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Boundary growth of Sobolev functions of monotone type for double phase functionals

Yoshihiro Mizuta and Tetsu Shimomura

Abstract. Our aim in this paper is to deal with boundary growth of spherical means of Sobolev functions of monotone type for the double phase functional $\Phi_{p,q}(x,t) = t^p + (b(x)t)^q$ in the unit ball $\mathbb{B}$ of $\mathbb{R}^n$, where $1 < p < q < \infty$ and $b(\cdot)$ is a non-negative bounded function on $\mathbb{B}$ which is Hölder continuous of order $\theta \in (0,1]$.

Kaksivaiheisen funktionaalin määrittelemien monotonisen tyypin Sobolevin funktioiden reunakasvu

Tiivistelmä. Tavoitteemme on käsitellä kaksivaiheisen funktionaalin $\Phi_{p,q}(x,t) = t^p + (b(x)t)^q$ määrittelemien monotonisen tyypin Sobolevin funktioiden pallokeskiarvojen reunakasvua avaruuden $\mathbb{R}^n$ yksikkökuulassa $\mathbb{B}$, kun $1 < p < q < \infty$ ja $b(\cdot)$ on kuulassa $\mathbb{B}$ määritelty ei-negatiivinen ja rajoitettu funktio, joka toteuttaa kertaluvun $\theta \in (0,1]$ Hölderin jatkuvuusehdon.

1. Introduction

We say that a continuous function u is monotone in the unit ball $\mathbb{B}$ of $\mathbb{R}^n$ $(n \geq 2)$ in the sense of Lebesgue, if both

$$\max_{D} u(x) = \max_{\partial D} u(x) \quad \text{and} \quad \min_{D} u(x) = \min_{\partial D} u(x)$$

hold for every relatively compact open set D with the closure $\overline{D} \subset \mathbb{B}$ (see [17]). Clearly, harmonic functions are monotone, and more generally, solutions of elliptic partial differential equations of second order and weak solutions for variational problems may be monotone. A function $u \in W^{1,p}(\mathbb{B})$ is $\mathcal{A}$-harmonic if it is a weak solution of equation

$$\operatorname{div}(\mathcal{A}(x, \nabla u)) = 0,$$

where $\mathcal{A}(x,\xi) \cdot \xi \approx |\xi|^p$ for some fixed $p \in (1,\infty)$, $\xi \in \mathbb{R}^n$ (see [14]). Note that $\mathcal{A}$-harmonic functions and hence coordinate functions of quasiregular mappings are monotone (see [10, 14, 16, 27, 30]), and thus the class of monotone functions is considerably wide. For these facts, see Gilbarg–Trudinger [12], Heinonen–Kilpeläinen–Martio [14], Serrin [28], Vuorinen [31, 32] and [9].

Regarding regularity theory of differential equations, Baroni, Colombo and Mingione [1, 2, 5, 6] studied a double phase functional:

$$\Phi(x,t) = \Phi_{p,q}(x,t) = t^p + (b(x)t)^q,$$

where $1 < p < q < \infty$ and $b(\cdot)$ is a non-negative bounded function on $\mathbb{B}$ which is Hölder continuous of order $\theta \in (0,1]$. See [4] for the eigenvalue problem, [18, 23] for

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Sobolev's inequality, [26] for Hardy–Sobolev inequalities and [21, 22] for Campanato–Morrey spaces for the double phase functionals. We refer to for instance [3, 7, 8, 13] and references therein for other recent works.

For a measurable function f on the unit sphere $\mathbf{S}$, we define the L^p mean by

$$S_p(f) = \left(\frac{1}{\sigma_n} \int_{\mathbf{S}} |f(\xi)|^p \, dS(\xi) \right)^{1/p},$$

where σ_n denotes the surface measure of $\mathbf{S}$.

In [24], we studied growth properties of spherical means of monotone Sobolev functions on $\mathbb{B}$ satisfying

$$\int_{\mathbb{B}} |\nabla u(x)|^p \, dx < \infty,$$

where ∇ denotes the gradient and $1 < p < \infty$. More precisely, it was shown that

$$\lim_{t \to 1} (1-t)^{(n-p)/p-(n-1)/\omega} S_\omega(u_t) = 0$$

when $n-1 < p < n$, $p < \omega < \infty$ and

$$\frac{n-p}{p} - \frac{n-1}{\omega} > 0,$$

where $u_t(\xi) = u(t\xi)$ for $\xi \in \mathbf{S}$. See also Gardiner [11], Stoll [29] and the first author [20].

We say that a continuous function u on $\mathbb{B}$ is of monotone type if there is $p > 1$ for which

$$|u(x) - u(y)|^p \le M r^{p-n} \int_{B(x_0,2r)} |\nabla u(y)|^p \, dy \tag{1.1}$$

holds whenever $x, y \in B(x_0, r)$ and $B(x_0, 2r) \subset \mathbb{B}$.

Clearly, harmonic functions on $\mathbb{B}$ are of monotone type by the mean value property. Moreover, by Lemma 2.1 below, monotone functions on $\mathbb{B}$ in the sense of Lebesgue are of monotone type because (1.1) is valid for $p > n-1$.

Our aim in this paper is to study growth properties of spherical means of monotone Sobolev functions for the double phase functional $\Phi(x,t)$, as an extension of [24].

Theorem 1.1. *Let u be a monotone type function on $\mathbb{B}$ satisfying (1.1) for $p > 1$ and*

$$\int_{\mathbb{B}} \Phi(x, |\nabla u(x)|) \, dx < \infty. \tag{1.2}$$

Suppose $1/p - 1/q \le \theta/n$, $\omega > q$,

$$\eta = \frac{n-q}{q} - \frac{n-1}{\omega} > 0$$

and

$$\eta_0 = \frac{n-p}{p} - \frac{n-1}{\omega} > 0.$$

Set

$$v(x) = (1-|x|)^{\eta_0} u(x) + (1-|x|)^{\eta} b(x) u(x).$$

Then

$$\lim_{t \to 1} S_\omega(v_t) = 0,$$

where $v_t(\xi) = v(t\xi)$ for $\xi \in \mathbf{S}$ and a function v on $\mathbb{B}$.

Remark 1.2. If $1/p - 1/q = \theta_1/n$, then

$$\begin{aligned}|b(x) - b(y)| &= |b(x) - b(y)|^{\theta_1/\theta}|b(x) - b(y)|^{1-\theta_1/\theta}\\ &\le C|b(x) - b(y)|^{\theta_1/\theta} \le C|x - y|^{\theta_1}.\end{aligned}$$

Thus in what follows we always assume that

$$1/p - 1/q = \theta/n.$$

Example 1.3. Let $x_0 \in \partial\mathbb{B}$ and $0 < \theta \le 1$. Examples of $b(x)$ are $|x - x_0|^\theta$ and $(1 - |x|)^\theta$.

Remark 1.4. In Theorem 1.1, as will be seen in Remark 4.3 given below, condition (1.2) can not be replaced by the single condition

$$\int_{\mathbb{B}} (b(x)|\nabla u(x)|)^q \, dx < \infty.$$

Hence Theorem 1.1 essentially depends on the double phase functional.

The sharpness of the exponent will be discussed in the last section (Remark 4.1).

It is well known that a coordinate function $u = f_i$ of a quasiregular mapping $f = (f_1, \dots, f_n)\colon \mathbb{B} \to \mathbb{R}^n$ is $\mathcal{A}$-harmonic (see [14, Theorem 14.39]) and monotone in $\mathbb{B}$ in the sense of Lebesgue. The key fact for monotone functions in the sense of Lebesgue is Lemma 2.1 below, so that Theorem 1.1 gives the following corollaries.

Corollary 1.5. *Let $1 \le p < n$, $1/p - 1/q = \theta/n$, $\omega > q$,*

$$\eta = \frac{n - (1+\theta)p}{p} - \frac{n-1}{\omega} = \frac{n-q}{q} - \frac{n-1}{\omega} > 0$$

and

$$\eta_0 = \frac{n-p}{p} - \frac{n-1}{\omega} = \eta + \theta.$$

Set

$$v(x) = (1 - |x|)^{\eta_0} u(x) + (1 - |x|)^{\eta} b(x) u(x).$$

If u is a harmonic function on $\mathbb{B}$ satisfying (1.2)*, then*

$$\lim_{t \to 1} S_\omega(v_t) = 0.$$

Corollary 1.6. *Let $n - 1 < p < n$, $1/p - 1/q = \theta/n$, $\omega > q$,*

$$\eta = \frac{n - (1+\theta)p}{p} - \frac{n-1}{\omega} = \frac{n-q}{q} - \frac{n-1}{\omega} > 0$$

and

$$\eta_0 = \frac{n-p}{p} - \frac{n-1}{\omega} = \eta + \theta.$$

Set

$$v(x) = (1 - |x|)^{\eta_0} u(x) + (1 - |x|)^{\eta} b(x) u(x).$$

If u is a coordinate function of a quasiregular mapping on $\mathbb{B}$ satisfying (1.2)*, then*

$$\lim_{t \to 1} S_\omega(v_t) = 0.$$

When $q = n$, Theorem 1.1 still holds for $\omega = \infty$ with a logarithmic weight instead of power weight.

Theorem 1.7. *Let $1 < p < n$, $1/p - 1/q = \theta/n$ and $q = n$. If u is a monotone type function on $\mathbb{B}$ satisfying* (1.1) *and* (1.2), *then*

$$\lim_{|x|\to 1} \left\{ (1-|x|)^{(n-p)/p} u(x) + \left(\log \frac{1}{1-|x|} \right)^{-1/q'} b(x) u(x) \right\} = 0.$$

Corollary 1.8. *Let $1 \le p < n$, $1/p - 1/q = \theta/n$ and $q = n$. If u is a harmonic function on $\mathbb{B}$ satisfying* (1.2), *then*

$$\lim_{|x|\to 1} \left\{ (1-|x|)^{(n-p)/p} u(x) + \left(\log \frac{1}{1-|x|} \right)^{-1/q'} b(x) u(x) \right\} = 0.$$

Corollary 1.9. *Let $n-1 < p < n$, $1/p - 1/q = \theta/n$ and $q = n$. If u is a coordinate function of a quasiregular mapping on $\mathbb{B}$ satisfying* (1.2), *then*

$$\lim_{|x|\to 1} \left\{ (1-|x|)^{(n-p)/p} u(x) + \left(\log \frac{1}{1-|x|} \right)^{-1/q'} b(x) u(x) \right\} = 0.$$

For the sharpness of the exponent in Theorem 1.7, see Remark 4.2. See [25] for p-precise function in the sense of Ziemer [33].

Throughout this paper, let C denote various constants independent of the variables in question.

2. Proof of Theorem 1.1

We use the notation $B(x,r)$ to denote the open ball centered at x of radius r. Let us begin with the key fact for monotone functions.

Lemma 2.1. *If u is a monotone function on $B(x_0, 2r)$ in the sense of Lebesgue and $p > n-1$, then*

$$|u(x) - u(y)|^p \le M r^{p-n} \int_{B(x_0,2r)} |\nabla u(y)|^p dy$$

whenever $x, y \in B(x_0, r)$.

For this, see e.g. [15, Lemma 7.1], [19, Remark, p. 9] and, for the case $p = n$, [32, Section 16].

We show the following result in the same manner as [24, Theorem 2].

Lemma 2.2. *If u is a monotone type function on $\mathbb{B}$ satisfying* (1.1) *for $1 < p < n$ and $\eta_0 = (n-p)/p - (n-1)/\omega > 0$ such that*

$$\int_{\mathbb{B}} |\nabla u(x)|^p \, dx < \infty, \tag{2.1}$$

then

$$S_\omega(u_s - u_t) \le C(1-t)^{-\eta_0} \left(\int_{B(0,1-(1-t)/2)\setminus B(0,1-3(1-t)/2)} |\nabla u(z)|^p dz \right)^{1/p}$$

when $1/3 < t - (1-t)/4 < s < t < 1$.

Proof. Let u be a monotone type function on B satisfying (1.1) and (2.1) for $1 < p < n$. By (1.1), we have

$$|u(x) - u(y)| \le C \left(r^{p-n} \int_{B(x_0,2r)} |\nabla u(z)|^p dz \right)^{1/p}$$

whenever $x, y \in B(x_0, r) \subset \mathbb{B}$. Hence, if $|s-t| \le (1-t)/4 = r$, then

$$S_\omega(u_s - u_t) \le C(1-t)^{(p-n)/p} \left(\int_S \left(\int_{B(t\xi,2r)} |\nabla u(z)|)^p \, dz \right)^{\omega/p} dS(\xi) \right)^{1/\omega}.$$

Minkowski's inequality for integral yields

$$\begin{aligned} &\left(\int_S \left(\int_{B(t\xi,2r)} |\nabla u(z)|^p \, dz \right)^{\omega/p} dS(\xi) \right)^{1/\omega} \\ &\le \left(\int_{B(0,t+2r)\setminus B(0,t-2r)} |\nabla u(z)|^p \left(\int_{\{\xi\in S:|\xi-z/t|<2r/t\}} dS(\xi) \right)^{p/\omega} dz \right)^{1/\omega} \\ &\le C(2r/t)^{(n-1)/\omega} \left(\int_{B(0,t+2r)\setminus B(0,t-2r)} |\nabla u(z)|^p dz \right)^{1/p}, \end{aligned}$$

so that

$$\begin{aligned} &S_\omega(u_s - u_t) \\ &\le C(1-t)^{(p-n)/p}((1-t)/t)^{(n-1)/\omega} \left(\int_{B(0,t+(1-t)/2)\setminus B(0,t-(1-t)/2)} |\nabla u(z)|^p \, dz \right)^{1/p}. \quad \square \end{aligned}$$

Corollary 2.3. *Let* $1 < p < n$, $p < \omega < \infty$ *and* $\eta_0 = (n-p)/p - (n-1)/\omega > 0$. *If* u *is a monotone type function on* $\mathbb{B}$ *satisfying* (1.1) *and* (2.1), *then*

$$\lim_{t\to 1} (1-t)^{\eta_0} S_\omega(u_t) = 0,$$

where $u_t(\xi) = u(t\xi)$ *for* $\xi \in \mathbf{S}$ *as before.*

Proof. Let $r_j = 2^{-j-1}$, $t_j = 1 - r_{j-1}$ and $A_j = B(0, 1-r_j) \setminus B(0, 1-3r_j)$ for $j = 1, 2, \ldots$. Then we find from Lemma 2.2

$$S_\omega(u_{t_j} - u_t) \le C r_{j+1}^{-\eta_0} \left(\int_{A_j} |\nabla u(z)|^p \, dz \right)^{1/p}$$

for $t_j \le t < t_j + r_{j+1}$,

$$S_\omega(u_t - u_s) \le C r_{j+2}^{-\eta_0} \left(\int_{A_j} |\nabla u(z)|^p \, dz \right)^{1/p}$$

for $t_j + r_{j+1} \le t < s < t_j + r_{j+1} + r_{j+2}$, and

$$S_\omega(u_s - u_{t_{j+1}}) \le C r_{j+2}^{-\eta_0} \left(\int_{A_{j+1}} |\nabla u(z)|^p \, dz \right)^{1/p}$$

for $t_j + r_{j+1} + r_{j+2} \le s < t_{j+1}$. Collecting these results, we have

$$\begin{aligned} &|S_\omega(u_{t_j}) - S_\omega(u_t)| \\ &\leqq M r_j^{-\eta_0} \left(\int_{A_j} |\nabla u(z)|^p \, dz \right)^{1/p} + M r_{j+1}^{-\eta_0} \left(\int_{A_{j+1}} |\nabla u(z)|^p \, dz \right)^{1/p} \end{aligned}$$

for $t_j \leqq t < t_{j+1}$. Hence it follows that

$$|S_\omega(u_{t_j}) - S_\omega(u_{t_{j+m}})| \leqq M \sum_{k=j}^{j+m} r_k^{-\eta_0} \left(\int_{A_k} |\nabla u(z)|^p \, dz \right)^{1/p}.$$

Since $A_i \cap A_k = \emptyset$ when $i \geqq k+2$, Hölder's inequality gives

$$\begin{aligned} |S_\omega(u_{t_j}) - S_\omega(u_{t_{j+m}})| &\leqq M \left(\sum_{k=j}^{j+m} r_k^{-p'\eta_0} \right)^{1/p'} \left(\sum_{k=j}^{j+m} \int_{A_k} |\nabla u(z)|^p \, dz \right)^{1/p} \\ &\leqq M r_{j+m}^{-\eta_0} \left(\int_{B(0,1-r_{j+m})-B(0,1-3r_j)} |\nabla u(z)|^p \, dz \right)^{1/p}. \end{aligned}$$

More generally, if $t_j \leqq t < 1$, then we take m such that $t_{j+m-1} \leqq t < t_{j+m}$, and establish

$$|S_\omega(u_{t_j}) - S_\omega(u_t)| \leqq M(1-t)^{-\eta_0} \left(\int_{\mathbb{B}-B(0,1-3r_j)} |\nabla u(z)|^p \, dz \right)^{1/p},$$

which implies that

$$\limsup_{t\to 1} \, (1-t)^{\eta_0} S_\omega(u_r) \leqq M \left(\int_{\mathbb{B}-B(0,1-3r_j)} |\nabla u(z)|^p \, dz \right)^{1/p}$$

for all j. Therefore it follows that

$$\lim_{t\to 1} \, (1-t)^\omega S_\omega(u,r) = 0,$$

as required. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let u be a monotone type function on B satisfying (1.1) and (1.2) for $1 < p < n$. By (1.1), we have

$$\begin{aligned} |b(x)u(x) - b(y)u(y)| &\leq |b(x)(u(x) - u(y))| + |u(y)(b(x) - b(y))| \\ &\leq Cb(x) \left(r^{p-n} \int_{B(x_0,2r)} |\nabla u(z)|^p \, dz \right)^{1/p} + Cr^\theta \, |u(y)| \end{aligned}$$

whenever $x, y \in B(x_0, r)$. Note here that

$$\begin{aligned} & b(x) \left(r^{p-n} \int_{B(x_0,2r)} |\nabla u(z)|^p \, dz \right)^{1/p} \\ & \leq C \left(r^{p-n} \int_{B(x_0,2r)} |(b(x) - b(z))|\nabla u(z)||^p \, dz \right)^{1/p} \\ & \quad + C \left(r^{p-n} \int_{B(x_0,2r)} (b(z)|\nabla u(z)|)^p \, dz \right)^{1/p} \\ & \leq C \left(r^{p-n+\theta p} \int_{B(x_0,2r)} |\nabla u(z)|^p \, dz \right)^{1/p} + C \left(r^{q-n} \int_{B(x_0,2r)} (b(z)|\nabla u(z)|)^q \, dz \right)^{1/q} \end{aligned}$$

by Jensen's inequality, so that

$$|b(x)u(x) - b(y)u(y)| \le Cr^{\theta} |u(y)| + C \left(r^{p-n+\theta p} \int_{B(x_0,2r)} |\nabla u(z)|^p \, dz \right)^{1/p} + C \left(r^{q-n} \int_{B(x_0,2r)} (b(z)|\nabla u(z)|)^q \, dz \right)^{1/q}$$

whenever $x, y \in B(x_0, r)$. Hence, if $|s - t| \le (1-t)/4 = r$, then

$$\begin{aligned} S_\omega((bu)_s - (bu)_t) &= \left(\frac{1}{\sigma_n} \int_{\mathbf{S}} |(bu)(s\xi) - (bu)(t\xi)|^\omega \, dS(\xi) \right)^{1/\omega} \\ &\le C(1-t)^{\theta} \left(\int_{\mathbf{S}} |u(t\xi)|^\omega \, dS(\xi) \right)^{1/\omega} \\ &\quad + C(1-t)^{(p-n+\theta p)/p} \left(\int_{\mathbf{S}} \left(\int_{B(t\xi,2r)} |\nabla u(z)|^p \, dz \right)^{\omega/p} dS(\xi) \right)^{1/\omega} \\ &\quad + C(1-t)^{(q-n)/q} \left(\int_{\mathbf{S}} \left(\int_{B(t\xi,2r)} (b(z)|\nabla u(z)|)^q \, dz \right)^{\omega/q} dS(\xi) \right)^{1/\omega}. \end{aligned}$$

Minkowski's inequality for integrals yields

$$\begin{aligned} &\left(\int_{\mathbf{S}} \left(\int_{B(t\xi,2r)} |\nabla u(z)|^p \, dz \right)^{\omega/p} dS(\xi) \right)^{1/\omega} \\ &\le \left(\int_{B(0,t+2r)\setminus B(0,t-2r)} |\nabla u(z)|^p \left(\int_{\{\xi \in S : |\xi - z/t| < 2r/t\}} dS(\xi) \right)^{p/\omega} dz \right)^{1/p} \\ &\le C(2r/t)^{(n-1)/\omega} \left(\int_{B(0,t+2r)\setminus B(0,t-2r)} |\nabla u(z)|^p \, dz \right)^{1/p}, \end{aligned}$$

so that

$$\begin{aligned} &S_\omega((bu)_s - (bu)_t) \\ &\le Cr^{\theta} \left(\int_{\mathbf{S}} |u(t\xi)|^\omega \, dS(\xi) \right)^{1/\omega} \\ &\quad + C(1-t)^{(p-n+\theta p)/p}((1-t)/t)^{(n-1)/\omega} \left(\int_{B(0,t+(1-t)/2)\setminus B(0,t-(1-t)/2)} |\nabla u(z)|^p \, dz \right)^{1/p} \\ &\quad + C(1-t)^{(q-n)/q}((1-t)/t)^{(n-1)/\omega} \left(\int_{B(0,t+(1-t)/2)\setminus B(0,t-(1-t)/2)} (b(z)|\nabla u(z)|)^q \, dz \right)^{1/q}. \end{aligned}$$

Let $r_j = 2^{-j-1}$, $t_j = 1 - r_{j-1}$ and $A_j = B(0, 1-r_j) \setminus B(0, 1-3r_j)$ for $j = 1, 2, \ldots$. Recall

$$\eta = (n - p - \theta p)/p - (n-1)/\omega = (n-q)/q - (n-1)/\omega > 0.$$

Then we find

$$S_\omega((bu)_{t_j} - (bu)_t) \le C r_{j+1}^{\theta} \left(\int_{\mathbf{S}} |u(t_j\xi)|^\omega \, dS(\xi) \right)^{1/\omega} + C r_{j+1}^{-\eta} \left(\int_{A_j} |\nabla u(z)|^p \, dz \right)^{1/p} + C r_{j+1}^{-\eta} \left(\int_{A_j} (b(z)|\nabla u(z)|)^q \, dz \right)^{1/q}$$

for $t_j \le t < t_j + r_{j+1}$,

$$S_\omega((bu)_t - (bu)_s) \le C r_{j+2}^{\theta} \left(\int_{\mathbf{S}} |u(t\xi)|^\omega \, dS(\xi) \right)^{1/\omega} + C r_{j+2}^{-\eta} \left(\int_{A_j} |\nabla u(z)|^p \, dz \right)^{1/p} + C r_{j+2}^{-\eta} \left(\int_{A_j} (b(z)|\nabla u(z)|)^q \, dz \right)^{1/q}$$

for $t_j + r_{j+1} \le t < s < t_j + r_{j+1} + r_{j+2}$, and

$$S_\omega((bu)_s - (bu)_{t_{j+1}}) \le C r_{j+2}^{\theta} \left(\int_{\mathbf{S}} |u(t_{j+1}\xi)|^\omega \, dS(\xi) \right)^{1/\omega} + C r_{j+2}^{-\eta} \left(\int_{A_{j+1}} |\nabla u(z)|^p \, dz \right)^{1/p} + C r_{j+2}^{-\eta} \left(\int_{A_{j+1}} (b(z)|\nabla u(z)|)^q \, dz \right)^{1/q}$$

for $t_j + r_{j+1} + r_{j+2} \le s < t_{j+1}$. Collecting these results, we have

$$\begin{aligned} & S_\omega((bu)_{t_j} - (bu)_t) \\ & \le C r_{j+1}^{\theta} \left(\int_{\mathbf{S}} |u(t_j\xi)|^\omega dS(\xi) \right)^{1/\omega} \\ & \quad + C r_j^{-\eta} \left(\int_{A_j} |\nabla u(z)|^p \, dz \right)^{1/p} + C r_{j+1}^{-\eta} \left(\int_{A_{j+1}} |\nabla u(z)|^p \, dz \right)^{1/p} \\ & \quad + C r_j^{-\eta} \left(\int_{A_j} (b(z)|\nabla u(z)|)^q \, dz \right)^{1/q} + C r_{j+1}^{-\eta} \left(\int_{A_{j+1}} (b(z)|\nabla u(z)|)^q \, dz \right)^{1/q} \end{aligned}$$

for $t_j \le t < t_{j+1}$. Hence it follows that

$$\begin{aligned} S_\omega((bu)_{t_j} - (bu)_t) & \le C \sum_{k=j}^{j+m} r_{k+1}^{\theta} S_\omega(u_{t_k}) \\ & \quad + C \sum_{k=j}^{j+m} r_k^{-\eta} \left(\int_{A_k} |\nabla u(z)|^p \, dz \right)^{1/p} \\ & \quad + C \sum_{k=j}^{j+m} r_k^{-\eta} \left(\int_{A_k} (b(z)|\nabla u(z)|)^q \, dz \right)^{1/q} \\ & = I_1 + I_2 + I_3 \end{aligned}$$

for $t_j \le t < t_{j+m}$. Since $A_i \cap A_k = \emptyset$ for $i \ge k+2$, Hölder's inequality gives

$$\begin{aligned} I_1 &\le C \sum_{k=j}^{j+m} r_{k+1}^{\theta} S_\omega(u_{t_k}) \le C \sum_{k=j}^{j+m} r_{j+m}^{-\eta} \sup_{t_j \le t < 1} (1-t)^{\eta_0} S_\omega(u_t) \\ &\le C r_{j+m}^{-\eta} \sup_{t_j \le t < 1} (1-t)^{\eta_0} S_\omega(u_t), \\ I_2 &\le C \left(\sum_{k=j}^{j+m} r_k^{-p'\eta} \right)^{1/p'} \left(\sum_{k=j}^{j+m} \int_{A_k} |\nabla u(z)|^p \, dz \right)^{1/p} \\ &\le C r_{j+m}^{-\eta} \left(\int_{B(0,1-r_{j+m}) \setminus B(0,1-3r_j)} |\nabla u(z)|^p \, dz \right)^{1/p} \end{aligned}$$

and

$$I_3 \le C r_{j+m}^{-\eta} \left(\int_{B(0,1-r_{j+m}) \setminus B(0,1-3r_j)} (b(z)|\nabla u(z)|)^q \, dz \right)^{1/q}.$$

Now, we establish

$$\begin{aligned} |S_\omega((bu)_{t_j}) - S_\omega((bu)_t)| &\le S_\omega((bu)_{t_j} - (bu)_t) \\ &\le C r_{j+m}^{-\eta} \sup_{t_j \le s < 1} (1-s)^{\eta_0} S_\omega(u_s) \\ &\quad + C r_{j+m}^{-\eta} \left(\int_{\mathbb{B} \setminus B(0,1-3r_j)} |\nabla u(z)|^p \, dz \right)^{1/p} \\ &\quad + C r_{j+m}^{-\eta} \left(\int_{\mathbb{B} \setminus B(0,1-3r_j)} (b(z)|\nabla u(z)|)^q \, dz \right)^{1/q} \end{aligned}$$

for $t_j \le t < t_{j+m}$. Letting $t_{j+m-1} \le t < t_{j+m}$ and $m \to \infty$, we obtain

$$\begin{aligned} \limsup_{t \to 1} (1-t)^{\eta} S_\omega((bu)_t) &\le C \sup_{t_j \le s < 1} (1-s)^{\eta_0} S_\omega(u_s) \\ &\quad + C \left(\int_{\mathbb{B} \setminus B(0,1-3r_j)} |\nabla u(z)|^p \, dz \right)^{1/p} \\ &\quad + C \left(\int_{\mathbb{B} \setminus B(0,1-3r_j)} (b(z)|\nabla u(z)|)^q \, dz \right)^{1/q}. \end{aligned}$$

Applying Corollary 2.3, we finally find

$$\lim_{t \to 1} (1-t)^{\eta} S_\omega((bu)_t) = 0,$$

and hence

$$\lim_{t \to 1} \{(1-t)^{\eta_0} S_\omega(u_t) + (1-t)^{\eta} S_\omega((bu)_t)\} = 0,$$

as required. □

3. Proof of Theorem 1.7

For a proof of Theorem 1.7, we prepare the following result.

Lemma 3.1. *Let $1 < p < n$. If u is a monotone type function on $\mathbb{B}$ satisfying* (1.1) *and* (2.1), *then*

$$\lim_{|x|\to 1} (1-|x|)^{(n-p)/p} u(x) = 0.$$

Proof. For $\xi \in \partial\mathbb{B}$ and $0 < 1/3 < t-(1-t)/4 < s < t < 1$, we have by (1.1)

$$|u(s\xi) - u(t\xi)| \le C\left(t^{p-n}\int_{B(0,1-(1-t)/2)\setminus B(0,1-3(1-t)/2))} |\nabla u(z)|^p\,dz\right)^{1/p}.$$

Let $r_j = 2^{-j-1}$, $t_j = 1 - r_{j-1}$ and $A_j = B(0, 1-r_j) \setminus B(0, 1-3r_j)$ for $j = 1, 2, \ldots$. Then we obtain

$$|u(s\xi) - u(t\xi)| \le C\left(r_j^{p-n}\int_{A_j} |\nabla u(z)|^p\,dz\right)^{1/p} + C\left(r_j^{p-n}\int_{A_{j+1}} |\nabla u(z)|^p\,dz\right)^{1/p}$$

for $t_j \le s < t \le t_{j+1}$. Hence it follows that

$$|u(s\xi) - u(t\xi)| \le C\sum_{k=j}^{j+m}\left(r_k^{p-n}\int_{A_k} |\nabla u(z)|^p\,dz\right)^{1/p}$$

when $t_j \le s < t < t_{j+m}$. Hölder's inequality gives

$$\begin{aligned} |u(s\xi) - u(t\xi)| &\le C\left(\sum_{k=j}^{j+m} r_k^{p'(p-n)/p}\right)^{1/p'}\left(\sum_{k=j}^{j+m}\int_{A_k} |\nabla u(z)|^p\,dz\right)^{1/p} \\ &\le C r_{j+m}^{(p-n)/p}\left(\sum_{k=j}^{j+m}\int_{A_k} |\nabla u(z)|^p\,dz\right)^{1/p} \end{aligned} \tag{3.1}$$

when $t_j \le s < t < t_{j+m}$. Consequently,

$$|u(s\xi) - u(t\xi)| \le C r_{j+m}^{(p-n)/p}\left(\int_{\mathbb{B}\setminus B(0,1-3r_j)} |\nabla u(z)|^p dz\right)^{1/p}$$

when $t_j \le s < t < t_{j+m}$, so that

$$\begin{aligned} \limsup_{t\to 1} (1-t)^{(p-n)/p} S_\infty(u_t) &\le \limsup_{t\to 1} (1-t)^{(p-n)/p} S_\infty(u_{t_j}) \\ &\quad + C\left(\int_{\mathbb{B}\setminus B(0,1-3r_j)} |\nabla u(z)|^p\,dz\right)^{1/p} \\ &= C\left(\int_{\mathbb{B}\setminus B(0,1-3r_j)} |\nabla u(z)|^p\,dz\right)^{1/p}, \end{aligned}$$

which proves the result by letting $j \to \infty$. □

Proof of Theorem 1.7. Along the lines in the proof of Theorem 1.1, we find

$$|b(x)u(x) - b(y)u(y)| \le Cr^{\theta}\,|u(y)| + C\left(r^{p-n+\theta p}\int_{B(x_0,2r)} |\nabla u(z)|^p\ dz\right)^{1/p} + C\left(r^{q-n}\int_{B(x_0,2r)} (b(z)|\nabla u(z)|)^q\ dz\right)^{1/q}$$

whenever $x, y \in B(x_0, r)$. Hence

$$\begin{aligned}|b(t\xi)u(t\xi) - b(t_{j+m}\xi)u(t_{j+m}\xi)| &\le C\sum_{k=j}^{j+m} r_k^{\theta}|u(t_k\xi)| + C\sum_{k=j}^{j+m}\left(\int_{A_k}|\nabla u(z)|^p\,dz\right)^{1/p} \\ &\quad + C\sum_{k=j}^{j+m}\left(\int_{A_k}(b(z)|\nabla u(z)|)^q\,dz\right)^{1/q} \\ &= I_1 + I_2 + I_3\end{aligned}$$

for $t_j \le t < t_{j+m}$. By Hölder's inequality, we obtain by (3.1)

$$\begin{aligned}I_1 &\le C\sum_{k=j}^{j+m} r_k^{\theta}\left(|u(t_j\xi)| + \sum_{\ell=j}^{k} r_\ell^{-\theta}\left(\int_{A_\ell}|\nabla u(z)|^p\,dz\right)^{1/p}\right) \\ &\le C\sum_{k=j}^{j+m} r_k^{\theta}|u(t_j\xi)| + C\sum_{k=j}^{j+m} r_k^{\theta}\left(\sum_{\ell=j}^{k} r_\ell^{-\theta}\left(\int_{A_\ell}|\nabla u(z)|^p\,dz\right)^{1/p}\right) \\ &\le C|u(t_j\xi)|\sum_{k=j}^{j+m} r_k^{\theta} + C\sum_{\ell=j}^{j+m} r_\ell^{-\theta}\left(\int_{A_\ell}|\nabla u(z)|^p\,dz\right)^{1/p}\left(\sum_{k=\ell}^{j+m} r_k^{\theta}\right) \\ &\le Cr_j^{\theta}|u(t_j\xi)| + C\sum_{\ell=j}^{j+m}\left(\int_{A_\ell}|\nabla u(z)|^p\,dz\right)^{1/p} \\ &\le Cr_j^{\theta}|u(t_j\xi)| + Cm^{1/p'}\left(\sum_{\ell=j}^{j+m}\int_{A_\ell}|\nabla u(z)|^p\,dz\right)^{1/p} \\ &\le Cr_j^{\theta}|u(t_j\xi)| + C\left(\log\frac{r_j}{r_{j+m}}\right)^{1/p'}\left(\int_{\mathbb{B}\setminus B(0,1-3r_j)}|\nabla u(z)|^p\,dz\right)^{1/p},\end{aligned}$$

$$\begin{aligned}I_2 &\le Cm^{1/p'}\left(\sum_{k=j}^{j+m}\int_{A_k}|\nabla u(z)|^p\,dz\right)^{1/p} \\ &\le C\left(\log\frac{r_j}{r_{j+m}}\right)^{1/p'}\left(\int_{B(0,1-r_{j+m})\setminus B(0,1-3r_j)}|\nabla u(z)|^p\,dz\right)^{1/p}\end{aligned}$$

and

$$I_3 \le C\left(\log\frac{r_j}{r_{j+m}}\right)^{1/q'}\left(\int_{B(0,1-r_{j+m})\setminus B(0,1-3r_j)}(b(z)|\nabla u(z)|)^q\,dz\right)^{1/q}.$$

Consequently, it follows that

$$\begin{aligned}
&|b(t\xi)u(t\xi) - b(t_{j+m}\xi)u(t_{j+m}\xi)| \\
&\le Cr_j^{\theta}|u(t_j\xi)| + C\left(\log\frac{r_j}{r_{j+m}}\right)^{1/p'}\left(\int_{\mathbb{B}\setminus B(0,1-3r_j)}|\nabla u(z)|^p\,dz\right)^{1/p} \\
&\quad + C\left(\log\frac{r_j}{r_{j+m}}\right)^{1/p'}\left(\int_{\mathbb{B}\setminus B(0,1-3r_j)}|\nabla u(z)|^p\,dz\right)^{1/p} \\
&\quad + C\left(\log\frac{r_j}{r_{j+m}}\right)^{1/q'}\left(\int_{\mathbb{B}\setminus B(0,1-3r_j)}(b(z)|\nabla u(z)|)^q\,dz\right)^{1/q}
\end{aligned}$$

for $t_j \le t < t_{j+m}$. Therefore we establish

$$\lim_{t\to 1}\left(\log\frac{1}{1-t}\right)^{-1/q'}(bu)_t(\xi) = 0,$$

as above. □

4. Sharpness

Remark 4.1. The exponent η in Theorem 1.1 is the best possible. To show this, for $x_0 \in \partial\mathbb{B}$, $\alpha > 0$ and $0 < \theta \le 1$, let us consider the functions

$$u(x) = |x - x_0|^{-\alpha} \quad \text{and} \quad b(x) = |x - x_0|^{\theta}.$$

Note here that u is a monotone function on $\mathbb{B}$ in the sense of Lebesgue and $|b(x) - b(y)| \le |x - y|^{\theta}$. For given $\varepsilon > 0$ we will find α such that

$$\lim_{t\to 1}(1-t)^{\eta-\varepsilon}S_\omega((bu)_t) = \infty.$$

For this, let us begin with

$$\int_{\mathbb{B}}|\nabla u(x)|^p\,dx < \infty$$

when $(-\alpha - 1)p + n > 0$. Similarly, we obtain

$$\int_{\mathbb{B}}(b(x)|\nabla u(x)|)^q dx < \infty$$

when $(-\alpha - 1 + \theta)q + n > 0$. Finally we see that

$$S_\omega((bu)_t) \ge C(1-t)^{\theta-\alpha+(n-1)/\omega},$$

where $\theta - \alpha + (n-1)/\omega = \{(-\alpha - 1 + \theta) + n/q\} - \{(n-q)/q - (n-1)/\omega\} < 0$.

Now, letting $0 < (-\alpha - 1) + n/p = (-\alpha - 1 + \theta) + n/q < \min\{\varepsilon, (n-q)/q - (n-1)/\omega\}$, we find

$$\lim_{t\to 1}(1-t)^{\eta-\varepsilon}S_\omega((bu)_t) = \infty,$$

which shows the best possibility as to the growth order.

Remark 4.2. The weight in Theorem 1.7 is the best possible. To show this, for $0 < \alpha < 1/n'$, let us consider the functions

$$u(x) = (\log(3/|x - e|))^{\alpha} \quad \text{and} \quad b(x) = 1,$$

where $e = (1, 0, \dots, 0)$. Note here that u is monotone in $\mathbb{B}$. If $\alpha + \varepsilon - 1/n' > 0$, then

$$\lim_{x\to e}(\log(3/|x - e|))^{\varepsilon - 1/n'}b(x)u(x) = \infty.$$

To show that u satisfies (1.2), note that

$$|\nabla u(x)| \le \alpha(\log(3/|x-e|))^{\alpha-1}|x-e|^{-1}$$

and hence

$$\int_{\mathbb{B}} |\nabla u(x)|^n \, dx < \infty,$$

since $(\alpha-1)n+1<0$.

Remark 4.3. We can find a monotone function u on $\mathbb{B}$ in the sense of Lebesgue and a non-negative bounded θ-Hölder continuous function b on $\mathbb{B}$ satisfying

$$\int_{\mathbb{B}} (b(x)|\nabla u(x)|)^q \, dx < \infty \tag{4.1}$$

and

$$\limsup_{r\to 1} \, (1-r)^\eta S_\omega(bu,r) = \infty, \tag{4.2}$$

when $\eta = (n-q)/q - (n-1)/\omega > 0$.

For this, let

$$\varphi(t) = \begin{cases} 1 & (t \le 1), \\ 2-t & (1 < t \le 2), \\ 0 & (t > 2). \end{cases}$$

Take $\xi = (1,0,\dots,0)$, $a>0$ and $c>1$, and consider

$$u(x) = |x-\xi|^{-a} \quad \text{and} \quad b(x) = \sum_{j=1}^{\infty} |x-x_j|^\theta \varphi(|x-x_j|2^{cj}),$$

where $x_j = (1-2^{-j})\xi$. Then

(1) $|b(x)-b(y)| \le C|x-y|^\theta$ for $x,y \in \mathbb{B}$.

In fact, if $x,y \in B(x_j, 2^{1-cj})$, then

$$\begin{aligned} |b(x)-b(y)| &\le \left||x-x_j|^\theta - |y-x_j|^\theta\right| + |y-x_j|^\theta \,||x-x_j| - |y-x_j||\, 2^{cj} \\ &\le |x-y|^\theta + |y-x_j|^\theta |x-y| 2^{cj} \\ &\le C|x-y|^\theta. \end{aligned}$$

This also holds for general $x,y \in \mathbb{B}$. Moreover,

(2) $\displaystyle\int_{\mathbb{B}} (b(x)|\nabla u(x)|)^q \, dx \le C\sum_{j=1}^{\infty} 2^{(1+a)qj} 2^{-(\theta q+n)cj}$;

(3) $S_\omega(bu, 1-2^{-j}) \ge C 2^{aj} 2^{-(\theta+(n-1)/\omega)cj}$.

If $(1+a)-(\theta+n/q)c<0$ and $-(n-q)/q+(n-1)/\omega+a-(\theta+(n-1)/\omega)c>0$, that is,

$$(n-q)/q - (n-1)/\omega + (\theta+(n-1)/\omega)c < a < -1 + (\theta+n/q)c,$$

then (2) and (3) imply (4.1) and (4.2), respectively.

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Yoshihiro Mizuta
Hiroshima University, Graduate School of
Advanced Science and Engineering
Department of Mathematics
Higashi-Hiroshima 739-8521, Japan
yomizuta@hiroshima-u.ac.jp

Tetsu Shimomura
Hiroshima University, Graduate School of
Humanities and Social Sciences
Department of Mathematics
Higashi-Hiroshima 739-8524, Japan
tshimo@hiroshima-u.ac.jp