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ADAMS' TRACE PRINCIPLE IN MORREY–LORENTZ SPACES ON β-HAUSDORFF DIMENSIONAL SURFACES

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Abstract. In this paper we strengthen to Morrey–Lorentz spaces the famous trace principle introduced by Adams. More precisely, we show that Riesz potential I_α is continuous

$$\|I_\alpha f\|_{\mathcal{M}^{\lambda_*}_{q,\infty}(d\mu)} \lesssim \|\mu\|_\beta^{1/q}\, \|f\|_{\mathcal{M}^{\lambda}_{p,\infty}(d\nu)}$$

if and only if the Radon measure $d\mu$ supported in $\Omega \subset \mathbf{R}^n$ is controlled by

$$[\![\mu]\!]_\beta = \sup_{x\in\mathbf{R}^n,\, r>0} r^{-\beta}\mu(B(x,r)) < \infty$$

provided that $1<p<q<\infty$ satisfies $n-\alpha p<\beta\leq n$, $\alpha=\frac{n}{\lambda}-\frac{\beta}{\lambda_*}$ and $\frac{\lambda_*}{q}\leq\frac{\lambda}{p}$. Our result provide a new class of functions spaces which is larger than previous ones, since we have strict continuous inclusions $\dot{B}^s_{p,\infty}\hookrightarrow L^{\lambda,\infty}\hookrightarrow\mathcal{M}^\lambda_p\hookrightarrow\mathcal{M}^\lambda_{p,\infty}$ as $1<p<\lambda<\infty$ and $s\in\mathbf{R}$ satisfies $\frac{1}{p}-\frac{s}{n}=\frac{1}{\lambda}$. If $d\mu$ is concentrated on $\partial\mathbf{R}^n_+$, as a byproduct we get Sobolev–Morrey trace inequality on half-spaces $\mathbf{R}^n_+$ which recovers the well-known Sobolev-trace inequality in $L^p(\mathbf{R}^n_+)$. Also, by a suitable analysis on non-doubling Calderón–Zygmund decomposition we show that

$$\|M_\alpha f\|_{\mathcal{M}^{\lambda}_{p,\ell}(d\mu)} \sim \|I_\alpha f\|_{\mathcal{M}^{\lambda}_{p,\ell}(d\mu)}$$

provided that $\mu(B_r(x))\sim r^\beta$ on support $\mathrm{spt}(\mu)$ and $n-\alpha<\beta\leq n$ with $0<\alpha<n$. This result extends the previous ones.

1. Introduction

It is well known that doubling property of measures μ plays an important role in many topics of research in analysis on euclidean spaces, essentially because Vitali covering lemma and Calderón–Zygmund decomposition depend of the doubling property $\mu(B_{2r}(x))\lesssim\mu(B_r(x))$ for all x on support $\mathrm{spt}(\mu)$ of measure μ and $r>0$. Recently it has been shown that fundamental results in harmonic analysis remain if doubling measures is replaced by a *growth condition*, namely,

$$\mu(B_r(x))\lesssim C\,r^\beta \quad\text{for all}\quad x\in\mathrm{spt}(\mu)\quad\text{and}\quad r>0, \tag{1.1}$$

where the implicit constant is independent of μ and $0<\beta\leq n$. For instance, we refer the pioneering work on Calderón–Zygmund theory for non-doubling measures [32, 33, 34] and [27]. According to Frostman's lemma [25, Chapter 1], a measure μ satisfying (1.1) is close to Hausdorff measure and Riesz capacity of Borel sets $\Omega\subset\mathbf{R}^n$. Essentially Frostman's lemma states that Hausdorff dimension of a Borel set $\Omega\subset\mathbf{R}^n$ is equal to

$$\begin{aligned}\dim_{\Lambda_\beta}\Omega &= \sup\{\beta\in(0,n]\colon \exists\,\mu\in\mathcal{M}(\Omega)\text{ such that (1.1) holds}\}\\ &= \sup\{\beta>0\colon \mathrm{cap}_\beta(\Omega)>0\}\end{aligned}$$

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where $\Lambda_\beta(\Omega)$ denotes the β-dimensional Hausdorff measure and $\mathrm{cap}_\beta(\Omega)$ denotes the Riesz capacity,

$$\mathrm{cap}_\beta(\Omega) = \sup\left\{[\mathrm{E}_\beta(\mu)]^{-1}\colon \mathrm{E}_\beta(\mu) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} |x-y|^{-\beta}\,d\mu(x)\,d\mu(y) \text{ for finite Borel measure } \mu\right\}.$$

The L^p-Riesz capacity on compact sets

$$\dot{c}_{\alpha,p}(E) = \inf\left\{\int_{\mathbf{R}^n} |f(x)|^p\,d\nu(x)\colon f\geq 0 \text{ and } I_\alpha f(x)\geq 1 \text{ on } E\right\},$$

plays an important role in potential analysis, where I_α is defined by

$$I_\alpha f(x) = C_{\alpha,n}\int_{\mathbf{R}^n} |x-y|^{\alpha-n} f(y)\,d\nu(y) \quad \text{a.e. } x\in\mathbf{R}^n \quad \text{as} \quad 0<\alpha<n$$

and $d\nu$ stands for Lebesgue measure in $\mathbf{R}^n$. It is well known from [6, Theorem 7.2.1] and [5, 11, Theorem 1] that a necessary and sufficient condition for Sobolev embedding

$$\dot{L}^\alpha_p(\mathbf{R}^n) \hookrightarrow L^q(\Omega,\mu)$$

on the "lower triangle" $1<p\leq q<\infty$, $0<\alpha<n$ and $p<n/\alpha$ is given by the isocapacitary inequality

$$\mu(E) \lesssim [\dot{c}_{\alpha,p}(E)]^{q/p}, \tag{1.2}$$

whenever E is a compact subset of $\mathbf{R}^n$ and μ is a Radon measure in Ω. Since $\dot{c}_{\alpha,p}(B_r(x)) \cong r^{n-\alpha p}$, then (1.2) implies the *growth condition* (1.1) with $\beta = q(n/p-\alpha)$. Capacity inequality is very difficult to verify even for compact sets, then one could ask: does the embedding $\dot{L}^\alpha_p(\mathbf{R}^n)\hookrightarrow L^q(\Omega,\mu)$ still hold if (1.2) is replaced by (1.1)? In [3, Theorem 2] Adams gave a positive answer to this question as $1<p<q<\infty$ and $\beta = q(n/p-\alpha)$ satisfies $0<\beta\leq n$ and $0<\alpha<n/p$. This theorem has a weak-Morrey version [4, Theorem 5.1] (see also [35, Lemma 2.1]) and a strong Morrey version [22, Theorem 1.1]. Let us be more precise. The Morrey space $\mathcal{M}^\ell_r(\Omega,d\mu)$ is defined by the space of μ-measurable functions $f\in L^r(\Omega\cap B_R)$ such that

$$\|f\|_{\mathcal{M}^\ell_r(\Omega,d\mu)} = \sup_{x\,\in\,\mathrm{spt}(\mu),\,R>0} R^{-\beta\left(\frac{1}{r}-\frac{1}{\ell}\right)}\left(\int_{B_R} |f(y)|^r\,d\mu\lfloor_\Omega\right)^{\frac{1}{r}} < \infty,$$

where the supremum is taken on balls $B_R(x)\subset\mathbf{R}^n$, $1\leq r\leq\ell<\infty$ and $\beta>0$ denotes the Housdorff dimension of Ω. In [22] the space $\mathcal{M}^\ell_r(d\mu)$ is denoted by $L^{r,\kappa}(d\mu)$ with $\kappa/r = n/\ell$ and in [13] is denoted by $\mathcal{M}_{r,\kappa}(d\mu)$ with $(n-\kappa)/r = n/\ell$. The Morrey–Lorentz space $\mathcal{M}^\ell_{r,s}(\Omega,d\mu)$ is defined by space of μ-measurable function $f\in L^{r,s}(\Omega\cap B_R)$ such that

$$\|f\|_{\mathcal{M}^\ell_{r,s}(\Omega,d\mu)} = \sup_{x\,\in\,\mathrm{spt}(\mu),\,R>0} R^{-\beta\left(\frac{1}{r}-\frac{1}{\ell}\right)}\|f\|_{L^{r,s}(\Omega\cap B_R)} < \infty, \tag{1.3}$$

where $L^{r,s}(\Omega\cap B_R)$ denotes the Lorentz space (see Section 2) defined by

$$\|f\|_{L^{r,s}(\mu\lfloor_\Omega(B_R))} = \left(r\int_0^{\mu\lfloor_\Omega(B_R)} [t^r d_f(t)]^{\frac{s}{r}}\,\frac{dt}{t}\right)^{\frac{1}{s}},$$

where $d_f(t) = \mu(\{x \in \Omega \cap B_R : |f(x)| > t\})$ and $\mu\lfloor_\Omega(B_R) = \mu(\Omega \cap B_R)$. According to [4, Theorem 5.1] if the growth condition (1.1) holds and $1 < p < q < \infty$ satisfies

$$\text{(1.4)} \qquad \frac{q}{\lambda_*} \le \frac{p}{\lambda}, \quad 0 < \alpha < \frac{n}{\lambda}, \quad n - \alpha p < \beta \le n \quad \text{and} \quad \frac{\beta}{\lambda_*} = \frac{n}{\lambda} - \alpha,$$

then

$$\text{(1.5)} \qquad I_\alpha \colon \mathcal{M}_p^\lambda(\mathbf{R}^n, d\nu) \to \mathcal{M}_{q,\infty}^{\lambda_*}(\Omega, d\mu)$$

is a bounded operator. Since Morrey space is not closed by *real interpolation*, the weak-trace theorem [4, Theorem 5.1] does not imply the strong trace version

$$\text{(1.6)} \qquad \|I_\alpha f\|_{\mathcal{M}_q^{\lambda_*}(d\mu)} \le C\|f\|_{\mathcal{M}_p^\lambda(d\nu)}.$$

However, by using Lemma 4.1-(i), continuity of fractional maximal function $M_\gamma \colon L^p(\mathbf{R}^n) \to L^{p\beta/(n-\gamma p)}(\Omega, d\mu)$ and atomic decomposition theorem in Hardy–Morrey space $\mathfrak{h}_p^\lambda(d\nu) = \mathcal{HM}_p^\lambda(\mathbf{R}^n, d\nu)$,

$$\|f\|_{\mathfrak{h}_p^\lambda} = \Big\| \sup_{t\in(0,\infty)} |\varphi_t * f| \Big\|_{\mathcal{M}_p^\lambda} < \infty \text{ with } \varphi_t = t^{-n}\varphi(x/t) \text{ for } \varphi \in \mathcal{S}(\mathbf{R}^n) \text{ and } f \in \mathcal{S}'(\mathbf{R}^n),$$

Liu and Xiao [22, Theorem 1.1] showed that $I_\alpha \colon \mathfrak{h}_p^\lambda(d\nu) \to \mathcal{M}_q^{\lambda_*}(d\mu)$ is continuous if and only if the Radon measure μ satisfy $[\![\mu]\!]_\beta < \infty$, provided that $1 \le p < q < \infty$ satisfies (1.4). In particular, Liu and Xiao shown the strong trace inequality (1.6) and since $\mathcal{M}_q^{\lambda_*}(d\mu) \subset \mathcal{M}_{q,\infty}^{\lambda_*}(d\mu)$ they get immediately a version of (1.6) with weak-Morrey norm in the left-hand side. However, according to Sawano et al. [17, Theorem 1.2] there is a function $g \in \mathcal{M}_{p,\infty}^\lambda(\mathbf{R}^n)$ such that $g \notin \mathcal{M}_p^\lambda(\mathbf{R}^n)$ and [22, Theorem 1.1] cannot recover this case. This motivates us to study trace inequality in Morrey–Lorentz spaces. In particular, under previous assumptions (1.4) we show that

$$I_\alpha \colon \mathcal{M}_{p,\infty}^\lambda(\mathbf{R}^n, d\nu) \to \mathcal{M}_{q,\infty}^{\lambda_*}(\Omega, d\mu)$$

is continuous if and only if the Radon measure μ satisfy $[\![\mu]\!]_\beta < \infty$. Then we provide a new class of data for the trace theorem (see Theorem 1.1). The Lorentz space $L^{p,\infty}$ and functions space based on $L^{p,\infty}$ have been successful applied to study existence and uniqueness of *mild solutions* for Navier–Stokes equations. The main effort in these works is to prove a bilinear estimate

$$\text{(1.7)} \qquad \|B(u,v)\|_{L^\infty((0,\infty);X)} \lesssim \|u\|_{L^\infty((0,\infty);X)} \, \|v\|_{L^\infty((0,\infty);X)}$$

without invoke Kato's approach, see [13] for weak-Morrey spaces, see [14] for Besov-weak-Morrey spaces and see [36] for weak-L^p spaces. For stationary Boussinesq equations, see [15] for Besov-weak-Morrey spaces and see [16] for weak-L^p spaces.

Choosing a specific $\mathfrak{h}_p^\lambda$-atom and using discrete Calderón reproducing formula in Hardy–Morrey spaces, from atomic decomposition theorem the authors [23] characterized the continuity of $I_\alpha \colon \mathfrak{h}_p^\lambda(d\nu) \to \mathfrak{h}_q^{\lambda_*}(d\mu)$ by using the growth condition $[\![\mu]\!]_\beta < \infty$, provided that $0 < p < q < 1$ satisfies (1.4). Meanwhile, it should be emphasized that $\mathcal{M}_{p,\infty}^\lambda \neq \mathfrak{h}_p^\lambda$. Indeed, according to the Fourier decaying $|\widehat{f}(\xi)| \lesssim |\xi|^{n(1/\lambda-1)}\|f\|_{\mathfrak{h}_p^\lambda(d\nu)}$ (see [2, Theorem 3.2]) every distribution $f \in \mathfrak{h}_p^\lambda$ satisfy $\int_{\mathbf{R}^n} f(x)\,dx = 0$ when $0 < p \le \lambda < 1$ which implies $|x|^{-n/\lambda} \notin \mathfrak{h}_p^\lambda$, however $|x|^{-n/\lambda} \in \mathcal{M}_{p,\infty}^\lambda$.

If $d\mu$ is a doubling measure and satisfy $[\![\mu]\!]_\beta < \infty$, the authors of [24, Theorem 1.1] showed that I_α is bounded from Besov space $\dot{B}_{p,\infty}^s(\mathbf{R}^n, d\nu)$ to Radon–Campanato space $\mathcal{L}_q^{\lambda_*}(\mu)$ for suitable parameters p, q, λ_* and $0 < s < 1$. Note that we have the continuous inclusions (see [10, pg. 154] and [21, Lemma 1.7])

$$\dot{H}_p^s \hookrightarrow \dot{B}_{p,\infty}^s \hookrightarrow L^{\lambda,\infty} \hookrightarrow \mathcal{M}_p^\lambda \hookrightarrow \mathcal{M}_{p,\infty}^\lambda, \tag{1.8}$$

where $1 < p < \lambda < \infty$ and $s \in \mathbf{R}$ satisfy $\frac{1}{p} - \frac{s}{n} = \frac{1}{\lambda}$. In fact, the inclusions in (1.8) are strict and then $\mathcal{M}_{p,\infty}^\lambda$ is strictly larger than Besov space $\dot{B}_{p,\infty}^s$. So, our Theorem 1.1 extends the previous trace results even when $d\mu$ is a doubling measure.

Theorem 1.1. *Let $1 < p \le \lambda < \infty$ and $1 < q \le \lambda_* < \infty$ be such that $q/\lambda_* \le p/\lambda$ for all $n - \delta p < \beta \le n$ and $1 < p < q < \infty$. Then*

$$\|I_\delta f\|_{\mathcal{M}_{q,s}^{\lambda_*}(d\mu)} \lesssim [\![\mu]\!]_\beta^{1/q} \|f\|_{\mathcal{M}_{p,\ell}^\lambda(d\nu)}$$

if and only if the Radon measure $d\mu$ satisfy $[\![\mu]\!]_\beta < \infty$, provided that $\delta = \frac{n}{\lambda} - \frac{\beta}{\lambda_*}$, $0 < \delta < n/\lambda$ and $1 \le \ell < s \le \infty$ or $s = \ell = \infty$.

A few remarks are in order.

Remark 1.2. (i) (Hardy–Littlewood–Sobolev) Theorem 1.1 implies

$$\|I_\delta f\|_{\mathcal{M}_{q,s}^{\lambda_*}} \lesssim [\![\nu]\!]_n^{1/q} \|f\|_{\mathcal{M}_{p,\ell}^\lambda}$$

for $d\mu = d\nu$ and $\beta = n$. So, our theorem extend Hardy–Littlewood–Sobolev [28, Theorem 9] for weak-Morrey spaces. However the optimality of $q/\lambda_* \le p/\lambda$ is known only for Morrey spaces [28, Theorem 10].

(ii) (Regularity on Morrey spaces) If u is a weak solution to fractional Laplace equation $(-\Delta)^{\frac{\delta}{2}} u = f$ in $\mathbf{R}^n$,

$$(-\Delta)^{\frac{\delta}{2}} u(x) := C(n, \delta)\, \mathrm{P.V.} \int_{\mathbf{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+\delta}}\, d\nu(y) \quad \text{with} \quad 0 < \delta < 2,$$

then $u \in \mathcal{M}_{q,s}^{\lambda_*}(\Omega, d\mu)$ if provided that $f \in \mathcal{M}_{p,\ell}^\lambda(\mathbf{R}^n, d\nu)$. Indeed, $u = I_\delta f$ is a weak solution of $(-\Delta_x)^{\frac{\delta}{2}} u = f$ because

$$\big\langle (-\Delta)^{\delta/2} u, \widehat{\varphi} \big\rangle = \int_{\mathbf{R}^n} \widehat{u}(\xi) |\xi|^\delta \varphi(\xi)\, d\xi = \int_{\mathbf{R}^n} \widehat{f}(\xi) \varphi(\xi)\, d\xi = \langle f, \widehat{\varphi} \rangle$$

for all $\varphi \in \mathcal{S}(\mathbf{R}^n)$. Then, Theorem 1.1 give us desired result.

(iii) (Adams' trace to surface-carried measures) Let Ω be a compact smooth surface with nonnegative second fundamental form and

$$\widehat{d\mu}(\xi) = \int_\Omega e^{-2\pi i x \cdot \xi}\, d\mu$$

the Fourier transform of a measure μ supported on Ω. If Ω has at least k non-vanishing principal curvatures at $\mathrm{spt}(\mu)$, the stationary phase method (see Stein and Shakarchi [31, Chapter 8]) gives the optimal decay

$$|\widehat{d\mu}(\xi)| \lesssim |\xi|^{-\frac{k}{2}} \quad \text{as} \quad |\xi| > 1.$$

Let $\phi \in \mathcal{S}(\mathbf{R}^n)$ be nonnegative, $\phi \gtrsim 1$ on $B(0,1)$ and $\widehat{\phi} = 0$ on $\mathbf{R}^n \backslash B(0,r)$ for some $r > 0$. Choosing $\phi_{x,r}(y) = \phi(\frac{x-y}{r})$ we have

$$\begin{aligned} |\mu(B_r(x))| &\lesssim \left| \int_{\mathbf{R}^n} \phi_{x,r}(y)\, d\mu(y) \right| = \left| \int_{\mathbf{R}^n} \widehat{\phi}_{x,r}(\xi) \widehat{\mu}(-\xi)\, d\xi \right| \\ &\le \int_{|\xi|\le r} |\widehat{\phi}(\xi)| |\widehat{\mu}(-\xi/r)|\, d\xi \lesssim r^{\frac{k}{2}} \int_{|\xi|\le r} |\widehat{\phi}(\xi)|\, |\xi|^{-\frac{k}{2}}\, d\xi \\ &\lesssim r^{k/2} \quad \text{for all } \ x \in \operatorname{spt}(\mu). \end{aligned}$$

It follows from Theorem 1.1 that $\|I_\delta f\|_{\mathcal{M}^{\lambda_*}_{q,s}(\Omega, d\mu)} \lesssim [\![\mu]\!]^{1/q}_{k/2} \|f\|_{\mathcal{M}^{\lambda}_{p,\ell}}$, if provided that $f \in \mathcal{M}^\lambda_{p,\ell}$.

Employing non-doubling Calderon–Zygmund decomposition [33] we obtain the suitable "good-λ inequality" (see (3.5))

$$\sum_j \mu(Q^t_j) \le \mu(\{x\colon (I_\alpha f)^\sharp(x) > 3\epsilon t/4\}) + \epsilon \sum_j \mu(Q^s_j) \ \text{ with } \ s = 4^{-n-2} t$$

provided that μ satisfy (1.1), where $(I_\alpha f)^\sharp$ denotes the (noncentered) sharp maximal function and $\{Q^t_j\}$ is a family of doubling cubes, see Section 3.1. Then, by a suitable analysis we have the norm equivalence (see Theorem 3.5)

$$\|M_\alpha f\|_{\mathcal{M}^\lambda_{p,\ell}(d\mu)} \sim \|I_\alpha f\|_{\mathcal{M}^\lambda_{p,\ell}(d\mu)} \tag{1.9}$$

provided that Radon measure μ satisfies $\mu(B_r(x)) \sim r^\beta$ for all $x \in \operatorname{spt}(\mu)$, where $0 < \alpha < n$ satisfy $n - \alpha < \beta \le n$ and M_α is defined as (centered fractional maximal function)

$$M_\alpha f(x) = \sup_{r>0} r^{\alpha - n} \int_{|y-x|<r} |f(y)|\, d\nu,$$

for all locally integrable function $f \in L^1_{\text{loc}}(\mathbf{R}^n, d\nu)$. It should be emphasized that (1.9) is understood in sense of trace, since M_α and I_α are defined for $f \in L^1_{\text{loc}}(\mathbf{R}^n, d\nu)$ with Lebesgue measure $d\nu$. In particular, when $d\mu$ coincides with Lebesgue measure $d\nu$, this equivalence recovers [7, Theorem 4.2] for Morrey spaces. The proof of (1.9) is involved, because it requires a suitable analysis of non-doubling Calderon–Zygmund decomposition to yield "good-λ inequality" (see Lemma 3.3) as well as the suitable pointwise estimate (see Lemma 3.4)

$$\overline{M}^\sharp I_\alpha f(x) \lesssim M_\alpha f(x)$$

whenever $\mu(B_r(x)) \sim r^\beta$, where $\overline{M}^\sharp$ denotes the (centered) sharp maximal function. Note that (1.9) and Theorem 1.1 yields a trace principle for M_δ if and only if $\mu(B_r(x)) \sim r^\beta$. However, the "if part" of trace principle for M_δ can be obtained directly from pointwise inequality $M_\delta f(x) \lesssim I_\delta |f(x)|$ and Theorem 1.1. The "only if part" is derived from the same technique used in Section 4.2.

Corollary 1.3. (Trace principle for M_δ) *Let $1 < p \le \lambda < \infty$ and $1 < q \le \lambda_* < \infty$ be such that $q/\lambda_* \le p/\lambda$ for all $n - \delta p < \beta \le n$ and $1 < p < q < \infty$. Then,*

$$M_\delta \colon \mathcal{M}^\lambda_{p,\ell}(\mathbf{R}^n, d\nu) \longrightarrow \mathcal{M}^{\lambda_*}_{q,s}(\Omega,\, d\mu) \ \text{ is continuous}$$

if and only if $[\![\mu]\!]_\beta < \infty$, for all $\delta = \frac{n}{\lambda} - \frac{\beta}{\lambda_*}$, $0 < \delta < n/\lambda$ and $1 \le \ell < s \le \infty$.

It is worth noting from an integral representation formula that $\|I_k f\|_{L^q(d\mu)} \lesssim [\![\mu]\!]_\beta \|f\|_{L^p(\mathbf{R}^n)}$ is equivalent to the trace inequality (see [26, Corollary, p. 67])

$$\left(\int_\Omega |f(x)|^q \, d\mu\right)^{\frac{1}{q}} \lesssim [\![\mu]\!]_\beta \|f\|_{W^{k,p}(\mathbf{R}^n)} \tag{1.10}$$

where $\|f\|_{W^{k,p}(\mathbf{R}^n)} = \sum_{|\gamma|\le k} \|D^\gamma f\|_{L^p(\mathbf{R}^n)}$ for all $1 < p < q < \infty$ and $\beta = q(n/p - k) > 0$ with $0 < k < n$. If Ω is a $W^{k,p}$-extension domain, that is, if there is a bounded linear operator $\mathcal{E}_k \colon W^{k,p}(\Omega) \to W^{k,p}(\mathbf{R}^n)$ such that $\mathcal{E}_k f|_\Omega = f$ for all $f \in W^{k,p}(\Omega)$, then (1.10) yields the Sobolev trace inequality

$$\left(\int_\Omega |f(x)|^q \, d\mu\right)^{\frac{1}{q}} \lesssim [\![\mu]\!]_\beta \|f\|_{W^{k,p}(\Omega)}$$

provided that μ is a measure on Ω such that $\sup_{x\in\mathbf{R}^n,\, r>0} r^{-\beta}\mu(\Omega \cap B_r(x)) < \infty$. From [30, Theorem 5, p. 181] it is known that Lipschitz domain is a $W^{k,p}$-extension domain. Moreover, it is also known that (ϵ, δ)-locally uniform domain is a $W^{k,p}$-extension domain for all $1 \le p \le \infty$ and $k \in \mathbf{N}$ (see [19] and [29]). Let us move to Sobolev–Morrey space $\mathcal{W}^{1,p}(\Omega)$ which is defined by

$$\|f\|_{\mathcal{W}^{1,p}(\Omega)} = \sup_{x\in\Omega, r>0} \left(r^{p-n} \int_{B_r(x)\cap\Omega} |\nabla f|^p \, d\nu\right)^{1/p}$$

for all $f \in L^1_{loc}(\Omega)$ and $p \in [1, n]$. Employing a slight modification to the extension operator $\mathcal{E}_k$ of Jones [19], the authors of [20] showed that an ϵ-uniform domain is a $\mathcal{W}^{1,p}$-extension domain. Since $\Omega = \mathbf{R}^n_+$ is a uniform domain, if $d\mu$ is supported on $\partial\mathbf{R}^n_+$ then from Theorem 1.1 (or [22, Theorem 1.1]) in Morrey spaces with $\beta = n - 1$ and integral representation formula [4, (3.5)] we obtain the Sobolev–Morrey trace inequality:

$$\|f(x', 0)\|_{\mathcal{M}_q^{\frac{\lambda(n-1)}{n-\lambda}}(\partial\mathbf{R}^n_+, dx')} \le C \, \|\nabla f\|_{\mathcal{M}_p^\lambda(\mathbf{R}^n_+)} \tag{1.11}$$

provided that $1 < p \le \lambda < n$ and $p < q \le \lambda(n-1)/(n-\lambda)$. However we cannot apply directly [20, Theorem 1.5(i)] to yield (1.11), since $\|f\|_{\mathcal{W}^{1,p}(\mathbf{R}^n_+)} = \|\nabla f\|_{\mathcal{M}_p^n(\mathbf{R}^n_+)}$ and $\lambda = n$. One could ask: does the Sobolev trace embedding (1.11) holds for Morrey spaces or weak-Morrey spaces? As a byproduct of Theorem 1.1 and Calderón-Stein's extension on half-spaces (see Lemma 5.1) we give a positive answer for this question.

Corollary 1.4. (Sobolev–Morrey trace) *Let* $1 < p \le \lambda < n$ *and* $1 < q \le \lambda_* < \infty$ *be such that* $\frac{n-1}{\lambda_*} = \frac{n}{\lambda} - 1$ *and* $q/\lambda_* \le p/\lambda$. *Then*

$$\|f(x', 0)\|_{\mathcal{M}_{q,s}^{\lambda_*}(\partial\mathbf{R}^n_+, dx')} \le C \, \|\nabla f\|_{\mathcal{M}_{p,d}^\lambda(\mathbf{R}^n_+)},$$

for all $1 < p < q < \infty$ *and* $1 \le d < s \le \infty$ *or* $s = \ell = \infty$.

The paper is organized as follows. In Section 2 we summarize properties of Lorentz spaces. In Section 3 we deal with non-doubling CZ-decomposition for polynomial growth measures and estimates for sharp maximal function. In Sections 4 and 5 we prove our main theorems.

2. The Lorentz spaces

Let $(\Omega, \mathcal{B}, \mu)$ be a measure space endowed by Borel regular measure $d\mu$. The Lorentz space $L^{p,d}(\Omega, \mu)$ is defined as the set of μ-measurable functions $f \colon \Omega \to \mathbf{R}$

such that

$$\|f\|^*_{L^{p,d}} = \left(\frac{d}{p}\int_0^{\mu(\Omega)} \left[t^{1/p} f^*(t)\right]^d \frac{dt}{t}\right)^{\frac{1}{d}} = \left(p\int_0^{\mu(\Omega)} [s^p d_f(s)]^{\frac{d}{p}} \frac{ds}{s}\right)^{\frac{1}{d}} < \infty \tag{2.1}$$

for all $1 \le p < \infty$ and $1 \le d < \infty$, where

$$f^*(t) = \inf\{s > 0\colon d_f(s) \le t\} \text{ and } d_f(s) = \mu(\{x \in \Omega : |f(x)| > s\}).$$

For $1 \le p \le \infty$ and $d = \infty$, the Lorentz space $L^{p,\infty}(\Omega, \mu)$ is defined by

$$\|f\|^*_{L^{p,\infty}} = \sup_{0<t<\mu(\Omega)} t^{1/p} f^*(t) = \sup_{0<s<\mu(\Omega)} [s^p d_f(s)]^{1/p}. \tag{2.2}$$

The Lorentz space $L^{p,d}(\Omega, d\mu)$ increases with the index d, that is,

$$L^{p,1} \hookrightarrow L^{p,d_1} \hookrightarrow L^p \hookrightarrow L^{p,d_2} \hookrightarrow L^{p,\infty}$$

provided that $1 \le d_1 \le p \le d_2 < \infty$. More precisely, we have the following lemma.

Lemma 2.1. (Calderón) *If $1 \le p < \infty$ and $0 < q < r \le \infty$, then $\|f\|_{L^{p,r}} \le (q/p)^{\frac{1}{q}-\frac{1}{r}}\|f\|_{L^{p,q}}$.*

The quantities (2.1) and (2.2) are not a norm, however

$$\|f\|^\natural_{L^{p,d}} = \left(\frac{d}{p}\int_0^{\mu(\Omega)} [t^{1/p} f^\natural(t)]^d \frac{dt}{t}\right)^{\frac{1}{d}} < \infty \text{ with } f^\natural(t) = \frac{1}{t}\int_0^t f^*(s)\,ds$$

defines a norm in $L^{p,d}(\Omega, d\mu)$ and one has

$$\|f\|^*_{L^{p,d}} \le \|f\|^\natural_{L^{p,d}} \le \frac{p}{p-1}\|f\|^*_{L^{p,d}}$$

for all $1 < p < \infty$ and $1 \le d \le \infty$. The following lemma is well-known in theory of Lorentz spaces.

Lemma 2.2. (Hunt's Theorem [9]) *Let (M_1, μ_1) and (M_2, μ_2) be measure spaces and let T be a sublinear operator such that*

$$\|Tf\|_{L^{q_i,s_i}(M_1,d\mu_1)} \le C_i\|f\|_{L^{p_i,r_i}(M_2,d\mu_2)} \quad \text{for} \quad i = 0, 1$$

for all $p_0 \ne p_1$ and $q_0 \ne q_1$. Let $0 < \theta < 1$ be such that $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$, then

$$\|Tf\|_{L^{q,s}(M_1,d\mu_1)} \le C_0^\theta C_1^{1-\theta}\|f\|_{L^{p,r}(M_2,d\mu_2)},$$

provided that $p \le q$ and $0 < r \le s \le \infty$, where $C_i > 0$ depends only on p_i, q_i, p, q.

3. Maximal functions and non-doubling measure

In this section we are interested in proving the estimate

$$\|I_\alpha f\|_{\mathcal{M}^\lambda_{p,\ell}(d\mu)} \lesssim \|M^\sharp I_\alpha f\|_{\mathcal{M}^\lambda_{p,\ell}(d\mu)}$$

for every Radon measure μ satisfying (1.1), where $M^\sharp f(x) := f^\sharp(x)$ denotes the uncentered sharp maximal function

$$f^\sharp(x) = \sup_{Q,\, x\in Q}\left\{\frac{1}{\mu(Q)}\int_Q |f - f_Q|\,d\mu\right\} \tag{3.1}$$

and $f_Q = \frac{1}{\mu(Q)}\int_Q f\,d\mu$. Also, we are interested in proving the estimate

$$\overline{M}^\sharp I_\alpha f(x) \lesssim M_\alpha f(x)$$

when the Radon measure μ satisfy $\mu(B_r(x)) \sim r^\beta$, where $\overline{M}^\sharp f(x) := f^{\overline{\sharp}}(x)$ denotes the centered sharp maximal function

$$f^{\overline{\sharp}}(x) = \sup_{r>0} \left\{ \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f - f_{B_r}| \, d\mu \right\}. \tag{3.2}$$

3.1. Non-doubling CZ-decomposition. Let us recall that a cube $Q \subset \mathbf{R}^n$ is called a (τ, γ)-doubling cube with respect to polynomial growth (1.1) of the measure μ, if $\mu(\tau Q) \leq \gamma \, \mu(Q)$ as $\tau > 1$ and $\gamma > \tau^\beta$. According to [33, Remark 2.1 and Remark 2.2] there are small/big (τ, γ)-doubling cubes in $\mathbf{R}^n$.

Lemma 3.1. [33] *Let μ be a Radon measure in $\mathbf{R}^n$ with growth condition* (1.1), *then*

(i) (Small doubling cubes) *Assume $\gamma > \tau^n$, then for μ-a.e. $x \in \mathbf{R}^n$ there exists a sequence $\{Q_j\}_j$ of (τ, γ)-doubling cubes centered at x such that $\ell(Q_j) \to 0$ as $j \to \infty$.*

(ii) (Big doubling cubes) *Assume $\gamma > \tau^\beta$, then for any $x \in spt(\mu)$ and $c > 0$, there exists a (τ, γ)-doubling cube Q centered at x such that $\ell(Q) > c$.*

Let $f \in L^1_{\text{loc}}(\mu)$ and $\lambda > \frac{1}{\mu(Q_0)} \|f\|_{L^1(Q_0)}$ be such that $\Omega_\lambda = \{x \in Q_0 \colon |f(x)| > \lambda\} \neq \varnothing$. From Lemma 3.1-(i) and Lebesgue differentiation theorem, there is a sequence of $(2, 2^{n+1})$-doubling cubes $\{Q_j(x)\}_j$ with $\ell(Q_j) \to 0$ such that

$$\frac{1}{\mu(Q_j)} \int_{Q_j} |f| \, d\mu > \lambda$$

for j sufficiently large. Since there are big $(2, 2^{n+1})$-doubling cubes Q_j, then

$$\frac{1}{\mu(Q_j)} \int_{Q_j} |f| \, d\mu \leq \frac{\|f\|_{L^1(\mu)}}{\mu(Q_j)} \leq \lambda \quad \text{for } \mu(Q_j) > c$$

sufficiently large. In other words, for μ-almost all $x \in \mathbf{R}^n$ such that $|f(x)| > \lambda$ there is a $(2, 2^{n+1})$-doubling cube $Q' \in \{Q_x\}_{x \in \Omega_\lambda}$ with center $x = x_Q$ such that

$$\frac{1}{\mu(2Q')} \int_{Q'} |f| \, d\mu \leq \lambda / 2^{n+1}.$$

Moreover, if $Q = Q(x)$ is a $(2, 2^{n+1})$-doubling cube with sidelength $\ell(Q) < \ell(Q')/2$ then

$$\frac{1}{\mu(Q)} \int_Q |f| \, d\mu > \lambda.$$

Hence, a non-doubling Calderón–Zygmund decomposition can be obtained. To simply, a doubling cube Q mean $(2, 2^{n+1})$-doubling cube.

Lemma 3.2. (Non-doubling CZ-decomposition [33]) *Let the Radon measure μ satisfy* (1.1). *Let Q be a doubling cube big so that $\lambda > \frac{1}{\mu(Q)} \int_Q |f| d\mu$ for $f \in L^1(\mu)(Q)$. Then, there is a sequence of doubling cubes $\{Q_j\}_j$ such that*

(i) $|f(x)| \leq \lambda$ *for* $x \in Q \setminus \bigcup_j Q_j$, μ-a.e.,

(ii) $\lambda < \frac{1}{\mu(Q_j)} \int_{Q_j} |f| \, d\mu \leq 4^{n+1} \lambda$,

(iii) $\bigcup_j Q_j = \bigcup_{k=1}^{\varepsilon_n} \bigcup_{Q_j^k \in \mathcal{F}_k} Q_j^k$,

where the family $\mathcal{F}_k = \{Q_j^k\}$ is pairwise disjoint.

Proof. This lemma is a consequence of Besicovitch's covering theorem and has been proved by Tolsa [33, Lemma 2.4]. Note that [33, Lemma 2.4] with $\eta = 4$ implies

$$\frac{1}{\mu(Q_j)}\int_{Q_j}|f|\,d\mu \le \frac{\mu(\eta Q_j)}{\mu(Q_j)}\left(\frac{1}{\mu(\eta Q_j)}\int_{\eta Q_j}|f|\,d\mu\right) \le \frac{\mu(\eta Q_j)}{\mu(Q_j)}\left(\frac{2^{n+1}}{\mu(2\eta Q_j)}\int_{\eta Q_j}|f|\,d\mu\right)$$
$$\le 4^{n+1}\lambda,$$

thanks to $\mu(2\eta Q_j) \le 2^{n+1}\mu(\eta Q_j)$ and $\mu(4Q_j) \le 4^{n+1}\mu(Q_j)$. □

3.2. Estimates for sharp maximal function. Inspired in [12, p. 153] we prove the following lemma.

Lemma 3.3. *Let μ be a Radon measure in $\mathbf{R}^n$ such that $[\![\mu]\!]_\beta < \infty$ for $0 < \beta \le n$. If $I_\alpha f \in L^1_{\mathrm{loc}}(d\mu)$ for $0 < \alpha < n$, then*

$$\|I_\alpha f\|_{\mathcal{M}^\lambda_{p,\ell}(d\mu)} \lesssim \|(I_\alpha f)^\sharp\|_{\mathcal{M}^\lambda_{p,\ell}(d\mu)}, \tag{3.3}$$

for every $1 \le p \le \lambda < \infty$ and $1 \le \ell \le \infty$.

Proof. Let $Q_0 \subseteq \mathbf{R}^n$ be a doubling cube. Applying Lemma 3.2 with $I_\alpha f \in L^1_{\mathrm{loc}}(\mu)(Q_0)$ and $t = \lambda$ we obtain a family of almost disjoint doubling cubes $\{Q^t_j\}$ so that

$$t < \frac{1}{\mu(Q^t_j)}\int_{Q^t_j}|I_\alpha f|\,d\mu \le 4^{n+1}t \tag{3.4}$$

and $I_\alpha f(x) \le t$ as $x \notin \bigcup_j Q^t_j$ μ-a.e. The main inequality to be proved reads as follows

$$\sum_j \mu(Q^t_j) \le \mu(\{x\colon (I_\alpha f)^\sharp(x) > 3\epsilon t/4\}) + \epsilon\sum_j \mu(Q^s_j) \quad \text{with} \quad s = 4^{-n-2}t \tag{3.5}$$

for all $\epsilon > 0$. Indeed, let $s = 4^{-n-2}t$ and $\mathcal{F}_1$ be the family of doubling cubes $\{Q^s_j\}$ of the CZ-decomposition associated to s and satisfying

$$Q^s_j \subset \left\{x \in Q_0\colon (I_\alpha f)^\sharp(x) > \frac{3\epsilon t}{4}\right\} \tag{3.6}$$

and let $\mathcal{F}_2$ be the family of doubling cubes such that $Q^s_j \not\subseteq \{x \in Q_0\colon (I_\alpha f)^\sharp(x) > 3\epsilon t/4\}$. If $Q \in \mathcal{F}_2$, obviously one has $(I_\alpha f)^\sharp(x) \le \frac{3\epsilon t}{4}$ for $x \in Q$ and right-hand side of (3.4) implies $(I_\alpha f)_Q = \frac{1}{\mu(Q)}\int_Q |I_\alpha f|\,d\mu \le 4^{n+1}s = t/4$. Now from left-hand side of (3.4) one has

$$\begin{aligned}
\sum_{Q^t_j\subset Q} t\mu(Q^t_j) &< \sum_{Q^t_j\subset Q}\int_{Q^t_j}|I_\alpha f(x)|\,d\mu \\
&\le \sum_{Q^t_j\subset Q}\int_{Q^t_j}|I_\alpha f(x) - (I_\alpha f)_Q|\,d\mu + (I_\alpha f)_Q \sum_{Q^t_j\subset Q}\mu(Q^t_j) \\
&\le \int_Q |I_\alpha f(x) - (I_\alpha f)_Q|\,d\mu + (I_\alpha f)_Q \sum_{Q^t_j\subset Q}\mu(Q^t_j) \\
&\le \frac{3\epsilon}{4}t\,\mu(Q) + \frac{t}{4}\sum_{Q^t_j\subset Q}\mu(Q^t_j).
\end{aligned}$$

Hence, summing over all cubes $Q \in \mathcal{F}_2$, we have

$$\sum_{Q\in\mathcal{F}_2}\sum_{Q_j^t\subset Q}\mu(Q_j^t) \le \epsilon \sum_{Q\in\mathcal{F}_2}\mu(Q). \tag{3.7}$$

If $Q \in \mathcal{F}_1$, trivially (3.6) gives us

$$\begin{aligned}\sum_{Q\in\mathcal{F}_1}\sum_{Q_j^t\subset Q}\mu(Q_j^t) &\le \sum_{Q\in\mathcal{F}_1}\mu\left(\left\{x\in Q_0\colon (I_\alpha f)^\sharp(x) > 3\epsilon t/4\right\}\cap Q\right)\\ &\le \mu\left(\left\{x\in Q_0\colon (I_\alpha f)^\sharp(x) > 3\epsilon t/4\right\}\right). \end{aligned}\tag{3.8}$$

Since

$$\sum_j \mu(Q_j^t) = \left(\sum_{Q\in\mathcal{F}_1} + \sum_{Q\in\mathcal{F}_2}\right)\sum_{Q_j^t\subset Q}\mu(Q_j^t),$$

from estimates (3.7) and (3.8) we obtain the good-λ inequality (3.5).

Now let $d_{I_\alpha f}(t) = \mu(\{x\in Q_0\colon I_\alpha f(x) > t\})$ be the distribution function of $I_\alpha f$, then by CZ-decomposition we have

$$d_{I_\alpha f}(t) \le \rho(t) := \sum_j \mu(Q_j^t)$$

thanks to Lemma 3.2-(i). Now fix $N = \mu\lfloor_\Omega(B_R)$ and invoke (3.5) in order to infer

$$\begin{aligned} &p\int_0^N t^{\ell-1}\left[\rho(t)\right]^{\frac{\ell}{p}}\,dt \lesssim p\int_0^N t^{\ell-1}\left[d_{(I_\alpha f)^\sharp}(3\epsilon t/4)\right]^{\frac{\ell}{p}}\,dt + p\int_0^N t^{\ell-1}[\epsilon\rho(4^{-n-2}t)]^{\frac{\ell}{p}}\,dt\\ &= (4/3\epsilon)^\ell p\int_0^{3\epsilon N/4} t^{\ell-1}\left[d_{(I_\alpha f)^\sharp}(t)\right]^{\frac{\ell}{p}}\,dt + 4^{(n+2)\ell}\epsilon^{\frac{\ell}{p}}p\int_0^{N4^{-n-2}} t^{\ell-1}[\rho(t)]^{\frac{\ell}{p}}\,dt. \end{aligned}$$

Now choosing $\epsilon > 0$ in such a way that $\epsilon^{\frac{\ell}{p}}4^{(n+2)\ell} = 1/2$ we obtain

$$\frac{p}{2}\int_0^N t^{\ell-1}\left[\rho(t)\right]^{\frac{\ell}{p}}\,dt \lesssim p\int_0^N t^{\ell-1}\left[d_{(I_\alpha f)^\sharp}(t)\right]^{\frac{\ell}{p}}\,dt.$$

Since $d_{I_\alpha f}(t) \le \rho(t)$ we estimate

$$\begin{aligned}\|I_\alpha f\|^\ell_{\mathcal{M}^\lambda_{p,\ell}(d\mu)} &= \sup_{x\in\operatorname{spt}(\mu),R>0} R^{-\ell\beta\left(\frac{1}{p}-\frac{1}{\lambda}\right)}\left(p\int_0^{\mu\lfloor_\Omega(B_R)} t^{\ell-1}\left[d_{I_\alpha f}(t)\right]^{\frac{\ell}{p}}\,dt\right)\\ &\lesssim \sup_{x\in\operatorname{spt}(\mu),R>0} R^{-\ell\beta\left(\frac{1}{p}-\frac{1}{\lambda}\right)}\left(p\int_0^{\mu\lfloor_\Omega(B_R)} t^{\ell-1}\left[d_{(I_\alpha f)^\sharp}(t)\right]^{\frac{\ell}{p}}\,dt\right)\\ &= \left\|(I_\alpha f)^\sharp\right\|^\ell_{\mathcal{M}^\lambda_{p,\ell}(d\mu)}, \end{aligned}$$

as required. The case $\ell = \infty$ is achieved without great effort. □

Lemma 3.4. *Let μ be a Radon measure such that $\mu(B_r(x)) \sim r^\beta$ for all $x \in \mathbf{R}^n$ and $r > 0$. If $f \in L^1_{\mathrm{loc}}(d\nu)$ is such that $I_\alpha f \in L^1_{\mathrm{loc}}(d\mu)$ when $0 < \alpha < n$ satisfy $n - \alpha < \beta \le n$, then*

$$\overline{M}^\sharp I_\alpha f(x) \lesssim M_\alpha f(x).$$

Proof. Taking $f' = f\chi_{B(x_0,2r)}$ and $f'' = \chi_{\mathbf{R}^n\setminus B(x_0,2r)}$ from Fubini's theorem and [6, Lemma 3.1.1] we estimate

$$\begin{aligned}
&\int_{|x-x_0|<r} |I_\alpha f'(x)|\, d\mu(x) \lesssim \int_{|x-x_0|<r} \left(\int_{|y-x_0|<2r} |y-x|^{\alpha-n}|f(y)|\, d\nu\right) d\mu(x) \\
&\le \int_{|y-x_0|<2r} \left(\int_{|y-x|<3r} |y-x|^{\alpha-n} d\mu(x)\right) |f(y)|\, d\nu \\
&\le \int_{|y-x_0|<2r} \left[(n-\alpha)\int_0^{3r} \frac{\mu(B(x,s))}{s^{n-\alpha}}\frac{ds}{s} + \frac{\mu(B(x,3r))}{(3r)^{n-\alpha}}\right] |f(y)|\, d\nu \\
&\lesssim [\![\mu]\!]_\beta r^\beta [2r]^{\alpha-n} \int_{|y-x_0|<2r} |f(y)|\, d\nu \lesssim [\![\mu]\!]_\beta\, \mu(B_r(x_0))\, M_\alpha f(x_0),
\end{aligned}$$

which yields $\overline{M}^\sharp I_\alpha f'(x_0) \lesssim [\![\mu]\!]_\beta M_\alpha f(x_0)$. Now from mean value theorem we have

$$\left||x-z|^{\alpha-n} - |y-z|^{\alpha-n}\right| \lesssim r\,|z-x_0|^{\alpha-n-1},$$

for $|x-x_0|<r$ and $|y-x_0|<r$. Hence, Fubini's theorem implies

$$\begin{aligned}
&\left|(I_\alpha f'')(x) - (I_\alpha f'')_{B_r(x_0)}\right| \\
&\le \frac{1}{\mu(B_r)} \int_{|z-x_0|>2r} \left\{ r\int_{|y-x_0|<r} |z-x_0|^{\alpha-n-1} d\mu(y)\right\} |f(z)|\, d\nu \\
&\lesssim r\int_{|z-x_0|>2r} |z-x_0|^{\alpha-n-1}|f(z)|d\nu \\
&= r\sum_{k=1}^{\infty} \int_{2^k r\le|z-x_0|<2^{k+1}r} |z-x_0|^{\alpha-n-1}|f(z)|\, d\nu \\
&\le \sum_{k=1}^{\infty} 2^{-(k+1)} M_\alpha f(x_0) \lesssim M_\alpha f(x_0),
\end{aligned}$$

which yields

$$\overline{M}^\sharp I_\alpha f''(x_0) = \sup_{r>0} \frac{1}{\mu(B_r(x_0))} \int_{|x-x_0|<r} \left|(I_\alpha f'')(x) - (I_\alpha f'')_{B(x_0,r)}\right| d\mu(x) \lesssim M_\alpha f(x_0),$$

as required. □

Theorem 3.5. (Trace-type equivalence) *Let μ be a Radon measure such that $\mu(B_r(x)) \sim r^\beta$ for all $x \in \mathbf{R}^n$ and $r>0$. If $f \in L^1_{loc}(d\nu)$ is such that $I_\alpha f \in L^1_{loc}(d\mu)$ whenever $0<\alpha<n$ satisfy $n-\alpha<\beta\le n$, then*

$$\|M_\alpha f\|_{\mathcal{M}^\lambda_{p,\ell}(d\mu)} \sim \|I_\alpha f\|_{\mathcal{M}^\lambda_{p,\ell}(d\mu)},$$

for all $1\le p\le\lambda<\infty$ and $1\le\ell\le\infty$.

Proof. Note that Lemma 3.3 is true with centered sharp maximal function $\overline{M}^\sharp I_\alpha f$. Since $M_\alpha f(x) \lesssim I_\alpha f(x)$, by Lemma 3.3 and 3.4 we obtain

$$\|M_\alpha f\|_{\mathcal{M}^\lambda_{p,\ell}} \lesssim \|I_\alpha f\|_{\mathcal{M}^\lambda_{p,\ell}} \overset{\text{Lemma 3.3}}{\lesssim} \left\|\overline{M}^\sharp I_\alpha f\right\|_{\mathcal{M}^\lambda_{p,\ell}} \overset{\text{Lemma 3.4}}{\lesssim} \|M_\alpha f\|_{\mathcal{M}^\lambda_{p,\ell}}, \tag{3.9}$$

which is the desired result. □

4. Proof of trace Theorem 1.1

Let us recall the pointwise estimate between Riesz potential and fractional maximal operator.

Lemma 4.1. *Let $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n, d\nu)$ and $B(x,r) \subset \mathbf{R}^n$ a ball with radius $r > 0$.*

(i) *If $0 \le \gamma < \delta < \alpha \le n$, then*

$$|I_\delta f(x)| \lesssim [M_\alpha f(x)]^{\frac{\delta-\gamma}{\alpha-\gamma}} [M_\gamma f(x)]^{1-\frac{\delta-\gamma}{\alpha-\gamma}}, \quad \forall x \in \mathbf{R}^n.$$

(ii) *If $1 \le p < \infty$ and $1 \le k \le \infty$, then*

$$[\nu(B(x,r))]^{\frac{1}{p}-1} \int_{B(x,r)} |f(y)|\, d\nu \lesssim \|f\|_{L^{p,k}(B(x,r))}.$$

In particular, $(M_{n/\lambda} f)(x) \lesssim \|f\|_{\mathcal{M}^\lambda_{p,k}(d\nu)}$, for all $p \le \lambda < \infty$.

Proof. The item (i) was obtained in [22, Lemma 4.1]. To show (ii), first let us recall the Hardy–Littlewood inequality

$$\int_{B(x,r)} |f(y)g(y)|\, d\nu \le \int_0^{\nu(B(x,r))} f^*(t) g^*(t)\, dt.$$

This inequality and Hölder's inequality in $L^k(\mathbf{R}, dt/t)$ give us

$$\begin{aligned}
\int_{B(x,r)} |f(x)|\, d\nu &\le \int_0^{\nu(B(x,r))} t^{1-\frac{1}{p}} \left(t^{\frac{1}{p}} f^*(t)\right) \frac{dt}{t} \\
&\le \left(\int_0^{\nu(B(x,r))} \left(t^{1-\frac{1}{p}}\right)^{k'} \frac{dt}{t} \right)^{\frac{1}{k'}} \left(\int_0^{\nu(B(x,r))} \left(t^{\frac{1}{p}} f^*(t)\right)^{k} \frac{dt}{t} \right)^{\frac{1}{k}} \\
&\lesssim [\nu(B(x,r))]^{1-\frac{1}{p}} \|f\|_{L^{p,k}(B(x,r))},
\end{aligned}$$

as desired. □

Now, we are in position to prove Theorem 1.1.

4.1. The condition $[\![\boldsymbol{\mu}]\!]_{\boldsymbol{\beta}} < \infty$ is sufficient. For $x \in B_\rho = B(x_0, \rho)$ with $\rho > 0$, let us write

$$\begin{aligned}
I_\delta f(x) &= \int_{|y-x|<\rho} |x-y|^{\delta-n} f(y)\, d\nu(y) + \int_{|y-x|\ge\rho} |x-y|^{\delta-n} f(y)\, d\nu(y) \\
&:= I_\delta f'(x) + I_\delta f''(x),
\end{aligned}$$

where $f' = \chi_{B(x_0,2\rho)} f$ and $f'' = f - f'$. If $y \in \mathbf{R}^n \backslash B(x_0, 2\rho)$, using integration by parts and Lemma 4.1-(ii), respectively, we have

$$\begin{aligned}
|I_\delta f''(x)| &\le \int_{2\rho}^\infty s^{\delta-n} \left(\int_{B(x,s)} |f(y)|\, d\nu \right) \frac{ds}{s} \\
&\lesssim \int_\rho^\infty s^{\delta-n} [\nu(B(x,s))]^{1-\frac{1}{p}} \|f\|_{L^{p,\ell}(B(x,s))} \frac{ds}{s} \\
&\le \left(\int_\rho^\infty s^{\delta-1-\frac{n}{\lambda}} ds \right) \|f\|_{\mathcal{M}^\lambda_{p,\ell}(d\nu)} \lesssim \rho^{\delta-\frac{n}{\lambda}} \|f\|_{\mathcal{M}^\lambda_{p,\ell}(d\nu)},
\end{aligned}$$

in view of $0<\delta<n/\lambda$. Therefore, $(I_\delta f'')^*(t)\lesssim \rho^{\delta-\frac{n}{\lambda}}\|f\|_{\mathcal{M}^\lambda_{p,\ell}(d\nu)}$ and we can estimate

$$\begin{aligned}\|I_\delta f''\|_{L^{q,s}(B(x_0,\rho),d\mu)} &\lesssim \rho^{\delta-\frac{n}{\lambda}}\|f\|_{\mathcal{M}^\lambda_{p,\ell}(\mathbf{R}^n,d\nu)}\left(\int_0^{\mu(B(x_0,\rho))} t^{\frac{s}{q}-1}\,dt\right)^{\frac{1}{s}}\\ &\le \rho^{\delta-\frac{n}{\lambda}}\mu(B(x_0,\rho))^{\frac{1}{q}}\|f\|_{\mathcal{M}^\lambda_{p,\ell}(d\nu)}\\ &\le \rho^{\frac{\beta}{q}-\frac{\beta}{\lambda_*}}\,[\![\mu]\!]_\beta^{\frac{1}{q}}\,\|f\|_{\mathcal{M}^\lambda_{p,\ell}(d\nu)},\end{aligned}\tag{4.1}$$

thanks to $\delta=\frac{n}{\lambda}-\frac{\beta}{\lambda_*}$ and $\mu(B(x_0,r))\le[\![\mu]\!]_\beta\, r^\beta$, for all $x_0\in\operatorname{spt}(\mu)$ and $r>0$. Since

$$\frac{n-\beta}{p}<\delta<\frac{n}{\lambda}$$

we can ensure the existence of γ such that $(n-\beta)/p<\gamma<\delta<n/\lambda$ which yields $n-\beta<\gamma p<np/\lambda\le n$. Hence, for $y\in B(x_0,2\rho)$ we invoke Lemma 4.1 with $\alpha=n/\lambda$ and estimate

$$\begin{aligned}\|I_\delta f'\|_{L^{q,s}(B(x_0,\rho),d\mu)} &\lesssim \|f\|^{\frac{\delta-\gamma}{\alpha-\gamma}}_{\mathcal{M}^\lambda_{p,\ell}(d\nu)}\left\|\,|M_\gamma f'|^{(1-\frac{\delta-\gamma}{\alpha-\gamma})}\right\|_{L^{q,s}(B(x_0,\rho),d\mu)}\\ &= \|f\|^{\frac{\delta-\gamma}{\alpha-\gamma}}_{\mathcal{M}^\lambda_{p,\ell}(d\nu)}\,\|M_\gamma f'\|^{1-\frac{\delta-\gamma}{\alpha-\gamma}}_{L^{q(1-\frac{\delta-\gamma}{\alpha-\gamma}),\,s(1-\frac{\delta-\gamma}{\alpha-\gamma})}(B(x_0,\rho),d\mu)},\end{aligned}$$

in view of $\|\,|g|^b\|_{L^{q,s}}=\|g\|^b_{L^{qb,sb}}$ for $b=1-\frac{\delta-\gamma}{\alpha-\gamma}$. Now, since $\ell<s$ and $b=(1-\frac{\delta-\gamma}{\alpha-\gamma})\in(0,1)$ we can choose γ close to δ such that $\ell\le sb$. It follows from Calderón's Lemma 2.1 that

$$\|I_\delta f'\|_{L^{q,s}(B(x_0,\rho),d\mu)}\lesssim \|f\|^{1-b}_{\mathcal{M}^\lambda_{p,\ell}(d\nu)}\,\|M_\gamma f'\|^b_{L^{qb,\,\ell}(B(x_0,\rho),d\mu)}\,.\tag{4.2}$$

Now from real interpolation (see Lemma 2.2) and trace principle [3, Theorem 2] in $L^p(d\mu)$ we will show that

$$\|M_\gamma f'\|_{L^{\overline{p},\ell}(B_\rho,d\mu)}\lesssim [\![\mu]\!]_\beta^{1/\overline{p}}\|f'\|_{L^{p,\ell}(\mathbf{R}^n,d\nu)}\tag{4.3}$$

for all $f'\in L^{p,\ell}(d\nu)$ whenever $1<p<\overline{p}=qb=\beta p/(n-\gamma p)$, $0<\beta\le n$ and $n-\beta<\gamma p<n$. Indeed, let $\theta\in(0,1)$, $p_0<p<p_1$ and $\bar p_0<\bar p<\bar p_1$ be such that

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}\quad\text{and}\quad\frac{1}{\bar p}=\frac{1-\theta}{\bar p_0}+\frac{\theta}{\bar p_1},$$

where $1<p_i<\bar p_i=\frac{\beta p_i}{n-\gamma p_i}$, $0<\beta\le n$ and $n-\beta<\gamma p_i<n$, $i=0,1$. Hence, from pointwise inequality $M_\gamma f'(x)\lesssim I_\gamma|f'(x)|$ and [3, Theorem 2] we have

$$\|M_\gamma f'\|_{L^{\bar p_i,\bar p_i}(B_\rho,d\mu)}\lesssim\|I_\gamma f'\|_{L^{\bar p_i,\bar p_i}(B_\rho,d\mu)}\le[\![\mu]\!]_\beta^{1/\bar p_i}\|f'\|_{L^{p_i,p_i}(\mathbf{R}^n,d\nu)},\quad i=0,1,$$

provided that the Radon measure μ satisfies $[\![\mu]\!]_\beta<\infty$. Therefore, thanks to Hunt's Theorem (see Lemma 2.2)

$$\|M_\gamma f'\|_{L^{\overline{p},\ell}(B_\rho,d\mu)}\lesssim[\![\mu]\!]_\beta^{(1-\theta)/\overline{p}_0}[\![\mu]\!]_\beta^{\theta/\overline{p}_1}\|f'\|_{L^{p,\ell}(\mathbf{R}^n,d\nu)}=[\![\mu]\!]_\beta^{1/\overline{p}}\|f'\|_{L^{p,\ell}(d\nu)}\text{ as }1\le\ell\le\infty,$$

where $1 < p < \overline{p} = \beta p/(n-\gamma p)$, $0 < \beta \le n$ and $n-\beta < \gamma p < n$, as required. Hence, we can inserting (4.3) into (4.2) to yield

$$\begin{aligned}
\|I_\delta f'\|_{L^{q,s}(B(x_0,\rho),d\mu)} &\lesssim \|f\|^{1-b}_{\mathcal{M}^{\lambda}_{p,\ell}(\mathbf{R}^n,\,d\nu)} \, [\![\mu]\!]^{b/\overline{p}}_{\beta} \, \|f'\|^{b}_{L^{p,\ell}(\mathbf{R}^n,d\nu)} \\
&= \|f\|^{1-b}_{\mathcal{M}^{\lambda}_{p,\ell}(\mathbf{R}^n,\,d\nu)} \, [\![\mu]\!]^{b/\overline{p}}_{\beta} \, \|f\|^{b}_{L^{p,\ell}(B(x_0,2\rho),d\nu)} \\
&\lesssim [\![\mu]\!]^{b/\overline{p}}_{\beta} \, \rho^{\left(\frac{n}{p}-\frac{n}{\lambda}\right)(1-\frac{\delta-\gamma}{\alpha-\gamma})} \|f\|_{\mathcal{M}^{\lambda}_{p,\ell}(\mathbf{R}^n,\,d\nu)} \\
&= [\![\mu]\!]^{\frac{1}{q}}_{\beta} \, \rho^{\beta\left(\frac{1}{q}-\frac{1}{\lambda_*}\right)} \|f\|_{\mathcal{M}^{\lambda}_{p,\ell}(\mathbf{R}^n,\,d\nu)}, \qquad (4.4)
\end{aligned}$$

where the equality (4.4) is a consequence of (4.5) and (4.6) below. Indeed, note that by $\alpha = n/\lambda$ and $\delta = n/\lambda - \beta/\lambda_*$, the request

$$qb = q\left(1 - \frac{\delta-\gamma}{\alpha-\gamma}\right) = \overline{p} = \frac{\beta p}{n-\gamma p} \qquad (4.5)$$

is equivalent to

$$\gamma p \beta \left(1 - \frac{q}{\lambda_*}\right) = n\beta\left(1-\frac{q}{\lambda_*}\right) - \beta n\left(1 - \frac{p}{\lambda}\right). \qquad (4.6)$$

Hence, we obtain

$$\begin{aligned}
\left(\frac{n}{p}-\frac{n}{\lambda}\right)\left(1-\frac{\delta-\gamma}{\alpha-\gamma}\right) &\overset{(4.5)}{=} \frac{\overline{p}}{q}\left(\frac{n}{p}-\frac{n}{\lambda}\right) = \frac{\overline{p}}{q}\frac{1}{p\beta}\left[n\beta\left(1-\frac{p}{\lambda}\right)\right] \\
&\overset{(4.6)}{=} \frac{1}{q}\frac{1}{n-\gamma p}\left[n\beta\left(1-\frac{q}{\lambda_*}\right) - \gamma p\beta\left(1-\frac{q}{\lambda_*}\right)\right] \\
&= \beta\left(\frac{1}{q}-\frac{1}{\lambda_*}\right).
\end{aligned}$$

Note that $(n-\beta)/p < \gamma < \delta < \alpha$ implies $0 < b < 1$. From estimates (4.1) and (4.4) we obtain

$$\rho^{-\beta\left(\frac{1}{q}-\frac{1}{\lambda_*}\right)} \|I_\delta f\|_{L^{q,s}(\mu\lfloor_\Omega(B_\rho))} \lesssim [\![\mu]\!]^{\frac{1}{q}}_{\beta} \, \|f\|_{\mathcal{M}^{\lambda}_{p,\ell}(d\nu)},$$

which is the desired continuity of the map $I_\delta \colon \mathcal{M}^{\lambda}_{p,\ell}(d\nu) \to \mathcal{M}^{\lambda_*}_{q,s}(\Omega, d\mu)$. □

4.2. The condition $[\![\mu]\!]_\beta < \infty$ is necessary. Let $B(x_0,r) \subset \mathbf{R}^n$ be a ball centered in x_0 and with radius $r > 0$. Choosing $f = \chi_{B(x_0,r)}$ when $x \in B(x_0,r)$ we can estimate

$$\begin{aligned}
(I_\delta f)(x) &= \int_{\mathbf{R}^n} |x-y|^{\delta-n} \chi_{B(x_0,r)}(y)\, d\nu(y) \\
&= \int_{|y-x_0|<r} |x-y|^{\delta-n}\, d\nu(y) \gtrsim r^{\delta-n}\nu(B(x_0,r)) = Cr^{\delta}
\end{aligned}$$

thanks to $|x-y| \le 2r$ for $y \in B(x_0,r)$. The previous argument implies that the estimate $(I_\delta f)^*(t) \gtrsim r^\delta$ hold for $0 < t < \mu(B(x_0,r))$. Hence, using (2.1) (see also (2.2)) we obtain

$$\|I_\delta f\|_{L^{q,s}(B(x_0,r),d\mu)} \gtrsim r^{\delta}\left(\int_0^{\mu(B(x_0,r))} t^{\frac{s}{q}-1} ds\right)^{\frac{1}{s}} = C\, r^{\frac{n}{\lambda}-\frac{\beta}{\lambda_*}}[\mu(B(x_0,r)]^{\frac{1}{q}}. \qquad (4.7)$$

Since $I_\delta\colon \mathcal{M}^{\lambda}_{p,\ell}(d\nu) \to \mathcal{M}^{\lambda_*}_{q,s}(d\mu)$ is bounded and $\|\chi_{B(x_0,r)}\|_{\mathcal{M}^{\lambda}_{p,\ell}(\mathbf{R}^n)} = Cr^{n/\lambda}$, then (4.7) implies that

$$r^{\frac{n}{\lambda}} \gtrsim \|I_\delta f\|_{\mathcal{M}^{\lambda_*}_{q,s}(d\mu)} \gtrsim r^{\beta\left(\frac{1}{\lambda_*}-\frac{1}{q}\right)}\|I_\delta f\|_{L^{q,s}(B(x_0,r),d\mu)} \gtrsim r^{\frac{n}{\lambda}-\frac{\beta}{q}}\mu(B(x_0,r))^{\frac{1}{q}}$$

which yields $\mu(B(x_0,r)) \lesssim r^\beta$ as desired. □

5. Proof of Corollary 1.4

The Calderón–Stein's extension operator $\mathcal{E}$ on Lipschitz domain Ω is defined by $\mathcal{E}f = f$ in $\overline{\Omega}$ and

$$\mathcal{E}f(x) = \int_1^\infty f(x', x_n + s\delta^*(x))\psi(s)\,ds \quad \text{on } \mathbf{R}^n\backslash\overline{\Omega}$$

where ψ is a continuous function on $[1,\infty)$ such that $\psi(s) = O(s^{-N})$ as $s \to \infty$ for every N,

$$\int_1^\infty \psi(s)\,ds = 1 \quad \text{and} \quad \int_1^\infty s^k\psi(s)\,ds = 0, \quad \text{for } k = 1, 2, \cdots$$

and $\delta^*(x) = 2c\Delta(x)$ is a C^∞-function comparable to $\delta(x) = \operatorname{dist}(x, \overline{\Omega})$, see [30, Theorem 2]. On half-space $\mathbf{R}^n_+$ one has $\delta^*(x) = 2x_n$ and we have

$$\mathcal{E}f(x', x_n) = \int_1^\infty f(x', (1-2s)x_n)\psi(s)\,ds \quad \text{if } x_n < 0 \tag{5.1}$$

provided that the above integral converges. The proof of the Lemma 5.1 below is similar to [1, Lemma 3.1], we include the proof for reader convenience.

Lemma 5.1. *Let $n \geq 2$ and $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n_+)$ such that $\nabla f \in \mathcal{M}^{\lambda}_{p,d}(\mathbf{R}^n_+)$, then*

$$\|\nabla\mathcal{E}f\|_{\mathcal{M}^{\lambda}_{p,d}(\mathbf{R}^n)} \leq C\|\nabla f\|_{\mathcal{M}^{\lambda}_{p,d}(\mathbf{R}^n_+)}$$

for $1 \leq p \leq \lambda < \infty$ and $d \in [1,\infty]$.

Proof. For each $x' \in \mathbf{R}^{n-1}$ fixed and multi-index α, the scaling property

$$\|D^\alpha f(\gamma\cdot)\|_{\mathcal{M}^{\lambda}_{p,d}} = \gamma^{|\alpha|-\frac{n}{\lambda}}\,\|f\|_{\mathcal{M}^{\lambda}_{p,d}}$$

yields

$$\|D^\alpha f(\cdot,(2s-1)x_n)\|_{\mathcal{M}^{\lambda}_{p,d}(\mathbf{R}^n_+)} = (2s-1)^{|\alpha|-\frac{1}{\lambda}}\|D^\alpha f(\cdot,x_n)\|_{\mathcal{M}^{\lambda}_{p,d}(\mathbf{R}^n_+)}.$$

It follows that

$$\begin{aligned}
\left\|\frac{\partial}{\partial x_n}\mathcal{E}f\mathbb{1}_{\{x_n<0\}}\right\|_{\mathcal{M}^{\lambda}_{p,d}(\mathbf{R}^n)} &= \left\|\int_1^\infty \partial_n f(x',(2s-1)x_n)\psi(s)\,ds\right\|_{\mathcal{M}^{\lambda}_{p,d}(\mathbf{R}^n_+)} \\
&\leq \int_1^\infty (2s-1)\,\|\partial_n f(x',(2s-1)x_n)\|_{\mathcal{M}^{\lambda}_{p,d}(\mathbf{R}^n_+)}\,|\psi(s)|\,ds \\
&\leq \left(\int_1^\infty (2s-1)^{2-\frac{1}{\lambda}}|\psi(s)|\,ds\right)\|\partial_n f\|_{\mathcal{M}^{\lambda}_{p,d}(\mathbf{R}^n_+)} \\
&\leq C\,\|\partial_n f\|_{\mathcal{M}^{\lambda}_{p,d}(\mathbf{R}^n_+)},
\end{aligned}$$

because $|\psi(s)| \leq Cs^{-N}$ for all N implies

$$\int_1^\infty (2s-1)^{2-\frac{1}{\lambda}}|\psi(s)|\,ds \leq C\int_1^\infty (s-1)^{\theta-1}s^{-\theta-(N-\theta)}\,ds = C\beta(\theta, N-\theta)$$

where $\beta(\theta, N-\theta)$ denotes the beta function and $\theta = 3 - 1/\lambda$. Since $\mathcal{E}f = f$ in $\overline{\mathbf{R}^n_+}$, then $\|\nabla \mathcal{E} f \mathbb{1}_{\{x_n \geq 0\}}\|_{\mathcal{M}^\lambda_{p,d}(\mathbf{R}^n)} = \|\nabla f\|_{\mathcal{M}^\lambda_{p,d}(\mathbf{R}^n_+)}$ and, moreover,

$$\begin{aligned}\left\|\partial_{x_j} \mathcal{E} f \mathbb{1}_{\{x_n<0\}}\right\|_{\mathcal{M}^\lambda_{p,d}(\mathbf{R}^n)} &\leq \left(\int_1^\infty (2s-1)^{1-\frac{1}{\lambda}} |\psi(s)|\, ds\right) \left\|\partial_{x_j} f\right\|_{\mathcal{M}^\lambda_{p,d}(\mathbf{R}^n_+)} \\ &\leq C \left\|\partial_{x_j} f\right\|_{\mathcal{M}^\lambda_{p,d}(\mathbf{R}^n_+)}\end{aligned}$$

for all $j = 1, \cdots, n-1$, as required. □

Thanks to Theorem 1.1 on $\partial \mathbf{R}^n_+$ with $\beta = n-1$, integral representation formula [4, (3.5)] and Lemma 5.1

$$\|f(x', 0)\|_{\mathcal{M}^{\lambda_*}_{q,s}(\partial \mathbf{R}^n_+)} \leq C \|\nabla \mathcal{E} f\|_{\mathcal{M}^\lambda_{p,d}(\mathbf{R}^n)} \leq C \|\nabla f\|_{\mathcal{M}^\lambda_{p,d}(\mathbf{R}^n_+)}$$

as desired.

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