}}{1-q^t}\right)\\ &= \frac{1-q^{-1}}{1-q^{t-1}} - \frac{(1-q^{-1})(1-q^{tn})}{1-q^{t-1}}\\ &= \frac{q^{tn}(1-q^{-1})}{1-q^{t-1}} \le \frac{1-q^{-1}}{1-q^{t-1}}|z|^{-t}.\end{aligned}$$ □

We are now ready to prove Lemma 8.1.

Proof of Lemma 8.1. The right side of equation (24) is equal to

$$\begin{aligned}\sum_{\xi\in\widehat{\mathbf{F}_q^\infty}} |\widehat{\mu}(\xi)|^2\widehat{r}_t(\xi) &= \lim_{n\to\infty}\sum_{|\xi|\le q^n} \widehat{\mu}(\xi)\overline{\widehat{\mu}(\xi)}\widehat{r}_t(\xi)\\ &= \lim_{n\to\infty}\sum_{|\xi|\le q^n}\iint_{\mathbf{F}_q^\infty\times\mathbf{F}_q^\infty} e((x-y)\cdot\xi)\,d\mu(x)\,d\mu(y)\widehat{r}_t(\xi)\\ &= \lim_{n\to\infty}\iint_{\mathbf{F}_q^\infty\times\mathbf{F}_q^\infty}\sum_{|\xi|\le q^n} e((x-y)\cdot\xi)\widehat{r}_t(\xi)\,d\mu(x)\,d\mu(y)\\ &= \lim_{n\to\infty}\iint_{\mathbf{F}_q^\infty\times\mathbf{F}_q^\infty} S_n r_t(x-y)\,d\mu(x)\,d\mu(y)\end{aligned}$$

Let Δ be the diagonal $\{(x,x)\colon x\in\mathbf{F}_q^\infty\}$. If $\mu\times\mu(\Delta)>0$, then the t-energy of μ is ∞ by definition. Furthermore, we have that

$$\begin{aligned}\iint_{\mathbf{F}_q^\infty\times\mathbf{F}_q^\infty} S_n r_t(x-y)\,d\mu(x)\,d\mu(y) &\ge \iint_{\Delta} S_n r_t(x-y)\,d\mu(x)\,d\mu(y)\\ &= \iint_{\Delta}\frac{q^{tn}(1-q^{-1})}{1-q^{t-1}}\,d\mu(x)\,d\mu(y)\\ &= \mu\times\mu(\Delta)\frac{q^{tn}(1-q^{-1})}{1-q^{t-1}} \to \infty.\end{aligned}$$

So the energy formula is correct in this case.

Next, assume that $\mu\times\mu(\Delta)=0$. In this case, we have

$$\lim_{n\to\infty}\iint_{\mathbf{F}_q^\infty\times\mathbf{F}_q^\infty} S_n r_t(x-y)\,d\mu(x)\,d\mu(y) = \lim_{n\to\infty}\iint_{(x,y)\notin\Delta} S_n r_t(x-y)\,d\mu(x)\,d\mu(y).$$

Assume now that the t-energy of μ is infinite. Then for any $K>0$, there exists $n_0(K)>0$ such that we have

$$\iint_{|x-y|>q^{-n_0}} r_t(x-y)\,d\mu(x)\,d\mu(y)\ge K.$$

But by Lemma 8.2, we have that for $n>n_0$, $S_n r_t(x-y)\ge r_t(x-y)\mathbf{1}_{B^c_{q^{-n}}}$. Therefore, for $n>n_0$, we have

$$\iint_{(x,y)\notin\Delta} S_n r_t(x-y)\,d\mu(x)\,d\mu(y)\ge K.$$

Because this can be done for any K, it follows that

$$\sum_{\xi \in \widehat{\mathbf{F}_q}} |\widehat{\mu}(\xi)|^2 \widehat{r}_t(\xi) = \infty$$

and the formula (24) is proven in this case.

It remains to verify the formula when the t-energy of μ is finite. In this case, we have that $S_n r_t(x-y)$ is nonnegative and uniformly bounded above by the $\mu \times \mu$-integrable function

$$\frac{1-q^{-1}}{1-q^{t-1}} r_t(x-y).$$

Therefore, the dominated convergence theorem applies, and we have

$$\lim_{n\to\infty} \iint_{(x,y)\notin\Delta} S_n r_t(x-y)\, d\mu(x)\, d\mu(y) = \iint_{(x,y)\notin\Delta} r_t(x-y)\, d\mu(x)\, d\mu(y),$$

which is precisely the t-energy of μ. □

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