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Reverse integral Hardy inequality on metric measure spaces

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Abstract. In this note, we obtain a reverse version of the integral Hardy inequality on metric measure spaces. Moreover, we give necessary and sufficient conditions for the weighted reverse Hardy inequality to be true. The main tool in our proof is a continuous version of the reverse Minkowski inequality. In addition, we present some consequences of the obtained reverse Hardy inequality on the homogeneous groups, hyperbolic spaces and Cartan–Hadamard manifolds.

Käänteinen Hardyn integraaliepäyhtälö metrisissä mitta-avaruuksissa

Tiivistelmä. Johdamme käänteisen Hardyn integraaliepäyhtälön metrisissä mitta-avaruuksissa ja annamme riittävät ja välttämättömät ehdot sen painolliselle versiolle. Todistuksemme päätyökalu on käänteisen Minkowskin epäyhtälön jatkuva versio. Lisäksi esitämme johtamamme käänteisen Hardyn epäyhtälön seurauksia homogeenissa ryhmissä, hyperbolisissa avaruuksissa ja Cartanin–Hadamardin monistoissa.

1. Introduction

In one of the pioneering papers of Hardy [10] and [11], he proved the following (direct) inequality:

$$\int_a^\infty \frac{1}{x^p}\left(\int_a^\infty f(t)\,dt\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_a^\infty f^p(x)\,dx, \tag{1.1}$$

where $f \ge 0$, $p > 1$, and $a > 0$. Today's literature on the development of the extensions of this integral Hardy inequality is very large, see e.g. [3, 5, 16, 17, 18, 19] and [20]. Note that the multi-dimensional version of the integral Hardy inequality was proved in [4].

In [2], the authors obtained the so-called reverse integral Hardy inequality in the following form:

$$\left(\int_a^b \left(\int_a^x f(t)\,dt\right)^q u(x)\,dx\right)^{\frac{1}{q}} \ge C\left(\int_a^b f^p(x)v(x)\,dx\right)^{\frac{1}{p}}, \tag{1.2}$$

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and the conjugate reverse integral Hardy inequality

$$\left(\int_a^b \left(\int_x^b f(t)dt\right)^q u(x)\,dx\right)^{\frac{1}{q}} \geq C\left(\int_a^b f^p(x)v(x)\,dx\right)^{\frac{1}{p}}, \tag{1.3}$$

where $f \geq 0$, for some positive weights u, v and $p, q < 0$. The reverse Hardy inequalities were also studied in [9, 15, 14] and [21].

The main aim of the present paper is to extend the reverse Hardy inequalities to general metric measure spaces. More specifically, we consider metric spaces $\mathbb{X}$ with a Borel measure dx allowing for the following *polar decomposition* at $a \in \mathbb{X}$: we assume that there is a locally integrable function $\lambda \in L^1_{\mathrm{loc}}$ such that for all $f \in L^1(\mathbb{X})$ we have

$$\int_{\mathbb{X}} f(x)\,dx = \int_0^\infty \int_{\Sigma_r} f(r,\omega)\lambda(r,\omega)\,d\omega\,dr, \tag{1.4}$$

for the set $\Sigma_r = \{x \in \mathbb{X}\colon d(x,a) = r\} \subset \mathbb{X}$ with a measure on it denoted by $d\omega$, and $(r,\omega) \to a$ as $r \to 0$.

As it can be seen, λ depends on the variables $x = (r,\omega)$, which yields a polar decomposition (1.4) which is rather general. We assume (1.4), because $\mathbb{X}$ does not have to have a differentiable structure. If there exits a differentiable structure on $\mathbb{X}$, the condition (1.4) can be obtained as the standard polar decomposition formula. Euclidean space $\mathbb{R}^N$ with $\lambda(r,\omega) = r^{N-1}$, homogeneous groups $\mathbb{G}$ with $\lambda(r,\omega) = r^{Q-1}$, (where Q is the homogeneous dimension of the group, see e.g., [7, 6]) and hyperbolic space $\mathbb{H}^n$ with $\lambda(r,\omega) = (\sinh r)^{n-1}$ are examples of $\mathbb{X}$ with the polar decomposition (1.4) with different $\lambda(r,\omega)$. Also, Cartan–Hadamard manifolds have the following polar decomposition:

$$\int_M f(x)\,dx = \int_0^\infty \int_{\mathbb{S}^{n-1}} f(\exp_a(\rho\omega))J(\rho,\omega)\rho^{n-1}\,d\rho\,d\omega,$$

where $a \in M$ is a fixed point, $\rho(x) = d(x,a)$ is the geodesic distance between x and a on M, $J(\rho,\omega)$ is the density function on M and $\exp_a$ is the exponential map from T_a to M (see for more detials, [8], [12] and [28]).

In [28] and [29], the (direct) integral Hardy inequalities on metric measure space were established for $1 < p \leq q < \infty$ and $0 < q < p$, $1 < p < \infty$, respectively, with applications on homogeneous Lie groups, hyperbolic spaces, Cartan–Hadamard manifolds with negative curvature and on general Lie groups with Riemannian distance. Also, on Riemannian manifolds the Hardy inequality was obtained in [31], and on homogeneous Lie groups the Hardy inequality was obtained in [13], [24]–[23] and [30]. In the present paper, we continue the analysis in the general setting of metric measure spaces as in [28] and show the reverse integral Hardy inequality with $q < 0$ and $p \in (0,1)$. We also discuss its consequences for homogeneous Lie groups, hyperbolic spaces and Cartan–Hadamard manifolds with negative curvature.

2. Main result

Let us recall briefly the reverse Hölder's inequality.

Theorem 2.1. [1, Theorem 2.12, p. 27] *Let $p \in (0,1)$, so that $p' = \frac{p}{p-1} < 0$. If non-negative functions satisfy $0 < \int_{\mathbb{X}} f^p(x)\,dx < +\infty$ and $0 < \int_{\mathbb{X}} g^{p'}(x)\,dx < +\infty$,* *we have*

$$\int_{\mathbb{X}} f(x)g(x)\,dx \geq \left(\int_{\mathbb{X}} f^p(x)\,dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{X}} g^{p'}(x)\,dx\right)^{\frac{1}{p'}}. \tag{2.1}$$

Let us present the reverse integral Minkowski inequality (or a continuous version of reverse Minkowski inequality).

Theorem 2.2. *Let $\mathbb{X}, \mathbb{Y}$ be metric measure spaces and let $F = F(x,y) \in \mathbb{X} \times \mathbb{Y}$ be a non-negative measurable function. Then we have*

$$\left[\int_{\mathbb{X}} \left(\int_{\mathbb{Y}} F(x,y)\,dy\right)^q dx\right]^{\frac{1}{q}} \geq \int_{\mathbb{Y}} \left(\int_{\mathbb{X}} F^q(x,y)\,dx\right)^{\frac{1}{q}} dy, \quad q < 0. \tag{2.2}$$

Proof. Let us consider the following function:

$$A(x) := \int_{\mathbb{Y}} F(x,y)\,dy, \tag{2.3}$$

so we have

$$A^q(x) = \left(\int_{\mathbb{Y}} F(x,y)\,dy\right)^q. \tag{2.4}$$

By integrating over $\mathbb{X}$ both sides and by using reverse Hölder's inequality (Theorem 2.1), we obtain

$$\begin{aligned}
\int_{\mathbb{X}} A^q(x)\,dx &= \int_{\mathbb{X}} A^{q-1}(x)A(x)\,dx \\
&= \int_{\mathbb{X}} A^{q-1}(x) \int_{\mathbb{Y}} F(x,y)\,dy\,dx = \int_{\mathbb{Y}} \int_{\mathbb{X}} A^{q-1}(x)F(x,y)\,dx\,dy \\
&\overset{(2.1)}{\geq} \int_{\mathbb{Y}} \left(\int_{\mathbb{X}} A^{q-1\frac{q}{q-1}}(x)\,dx\right)^{\frac{q-1}{q}} \left(\int_{\mathbb{X}} F^q(x,y)dx\right)^{\frac{1}{q}} dy \\
&= \left(\int_{\mathbb{X}} A^q(x)\,dx\right)^{\frac{q-1}{q}} \int_{\mathbb{Y}} \left(\int_{\mathbb{X}} F^q(x,y)\,dx\right)^{\frac{1}{q}} dy.
\end{aligned} \tag{2.5}$$

From this, we get

$$\left[\int_{\mathbb{X}} \left(\int_{\mathbb{Y}} F(x,y)dy\right)^q dx\right]^{\frac{1}{q}} \geq \int_{\mathbb{Y}} \left(\int_{\mathbb{X}} F^q(x,y)\,dx\right)^{\frac{1}{q}} dy, \tag{2.6}$$

proving (2.2). □

Remark 2.3. In our sense, the negative exponent q of 0, we understand in the following form:

$$0^q = (+\infty)^{-q} = +\infty \quad \text{and} \quad 0^{-q} = (+\infty)^q = 0. \tag{2.7}$$

We denote by $B(a,r)$ the ball in $\mathbb{X}$ with centre a and radius r, i.e.

$$B(a,r) := \{x \in \mathbb{X} \colon d(x,a) < r\},$$

where d is the metric on $\mathbb{X}$. Once and for all we will fix some point $a \in \mathbb{X}$, and we will write

$$|x|_a := d(a,x). \tag{2.8}$$

Now we prove the reverse integral Hardy inequality on a metric measure space.

Theorem 2.4. *Assume that $p \in (0,1)$ and $q < 0$. Let $\mathbb{X}$ be a metric measure space with a polar decomposition at $a \in \mathbb{X}$. Suppose that $u, v > 0$ and $u, v^{1-p'}$ are locally integrable functions on $\mathbb{X}$. Then the inequality*

$$\left[\int_{\mathbb{X}}\left(\int_{B(a,|x|_a)} f(y)\,dy\right)^q u(x)\,dx\right]^{\frac{1}{q}} \geq C(p,q)\left(\int_{\mathbb{X}} f^p(x)v(x)\,dx\right)^{\frac{1}{p}} \tag{2.9}$$

holds for some $C(p,q) > 0$ and for all non-negative real-valued measurable functions f, if and only if

$$0 < D_1 := \inf_{x\neq a}\left[\left(\int_{\mathbb{X}\setminus B(a,|x|_a)} u(y)\,dy\right)^{\frac{1}{q}}\left(\int_{B(a,|x|_a)} v^{1-p'}(y)\,dy\right)^{\frac{1}{p'}}\right]. \tag{2.10}$$

Moreover, the biggest constant $C(p,q)$ in (2.9) has the following relation to D_1:

$$D_1 \geq C(p,q) \geq \left(\frac{p'}{p'+q}\right)^{-\frac{1}{q}}\left(\frac{q}{p'+q}\right)^{-\frac{1}{p'}} D_1. \tag{2.11}$$

Proof. Let us divide proof of this theorem in several steps.

Step 1. Let us denote $g(x) := f(x)v^{\frac{1}{p}}(x)$. Let $\alpha \in \left(0, -\frac{1}{p'}\right)$ and $z(x) = v^{-\frac{1}{p}}(x)$ where $\frac{1}{p} + \frac{1}{p'} = 1$. Let us denote, using (1.4),

$$V(x) := \int_{B(a,|x|_a)} v^{-\frac{p'}{p}}(y)\,dy = \int_{B(a,|x|_a)} z^{p'}(y)\,dy, \tag{2.12}$$

$$H_1(s) := \int_{\sum_s} \lambda(s,\sigma)g(s,\sigma)z(s,\sigma)\,d\sigma, \tag{2.13}$$

$$H_2(s) := \int_{\sum_s} \lambda(s,\sigma)z^{p'}(s,\sigma)V^{\alpha p'}(s,\sigma)\,d\sigma, \tag{2.14}$$

$$H_3(s) := \int_{\sum_s} \lambda(s,\sigma)g^p(s,\sigma)V^{-\alpha p}(s,\sigma)\,d\sigma, \tag{2.15}$$

$$U(r) := \int_{\sum_r} \lambda(r,\omega)u(r,\omega)\,d\omega. \tag{2.16}$$

After some calculation, we compute, using reverse Hölder's inequality (Theorem 2.1),

$$\begin{aligned}
A &:= \int_{\mathbb{X}}\left(\int_{B(a,|x|_a)} f(y)\,dy\right)^q u(x)\,dx \\
&= \int_{\mathbb{X}}\left(\int_{B(a,|x|_a)} g(y)z(y)\,dy\right)^q u(x)\,dx \\
&= \int_{\mathbb{X}}\left(\int_{B(a,|x|_a)} g(y)z(y)\,dy\right)^p\left(\int_{B(a,|x|_a)} g(y)z(y)\,dy\right)^{q-p} u(x)\,dx \\
&= \int_{\mathbb{X}}\left(\int_{B(a,|x|_a)} g(y)V^{-\alpha}(y)V^{\alpha}(y)z(y)\,dy\right)^p\left(\int_{B(a,|x|_a)} g(y)z(y)\,dy\right)^{q-p} u(x)\,dx
\end{aligned}$$

$$
\begin{aligned}
(2.17)\qquad & \geq \int_{\mathbb{X}} \left(\int_{B(a,|x|_a)} g^p(y)V^{-\alpha p}(y)\,dy\right)\left(\int_{B(a,|x|_a)} z^{p'}(y)V^{\alpha p'}(y)\,dy\right)^{\frac{p}{p'}} \\
& \times \left(\int_{B(a,|x|_a)} g(y)z(y)\,dy\right)^{q-p} u(x)\,dx \\
& = \int_0^\infty U(r)\left(\int_0^r H_1(s)\,ds\right)^{q-p}\left(\int_0^r H_2(s)\,ds\right)^{\frac{p}{p'}}\left(\int_0^r H_3(s)\,ds\right)dr.
\end{aligned}
$$

Let us denote by $\tilde{H}_2(s) := \int_{\sum_s} \lambda(s,\sigma)z^{p'}(s,\sigma)\,d\sigma$. Then we have

$$
\begin{aligned}
\left(\int_0^r H_2(s)ds\right)^{\frac{p}{p'}} &= \left(\int_0^r \int_{\sum_s} \lambda(s,\sigma)z^{p'}(s,\sigma)V^{\alpha p'}(s,\sigma)\,ds\,d\sigma\right)^{\frac{p}{p'}} \\
&= \left(\int_0^r\int_{\sum_s}\lambda(s,\sigma)z^{p'}(s,\sigma)\left(\int_0^s\int_{\sum_\rho}\lambda(\rho,\sigma_1)z^{p'}(\rho,\sigma_1)\,d\rho\,d\sigma_1\right)^{\alpha p'} ds\,d\sigma\right)^{\frac{p}{p'}} \\
(2.18)\qquad &= \left(\int_0^r \tilde{H}_2(s)\left(\int_0^s \tilde{H}_2(\rho)d\rho\right)^{\alpha p'} ds\right)^{\frac{p}{p'}} \\
&= \left(\int_0^r \left(\int_0^s \tilde{H}_2(\rho)\,d\rho\right)^{\alpha p'} d_s\left(\int_0^s \tilde{H}_2(\rho)\,d\rho\right)\right)^{\frac{p}{p'}} \\
&\overset{1+\alpha p'>0}{=} \frac{1}{(1+\alpha p')^{\frac{p}{p'}}}\left(\left(\int_0^s \tilde{H}_2(\rho)\,d\rho\right)^{1+\alpha p'}\Big|_0^r\right)^{\frac{p}{p'}} \\
&\overset{1+\alpha p'>0}{=} \frac{1}{(1+\alpha p')^{\frac{p}{p'}}}\left(\int_0^r \tilde{H}_2(\rho)\,d\rho\right)^{\frac{p(1+\alpha p')}{p'}} \frac{V_1^{\frac{p(1+\alpha p')}{p'}}(r)}{(1+\alpha p')^{\frac{p}{p'}}},
\end{aligned}
$$

where $V_1(r) = \int_0^r \tilde{H}_2(\rho)\,d\rho$. By using this fact and reverse Hölder's inequality with $\frac{p}{q} + \frac{q-p}{q} = 1$, we obtain

$$
\begin{aligned}
A &\geq \int_0^\infty \left(\int_0^r H_3(s)\,ds\right)U(r)\left(\int_0^r H_1(s)\,ds\right)^{q-p}\left(\int_0^r H_2(s)\,ds\right)^{\frac{p}{p'}} dr \\
&\overset{(2.18)}{=} \frac{1}{(1+\alpha p')^{\frac{p}{p'}}}\int_0^\infty \left(\int_0^r H_3(s)\,ds\right)U(r)\left(\int_0^r H_1(s)\,ds\right)^{q-p} V_1^{\frac{p(1+\alpha p')}{p'}}(r)\,dr \\
&= \frac{1}{(1+\alpha p')^{\frac{p}{p'}}}\int_0^\infty U^{\frac{p}{q}}(r)\left(\int_0^r H_3(s)\,ds\right)V_1^{\frac{p(1+\alpha p')}{p'}}(r)\left(\int_0^r H_1(s)\,ds\right)^{q-p} U^{\frac{q-p}{q}}\,dr \\
&\geq \frac{1}{(1+\alpha p')^{\frac{p}{p'}}}\left(\int_0^\infty \left(\int_0^r H_3(s)\,ds\right)^{\frac{q}{p}} U(r)V_1^{\frac{q(1+\alpha p')}{p'}}(r)\,dr\right)^{\frac{p}{q}} \\
&\quad\times\left(\int_0^\infty\left(\int_0^r H_1(s)\,ds\right)^q U(r)\,dr\right)^{\frac{q-p}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{(1+\alpha p')^{\frac{p}{p'}}} \left(\int_0^\infty U(r) \left(\int_0^r H_3(s)\, ds \right)^{\frac{q}{p}} V_1^{\frac{q(1+\alpha p')}{p'}}(r)\, dr \right)^{\frac{p}{q}} \\
&\times \left(\int_{\mathbb{X}} \left(\int_{B(a,|x|_a)} g(y) z(y)\, dy \right)^q u(x)\, dx \right)^{\frac{q-p}{q}} \\
&= \frac{A^{\frac{q-p}{q}}}{(1+\alpha p')^{\frac{p}{p'}}} \left(\int_0^\infty U(r) \left(\int_0^r H_3(s)\, ds \right)^{\frac{q}{p}} V_1^{\frac{q(1+\alpha p')}{p'}}(r)\, dr \right)^{\frac{p}{q}}.
\end{aligned}
$$

Therefore,

$$
A^{\frac{p}{q}} \geq \frac{1}{(1+\alpha p')^{\frac{p}{p'}}} \left(\int_0^\infty U(r) \left(\int_0^r H_3(s)\, ds \right)^{\frac{q}{p}} V_1^{\frac{q(1+\alpha p')}{p'}}(r)\, dr \right)^{\frac{p}{q}}.
$$

Let us treat the following integral with the reverse Minkowski inequality with exponent $\frac{q}{p} < 0$, so that we obtain

$$
\begin{aligned}
&\left(\int_0^\infty U(r) \left(\int_0^r H_3(s)\, ds \right)^{\frac{q}{p}} V_1^{\frac{q(1+\alpha p')}{p'}}(r)\, dr \right)^{\frac{p}{q}} \\
&= \left(\int_0^\infty \left(\int_0^r U^{\frac{p}{q}}(r) H_3(s) V^{\frac{(1+\alpha p')p}{p'}}(r)\, ds \right)^{\frac{q}{p}} dr \right)^{\frac{p}{q}} \\
&= \left(\int_0^\infty \left(\int_0^\infty U^{\frac{p}{q}}(r) H_3(s) V_1^{\frac{(1+\alpha p')p}{p'}}(r) \chi_{\{s<r\}}\, ds \right)^{\frac{q}{p}} dr \right)^{\frac{p}{q}} \\
&\overset{(2.2)}{\geq} \int_0^\infty H_3(s) \left(\int_s^\infty U(r) V_1^{\frac{q(1+\alpha p')}{p'}}(r)\, dr \right)^{\frac{p}{q}} ds \\
&= \int_{\mathbb{X}} g^p(y) V^{-\alpha p}(y) \left(\int_{\mathbb{X}\setminus B(a,|y|_a)} u(x) V^{\frac{q(1+\alpha p')}{p'}}(x)\, dx \right)^{\frac{p}{q}} dy \\
&\geq D^p(\alpha) \int_{\mathbb{X}} g^p(y)\, dy,
\end{aligned}
$$

where $D(\alpha) := \inf_{x\neq a} D(x,\alpha) = \inf_{x\neq a} V^{-\alpha}(x) \big(\int_{\mathbb{X}\setminus B(a,|x|_a)} u(y) V^{\frac{q(1+\alpha p')}{p'}}(y)\, dy \big)^{\frac{1}{q}}$ and χ is the cut-off function. Then we obtain

$$
\begin{aligned}
A^{\frac{p}{q}} &= \left(\int_{\mathbb{X}} \left(\int_{B(a,|x|_a)} f(y)\, dy \right)^q u(x)\, dx \right)^{\frac{p}{q}} \geq \frac{D^p(\alpha)}{(1+\alpha p')^{\frac{p}{p'}}} \int_{\mathbb{X}} g^p(y)\, dy \\
&= \frac{D^p(\alpha)}{(1+\alpha p')^{\frac{p}{p'}}} \int_{\mathbb{X}} f^p(y) v(y)\, dy.
\end{aligned}
$$

Step 2. Let us recall D_1, given in the following form:

$$
0 < D_1 = \inf_{x\neq a} \left[\left(\int_{\mathbb{X}\setminus B(a,|x|_a)} u(x)\, dx \right)^{\frac{1}{q}} \left(\int_{B(a,|x|_a)} v^{1-p'}(y)\, dy \right)^{\frac{1}{p'}} \right]. \tag{2.19}
$$

Let us note a relation between V and V_1, where $V = V(x)$ is defined by (2.12),

$$
\begin{aligned}
V(x) = \int_{B(a,|x|_a)} v^{-\frac{p'}{p}}(y)\,dy &= \int_{B(a,|x|_a)} z^{p'}(y)\,dy \\
&= \int_0^{|x|_a} \int_{\sum_r} z^{p'}(r,\omega)\lambda(r,\omega)\,dr\,d\omega \\
&= \int_0^{|x|_a} \tilde{H}_2(r)\,dr =: V_1(|x|_a),
\end{aligned}
\tag{2.20}
$$

where, as before, $\tilde{H}_2(r) = \int_{\sum_r} z^{p'}(r,\omega)\lambda(r,\omega)\,d\omega$. Then let us calculate and estimate the following integral:

$$
\begin{aligned}
I &= \int_{\mathbb{X}\setminus B(a,|x|_a)} u(y) V^{\frac{q(1+\alpha p')}{p'}}(y)\,dy = \int_{|x|_a}^{\infty}\int_{\sum_r} \lambda(r,\omega)u(r,\omega)V_1^{\frac{q(1+\alpha p')}{p'}}(r)\,dr\,d\omega \\
&= \int_{|x|_a}^{\infty} U(r) V_1^{\frac{q(1+\alpha p')}{p'}}(r)\,dr = \int_{|x|_a}^{\infty} V_1^{\frac{q(1+\alpha p')}{p'}}(r)\,d_r\left(-\int_r^{\infty} U(s)\,ds\right) \\
&= -V_1^{\frac{q(1+\alpha p')}{p'}}(r)\int_r^{\infty} U(s)\,ds\Big|_{|x|_a}^{\infty} \\
&\quad + \frac{q(1+\alpha p')}{p'}\int_{|x|_a}^{\infty}\left(\int_r^{\infty} U(s)\,ds\right) V_1^{\frac{q(1+\alpha p')}{p'}-1}(r)\,dV_1(r) \\
&\overset{\frac{q}{p'}>0}{=} V_1^{\frac{q(1+\alpha p')}{p'}}(|x|_a)\int_{|x|_a}^{\infty} U(s)\,ds \\
&\quad + \frac{q(1+\alpha p')}{p'}\int_{|x|_a}^{\infty}\left(\int_r^{\infty} U(s)\,ds\right) V_1^{\frac{q(1+\alpha p')}{p'}-1}(r)\,dV_1(r) \\
&= V_1^{\frac{q}{p'}}(|x|_a)\left(\int_{|x|_a}^{\infty} U(s)\,ds\right) V_1^{\alpha q}(|x|_a) \\
&\quad + \frac{q(1+\alpha p')}{p'}\int_{|x|_a}^{\infty}\left(\int_r^{\infty} U(s)\,ds\right) V_1^{\frac{q}{p'}}(r) V_1^{\alpha q-1}(r)\,dV_1(r) \\
&\leq D_1^q V_1^{\alpha q}(|x|_a) + \frac{q(1+\alpha p')D_1^q}{p'}\int_{|x|_a}^{\infty} V_1^{\alpha q-1}(r)\,dV_1(r) \\
&= D_1^q V_1^{\alpha q}(|x|_a) + \frac{(1+\alpha p')D_1^q}{\alpha p'} V_1^{\alpha q}(r)\Big|_{|x|_a}^{\infty} \\
&= D_1^q V_1^{\alpha q}(|x|_a) + \lim_{r\to\infty}\frac{(1+\alpha p')D_1^q}{\alpha p'} V_1^{\alpha q}(r) - \frac{(1+\alpha p')D_1^q}{\alpha p'} V_1^{\alpha q}(|x|_a) \\
&\overset{\frac{(1+\alpha p')D_1^q}{\alpha p'}<0}{\leq} D_1^q V_1^{\alpha q}(|x|_a) - \frac{(1+\alpha p')D_1^q}{\alpha p'} V_1^{\alpha q}(|x|_a) \\
&\overset{(2.20)}{=} -\frac{1}{\alpha p'} D_1^q V^{\alpha q}(x).
\end{aligned}
\tag{2.21}
$$

Then we have $I = D^q(x,\alpha)V^{\alpha q}(x) \leq -\frac{1}{\alpha p'}D_1^q V^{\alpha q}(x)$. Consequently,

$$
D(x,\alpha) \geq (-\alpha p')^{-\frac{1}{q}} D_1,
$$

it means

$$
D(\alpha) \geq (-\alpha p')^{-\frac{1}{q}} D_1.
$$

Finally, we obtain

$$A^{\frac{1}{q}} \geq \frac{D_1(-\alpha p')^{-\frac{1}{q}}}{(1+\alpha p')^{\frac{1}{p'}}} \left(\int_{\mathbb{X}} f^p(y) v(y)\, dy\right)^{\frac{1}{p}}.$$

Let us consider the function $k(\alpha) := \frac{(-\alpha p')^{-\frac{1}{q}}}{(1+\alpha p')^{\frac{1}{p'}}} = (-\alpha p')^{-\frac{1}{q}}(1+\alpha p')^{-\frac{1}{p'}}$, where $\alpha \in \left(0, -\frac{1}{p'}\right)$. Firstly, let us find extremum of this function. We have

$$\begin{aligned}
\frac{dk(\alpha)}{d\alpha} &= -\frac{1}{q}(-p')(-\alpha p')^{-\frac{1}{q}-1}(1+\alpha p')^{-\frac{1}{p'}} \\
&\quad + \left(-\frac{1}{p'}\right) p'(1+\alpha p')^{-\frac{1}{p'}-1}(-\alpha p')^{-\frac{1}{q}} \\
&= p'(-\alpha p')^{-\frac{1}{q}-1}(1+\alpha p')^{-\frac{1}{p'}-1}\left(\frac{(1+\alpha p')}{q} + \alpha\right) \\
&= \frac{p'}{q}(-\alpha p')^{-\frac{1}{q}-1}(1+\alpha p')^{-\frac{1}{p'}-1}\left(\alpha(p'+q)+1\right) = 0,
\end{aligned} \tag{2.22}$$

which implies that its solution is

$$\alpha_1 = -\frac{1}{p'+q} \in \left(0, -\frac{1}{p'}\right).$$

By taking the second derivative of $k(\alpha)$ at the point α_1 and by denoting $k_1(\alpha) = (-\alpha p')^{-\frac{1}{q}-1}(1+\alpha p')^{-\frac{1}{p'}-1}$, we obtain

$$\frac{d^2k(\alpha)}{d\alpha^2}\Big|_{\alpha=\alpha_1} = \frac{p'(p'+q)}{q}\left(\frac{p'}{p'+q}\right)^{-\frac{1}{q}-1}\left(\frac{q}{p'+q}\right)^{-\frac{1}{p'}-1} < 0. \tag{2.23}$$

This means that, function $k(\alpha)$ has supremum at the point $\alpha = \alpha_1$. Then, the biggest constant has the following relationship $C(p,q) \geq \left(\frac{p'}{p'+q}\right)^{-\frac{1}{q}}\left(\frac{q}{p'+q}\right)^{-\frac{1}{p'}} D_1$.

Step 3. Let us give a necessity condition of inequality (2.9). By using (2.9) and test function $f(x) = v^{-\frac{p'}{p}}(x)\chi_{\{(0,t)\}}(|x|_a)$, we can estimate as follows

$$\begin{aligned}
C(p,q) &\leq \left[\int_{\mathbb{X}}\left(\int_{B(a,|x|_a)} f(y)\,dy\right)^q u(x)\,dx\right]^{\frac{1}{q}}\left[\int_{\mathbb{X}} f^p(y)v(y)\,dx\right]^{-\frac{1}{p}} \\
&= \left[\int_{\mathbb{X}}\left(\int_{|y|_a\leq t} v^{1-p'}(y)\,dy\right)^q u(x)\,dx\right]^{\frac{1}{q}}\left[\int_{|y|_a\leq t} v^{-p'}(y)v(y)\,dx\right]^{-\frac{1}{p}} \\
&\overset{q<0}{\leq} \left[\int_{|x|_a\geq t}\left(\int_{|y|_a\leq t} v^{1-p'}(y)\,dy\right)^q u(x)\,dx\right]^{\frac{1}{q}}\left[\int_{|y|_a\leq t} v^{1-p'}(y)\,dx\right]^{-\frac{1}{p}} \\
&= \left[\int_{|x|_a\geq t} u(x)\,dx\right]^{\frac{1}{q}}\left[\int_{|y|_a\leq t} v^{1-p'}(y)\,dx\right]^{\frac{1}{p'}},
\end{aligned} \tag{2.24}$$

which gives $D_1 \geq C(p,q)$. □

Let us give conjugate reverse integral Hardy inequality.

Theorem 2.5. *Assume that $p \in (0,1)$ and $q < 0$. Let $\mathbb{X}$ be a metric measure space with a polar decomposition at a. Suppose that $u, v > 0$ and $u, v^{1-p'}$ are locally* *integrable functions on $\mathbb{X}$. Then the inequality*

$$\left[\int_{\mathbb{X}}\left(\int_{\mathbb{X}\backslash B(a,|x|_a)} f(y)\,dy\right)^q u(x)\,dx\right]^{\frac{1}{q}} \geq C(p,q)\left(\int_{\mathbb{X}} f^p(x)v(x)\,dx\right)^{\frac{1}{p}} \tag{2.25}$$

holds for some $C(p,q)>0$ and for all non-negative real-valued measurable functions f, if only if

$$0 < D_2 := \inf_{x\neq a}\left[\left(\int_{B(a,|x|_a)} u(y)\,dy\right)^{\frac{1}{q}}\left(\int_{\mathbb{X}\backslash B(a,|x|_a)} v^{1-p'}(y)\,dy\right)^{\frac{1}{p'}}\right]. \tag{2.26}$$

Moreover, the biggest constant $C(p,q)$ in (2.25) *has the following relation to D_2:*

$$D_2 \geq C(p,q) \geq \left(\frac{p'}{p'+q}\right)^{-\frac{1}{q}}\left(\frac{q}{p'+q}\right)^{-\frac{1}{p'}} D_2. \tag{2.27}$$

Proof. The proof of this theorem is step by step similar to that of Theorem 2.4 so we omit the details. ☐

3. Consequences

In this section, we consider some consequences of the reverse integral Hardy inequality.

3.1. Homogeneous groups. Let us recall that a Lie group (on $\mathbb{R}^n$) $\mathbb{G}$ with the dilation

$$D_\lambda(x) := (\lambda^{\nu_1}x_1,\ldots,\lambda^{\nu_n}x_n), \quad \nu_1,\ldots,\nu_n>0, \quad D_\lambda\colon \mathbb{R}^n\to\mathbb{R}^n,$$

which is an automorphism of the group $\mathbb{G}$ for each $\lambda>0$, is called a *homogeneous (Lie) group*. For simplicity, throughout this paper we use the notation λx for the dilation D_λ. The homogeneous dimension of the homogeneous group $\mathbb{G}$ is denoted by $Q:=\nu_1+\ldots+\nu_n$. Also, in this note we denote a homogeneous quasi-norm on $\mathbb{G}$ by $|x|$, which is a continuous non-negative function

$$\mathbb{G}\ni x\mapsto |x|\in[0,\infty), \tag{3.1}$$

with the properties

i) $|x|=|x^{-1}|$ for all $x\in\mathbb{G}$,
ii) $|\lambda x|=\lambda|x|$ for all $x\in\mathbb{G}$ and $\lambda>0$,
iii) $|x|=0$ iff $x=0$.

Moreover, the following polarisation formula on homogeneous Lie groups will be used in our proofs: there is a (unique) positive Borel measure σ on the unit quasi-sphere $\mathfrak{S}:=\{x\in\mathbb{G}:\,|x|=1\}$, so that for every $f\in L^1(\mathbb{G})$ we have

$$\int_{\mathbb{G}} f(x)\,dx = \int_0^\infty\int_{\mathfrak{S}} f(ry)r^{Q-1}d\sigma(y)dr. \tag{3.2}$$

We refer to [7] for the original appearance of such groups, and to [6] for a recent comprehensive treatment. Let us define the quasi-ball centered at x with radius r in the following form:

$$B(x,r) := \{y\in\mathbb{G}\colon |x^{-1}y|<r\}. \tag{3.3}$$

Then we have the following reverse integral Hardy inequality on homogeneous Lie groups.

Corollary 3.1. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q with a quasi-norm $|\cdot|$. Assume that $q<0$, $p\in(0,1)$ and $\alpha,\beta\in\mathbb{R}$. Then the reverse integral Hardy inequality*

$$\left[\int_{\mathbb{G}}\left(\int_{B(0,|x|)} f(y)\,dy\right)^{q}|x|^{\alpha}\,dx\right]^{\frac{1}{q}} \geq C\left(\int_{\mathbb{G}} f^{p}(x)|x|^{\beta}\,dx\right)^{\frac{1}{p}}, \tag{3.4}$$

holds for $C>0$ and for all non-negative measurable functions f, if only if

$$\alpha+Q<0,\quad \beta(1-p')+Q>0 \quad \text{and} \quad \frac{Q+\alpha}{q}+\frac{Q+\beta(1-p')}{p'}=0. \tag{3.5}$$

Moreover, the biggest constant C for (3.4) satisfies

$$\begin{aligned} &\left(\frac{|\mathfrak{S}|}{|\alpha+Q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}} \geq C \\ &\geq \left(\frac{|\mathfrak{S}|}{|\alpha+Q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}}\left(\frac{p'}{p'+q}\right)^{-\frac{1}{q}}\left(\frac{q}{p'+q}\right)^{-\frac{1}{p'}}, \end{aligned} \tag{3.6}$$

where $|\mathfrak{S}|$ is the area of unit sphere with respect to $|\cdot|$.

Proof. Let us check condition (2.10) with $u(x)=|x|^{\alpha}$, $v(x)=|x|^{\beta}$ and with $a=0$. Let us calculate the first integral in (2.10):

$$\begin{aligned} \int_{\mathbb{G}\backslash B(0,|x|)} u(y)\,dy &= \int_{\mathbb{G}\backslash B(0,|x|)} |y|^{\alpha}\,dy \stackrel{(3.2)}{=} \int_{|x|}^{\infty}\int_{\mathfrak{S}} \rho^{\alpha}\rho^{Q-1}\,d\rho\,d\sigma(\omega) \\ &= |\mathfrak{S}|\int_{|x|}^{\infty}\rho^{Q+\alpha-1}\,d\rho \stackrel{Q+\alpha<0}{=} -\frac{|\mathfrak{S}|}{Q+\alpha}|x|^{Q+\alpha} = \frac{|\mathfrak{S}|}{|Q+\alpha|}|x|^{Q+\alpha}, \end{aligned} \tag{3.7}$$

where $|\mathfrak{S}|$ is the area of the unit quasi-sphere in $\mathbb{G}$. Then,

$$\begin{aligned} \int_{B(0,|x|)} v^{1-p'}(y)\,dy &= \int_{B(0,|x|)} |y|^{\beta(1-p')}\,dy \stackrel{(3.2)}{=} \int_{0}^{|x|}\int_{\mathfrak{S}} \rho^{\beta(1-p')}\rho^{Q-1}\,d\rho\,d\sigma(\omega) \\ &= |\mathfrak{S}|\int_{0}^{|x|}\rho^{Q+\beta(1-p')-1}d\rho \\ &\stackrel{Q+\beta(1-p')>0}{=} \frac{|\mathfrak{S}|}{Q+\beta(1-p')}|x|^{Q+\beta(1-p')}. \end{aligned} \tag{3.8}$$

Finally by summarising above facts with $\frac{Q+\alpha}{q}+\frac{Q+\beta(1-p')}{p'}=0$, we have

$$\begin{aligned} D_1 &= \left(\frac{|\mathfrak{S}|}{|\alpha+Q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}}\inf_{r>0} r^{\frac{Q+\alpha}{q}+\frac{Q+\beta(1-p')}{p'}} \\ &= \left(\frac{|\mathfrak{S}|}{|\alpha+Q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}} > 0. \end{aligned} \tag{3.9}$$

Then by using (2.11), we obtain

$$\begin{aligned} &\left(\frac{|\mathfrak{S}|}{|\alpha+Q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}} \geq C \\ &\geq \left(\frac{|\mathfrak{S}|}{|\alpha+Q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}}\left(\frac{p'}{p'+q}\right)^{-\frac{1}{q}}\left(\frac{q}{p'+q}\right)^{-\frac{1}{p'}}, \end{aligned} \tag{3.10}$$

completing the proof. ∎

Then we have conjugate reverse integral Hardy inequality on homogeneous Lie groups.

Corollary 3.2. *Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension Q with a quasi-norm $|\cdot|$. Assume that $q<0$, $p\in(0,1)$ and $\alpha,\beta\in\mathbb{R}$. Then the conjugate reverse integral Hardy inequality*

$$\left[\int_{\mathbb{G}}\left(\int_{\mathbb{G}\setminus B(0,|x|)} f(y)\,dy\right)^q |x|^\alpha\,dx\right]^{\frac{1}{q}} \geq C\left(\int_{\mathbb{G}} f^p(x)|x|^\beta\,dx\right)^{\frac{1}{p}}, \tag{3.11}$$

holds for $C>0$ and for all non-negative measurable functions f, if only if

$$\alpha+Q>0,\quad \beta(1-p')+Q<0 \quad \text{and} \quad \frac{Q+\alpha}{q}+\frac{Q+\beta(1-p')}{p'}=0. \tag{3.12}$$

Moreover, the biggest constant C for (3.11) satisfies

$$\begin{aligned}&\left(\frac{|\mathfrak{S}|}{\alpha+Q}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{|Q+\beta(1-p')|}\right)^{\frac{1}{p'}} \geq C\\ &\geq \left(\frac{|\mathfrak{S}|}{\alpha+Q}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{|Q+\beta(1-p')|}\right)^{\frac{1}{p'}}\left(\frac{p'}{p'+q}\right)^{-\frac{1}{q}}\left(\frac{q}{p'+q}\right)^{-\frac{1}{p'}},\end{aligned} \tag{3.13}$$

where $|\mathfrak{S}|$ is the area of unit sphere with respect to $|\cdot|$.

Proof. Proof of this corollary is similar to the previous case. □

3.2. Hyperbolic space. Let $\mathbb{H}^n$ be the hyperbolic space of dimension n and let $a\in\mathbb{H}^n$. Let us set

$$u(x)=(\sinh|x|_a)^\alpha,\quad v(x)=(\sinh|x|_a)^\beta. \tag{3.14}$$

Then we have the following result of this subsection.

Corollary 3.3. *Let $\mathbb{H}^n$ be the hyperbolic space of dimension n and let $a\in\mathbb{H}^n$. Assume that $q<0$, $p\in(0,1)$ and $\alpha,\beta\in\mathbb{R}$. Then the reverse integral Hardy inequality*

$$\left[\int_{\mathbb{H}^n}\left(\int_{B(a,|x|_a)} f(y)\,dy\right)^q (\sinh|x|_a)^\alpha\,dx\right]^{\frac{1}{q}} \geq C\left(\int_{\mathbb{H}^n} f^p(x)(\sinh|x|_a)^\beta\,dx\right)^{\frac{1}{p}}, \tag{3.15}$$

holds for $C>0$ and for all non-negative measurable functions f, if

$$0\leq\alpha+n<1,\quad \beta(1-p')+n>0 \quad \text{and} \quad \frac{\alpha+n}{q}+\frac{\beta(1-p')+n}{p'}\geq\frac{1}{q}+\frac{1}{p'}. \tag{3.16}$$

Proof. Let us check condition (2.10). By using the polar decomposition for the hyperbolic space, we have

$$D_1=\inf_{x\neq a}\left(\int_{|x|_a}^{\infty}(\sinh\rho)^{\alpha+n-1}\,d\rho\right)^{\frac{1}{q}}\left(\int_0^{|x|_a}(\sinh\rho)^{\beta(1-p')+n-1}\,d\rho\right)^{\frac{1}{p'}}. \tag{3.17}$$

If $\alpha+n<1$ and $\beta(1-p')+n>0$, then (3.17) is integrable. Let us check the finiteness and positiveness of the infimum (3.17). Let us divide the proof in two cases.

First case, $|x|_a \gg 1$. Then $\sinh |x|_a \approx \exp |x|_a$ if $|x|_a \gg 1$. Then we obtain,

$$\begin{aligned}
D_1^1 &= \inf_{|x|_a\gg 1}\left(\int_{|x|_a}^{\infty}(\sinh\rho)^{\alpha+n-1}\,d\rho\right)^{\frac{1}{q}}\left(\int_0^{|x|_a}(\sinh\rho)^{\beta(1-p')+n-1}\,d\rho\right)^{\frac{1}{p'}}\\
&\simeq \inf_{|x|_a\gg 1}\left(\int_{|x|_a}^{\infty}(\exp\rho)^{\alpha+n-1}\,d\rho\right)^{\frac{1}{q}}\left(\int_0^{|x|_a}(\exp\rho)^{\beta(1-p')+n-1}\,d\rho\right)^{\frac{1}{p'}}\\
&= \inf_{|x|_a\gg 1}\left((\exp|x|_a)^{\alpha+n-1}\right)^{\frac{1}{q}}\left((\exp|x|_a)^{\beta(1-p')+n-1}\right)^{\frac{1}{p'}}\\
&= \inf_{|x|_a\gg 1}(\exp|x|_a)^{\frac{\alpha+n-1}{q}+\frac{\beta(1-p')+n-1}{p'}},
\end{aligned} \tag{3.18}$$

infimum of the last term is positive, if only if $\frac{\alpha+n-1}{q}+\frac{\beta(1-p')+n-1}{p'} \geq 0$, i.e., $\frac{\alpha+n}{q}+\frac{\beta(1-p')+n}{p'} \geq \frac{1}{q}+\frac{1}{p'}$, then $D_1^1>0$.

Let us consider the another case $|x|_a \ll 1$. For $|x|_a \ll 1$ we have $\sinh \rho_{\{0\leq\rho<|x|_a\}} \approx \rho$, then we calculate

$$\begin{aligned}
&\inf_{|x|_a\ll 1}\left(\int_{|x|_a}^{\infty}(\sinh\rho)^{\alpha+n-1}\,d\rho\right)^{\frac{1}{q}}\left(\int_0^{|x|_a}(\sinh\rho)^{\beta(1-p')+n-1}\,d\rho\right)^{\frac{1}{p'}}\\
&\simeq \inf_{|x|_a\ll 1}\left(\int_{|x|_a}^{R}(\sinh\rho)^{\alpha+n-1}\,d\rho+\int_R^{\infty}(\sinh\rho)^{\alpha+n-1}d\rho\right)^{\frac{1}{q}}\\
&\quad\cdot\left(\int_0^{|x|_a}\rho^{\beta(1-p')+n-1}\,d\rho\right)^{\frac{1}{p'}}\\
&\simeq \inf_{|x|_a\ll 1}\left(\int_{|x|_a}^{R}(\sinh\rho)^{\alpha+n-1}\,d\rho+\int_R^{\infty}(\sinh\rho)^{\alpha+n-1}d\rho\right)^{\frac{1}{q}}|x|_a^{\frac{\beta(1-p')+n}{p'}}.
\end{aligned} \tag{3.19}$$

Similarly, for small R we have $\sinh \rho_{\{|x|_a\leq\rho<R\}} \approx \rho$, so that we obtain

$$\begin{aligned}
&\inf_{|x|_a\ll 1}\left(\int_{|x|_a}^{\infty}(\sinh\rho)^{\alpha+n-1}\,d\rho\right)^{\frac{1}{q}}\left(\int_0^{|x|_a}(\sinh\rho)^{\beta(1-p')+n-1}d\rho\right)^{\frac{1}{p'}}\\
&\simeq \inf_{|x|_a\ll 1}\left(\int_{|x|_a}^{R}(\sinh\rho)^{\alpha+n-1}\,d\rho+\int_R^{\infty}(\sinh\rho)^{\alpha+n-1}\,d\rho\right)^{\frac{1}{q}}|x|_a^{\frac{\beta(1-p')+n}{p'}}\\
&\simeq \inf_{|x|_a\ll 1}\left(\int_{|x|_a}^{R}\rho^{\alpha+n-1}\,d\rho+\int_R^{\infty}(\sinh\rho)^{\alpha+n-1}\,d\rho\right)^{\frac{1}{q}}|x|_a^{\frac{\beta(1-p')+n}{p'}}\\
&\simeq \inf_{|x|_a\ll 1}\left(|x|_a^{\alpha+n}+C_R\right)^{\frac{1}{q}}|x|_a^{\frac{\beta(1-p')+n}{p'}}.
\end{aligned} \tag{3.20}$$

If $\alpha+n\geq 0$, we have $\frac{\alpha+n}{q}\leq 0$, then we have

$$\begin{aligned}
D_1^2 &= \inf_{|x|_a\ll 1}\left(\int_{|x|_a}^{\infty}(\sinh\rho)^{\alpha+n-1}\,d\rho\right)^{\frac{1}{q}}\left(\int_0^{|x|_a}(\sinh\rho)^{\beta(1-p')+n-1}\,d\rho\right)^{\frac{1}{p'}}\\
&\simeq \inf_{|x|_a\ll 1}\left(|x|_a^{\alpha+n}+C_R\right)^{\frac{1}{q}}|x|_a^{\frac{\beta(1-p')+n}{p'}} \simeq \inf_{|x|_a\ll 1}|x|_a^{\frac{\beta(1-p')+n}{p'}}>0,
\end{aligned} \tag{3.21}$$

and infimum is positive, if only if $\frac{\beta(1-p')+n}{p'} < 0$, i.e., $\beta(1-p')+n > 0$. □

Let us give the reverse conjugate integral Hardy's inequality in hyperbolic spaces:

Corollary 3.4. *Let $\mathbb{H}^n$ be the hyperbolic space of dimension n and $a \in \mathbb{H}^n$. Assume that $q < 0$, $p \in (0,1)$ and let $\alpha, \beta \in \mathbb{R}$. Then the reverse conjugate integral Hardy inequality*

$$\left[\int_{\mathbb{H}^n}\left(\int_{\mathbb{X}\setminus B(a,|x|_a)} f(y)\,dy\right)^q (\sinh|x|_a)^\alpha\,dx\right]^{\frac{1}{q}} \geq C\left(\int_{\mathbb{H}^n} f^p(x)(\sinh|x|_a)^\beta\,dx\right)^{\frac{1}{p}}, \tag{3.22}$$

holds for all non-negative measurable functions f, if

$$\alpha+n>0, \quad 1>\beta(1-p')+n\geq 0 \quad \text{and} \quad \frac{\alpha+n}{q}+\frac{\beta(1-p')+n}{p'} \geq \frac{1}{q}+\frac{1}{p'}.$$

Proof. Similarly to the previous case, check condition (2.26) and then, we have

$$D_2 = \inf_{x\neq a}\left(\int_0^{|x|_a}(\sinh\rho)^{\alpha+n-1}\,d\rho\right)^{\frac{1}{q}}\left(\int_{|x|_a}^\infty(\sinh\rho)^{\beta(1-p')+n-1}\,d\rho\right)^{\frac{1}{p'}}. \tag{3.23}$$

If $\alpha+n>0$ and $\beta(1-p')+n<1$, then the integrals in (3.23) are finite for each $|x|_a$. If $|x|_a \gg 1$, we obtain,

$$\begin{aligned}
D_2^1 &= \inf_{|x|_a\gg 1}\left(\int_{|x|_a}^\infty(\sinh\rho)^{\alpha+n-1}\,d\rho\right)^{\frac{1}{q}}\left(\int_0^{|x|_a}(\sinh\rho)^{\beta(1-p')+n-1}\,d\rho\right)^{\frac{1}{p'}}\\
&\simeq \inf_{|x|_a\gg 1}\left(\int_{|x|_a}^\infty(\exp\rho)^{\alpha+n-1}\,d\rho\right)^{\frac{1}{q}}\left(\int_0^{|x|_a}(\exp\rho)^{\beta(1-p')+n-1}\,d\rho\right)^{\frac{1}{p'}}\\
&= \inf_{|x|_a\gg 1}\left((\exp|x|_a)^{\alpha+n-1}\right)^{\frac{1}{q}}\left((\exp|x|_a)^{\beta(1-p')+n-1}\right)^{\frac{1}{p'}}\\
&= \inf_{|x|_a\gg 1}(\exp|x|_a)^{\frac{\alpha+n-1}{q}+\frac{\beta(1-p')+n-1}{p'}},
\end{aligned} \tag{3.24}$$

infimum of the last term is positive, if only if $\frac{\alpha+n-1}{q}+\frac{\beta(1-p')+n-1}{p'} \geq 0$, i.e., $\frac{\alpha+n}{q}+\frac{\beta(1-p')+n}{p'} \geq \frac{1}{q}+\frac{1}{p'}$, then $D_1^1 > 0$.

If $|x|_a \ll 1$, we obtain

$$\begin{aligned}
&\inf_{|x|_a\ll 1}\left(\int_0^{|x|_a}(\sinh\rho)^{\alpha+n-1}\,d\rho\right)^{\frac{1}{q}}\left(\int_{|x|_a}^\infty(\sinh\rho)^{\beta(1-p')+n-1}\,d\rho\right)^{\frac{1}{p'}}\\
&\simeq \inf_{|x|_a\ll 1}\left(\int_0^{|x|_a}\rho^{\alpha+n-1}\,d\rho\right)^{\frac{1}{q}}\\
&\quad\cdot\left(\int_{|x|_a}^R(\sinh\rho)^{\beta(1-p')+n-1}\,d\rho+\int_R^\infty(\sinh\rho)^{\beta(1-p')+n-1}\,d\rho\right)^{\frac{1}{p'}}\\
&\simeq \inf_{|x|_a\ll 1}\left(\int_{|x|_a}^R(\sinh\rho)^{\beta(1-p')+n-1}\,d\rho+\int_R^\infty(\sinh\rho)^{\beta(1-p')+n-1}\,d\rho\right)^{\frac{1}{p'}}|x|_a^{\frac{\alpha+n}{q}}.
\end{aligned} \tag{3.25}$$

Similarly, for small R we have $\sinh \rho_{\{|x|_a \leq \rho < R\}} \approx \rho$, so that we obtain

(3.26)
$$\begin{aligned}
&\inf_{|x|_a \ll 1} \left(\int_0^{|x|_a} (\sinh \rho)^{\alpha+n-1} \, d\rho \right)^{\frac{1}{q}} \left(\int_{|x|_a}^{\infty} (\sinh \rho)^{\beta(1-p')+n-1} \, d\rho \right)^{\frac{1}{p'}} \\
&\simeq \inf_{|x|_a \ll 1} \left(\int_{|x|_a}^{R} (\sinh \rho)^{\beta(1-p')+n-1} \, d\rho + \int_R^{\infty} (\sinh \rho)^{\beta(1-p')+n-1} \, d\rho \right)^{\frac{1}{p'}} |x|_a^{\frac{\alpha+n}{q}} \\
&\simeq \inf_{|x|_a \ll 1} \left(|x|_a^{\beta(1-p')+n} + C'_R \right)^{\frac{1}{q}} |x|_a^{\frac{\alpha+n}{q}}.
\end{aligned}$$

If $\beta(1-p')+n \geq 0$, we have $\frac{\beta(1-p')+n}{q} \leq 0$, then we have

$$\begin{aligned}
D_2^2 &= \inf_{|x|_a \ll 1} \left(\int_0^{|x|_a} (\sinh \rho)^{\alpha+n-1} \, d\rho \right)^{\frac{1}{q}} \left(\int_{|x|_a}^{\infty} (\sinh \rho)^{\beta(1-p')+n-1} \, d\rho \right)^{\frac{1}{p'}} \\
&\simeq \inf_{|x|_a \ll 1} \left(|x|_a^{\beta(1-p')+n} + C'_R \right)^{\frac{1}{q}} |x|_a^{\frac{\alpha+n}{q}} \simeq \inf_{|x|_a \ll 1} |x|_a^{\frac{\alpha+n}{q}},
\end{aligned} \tag{3.27}$$

and infimum is positive, if only if $\frac{\alpha+n}{q} < 0$, i.e., $\alpha + n > 0$. □

3.3. Cartan–Hadamard manifolds. Let (M, g) be the Cartan–Hadamard manifold with curvature K_M. If $K_M = 0$ then $J(t, \omega) = 1$ and we set

$$u(x) = |x|_a^{\alpha}, \quad v(x) = |x|_a^{\beta}, \quad \text{when } K_M = 0. \tag{3.28}$$

If $K_M < 0$ then $J(t, \omega) = \left(\frac{\sinh \sqrt{b}t}{\sqrt{b}t} \right)^{n-1}$ and we set

$$u(x) = (\sinh \sqrt{-K_M} |x|_a)^{\alpha}, \quad v(x) = (\sinh \sqrt{-K_M} |x|_a)^{\beta}, \quad \text{when } K_M < 0. \tag{3.29}$$

Then we have the following result of this subsection.

Corollary 3.5. *Assume that (M, g) be the Cartan–Hadamard manifold of dimension n and with curvature K_M. Assume that $q < 0$, $p \in (0,1)$ and $\alpha, \beta \in \mathbb{R}$. Then we have*

i) *if $K_M = 0$, $u(x) = |x|_a^{\alpha}, v(x) = |x|_a^{\beta}$, then*

$$\left[\int_M \left(\int_{B(a,|x|_a)} f(y) \, dy \right)^q |x|_a^{\alpha} \, dx \right]^{\frac{1}{q}} \geq C \left(\int_M f^p(x) |x|^{\beta} \, dx \right)^{\frac{1}{p}}, \tag{3.30}$$

holds for $C > 0$ and for non-negative measurable functions f, if only if $\alpha + n < 0$, $\beta(1-p') + n > 0$ and $\frac{n+\alpha}{q} + \frac{n+\beta(1-p')}{p'} = 0$;

ii) *if $K_M = 0$, $u(x) = |x|_a^{\alpha}, v(x) = |x|_a^{\beta}$, then*

$$\left[\int_M \left(\int_{M \setminus B(a,|x|_a)} f(y) \, dy \right)^q |x|^{\alpha} \, dx \right]^{\frac{1}{q}} \geq C \left(\int_M f^p(x) |x|^{\beta} \, dx \right)^{\frac{1}{p}}, \tag{3.31}$$

holds for $C > 0$ and for non-negative measurable functions f, if only if $\alpha + n > 0$, $\beta(1-p') + n < 0$ and $\frac{n+\alpha}{q} + \frac{n+\beta(1-p')}{p'} = 0$;

iii) *if* $K_M < 0$, $u(x) = (\sinh\sqrt{-K_M}|x|_a)^\alpha, v(x) = (\sinh|x|_a)^\beta$, *then*

$$\begin{aligned}\left[\int_M \left(\int_{B(a,|x|_a)} f(y)\,dy\right)^q (\sinh\sqrt{-K_M}|x|_a)^\alpha\,dx\right]^{\frac{1}{q}} \\ \geq C\left(\int_M f^p(x)(\sinh\sqrt{-K_M}|x|_a)^\beta\,dx\right)^{\frac{1}{p}},\end{aligned} \tag{3.32}$$

holds for $C > 0$ *and for all non-negative measurable functions* f, *if* $0 \leq \alpha+n < 1$, $\beta(1-p')+n > 0$ *and* $\frac{\alpha+n}{q} + \frac{\beta(1-p')+n}{p'} \geq \frac{1}{q} + \frac{1}{p'}$;

iv) *if* $K_M < 0$, $u(x) = (\sinh\sqrt{-K_M}|x|_a)^\alpha, v(x) = (\sinh\sqrt{-K_M}|x|_a)^\beta$, *then*

$$\begin{aligned}\left[\int_M \left(\int_{M\setminus B(a,|x|_a)} f(y)\,dy\right)^q (\sinh\sqrt{-K_M}|x|_a)^\alpha\,dx\right]^{\frac{1}{q}} \\ \geq C\left(\int_M f^p(x)(\sinh\sqrt{-K_M}|x|_a)^\beta\,dx\right)^{\frac{1}{p}},\end{aligned} \tag{3.33}$$

holds for $C > 0$ *and for all non-negative measurable functions* f, *if* $\alpha+n > 0$, $1 > \beta(1-p')+n \geq 0$ *and* $\frac{\alpha+n}{q} + \frac{\beta(1-p')+n}{p'} \geq \frac{1}{q} + \frac{1}{p'}$.

Proof. In the case $K_M = 0$ the proof of this corollary is similar to Corollary 3.1 and 3.2. Also, in the case $K_M < 0$ the proof is similar to hyperbolic space case. □

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