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# SINGULARITIES IN $\mathscr{L}^p$-QUASIDISKS

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**Abstract.** We study planar domains with exemplary boundary singularities of the form of cusps. A natural question is how much elastic energy is needed to flatten these cusps; that is, to remove singularities. We give, in a connection of quasidisks, a sharp integrability condition for the distortion function to answer this question.

## 1. Introduction and overview

The subject matter emerge most clearly when the setting is more general than we actually present it here. Thus we suggest, as a possibility, to consider two planar sets $\mathbf{X}, \mathbf{Y} \subset \mathbf{C}$ of the same global topological configuration, meaning that there is a sense preserving homeomorphism $f\colon \mathbf{C} \xrightarrow{\text{onto}} \mathbf{C}$ which takes $\mathbf{X}$ onto $\mathbf{Y}$. Clearly $f\colon \mathbf{C} \setminus \mathbf{X} \xrightarrow{\text{onto}} \mathbf{C} \setminus \mathbf{Y}$. Throughout this paper homeomorphisms $f\colon \mathbf{C} \xrightarrow{\text{onto}} \mathbf{C}$ are sense preserving.

**1.1. Mappings of finite distortion.** In the last 20 years, there have been an increasing interest in the Geometric Function Theory (GFT) to extend the theory of quasiconformal mappings to the class of *mappings of finite distortion* [3, 10, 12].

**Definition 1.1.** A homeomorphism $f \in \mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbf{C}, \mathbf{C})$ is said to have *finite distortion* if there is a measurable function $K\colon \mathbf{C} \to [1, \infty)$ such that

$$|Df(z)|^2 \leqslant K(z) J_f(z) \quad \text{for almost every } z \in \mathbf{C}. \tag{1.1}$$

Hereafter $|Df(z)|$ stands for the operator norm of the differential matrix $Df(z) \in \mathbf{R}^{2\times 2}$, and $J_f(z)$ for its determinant. The smallest function $K(z) \geqslant 1$ for which (1.1) holds is called the *distortion* of $f$, denoted by $K_f = K_f(z)$. In terms of d'Alembert complex derivatives, we have $|Df(z)| = |f_z| + |f_{\bar z}|$ and $J_f(z) = |f_z|^2 - |f_{\bar z}|^2$. Thus $f$ can be viewed as a *very weak solution* to the *Beltrami equation*:

$$\frac{\partial f}{\partial \bar z} = \mu(z) \frac{\partial f}{\partial z}, \quad \text{where } |\mu(z)| = \frac{K_f(z) - 1}{K_f(z) + 1} < 1. \tag{1.2}$$

The distortion inequality (1.1) asks that $Df(z) = 0 \in \mathbf{R}^{2\times 2}$ at the points where the Jacobian $J_f(z) = \det Df(z)$ vanishes. We obtain *quasiconformal* mappings if $K_f \in \mathscr{L}^\infty(\mathbf{C})$.

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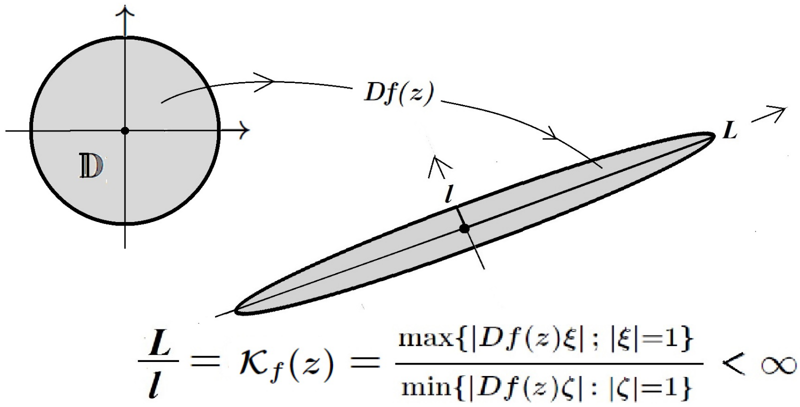


Figure 1. The ratio $L/l$, which measures the infinitesimal distortion of the material structure at the point $z$, is allowed to be arbitrarily large. Nevertheless, $L/l$ has to be finite almost everywhere.

**1.2. Quasiconformal equivalence.** It should be pointed out that the inverse map $f^{-1}\colon \mathbf{C} \xrightarrow{\text{onto}} \mathbf{C}$ is also $K$-quasiconformal and a composition $f \circ g$ of $K_1$ and $K_2$-quasiconformal mappings is $K_1 \cdot K_2$-quasiconformal. These special features of quasiconformal mappings furnish an equivalence relation between subsets of $\mathbf{C}$.

**Definition 1.2.** We say that $\mathbf{X} \subset \mathbf{C}$ is *quasiconformal equivalent* to $\mathbf{Y} \subset \mathbf{C}$, and write $\mathbf{X} \overset{\text{quasi}}{=\!=} \mathbf{Y}$, if $Y = f(\mathbf{X})$ for some quasiconformal mapping $f\colon \mathbf{C} \xrightarrow{\text{onto}} \mathbf{C}$.

**1.3. Quasidisks.** Quasidisks are domains which are quasiconformal equivalent to the unit disk $\mathbf{D} \subset \mathbf{C}$. Thus we introduce the following:

**Definition 1.3.** A domain $\mathbf{X} \subset \mathbf{C}$ is called *quasidisk* if it admits a quasiconformal mapping $f\colon \mathbf{C} \xrightarrow{\text{onto}} \mathbf{C}$ which takes $\mathbf{X}$ onto $\mathbf{D}$. In symbols, we have $\mathbf{X} \overset{\text{quasi}}{=\!=} \mathbf{D}$.

Quasidisks have been studied intensively for many years because of their exceptional functional theoretical properties, relationships with Teichmüller theory and Kleinian groups and interesting applications in complex dynamics, see [6] for an elegant survey. Perhaps the best know geometric characterization for a quasidisk is the *Ahlfors' condition* [1].

**Theorem 1.4.** (Ahlfors) *Let* $\mathbf{X}$ *be a (simply connected) Jordan domain in the plane. Then* $\mathbf{X}$ *is a quasidisk if and only if there is a constant* $1 \leqslant \gamma < \infty$, *such that for each pair of distinct points* $a, b \in \partial\mathbf{X}$ *we have*

$$\operatorname{diam}\Gamma \leqslant \gamma |a-b| \tag{1.3}$$

*where* $\Gamma$ *is the component of* $\partial\mathbf{X} \setminus \{a, b\}$ *with smallest diameter.*

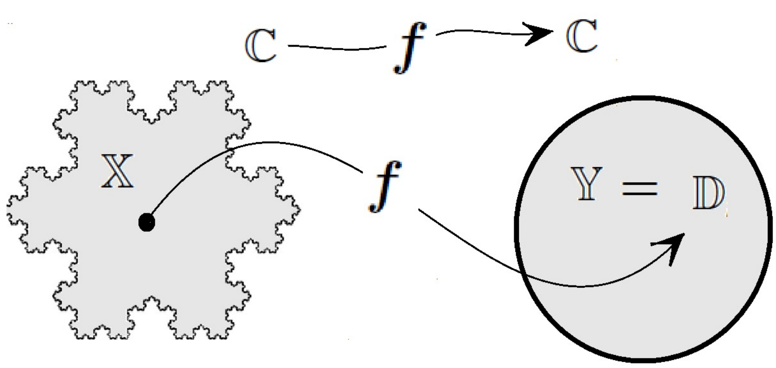


Figure 2. Koch snowflake reveals complexity of a quasidisk.

One should infer from the Ahlfors' condition (1.3) that:

*Quasidisks do not allow for cusps in the boundary.*

That is to say, unfortunately, the point-wise inequality $K_f(z) \leqslant K < \infty$ precludes $f$ from smoothing even basic singularities. It is therefore of interest to look for more general deformations $f\colon \mathbf{C} \xrightarrow{\text{onto}} \mathbf{C}$. We shall see, and it will become intuitively clear, that the act of deviating from conformality should be measured by integral-mean distortions rather than point-wise distortions. More general class of mappings, for which one might hope to build a viable theory, consists of homeomorphisms with locally $\mathscr{L}^p$-integrable distortion, $1 \leqslant p < \infty$.

**Definition 1.5.** The term *mapping of $\mathscr{L}^p$-distortion*, $1 \leqslant p < \infty$, refers to a homeomorphism $f\colon \mathbf{C} \to \mathbf{C}$ of class $\mathscr{W}^{1,1}_{\text{loc}}(\mathbf{C}, \mathbf{C})$ with $K_f \in \mathscr{L}^p_{\text{loc}}(\mathbf{C})$.

Now, we generalize the notion of quasidisks; simply, replacing the assumption $K_f \in \mathscr{L}^\infty(\mathbf{C})$ by $K_f \in \mathscr{L}^p_{\text{loc}}(\mathbf{C})$.

**Definition 1.6.** A domain $\mathbf{X} \subset \mathbf{C}$ is called an *$\mathscr{L}^p$-quasidisk* if it admits a homeomorphism $f\colon \mathbf{C} \to \mathbf{C}$ of $\mathscr{L}^p$-distortion such that $f(\mathbf{X}) = \mathbf{D}$.

Clearly, $\mathscr{L}^p$-quasidisks are Jordan domains. Surprisingly, the $\mathscr{L}^1_{\text{loc}}$-integrability of the distortion seems not to cause any geometric constraint on $\mathbf{X}$. We confirm this observation for domains with rectifiable boundary.

**Theorem 1.7.** *Simply-connected Jordan domains with rectifiable boundary are $\mathscr{L}^1$-quasidisks.*

The $\mathscr{L}^p$-quasidisks with $p > 1$ can be characterized by model singularities at their boundaries. The most specific singularities, which fail to satisfy the Ahlfors' condition (1.3), are cusps. Let us consider the power-type inward and outward cusp domains, see Figure 3. For $\beta > 1$ we consider a disk with inward cusp defined by

$$\mathbf{D}^{\prec}_\beta = \mathbf{B}(1-\beta, r_\beta) \setminus \{z = x + iy \in \mathbf{C}\colon x \geqslant 0, |y| \leqslant x^\beta\} \quad r_\beta = \sqrt{\beta^2+1}.$$

Whereas a disk with outer cusp will be defined by

$$\mathbf{D}^{\succ}_\beta = \{z = x + iy \in \mathbf{C}\colon 0 < x < 1, |y| < x^\beta\} \cup \mathbf{B}(1+\beta, r_\beta).$$

Here, $r_\beta = \sqrt{\beta^2+1}$.

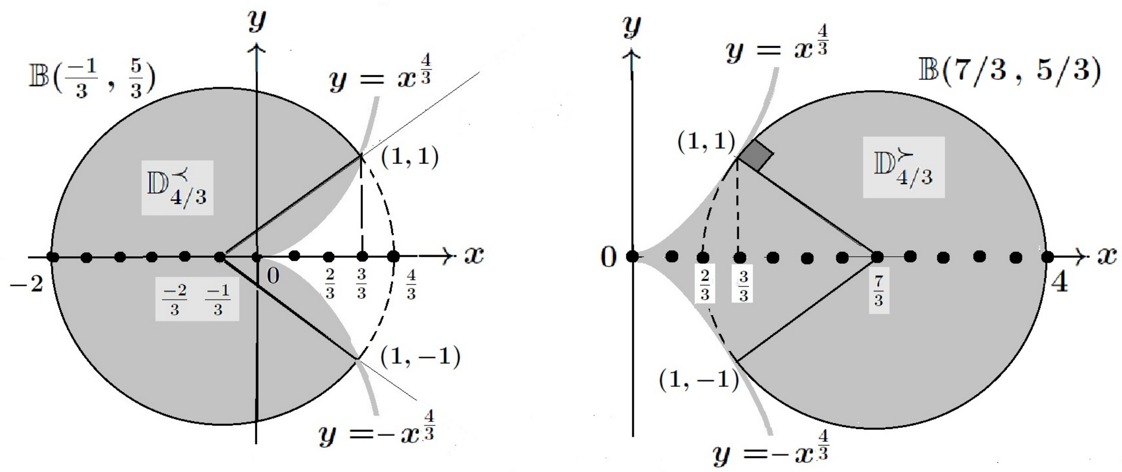


Figure 3. The inner and outer *power cusps* in the disks $\mathbf{D}^{\prec}_\beta$ and $\mathbf{D}^{\succ}_\beta$, with $\beta = \frac{4}{3}$.

Note, all of these domains fail to satisfy the Ahlfors' condition (1.3). However, replacing $|a-b|$ in (1.3) by $|a-b|^\alpha$ we obtain:

**Definition 1.8.** The boundary of a Jordan domain $\mathbf{X} \subset \mathbf{C}$ is *$\alpha$-Hölder bounded turning*, with $\alpha \in (0,1]$, if there is a constant $1 \leqslant \gamma < \infty$ such that for each pair of distinct points $a, b \in \partial\mathbf{X}$ we have

$$\operatorname{diam} \Gamma \leqslant \gamma\, |a-b|^\alpha \tag{1.4}$$

where $\Gamma$ is the component of $\partial\mathbf{X} \setminus \{a,b\}$ with smallest diameter.

**Theorem 1.9.** *Let $\mathbf{X}$ be either $\mathbf{D}_\beta^{\prec}$ or $\mathbf{D}_\beta^{\succ}$ and $1 < p < \infty$. Then $\mathbf{X}$ is a $\mathscr{L}^p$-quasidisk if and only if $\beta < \frac{p+3}{p-1}$; equivalently, $p < \frac{\beta+3}{\beta-1}$.*

This simply means that $\partial\mathbf{X}$ is $\frac{1}{\beta}$-Hölder bounded turning. Theorem 1.9 tells us how much the distortion of a homeomorphism $f\colon \mathbf{C} \to \mathbf{C}$ is needed to flatten (or smoothen) the power type cusp $t^\beta$. It turns out that a lot more distortion is needed to create a cusp than to smooth it back. Indeed, in a series of papers [14, 15, 16], Koskela and Takkinen raised such an inverse question. For which cusps does there exist a homeomorphism $h\colon \mathbf{C} \to \mathbf{C}$ of finite distortion $1 \leqslant K_h < \infty$ which takes $\mathbf{D}$ onto $\mathbf{D}_\beta^{\prec}$? A necessary condition turns out to be that $e^{K_h} \notin \mathscr{L}^p_{\mathrm{loc}}(\mathbf{C})$ with $p > \frac{2}{\beta-1}$. However, if $p < \frac{2}{\beta-1}$ there is such a homeomorphism. Especially, each power-type cusp domain can be obtained as the image of an open disk by a homeomorphism $h\colon \mathbf{C} \to \mathbf{C}$ with $K_h \in \mathscr{L}^p_{\mathrm{loc}}(\mathbf{C})$ for all $p < \infty$. Combining this with Theorem 1.9 boils down to the following postulate:

*Creating singularities takes almost no efforts (just allow for a little distortion) while tidying them up is a whole new story.*

In the spirit of extremal quasiconformal mappings in Teichmüller spaces, one might be interested in studying homeomorphisms $f\colon \mathbf{C} \xrightarrow{\text{onto}} \mathbf{C}$ of smallest $\mathscr{L}^p$-energy, subject to the condition $f(\mathbf{X}) = \mathbf{Y}$. Here the given pair $\mathbf{X}, \mathbf{Y}$ of subsets in $\mathbf{C}$ is assumed to admit at least one such homeomorphism of finite energy. To look at a more specific situation, take for $\mathbf{X}$ an $\mathscr{L}^p$-quasidisk from Theorem 1.9, and the unit disk $\mathbf{D}$ for $\mathbf{Y}$. What is then the energy-minimal map $f\colon \mathbf{C} \xrightarrow{\text{onto}} \mathbf{C}$? Polyconvexity of the integrand will certainly help us find what conditions are needed for the existence of energy-minimal mappings. We shall not enter these topics here, but refer to [2, 13, 19] for related results.

**1.4. The main result.** Since a simply connected Jordan domain is conformally equivalent with the unit disk, it is natural to consider special $\mathscr{L}^p$-quasidisks; namely, the domains $\mathbf{X}$ which can be mapped onto an open disk under a homeomorphism $f\colon \mathbf{C} \to \mathbf{C}$ with $\mathscr{L}^p$-integrable distortion and to be quasiconformal when restricted to $\mathbf{X}$.

The answer to this question when $\mathbf{X}$ is a power type cusp domain can be inferred from our main result (take $(\infty, p)$ or $(p, \infty)$) which also generalizes Theorem 1.9 (take $(p,p)$).

**Theorem 1.10.** (Main theorem) *Consider power-type inward cusp domains $\mathbf{X} = \mathbf{D}_\beta^{\prec}$ with $\beta > 1$. Given a pair $(q,p)$ of exponents $1 \leqslant q \leqslant \infty$ (for $\mathbf{X}$) and $1 < p \leqslant \infty$*

*(for the complement of* $\mathbf{X}$*), define the so-called critical power of inward cusps*

$$\beta_{\mathrm{cr}} \stackrel{\mathrm{def}}{=\!=} \begin{cases} \frac{pq+2p+q}{pq-q}, & \text{if } 1<p<\infty \text{ and } q<\infty \\ \frac{2}{q}+1, & \text{if } p=\infty \text{ and } q<\infty \\ \frac{p+1}{p-1}, & \text{if } 1<p<\infty \text{ and } q=\infty \end{cases} \tag{1.5}$$

*Then there exists a Sobolev homeomorphism* $f\colon \mathbf{C}\to\mathbf{C}$ *which takes* $\mathbf{X}$ *onto* $\mathbf{D}$ *such that*

- $K_f \in \mathscr{L}^q(\mathbf{X})$

*and*

- $K_f \in \mathscr{L}^p(\mathbf{B}_R \setminus \overline{\mathbf{X}})$ *for every* $R>2$,

*if and only if* $\beta < \beta_{\mathrm{cr}}$.

Here and what follows $\mathbf{B}_R = \{z\in\mathbf{C}\colon |z|<R\}$ for $R>0$.

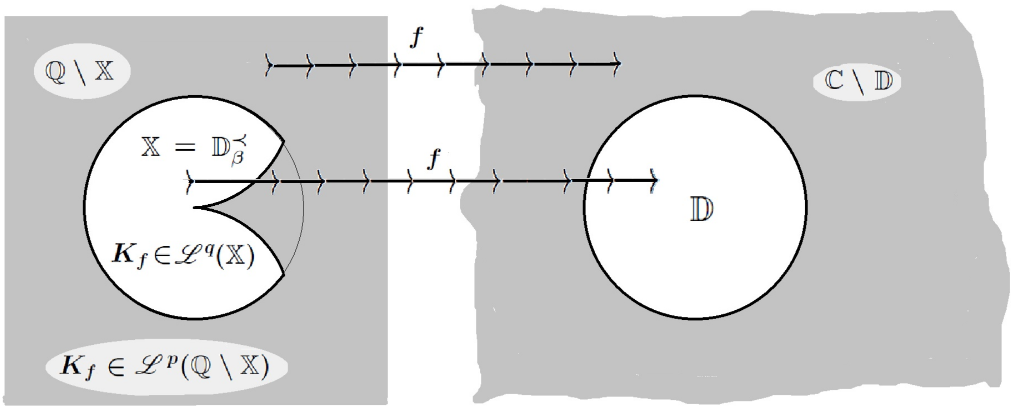


Figure 4. An $\mathscr{L}^{q,p}$-quasidisk.

Applying the standard inversion of unit disk, Theorem 1.10 extends to the power-type outer cusp domains as well. In this case the roles of $p$ and $q$ are interchanged. The reader interested in learning more about the conformal case $f\colon \mathbf{D}_\beta^{\prec} \xrightarrow{\text{onto}} \mathbf{D}$ is referred to [22].

Our proof of Theorem 1.10 is self-contained. The "only if" part of Theorem 1.10 relies on a regularity estimate of a reflection in $\partial\mathbf{D}_\beta^{\prec}$. Such a reflection is defined and examined in the boundary of an arbitrary $\mathscr{L}^p$-quasidisk. In this connection we recall a classical result of Kühnau [18] which tells us that a Jordan domain is a quasidisk if and only if it admits a quasiconformal reflection in its boundary. Before going into details about the boundary reflection procedures (Section 3) we need some preliminaries.

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## 2. Preliminaries

First we recall a well-known theorem of Gehring and Lehto [9] which asserts that a planar open mapping with finite partial derivatives at almost every point is differentiable at almost every point. For homeomorphisms the result was earlier established by Menchoff [20].

**Lemma 2.1.** *Suppose that* $f\colon \mathbf{C}\to\mathbf{C}$ *is a homeomorphism in the class* $\mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbf{C},\mathbf{C})$*. Then* $f$ *is differentiable almost everywhere.*

It is easy to see, at least formally, applying a change of variables that the integral of distortion function equals the Dirichlet integral of inverse mapping. This observation is the key to the fundamental identity which we state next, see [10, 11, 21].

**Lemma 2.2.** *Suppose that a homeomorphism* $f\colon \mathbf{C} \xrightarrow{\text{onto}} \mathbf{C}$ *of Sobolev class* $\mathscr{W}^{1,1}_{\text{loc}}(\mathbf{C},\mathbf{C})$. *Then* $f$ *is a mapping of* $\mathscr{L}^1$*-distortion if and only if the inverse* $h \overset{\text{def}}{=\!=} f^{-1} \in W^{1,2}_{\text{loc}}(\mathbf{C},\mathbf{C})$. *Furthermore, then for every bounded domain* $\mathbf{U} \subset \mathbf{C}$ *we have*

$$\int_{f(\mathbf{U})} |Dh(y)|^2 \,\mathrm{d}y = \int_{\mathbf{U}} K_f(x)\,\mathrm{d}x$$

*and* $J_f(x) > 0$ *a.e.*

At least formally the identity $(h \circ f)(x) = x$, after differentiation, implies that $Dh(f(x))Df(x) = \mathbf{I}$. The validity of such identity under minimal regularity assumptions on the mappings is the essence of the following lemma, see [10, Lemma A.29].

**Lemma 2.3.** *Let* $f\colon \mathbf{X} \to \mathbf{Y}$ *be a homeomorphism which is differentiable at* $x \in \mathbf{X}$ *with* $J_f(x) > 0$. *Let* $h\colon \mathbf{Y} \to \mathbf{X}$ *be the inverse of* $f$. *Then* $h$ *is differentiable at* $f(x)$ *and* $Dh(f(x)) = (Df(x))^{-1}$.

Next we state a crucial version of the area formula for us.

**Lemma 2.4.** *Let* $\mathbf{X},\mathbf{Y} \subset \mathbf{C}$ *be domains and* $g\colon \mathbf{X} \xrightarrow{\text{onto}} \mathbf{Y}$ *a homeomorphism. Suppose that* $\mathbf{V} \subset \mathbf{X}$ *is a measurable set and* $g$ *is differentiable at every point of* $\mathbf{V}$. *If* $\eta$ *is a nonnegative Borel measurable function, then*

$$\int_{\mathbf{V}} \eta(g(x))|J_g(x)|\,\mathrm{d}x \leqslant \int_{g(\mathbf{V})} \eta(y)\,\mathrm{d}y. \tag{2.1}$$

This follows from [5, Theorem 3.1.8] together with the area formula for Lipschitz mappings.

The circle is uniquely characterized by the property that among all closed Jordan curves of given length $L$, the circle of circumference $L$ encloses maximum area. This property is expressed in the well-known isoperimetric inequality.

**Lemma 2.5.** *Suppose* $\mathbf{U}$ *is a bounded Jordan domain with rectifiable boundary* $\partial\mathbf{U}$. *Then*

$$|\mathbf{U}| \leqslant \frac{1}{4\pi}[\ell(\partial\mathbf{U})]^2 \tag{2.2}$$

*where* $|\mathbf{U}|$ *is the area of* $\mathbf{U}$ *and* $\ell(\partial\mathbf{U})$ *is the length of* $\partial\mathbf{U}$.

## 3. Reflection

We denote the one point compactification of the complex plane by $\widehat{\mathbf{C}} \overset{\text{def}}{=\!=} \mathbf{C}\cup\{\infty\}$.

**Definition 3.1.** A domain $\Omega \subset \widehat{\mathbf{C}}$ admits a *reflection in its boundary* $\partial\Omega$ if there exists a homeomorphism $g$ of $\widehat{\mathbf{C}}$ such that

- $g(\Omega) = \widehat{\mathbf{C}} \setminus \overline{\Omega}$, and
- $g(z) = z$ for $z \in \partial\Omega$.

A domain $\Omega \subset \widehat{\mathbf{C}}$ is a Jordan domain if and only if it admits a reflection in its boundary, see [7]. In this section we raise a question what else can we say about the reflection if the domain is an $\mathscr{L}^p$-quasidisk. A classical result of Kühnau [18] tells us that $\Omega \subset \widehat{\mathbf{C}}$ is a quasidisk if and only if it admits a quasiconformal reflection in $\partial\Omega$. Let $\mathbf{X} \subset \mathbf{C}$ be an $\mathscr{L}^p$-quasidisk. Then there exists a homeomorphism $f\colon \mathbf{C} \xrightarrow{\text{onto}} \mathbf{C}$

such that $f(\mathbf{X}) = \mathbf{D}$. We extend $f$ by setting $f(\infty) = \infty$ and still denote the extended mapping by $f$. This way we obtain a homeomorphism $f\colon \widehat{\mathbf{C}} \xrightarrow{\text{onto}} \widehat{\mathbf{C}}$. We also denote its inverse by $h\colon \widehat{\mathbf{C}} \xrightarrow{\text{onto}} \widehat{\mathbf{C}}$.

The circle inversion map $\Psi\colon \widehat{\mathbf{C}} \xrightarrow{\text{onto}} \widehat{\mathbf{C}}$,

$$\Psi(z) \overset{\text{def}}{=\!=} \begin{cases} \frac{z}{|z|^2} & \text{if } z \neq 0, \\ \infty & \text{if } z = 0 \end{cases},$$

is anticonformal, which means that at every point it preserves angles and reverses orientation. The circle inversion defines a reflection in $\partial\mathbf{X}$ by the rule

$$g\colon \widehat{\mathbf{C}} \xrightarrow{\text{onto}} \widehat{\mathbf{C}} \qquad g(x) \overset{\text{def}}{=\!=} h \circ \Psi \circ f(x). \tag{3.1}$$

**Theorem 3.2.** *Let* $\mathbf{X}$ *be an* $\mathscr{L}^p$*-quasidisk and* $g$ *the reflection in* $\partial\mathbf{X}$ *given by* (3.1). *Then for a bounded domain* $\mathbf{U} \subset \mathbf{C}$ *such that* $h(0) \notin \overline{\mathbf{U}}$ *we have* $g \in \mathscr{W}^{1,1}(\mathbf{U}, \mathbf{C})$ *and*

$$\int_{\mathbf{U}} \frac{|Dg(x)|^p}{|J_g(x)|^{\frac{p-1}{2}}}\, \mathrm{d}x \leqslant \left(\int_{g(\mathbf{U})} K_f^p(x)\, \mathrm{d}x\right)^{\frac{1}{2}} \cdot \left(\int_{\mathbf{U}} K_f^p(x)\, \mathrm{d}x\right)^{\frac{1}{2}}. \tag{3.2}$$

*Proof.* Let $\mathbf{U}$ be a bounded domain in $\mathbf{C}$ such that $h(0) \notin \overline{\mathbf{U}}$. For $x \in \mathbf{U}$ we denote

$$\tilde{f}(x) \overset{\text{def}}{=\!=} \Psi \circ f(x) \quad \text{and} \quad \tilde{h}(y) \overset{\text{def}}{=\!=} (\tilde{f})^{-1}(y).$$

We write

$$\mathbf{V} \overset{\text{def}}{=\!=} \{x \in \mathbf{U}\colon f \text{ is differentiable at } x \quad \text{and} \quad J_f(x) > 0\}.$$

Then by Lemma 2.1 and Lemma 2.2 we obtain $|\mathbf{V}| = |\mathbf{U}|$.

Fix $x \in \mathbf{V}$. Then $\tilde{f}$ is differentiable at $x$. Furthermore, $h$ is differentiable at $f(x)$, see Lemma 2.3. Therefore, for $x \in \mathbf{V}$ the chain rule gives

$$|Dg(x)| \leqslant |Dh(\tilde{f}(x))|\,|D\tilde{f}(x)| \;\text{ and } J_g(x) = J_h(\tilde{f}(x))J_{\tilde{f}}(x). \tag{3.3}$$

Hence, applying Hölder's inequality we have

$$\begin{aligned} \int_{\mathbf{U}} \frac{|Dg(x)|^p}{|J_g(x)|^{\frac{p-1}{2}}}\, \mathrm{d}x &= \int_{\mathbf{V}} \frac{|Dg(x)|^p}{|J_g(x)|^{\frac{p-1}{2}}}\, \mathrm{d}x \leqslant \int_{\mathbf{V}} \frac{|Dh(\tilde{f}(x))|^p}{|J_h(\tilde{f}(x))|^{\frac{p-1}{2}}} \frac{|D\tilde{f}(x)|^p}{|J_{\tilde{f}}(x)|^{\frac{p-1}{2}}}\, \mathrm{d}x \\ &\leqslant \left(\int_{\mathbf{V}} \frac{|Dh(\tilde{f}(x))|^{2p}}{|J_h(\tilde{f}(x))|^{p-1}} |J_{\tilde{f}}(x)|\, \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathbf{V}} \frac{|D\tilde{f}(x)|^{2p}}{|J_{\tilde{f}}(x)|^p}\, \mathrm{d}x\right)^{\frac{1}{2}}. \end{aligned} \tag{3.4}$$

According to Lemma 2.4 we obtain

$$\int_{\mathbf{V}} \frac{|Dh(\tilde{f}(x))|^{2p}}{|J_h(\tilde{f}(x))|^{p-1}} |J_{\tilde{f}}(x)|\, \mathrm{d}x \leqslant \int_{\tilde{f}(\mathbf{V})} \frac{|Dh(y)|^{2p}}{[J_h(y)]^{p-1}} \mathrm{d}y. \tag{3.5}$$

Applying Lemma 2.4 again this time for $h$, we have

$$\int_{\tilde{f}(\mathbf{V})} \frac{|Dh(y)|^{2p}}{[J_h(y)]^p} J_h(y) \mathrm{d}y \leqslant \int_{g(\mathbf{V})} |Dh\big(f(x)\big)|^{2p} [J_f(x)]^p \mathrm{d}x.$$

This together with Lemma 2.3 gives

$$\int_{\tilde{f}(\mathbf{V})} \frac{|Dh(y)|^{2p}}{[J_h(y)]^p} J_h(y)\, \mathrm{d}y \leqslant \int_{g(\mathbf{V})} |(Df(x))^{-1}|^{2p} [J_f(x)]^p\, \mathrm{d}x.$$

The familiar Cramer's rule implies

$$\int_{g(\mathbf{V})} |(Df(x))^{-1}|^{2p}[J_f(x)]^p \,\mathrm{d}x = \int_{g(\mathbf{V})} \frac{|Df(x)|^{2p}}{[J_f(x)]^p}\,. \tag{3.6}$$

Combining the estimate (3.5) with (3.6) we have

$$\int_{\mathbf{V}} \frac{|Dh(\tilde f(x))|^{2p}}{|J_h(\tilde f(x))|^{p-1}} |J_{\tilde f}(x)| \,\mathrm{d}x \leqslant \int_{g(\mathbf{U})} K_f^p(x)\,\mathrm{d}x. \tag{3.7}$$

Estimating the second term on the right hand side of (3.4) we simply note that $|D\Psi(z)|^2 = J(z,\Psi)$ for $z \in \mathbf{C}\setminus\{0\}$ and so

$$\int_{\mathbf{V}} \frac{|D\tilde f(x)|^{2p}}{|J_{\tilde f}(x)|^p}\,\mathrm{d}x = \int_{\mathbf{V}} K_f^p(x)\,\mathrm{d}x \leqslant \int_{\mathbf{U}} K_f^p(x)\,\mathrm{d}x\,. \tag{3.8}$$

The claim follows from (3.4), (3.7) and (3.8). □

## 4. Proof of Theorem 1.7

The proof is based on a Sobolev variant of the Jordan–Schönflies theorem.

**Lemma 4.1.** *Let* $\mathbf{Y}$ *be bounded simply connected Jordan domains,* $\partial\mathbf{Y}$ *being rectifiable. A boundary homeomorphism* $\phi\colon \partial\mathbf{D} \xrightarrow{\text{onto}} \partial\mathbf{Y}$ *satisfying*

$$\int_{\partial\mathbf{Y}}\int_{\partial\mathbf{Y}} \Big|\log|\phi^{-1}(\xi)-\phi^{-1}(\eta)|\Big|\,|\mathrm{d}\xi||\mathrm{d}\eta| < \infty \tag{4.1}$$

*admits a homeomorphic extension* $h\colon \mathbf{C}\to\mathbf{C}$ *of Sobolev class* $\mathscr{W}^{1,2}_{\mathrm{loc}}(\mathbf{C},\mathbf{C})$.

This result is from [17, Theorem 1.6]. A necessary condition is that the mapping $\phi$ is the Sobolev trace of some (possibly non-homeomorphic) mapping in $\mathscr{W}^{1,2}(\mathbf{D},\mathbf{C})$. The class of boundary functions which admit a harmonic extension with finite Dirichlet energy was characterized by Douglas [4]. The Douglas condition for a function $\phi\colon \partial\mathbf{D} \xrightarrow{\text{onto}} \partial\mathbf{Y}$ reads as

$$\int_{\partial\mathbf{D}}\int_{\partial\mathbf{D}} \left|\frac{\phi(\xi)-\phi(\eta)}{\xi-\eta}\right|^2 |\mathrm{d}\xi||\mathrm{d}\eta| < \infty. \tag{4.2}$$

In [2] it was shown that for $\mathscr{C}^1$-smooth $\mathbf{Y}$ the Douglas condition (4.2) can be equivalently given in terms of the inverse mapping $\phi^{-1}\colon \partial\mathbf{Y} \xrightarrow{\text{onto}} \partial\mathbf{D}$ by (4.1). Beyond the $\mathscr{C}^1$-smooth domains, if $\mathbf{Y}$ is Lipschitz regular, then a boundary homeomorphism $\phi\colon \partial\mathbf{D} \xrightarrow{\text{onto}} \partial\mathbf{Y}$ admits a homeomorphic extension $h\colon \overline{\mathbf{D}} \xrightarrow{\text{onto}} \overline{\mathbf{Y}}$ in $\mathscr{W}^{1,2}(\mathbf{D},\mathbf{C})$ if and only if $\phi$ satisfies the Douglas condition. There is, however, an inner chordarc domain $\mathbf{Y}$ and a homeomorphism $\phi\colon \partial\mathbf{D} \xrightarrow{\text{onto}} \partial\mathbf{Y}$ satisfying the Douglas condition which does not admit a homeomorphic extension $h\colon \overline{\mathbf{D}} \xrightarrow{\text{onto}} \overline{\mathbf{Y}}$ with finite Dirichlet energy. Recall that $\mathbf{Y}$ is an inner chordarc domain if there exists a homeomorphism $\Upsilon\colon \overline{\mathbf{Y}} \xrightarrow{\text{onto}} \overline{\mathbf{D}}$ which is $\mathscr{C}^1$-diffeomorphic in $\mathbf{Y}$ with bounded gradient matrices $D\Upsilon$ and $(D\Upsilon)^{-1}$. These and more about Sobolev homeomorphic extension results we refer to [17].

*Proof of Theorem 1.7.* Let $\mathbf{X}\subset\mathbf{C}$ be a simply connected Jordan domain, $\partial\mathbf{X}$ being rectifiable. According to Lemma 2.2, $\mathbf{X}$ is an $\mathscr{L}^1$-quasidisk if and only if there exists a homeomorphism $h\colon \mathbf{C} \xrightarrow{\text{onto}} \mathbf{C}$ in $\mathscr{W}^{1,2}_{\mathrm{loc}}(\mathbf{C},\mathbf{C})$ such that $h(\mathbf{D})=\mathbf{X}$. Therefore, by Lemma 4.1 it suffices to construct a boundary homeomorphism $\phi\colon \partial\mathbf{D} \xrightarrow{\text{onto}} \partial\mathbf{X}$ which satisfies

$$\int_{\partial\mathbf{X}}\int_{\partial\mathbf{X}} \Big|\log|\phi^{-1}(\xi)-\phi^{-1}(\eta)|\Big|\,|\mathrm{d}\xi||\mathrm{d}\eta| < \infty.$$

Let $\xi, \eta \in \partial\mathbf{X}$ be arbitrary. We denote by $\gamma_{\xi\eta}$ the subcurve of $\partial\mathbf{X}$, connecting $\xi$ and $\eta$. The curve $\gamma_{\xi\eta}$ is parametrized counterclockwise. Setting $z_\xi = 1$. For arbitrary $z \in \partial\mathbf{D}$ let $\widehat{z_\xi z} \subset \partial\mathbf{D}$ be the circular arc starting from $z_\xi$ ending at $z$. The arc is parametrized counterclockwise. For $\eta \in \partial\mathbf{X}$, there exists a unique $z_\eta \in \partial\mathbf{D}$ with

$$\frac{\ell(\gamma_{\xi\eta})}{\ell(\partial\mathbf{X})} = \frac{\ell(\widehat{z_\xi z_\eta})}{\ell(\partial\mathbf{D})}.$$

Now, we define the boundary homeomorphism $\phi\colon \partial\mathbf{D} \to \partial\mathbf{X}$ by setting $\phi(z_\eta) = \eta$.

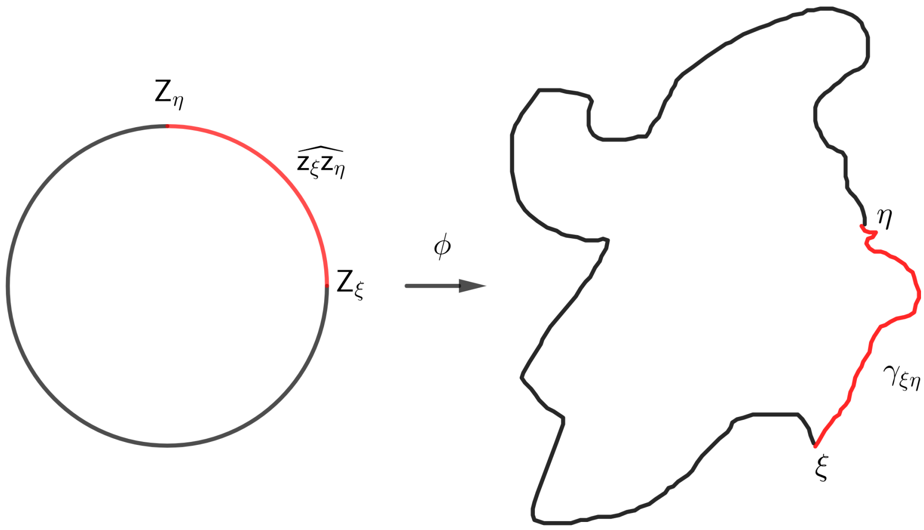


Figure 5.

First, we observe that $|\phi'(z)| = \frac{\ell(\partial\mathbf{X})}{\ell(\partial\mathbf{D})}$ for every $z \in \partial\mathbf{D}$. Furthermore since the length of the shorter circular arc between two points in $\partial\mathbf{D}$ is comparable to their Euclidean distance the change of variables formula gives

$$\begin{aligned}\int_{\partial\mathbf{X}} |\log|\phi^{-1}(\xi) - \phi^{-1}(\eta)||\,|\mathrm{d}\eta| &\leqslant C \int_{\partial\mathbf{D}} |\log|\phi^{-1}(\xi) - \phi^{-1}(\eta)||\,|\mathrm{d}\phi^{-1}(\eta)| \\ &\leqslant C \int_0^{2\pi} |\log t|\,\mathrm{d}t < \infty. \qquad \square\end{aligned}$$

## 5. Proof of Theorem 1.10

Before jumping into the proof we fix a few notation and prove two auxiliary results. Fix a power-type inward cusp domain $\mathbf{D}_\beta^{\prec}$. For $0 < t < 1$ we write

$$\mathbf{I}_t \overset{\text{def}}{=\!=} \{t + iy \in \mathbf{C} \colon 0 \leqslant |y| < t^\beta\}$$

and

$$\mathbf{U}_t \overset{\text{def}}{=\!=} \{x + iy \in \mathbf{C} \colon 0 < x < t \text{ and } 0 \leqslant |y| < x^\beta\}.$$

The area of $\mathbf{U}_t$ is given by

$$|\mathbf{U}_t| = \int_0^t \int_{-s^\beta}^{s^\beta} 1\,\mathrm{d}y\,\mathrm{d}s = \frac{2t^{\beta+1}}{\beta+1}.$$

Suppose the cusp domain $\mathbf{D}_\beta^{\prec}$ is an $\mathscr{L}^s$-quasidisk for $1 \leqslant s < \infty$. Note that according to Theorem 1.7 the domain $\mathbf{D}_\beta^{\prec}$ is always an $\mathscr{L}^1$-quasidisk for every $\beta$. Therefore, there exists a homeomorphism $f\colon \mathbf{C} \xrightarrow{\text{onto}} \mathbf{C}$ of $\mathscr{L}^1$-distortion such that $f(\mathbf{D}_\beta^{\prec}) = \mathbf{D}$. We denote the inverse of $f$ by $h\colon \mathbf{C} \xrightarrow{\text{onto}} \mathbf{C}$. After first extending the homeomorphisms $f$ and $h$ by $f(\infty) = \infty = h(\infty)$ we define a homeomorphism $g\colon \widehat{\mathbf{C}} \xrightarrow{\text{onto}} \widehat{\mathbf{C}}$ by the formula (3.1). The mapping $g$ gives a reflection in the boundary of $\mathbf{D}_\beta^{\prec}$; that is,

- $g(\mathbf{D}_\beta^{\prec}) = \widehat{\mathbf{C}} \setminus \overline{\mathbf{D}_\beta^{\prec}}$,
- $g(\widehat{\mathbf{C}} \setminus \overline{\mathbf{D}_\beta^{\prec}}) = \mathbf{D}_\beta^{\prec}$ and
- $g(x) = x$ for $x \in \partial \mathbf{D}_\beta^{\prec}$.

**Lemma 5.1.** *Let $\epsilon_n = 2^{-n}$ for $n \in \mathbf{N}$. Then there exists a subsequence $\{\epsilon_{n_k}\}$ of $\{\epsilon_n\}$ such that for every $k \in \mathbf{N}$ we have either*

- $|g(\mathbf{U}_{\epsilon_{n_k}})| \leqslant \epsilon_{n_k}^2$ *or*
- $|g(\mathbf{U}_{\epsilon_{n_k}})| \leqslant 5|g(\mathbf{U}_{\epsilon_{n_k+1}})|$ *and* $|g(\mathbf{U}_{\epsilon_{n_k}})| > \epsilon_{n_k}^2$.

*Proof.* Assume to the contrary that the claim is not true, then there exists $n_o \in \mathbf{N}$ such that for every $i \geqslant n_o$, we have $|g(U_{\epsilon_i})| > \epsilon_i^2$ and $|g(U_{\epsilon_i})| > 5|g(U_{\epsilon_{i+1}})|$. Hence we have

$$|g(U_{\epsilon_{n_o}})| > 5|g(U_{\epsilon_{n_o+1}})| > \ldots > 5^n |g(U_{\epsilon_{n_0+n}})| > \ldots$$

which implies that for every $n \in \mathbf{N}$, we have

$$|g(U_{\epsilon_{n_o}})| > \left(\frac{5}{4}\right)^n 4^{-n_o}. \tag{5.1}$$

Letting $n \to \infty$ the term on the right hand side of (5.1) converges to $\infty$ which contradicts with $|g(U_{\epsilon_{n_o}})| < |\mathbf{D}_\beta^{\prec}| < \infty$. □

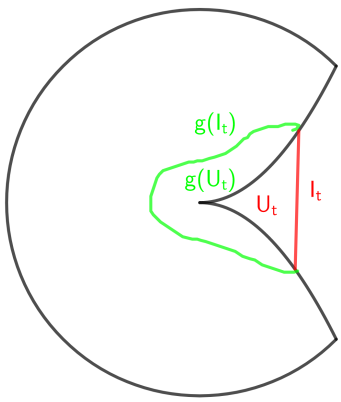


Figure 6.

The key observation to show that $\mathbf{D}_\beta^{\prec}$, $\beta > 1$, is not an $\mathscr{L}^s$-quasidisk for sufficiently large $s > 1$ is to compare the length of curves $g(\mathbf{I}_t)$ and $\mathbf{I}_t$, see Figure 6.

**Lemma 5.2.** *Suppose that $\mathbf{D}_\beta^{\prec}$ is an $\mathscr{L}^s$-quasidisk for $1 < s < \infty$. Then for almost every $0 < t < 1$ we have*

$$\ell(g(\mathbf{I}_t)) \leqslant \left(\int_{\mathbf{I}_t} \frac{|Dg(x)|^s}{|J_g(x)|^{\frac{s-1}{2}}} \,\mathrm{d}x\right)^{\frac{1}{s}} \left(\int_{\mathbf{I}_t} |J_g(x)|^{\frac{1}{2}} \,\mathrm{d}x\right)^{\frac{s-1}{s}}. \tag{5.2}$$

*Proof.* The estimate in (5.2) follows immediately from Hölder's inequality

$$\ell(g(\mathbf{I}_t)) \leqslant \int_{\mathbf{I}_t} |Dg(x)|\,\mathrm{d}x \leqslant \int_{\mathbf{I}_t} \frac{|Dg(x)|}{|J_g(x)|^{\frac{s-1}{2s}}} \cdot |J_g(x)|^{\frac{s-1}{2s}}\,\mathrm{d}x$$
$$\leqslant \left(\int_{\mathbf{I}_t} \frac{|Dg(x)|^s}{|J_g(x)|^{\frac{s-1}{2}}}\,\mathrm{d}x\right)^{\frac{1}{s}} \left(\int_{\mathbf{I}_t} |J_g(x)|^{\frac{1}{2}} dx\right)^{\frac{s-1}{s}}. \qquad \square$$

Now, we are ready to prove our main result Theorem 1.10.

**5.1. The nonexistence part.** In this section we prove the nonexistence part of Theorem 1.10. We restate it as the following proposition.

**Proposition 5.3.** *Let $\mathbf{D}_\beta^\prec$ be a power-type inward cusp domain with $\beta > 1$. Given a pair $(p,q)$ of exponents, $1 \leqslant q \leqslant \infty$ and $1 < p \leqslant \infty$, and the critical power of inward cusps $\beta_{\mathrm{cr}}$ is given by the formula* (1.5). *Then there is no homeomorphism $f\colon \mathbf{C} \to \mathbf{C}$ of finite distortion with $f(\mathbf{D}_\beta^\prec) = \mathbf{D}$ and $K_f \in \mathscr{L}^p(\mathbf{B}_R \setminus \overline{\mathbf{D}_\beta^\prec}) \cap \mathscr{L}^q(\mathbf{D}_\beta^\prec)$ for every $R > 2$.*

*Proof.* Suppose to the contrary that there exists such a homeomorphism. Write

$$s \overset{\mathrm{def}}{=\!=} \min\{p,q\} > 1.$$

We will split our argument into two parts. According to Lemma 5.1 (we denote $\mathscr{J} = \{n_k \in \mathbf{N} \colon k \in \mathbf{N}\}$) there exists a set $\mathscr{J} \subset \mathbf{N}$ and a decreasing sequence $\epsilon_j$ such that $\epsilon_j \to 0$ as $j \to \infty$ and for every $j \in \mathscr{J}$ we have either

(i) $|g(\mathbf{U}_{\epsilon_j})| \leqslant \epsilon_j^2$ or

(ii) $|g(\mathbf{U}_{\epsilon_j})| \leqslant 5|g(\mathbf{U}_{\epsilon_{j+1}})|$, $|g(\mathbf{U}_{\epsilon_j})| > \epsilon_j^2$ and $\epsilon_j = 2\epsilon_{j+1}$.

We simplify the notation a little bit and write $\mathbf{U}_j = \mathbf{U}_{\epsilon_j}$. In both cases we will integrate the inequality (5.2) with respect to the variable $t$ and then bound the right hand side by the following basic estimate.

$$\left(\int_{\mathbf{U}_j} \frac{|Dg(x)|^s}{|J_g(x)|^{\frac{s-1}{2}}}\,\mathrm{d}x\right)^{\frac{2}{s}} \left(\int_{\mathbf{U}_j} |J_g(x)|^{\frac{1}{2}}\,\mathrm{d}x\right)^{\frac{2(s-1)}{s}}$$
$$\leqslant \begin{cases} C_1(\epsilon_j)\,|\mathbf{U}_j|^{\frac{p-1}{p}} \cdot |g(\mathbf{U}_j)|^{\frac{q-1}{q}} & \text{when } q\,, p < \infty, \\ C_2(\epsilon_j)\,|\mathbf{U}_j| \cdot |g(\mathbf{U}_j)|^{\frac{q-1}{q}} & \text{when } p = \infty, \\ C_3(\epsilon_j)\,|\mathbf{U}_j|^{\frac{p-1}{p}} \cdot |g(\mathbf{U}_j)| & \text{when } q = \infty. \end{cases} \tag{5.3}$$

Here the functions $C_1(\epsilon_j)$, $C_2(\epsilon_j)$ and $C_3(\epsilon_j)$ converge to 0 as $j \to \infty$. *Proof of* (5.3). Since $f$ is a mapping of $\mathscr{L}^s$-distortion and $h(0) = f^{-1}(0) \notin \overline{\mathbf{U}_j}$ applying Theorem 3.2 we have

$$\int_{\mathbf{U}_j} \frac{|Dg(x)|^s}{|J_g(x)|^{\frac{s-1}{2}}}\,\mathrm{d}x \leqslant \left(\int_{g(\mathbf{U}_j)} K_f^s(x)\,\mathrm{d}x\right)^{\frac{1}{2}} \cdot \left(\int_{\mathbf{U}_j} K_f^s(x)\,\mathrm{d}x\right)^{\frac{1}{2}}. \tag{5.4}$$

Especially, Theorem 3.2 tells us that $g \in \mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbf{C},\mathbf{C})$. Therefore, Lemma 2.1 and Lemma 2.4 give

$$\int_{\mathbf{U}_j} |J_g(x)|\,\mathrm{d}x \leqslant |g(\mathbf{U}_j)|. \tag{5.5}$$

This together with Hölder's inequality implies

$$\int_{\mathbf{U}_j} |J_g(x)|^{\frac{1}{2}}\,\mathrm{d}x \leqslant |\mathbf{U}_j|^{\frac{1}{2}} |g(\mathbf{U}_j)|^{\frac{1}{2}}. \tag{5.6}$$

Combining (5.4) and (5.6) we conclude that

$$\tag{5.7} \begin{aligned} &\left(\int_{\mathbf{U}_j} \frac{|Dg(x)|^s}{|J_g(x)|^{\frac{s-1}{2}}}\,\mathrm{d}x\right)^{\frac{2}{s}} \left(\int_{\mathbf{U}_j} |J_g(x)|^{\frac12}\,\mathrm{d}x\right)^{\frac{2(s-1)}{s}} \\ &\leqslant \left(\int_{g(\mathbf{U}_j)} K_f^s(x)\,\mathrm{d}x \cdot \int_{\mathbf{U}_j} K_f^s(x)\,\mathrm{d}x\right)^{\frac1s} (|\mathbf{U}_j|\cdot|g(\mathbf{U}_j)|)^{\frac{s-1}{s}}. \end{aligned}$$

Recall that $1 < s = \min\{p, q\} < \infty$. Now the claimed inequality (5.3) follows from the estimate (5.7) after applying Hölder's inequality with

$$\tag{5.8} \begin{aligned} C_1(\epsilon_j) &\stackrel{\text{def}}{=\!=} ||K_f||_{\mathscr{L}^p(\mathbf{U}_j)}||K_f||_{\mathscr{L}^q(g(\mathbf{U}_j))}, \\ C_2(\epsilon_j) &\stackrel{\text{def}}{=\!=} ||K_f||_{\mathscr{L}^\infty(\mathbf{U}_j)}||K_f||_{\mathscr{L}^q(g(\mathbf{U}_j))}, \\ C_3(\epsilon_j) &\stackrel{\text{def}}{=\!=} ||K_f||_{\mathscr{L}^p(\mathbf{U}_j)}||K_f||_{\mathscr{L}^\infty(g(\mathbf{U}_j))}. \end{aligned}$$

□

5.1.1. *Case* (i). Recall that in this case we assume that $|g(\mathbf{U}_j)| \leqslant \epsilon_j^2$. The homeomorphism $f$ is a mapping of $\mathscr{L}^s$-distortion, Lemma 5.2 implies that for almost every $0 < t < 1$ we have

$$\tag{5.9} \ell(g(\mathbf{I}_t)) \leqslant \left(\int_{\mathbf{I}_t} \frac{|Dg(x)|^s}{|J_g(x)|^{\frac{s-1}{2}}}\,\mathrm{d}x\right)^{\frac1s} \left(\int_{\mathbf{I}_t} |J_g(x)|^{\frac12}\,\mathrm{d}x\right)^{\frac{s-1}{s}}.$$

Since the curve $g(\mathbf{I}_t)$ connects the points $(t, t^\beta)$ and $(t, -t^\beta)$ staying in $\mathbf{D}_\beta^{\prec}$, the length of $g(\mathbf{I}_t)$ is at least $2t$. Therefore,

$$\tag{5.10} 2t \leqslant \left(\int_{\mathbf{I}_t} \frac{|Dg(x)|^s}{|J_g(x)|^{\frac{s-1}{2}}}\,\mathrm{d}x\right)^{\frac1s} \left(\int_{\mathbf{I}_t} |J_g(x)|^{\frac12}\,\mathrm{d}x\right)^{\frac{s-1}{s}}.$$

Integrating this estimate from 0 to $\epsilon_j$ with respect to the variable $t$ and applying Hölder's inequality we obtain

$$\tag{5.11} \epsilon_j^2 \leqslant \left(\int_{\mathbf{U}_j} \frac{|Dg(x)|^s}{|J_g(x)|^{\frac{s-1}{2}}}\,\mathrm{d}x\right)^{\frac1s} \left(\int_{\mathbf{U}_j} |J_g(x)|^{\frac12}\,\mathrm{d}x\right)^{\frac{s-1}{s}}.$$

After squaring this and applying the basic estimate (5.3) we conclude that

$$\epsilon_j^4 \leqslant \begin{cases} C_1(\epsilon_j)\,|\mathbf{U}_j|^{\frac{p-1}{p}}\cdot|g(\mathbf{U}_j)|^{\frac{q-1}{q}} & \text{when } q\,,p<\infty, \\ C_2(\epsilon_j)\,|\mathbf{U}_j|\cdot|g(\mathbf{U}_j)|^{\frac{q-1}{q}} & \text{when } p=\infty, \\ C_3(\epsilon_j)\,|\mathbf{U}_j|^{\frac{p-1}{p}}\cdot|g(\mathbf{U}_j)| & \text{when } q=\infty. \end{cases}$$

Now, since $|\mathbf{U}_j| = \frac{2\epsilon_j^{\beta+1}}{\beta+1} \leqslant \epsilon_j^{\beta+1}$ and $|g(\mathbf{U}_j)| \leqslant \epsilon_j^2$ we have

$$1 \leqslant \begin{cases} C_1(\epsilon_j)\,\epsilon_j^{\frac{(\beta-\beta_{\rm cr})(pq-q)}{pq}} & \text{when } q\,,p<\infty, \\ C_2(\epsilon_j)\,\epsilon_j^{\beta-\beta_{\rm cr}} & \text{when } p=\infty, \\ C_3(\epsilon_j)\,\epsilon_j^{\frac{(\beta-\beta_{\rm cr})(p-1)}{p}} & \text{when } q=\infty. \end{cases}$$

Note that $C_1(\epsilon_j)$, $C_2(\epsilon_j)$ and $C_3(\epsilon_j)$ converge to 0 as $j \to \infty$. Therefore, $\beta < \beta_{\rm cr}$, this finishes the proof of Theorem 1.10 in Case (i).

5.1.2. *Case* (ii). As in the previous case applying Lemma 5.2 for almost every $0<t<1$ we have

$$\ell(g(\mathbf{I}_t)) \leqslant \left(\int_{\mathbf{I}_t} \frac{|Dg(x)|^s}{|J_g(x)|^{\frac{s-1}{2}}}\,\mathrm{d}x\right)^{\frac{1}{s}} \left(\int_{\mathbf{I}_t} |J_g(x)|^{\frac12}\,\mathrm{d}x\right)^{\frac{s-1}{s}}. \tag{5.12}$$

Now, we first note that $2\,\ell(g(\mathbf{I}_t)) \geqslant \ell\big(\partial g(\mathbf{U}_t)\big)$ and then apply the isoperimetric inequality, Lemma 2.5 we get

$$|g(\mathbf{U}_t)|^{\frac12} \leqslant \left(\int_{\mathbf{I}_t} \frac{|Dg(x)|^s}{|J_g(x)|^{\frac{s-1}{2}}}\,\mathrm{d}x\right)^{\frac{1}{s}} \left(\int_{\mathbf{I}_t} |J_g(x)|^{\frac12}\,\mathrm{d}x\right)^{\frac{s-1}{s}}. \tag{5.13}$$

Integrating from $\epsilon_{j+1}$ to $\epsilon_j$ with respect to $t$ we obtain

$$(\epsilon_j-\epsilon_{j+1})|g(\mathbf{U}_{j+1})|^{\frac12} \leqslant \left(\int_{\mathbf{U}_j} \frac{|Dg(x)|^s}{|J_g(x)|^{\frac{s-1}{2}}}\,\mathrm{d}x\right)^{\frac{1}{s}} \left(\int_{\mathbf{U}_j} |J_g(x)|^{\frac12}\,\mathrm{d}x\right)^{\frac{s-1}{s}}.$$

Since by the assumptions of Case (ii), $|g(\mathbf{U}_j)| \leqslant 5|g(\mathbf{U}_{j+1})|$ and $\epsilon_j = 2\epsilon_{j+1}$ we have

$$\epsilon_j|g(\mathbf{U}_j)|^{\frac12} \leqslant 10\left(\int_{\mathbf{U}_j} \frac{|Dg(x)|^s}{|J_g(x)|^{\frac{s-1}{2}}}\,\mathrm{d}x\right)^{\frac{1}{s}} \left(\int_{\mathbf{U}_j} |J_g(x)|^{\frac12}\,\mathrm{d}x\right)^{\frac{s-1}{s}}.$$

Combining this with (5.3) we obtain

$$\epsilon_j^2|g(\mathbf{U}_j)| \leqslant 100\cdot\begin{cases} C_1(\epsilon_j)\,|\mathbf{U}_j|^{\frac{p-1}{p}}\cdot|g(\mathbf{U}_j)|^{\frac{q-1}{q}} & \text{when } q\,,p<\infty,\\ C_2(\epsilon_j)\,|\mathbf{U}_j|\cdot|g(\mathbf{U}_j)|^{\frac{q-1}{q}} & \text{when } p=\infty,\\ C_3(\epsilon_j)\,|\mathbf{U}_j|^{\frac{p-1}{p}}\cdot|g(\mathbf{U}_j)| & \text{when } q=\infty.\end{cases}$$

Therefore,

$$\epsilon_j^2 \leqslant 100\cdot\begin{cases} C_1(\epsilon_j)\,|\mathbf{U}_j|^{\frac{p-1}{p}}\cdot|g(\mathbf{U}_j)|^{-\frac{1}{q}} & \text{when } q\,,p<\infty,\\ C_2(\epsilon_j)\,|\mathbf{U}_j|\cdot|g(\mathbf{U}_j)|^{-\frac{1}{q}} & \text{when } p=\infty,\\ C_3(\epsilon_j)\,|\mathbf{U}_j|^{\frac{p-1}{p}} & \text{when } q=\infty.\end{cases}$$

This time $|\mathbf{U}_j| = \frac{2\epsilon_j^{\beta+1}}{\beta+1} \leqslant \epsilon_j^{\beta+1}$ and $|g(\mathbf{U}_j)| > \epsilon_j^2$. Therefore,

$$1 \leqslant 100\cdot\begin{cases} C_1(\epsilon_j)\,\epsilon_j^{\frac{(\beta-\beta_{\mathrm{cr}})(pq-q)}{pq}} & \text{when } q\,,p<\infty,\\ C_2(\epsilon_j)\,\epsilon_j^{\beta-\beta_{\mathrm{cr}}} & \text{when } p=\infty,\\ C_3(\epsilon_j)\,\epsilon_j^{\frac{(\beta-\beta_{\mathrm{cr}})(p-1)}{p}} & \text{when } q=\infty.\end{cases}$$

Therefore $\beta<\beta_{\mathrm{cr}}$. This finishes the proof of nonexistence part of Theorem 1.10. □

**5.2. The existence part.** In this section we prove the existence part of Theorem 1.10. We restate it as the following proposition.

**Proposition 5.4.** *Let $\mathbf{D}_\beta^{\prec}$ be a power-type inward cusp domain with $\beta>1$. Given a pair $(p,q)$ of exponents, $1\leqslant q\leqslant\infty$ and $1<p\leqslant\infty$. The critical power of inward cusps $\beta_{\mathrm{cr}}$ is given by the formula* (1.5). *Then there is a homeomorphism of finite distortion $f\colon \mathbf{C}\to\mathbf{C}$ with $f(\mathbf{D}_\beta^{\prec})=\mathbf{D}$ and $K_f\in\mathscr{L}^p(\mathbf{B}_R\setminus\overline{\mathbf{D}_\beta^{\prec}})\cap\mathscr{L}^q(\mathbf{D}_\beta^{\prec})$ for every $R>2$, whenever $1\leqslant\beta<\beta_{\mathrm{cr}}$.*

*Proof.* Simplifying the construction we will replace the unit disk $\mathbf{D}$ by $\mathbf{D}_1^{\prec}$. This causes no loss of generality because the domains $\mathbf{D}_1^{\prec}$ and $\mathbf{D}$ are bi-Lipschitz equivalent. Especially, $\mathbf{D}_1^{\prec}$ is a quasidisk. Hence we may also assume the strict inequality $1 < \beta < \beta_{\text{cr}}$ in the construction.

In addition to these we will construct a self-homeomorphism of the unit disk onto itself which coincide with identity on the boundary. Therefore, the constructed homeomorphism can be extended as the identity map to the complement of unit disk. In summary, it suffices to construct a homeomorphism $f\colon \mathbf{D} \xrightarrow{\text{onto}} \mathbf{D}$, $f(z) = z$ on $\partial\mathbf{D}$, $f(\mathbf{D}_\beta^{\prec}) = \mathbf{D}_1^{\prec}$ and $K_f \in \mathscr{L}^p(\mathbf{D} \setminus \overline{\mathbf{D}_\beta^{\prec}}) \cap \mathscr{L}^q(\mathbf{D}_\beta^{\prec})$. We will use the polar coordinates $(r,\theta)$ and write $f\colon \mathbf{D} \to \mathbf{D}$ in the form $f(r,\theta) = (\tilde{r}(r), \tilde{\theta}(\theta, r))$. Here $\tilde{r} : [0,1] \xrightarrow{\text{onto}} [0,1]$ is a strictly increasing function defined by

$$\tilde{r}(r) \overset{\text{def}}{=\!=} \begin{cases} \frac{e}{\exp\left(\left(\frac{1}{r}\right)^{\gamma_\beta}\right)} & \text{when } q < \infty, \\ r & \text{when } q = \infty. \end{cases} \tag{5.14}$$

The value $\gamma_\beta$ is chosen so that

$$\begin{cases} \max\left\{\frac{\beta(p-1)-(p+1)}{p}, 0\right\} < \gamma_\beta < \frac{2}{q} & \text{when } p < \infty, \\ \gamma_\beta = \beta - 1 & \text{when } p = \infty. \end{cases} \tag{5.15}$$

For every $0 < r < 1$ we choose $a_r, b_r \in S(0,r) \cap \partial\mathbf{D}_\beta^{\prec}$ such that $\operatorname{Im} a_r > 0$ and $\operatorname{Im} b_r < 0$. Here and in what follows we write $S(0,r) = \partial\mathbf{D}(0,r)$. Respectively, we choose $\tilde{a}_{\tilde{r}(r)}, \tilde{b}_{\tilde{r}(r)} \in S(0,\tilde{r}(r)) \cap \partial\mathbf{D}_1^{\prec}$ such that $\operatorname{Im} \tilde{a}_{\tilde{r}(r)} > 0$ and $\operatorname{Im} \tilde{b}_{\tilde{r}(r)} < 0$. We define the argument function $\tilde{\theta}(r,\theta)$ so that it satisfies the following three properties

(1) $f(a_r) = \tilde{a}_{\tilde{r}(r)}$ and $f(b_r) = \tilde{b}_{\tilde{r}(r)}$.
(2) $f$ maps the circular arc $S(0,r) \cap \mathbf{D}_\beta^{\prec}$ onto the circular arc $S(0,\tilde{r}(r)) \cap \mathbf{D}_1^{\prec}$ linearly as a function of $\theta$.
(3) $f$ maps the circular arc $S(0,r) \cap \left(\mathbf{D} \setminus \overline{\mathbf{D}_\beta^{\prec}}\right)$ onto the circular arc $S(0,\tilde{r}(r)) \cap \left(\mathbf{D} \setminus \overline{\mathbf{D}_1^{\prec}}\right)$ linearly as a function of $\theta$.

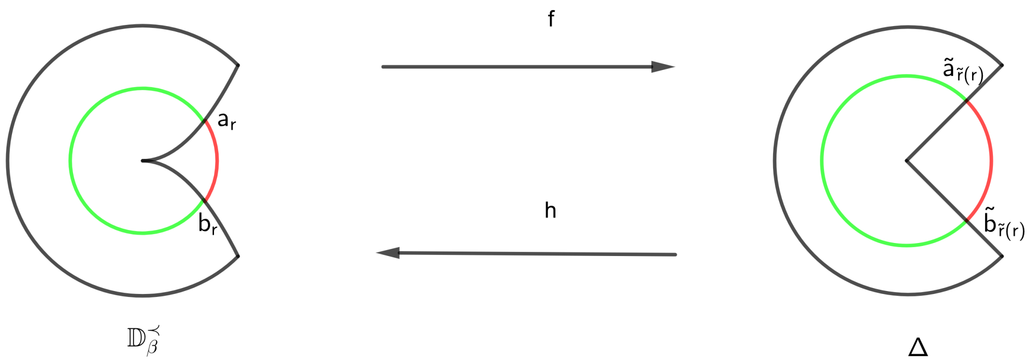


Figure 7.

We have

$$\mathbf{D} \cap \mathbf{D}_\beta^{\prec} = \left\{(r,\theta) \in \mathbf{C} \colon 0 < r < 1 \ \text{ and } \ \arctan t^{\beta-1} < \theta < 2\pi - \arctan t^{\beta-1}\right\}$$

and

$$\mathbf{D} \setminus \overline{\mathbf{D}_\beta^{\prec}} = \left\{(r,\theta) \in \mathbf{C} \colon 0 < r < 1 \ \text{ and } \ -\arctan t^{\beta-1} < \theta < \arctan t^{\beta-1}\right\}.$$

Here $t > 0$ and solves the equation $t^2 + t^{2\beta} = r^2$. We also have

$$\mathbf{D} \cap \mathbf{D}_1^{\prec} = \left\{(\tilde{r},\tilde{\theta}) \in \mathbf{C} \colon 0 < \tilde{r} < 1 \ \text{ and } \ \frac{\pi}{4} < \tilde{\theta} < \frac{7\pi}{4}\right\}$$

and

$$\mathbf{D} \setminus \overline{\mathbf{D}_1^{\prec}} = \left\{(\tilde{r}, \tilde{\theta}) \in \mathbf{C} \colon 0 < \tilde{r} < 1 \text{ and } \frac{-\pi}{4} < \tilde{\theta} < \frac{\pi}{4}\right\}.$$

Using the polar coordinates we have

$$\tilde{\theta}(\theta, r) = \begin{cases} \frac{3\pi\theta}{4\left(\pi - \arctan t^{\beta-1}\right)} + \left(\frac{\pi}{4} - \frac{3\pi \arctan t^{\beta-1}}{4(\pi - \arctan t^{\beta-1})}\right) & \text{when } (r, \theta) \in \mathbf{D}_\beta^{\prec}, \\ \frac{\pi\theta}{4 \arctan t^{\beta-1}} & \text{when } (r, \theta) \in \mathbf{D} \setminus \overline{\mathbf{D}_\beta^{\prec}}. \end{cases}$$

For $(r, \theta) \in \mathbf{D}$, the differential matrix of $f$ reads as

$$Df(r, \theta) = \begin{pmatrix} \frac{\partial}{\partial r}\tilde{r}(r) & 0 \\ \tilde{r}(r)\frac{\partial}{\partial r}\tilde{\theta}(r, \theta) & \frac{\tilde{r}(r)}{r}\frac{\partial}{\partial \theta}\tilde{\theta}(r, \theta) \end{pmatrix}.$$

Computing the derivative of radial part $\tilde{r}(r)$ we have

$$\frac{\partial}{\partial r}\tilde{r}(r) = \begin{cases} \gamma_\beta \left(\frac{1}{r}\right)^{\gamma_\beta + 1} \tilde{r}(r) & \text{when } q < \infty, \\ 1 & \text{when } q = \infty. \end{cases} \tag{5.16}$$

5.2.1. *Proof of* $K_f \in \mathscr{L}^q(\mathbf{D}_\beta^{\prec})$. For $(r, \theta) \in \mathbf{D}_\beta^{\prec}$, we have

$$\tilde{r}(r)\frac{\partial}{\partial r}\tilde{\theta}(r, \theta) = \tilde{r}(r)\frac{\partial}{\partial r}\left[\frac{3\pi\theta}{4\left(\pi - \arctan t^{\beta-1}\right)} + \left(\pi - \frac{3\pi^2}{4\left(\pi - \arctan t^{\beta-1}\right)}\right)\right]$$

and

$$\frac{\tilde{r}(r)}{r}\frac{\partial}{\partial\theta}\tilde{\theta}(r, \theta) = \frac{\tilde{r}(r)}{r}\frac{3\pi}{4\left(\pi - \arctan t^{\beta-1}\right)}.$$

Since $t > 0$ solves the equation $t^2 + t^{2\beta} = r^2$, for $0 < r < 1$, we have $\frac{\partial t}{\partial r} \approx 1$ and $0 < \arctan t^{\beta-1} < \frac{\pi}{4}$. Here and in what follows the notation $A \approx B$ is a shorter form of two inequalities $A \leqslant cB$ and $B \leqslant cA$ for some positive constant $c$. Therefore, there exists a constant $C > 1$ independent of $r$ and $\theta$, such that

$$|\tilde{r}(r)\frac{\partial}{\partial r}\tilde{\theta}(r, \theta)| \leqslant C \cdot \begin{cases} \left(\frac{1}{r}\right)^{\gamma_\beta + 1} \tilde{r}(r) & \text{when } q < \infty \\ 1 & \text{when } q = \infty. \end{cases}$$

and

$$\frac{\tilde{r}(r)}{r}\frac{\partial}{\partial\theta}\tilde{\theta}(r, \theta) \approx \begin{cases} \frac{\tilde{r}(r)}{r} & \text{when } q < \infty \\ 1 & \text{when } q = \infty. \end{cases}$$

Now, we have

$$K_f(r, \theta) \leqslant C \cdot \begin{cases} r^{-\gamma_\beta} & \text{when } q < \infty \\ 1 & \text{when } q = \infty. \end{cases}$$

for some constant $C > 0$.

Since $\gamma_\beta$ is chosen so that $0 < \gamma_\beta < \frac{2}{q}$ for $q < \infty$, we have $K_f \in \mathscr{L}^q(\mathbf{D}_\beta^{\prec})$. Also if $q = \infty$, then the distortion function $K_f \in \mathscr{L}^\infty(\mathbf{D}_\beta^{\prec})$, as claimed.

5.2.2. *Proof of* $K_f \in \mathscr{L}^p(\mathbf{D} \setminus \overline{\mathbf{D}_\beta^{\prec}})$. For $(r, \theta) \in \mathbf{D} \setminus \overline{\mathbf{D}_\beta^{\prec}}$, we have

$$\tilde{r}(r)\frac{\partial}{\partial r}\tilde{\theta}(r, \theta) = \tilde{r}(r)\frac{\partial}{\partial r}\left(\frac{\pi\theta}{4 \arctan t^{\beta-1}}\right)$$

and

$$\frac{\tilde{r}(r)}{r}\frac{\partial}{\partial\theta}\tilde{\theta}(r, \theta) = \frac{\tilde{r}(r)}{r}\frac{\pi}{4 \arctan t^{\beta-1}}.$$

Recall that since $t>0$ solves the equation $t^2+t^{2\beta}=r^2$, for $0<r<1$, we have $\frac{\partial t}{\partial r}\approx 1$. In this case, $-\arctan t^{\beta-1}<\theta<\arctan t^{\beta-1}$, therefore there exists a constant $C>0$ such that

$$\left|\tilde{r}(r)\frac{\partial}{\partial r}\tilde{\theta}(r,\theta)\right|\leqslant C\left(\frac{1}{r}\right)^{\gamma_\beta+1}\tilde{r}(r).$$

Since

$$\lim_{t\to 0^+}\frac{\arctan t^{\beta-1}}{t^{\beta-1}}=1 \quad\text{and}\quad t<r<2t,$$

we have

$$\frac{\pi}{4\arctan t^{\beta-1}}\frac{\tilde{r}(r)}{r}\approx\frac{\tilde{r}(r)}{r^\beta}.$$

Therefore,

$$K_f(r,\theta)\leqslant\frac{C}{r^{|\beta-\gamma_\beta-1|}}\quad\text{when } (r,\theta)\in\mathbf{D}\setminus\overline{\mathbf{D}_\beta^{\prec}}$$

For $p=\infty$, since $\gamma_\beta=\beta-1$, we have $K_f\in\mathscr{L}^\infty(\mathbf{D}\setminus\mathbf{D}_\beta^{\prec})$. For $p<\infty$, $\beta$ is chosen so that $1<\beta<\beta_{\mathrm{cr}}$. When $q<\infty$, $\gamma_\beta$ is chosen so that

$$\max\left\{\frac{\beta(p-1)-(p+1)}{p},0\right\}<\gamma_\beta<\frac{2}{q},$$

and when $q=\infty$, $\gamma_\beta$ is set to be 0. Since $|\gamma_\beta+1-\beta|<\frac{2}{p}$ we have

$$\int_{\mathbf{D}\setminus\overline{\mathbf{D}_\beta^{\prec}}}K_f^p(x)\,\mathrm{d}x\leqslant\int_0^{2\pi}\int_0^1\frac{1}{r^{p|\beta-\gamma_\beta-1|-1}}\,\mathrm{d}r\,\mathrm{d}\theta<\infty.$$

□

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