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The Bohr phenomenon for analytic functions on shifted disks

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Abstract. In this paper, we investigate the Bohr phenomenon for the class of analytic functions defined on the simply connected domain

$$\Omega_\gamma = \left\{ z \in \mathbb{C} \colon \left| z + \frac{\gamma}{1-\gamma} \right| < \frac{1}{1-\gamma} \right\} \quad \text{for} \ \ 0 \le \gamma < 1.$$

We study improved Bohr radius, Bohr–Rogosinski radius and refined Bohr radius for the class of analytic functions defined in Ω_γ, and obtain several sharp results.

Bohrin ilmiö analyyttisille funktioille siirretyissä kiekoissa

Tiivistelmä. Tarkastelemme Bohrin ilmiötä yhdesti yhtenäisessä alueessa

$$\Omega_\gamma = \left\{ z \in \mathbb{C} \colon \left| z + \frac{\gamma}{1-\gamma} \right| < \frac{1}{1-\gamma} \right\}$$

määriteltyjen analyyttisten funktioiden luokassa, missä $0 \le \gamma < 1$. Tutkimme parannettua Bohrin sädettä, Bohrin–Rogosinskin sädettä sekä tarkennettua Bohrin sädettä alueessa Ω_γ määriteltyjen analyyttisten funktioiden luokassa ja saamme useita tarkkoja tuloksia.

1. Introduction and preliminaries

Let $\mathcal{B}(\mathbb{D})$ be the class of analytic functions in unit disk $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ such that $f(\mathbb{D}) \subseteq \overline{\mathbb{D}}$. The classical Bohr theorem for functions $f \in \mathcal{B}(\mathbb{D})$ says that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then its associated majorant series $M_f(r)$ satisfies the following inequality

$$M_f(r) := \sum_{n=0}^{\infty} |a_n| r^n \le 1 \quad \text{for} \ \ |z| = r \le \frac{1}{3} \tag{1.1}$$

and the constant 1/3, called Bohr radius for the class $\mathcal{B}(\mathbb{D})$, cannot be improved. The inequality (1.1) is known as classical Bohr inequality (1.1) for the class $\mathcal{B}(\mathbb{D})$. The Bohr inequality was first obtained by Harald Bohr [24] in 1914 with the constant 1/6. The optimal value 1/3, which is called the Bohr radius for disk case was later established independently by Weiner, Riesz and Schur. For the proofs we refer to [41] and [42]. The notion of Bohr inequality has been generalized to several complex variables by finding the multidimensional Bohr radius. We refer the reader to the articles [6, 8, 23, 37]. For more information and intriguing aspects on Bohr phenomenon, we suggest the reader to glance through the articles [1]–[5], [7]–[13] and [16]–[18]. Bohr phenomenon for operator valued functions have been extensively studied by Bhowmik and Das (see [21, 22]).

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The main aim of this article is to study the Bohr inequality for the class of analytic functions that are defined in a general simply connected domain in the complex plain. Let Ω be a simply connected domain containing $\mathbb{D}$ and $\mathcal{B}(\Omega)$ be the class of analytic functions in Ω such that $f(\Omega) \subseteq \overline{\mathbb{D}}$. We define the Bohr radius $B = B_\Omega$ for the class $\mathcal{B}(\Omega)$ by

$$B := \sup\left\{r \in (0,1)\colon \sum_{n=0}^{\infty} |a_n| r^n \leq 1 \text{ for all } f \in \mathcal{B}(\Omega) \text{ with} f(z) = \sum_{n=0}^{\infty} a_n z^n,\ z \in \mathbb{D}\right\}.$$

In particular, if $\Omega = \mathbb{D}$, then $B_{\mathbb{D}} = 1/3$, which is the classical Bohr radius for the class $\mathcal{B}(\mathbb{D})$. Let $\mathbb{D}(a,r) := \{z \in \mathbb{C}\colon |z-a| < r\}$. Clearly, $\mathbb{D} := \mathbb{D}(0,1)$. Let $0 \leq \gamma < 1$. We consider the disk Ω_γ defined by

$$\Omega_\gamma := \left\{z \in \mathbb{C}\colon \left|z + \frac{\gamma}{1-\gamma}\right| < \frac{1}{1-\gamma}\right\}.$$

It is easy to see that Ω_γ always contains the unit disk $\mathbb{D}$. In 2010, the notion of classical Bohr inequality (1.1) has been generalized by Fournier and Ruscheweyh [28] to the class $\mathcal{B}(\Omega_\gamma)$. More precisely,

Theorem 1.2. [28] *For $0 \leq \gamma < 1$, let $f \in \mathcal{B}(\Omega_\gamma)$, with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathbb{D}$. Then,*

$$\sum_{n=0}^{\infty} |a_n| r^n \leq 1 \quad \text{for } r \leq \rho := \frac{1+\gamma}{3+\gamma}.$$

Moreover, $\sum_{n=0}^{\infty} |a_n| \rho^n = 1$ holds for a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathcal{B}(\Omega_\gamma)$ if, and only if, $f(z) = c$ with $|c| = 1$.

In this article, we study the Bohr–Rogosinski radius for the class $\mathcal{B}(\Omega_\gamma)$. In 2017, Kayumov and Ponnusamy [34] introduced Bohr–Rogosinski radius motivated from Rogosinski radius for bounded analytic functions in $\mathbb{D}$. Rogosinski radius is defined as follows: Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $\mathbb{D}$ and its corresponding partial sum of f is defined by $S_N(z) := \sum_{n=0}^{N-1} a_n z^n$. Then, for every $N \geq 1$, we have $|\sum_{n=0}^{N-1} a_n z^n| < 1$ in the disk $|z| < 1/2$ and the radius $1/2$ is sharp. Motivated by Rogosinski radius, Kayumov and Ponnusamy have considered the Bohr–Rogosinski sum $R_N^f(z)$ is defined by

$$R_N^f(z) := |f(z)| + \sum_{n=N}^{\infty} |a_n||z|^n. \tag{1.3}$$

It is worth to point out that $|S_N(z)| = \left|f(z) - \sum_{n=N}^{\infty} a_n z^n\right| \leq |R_N^f(z)|$. Thus, it is easy to see that the validity of Bohr-type radius for $R_N^f(z)$, which is related to the classical Bohr sum (Majorant series) in which $f(0)$ is replaced by $f(z)$, gives Rogosinski radius in the case of bounded analytic functions in $\mathbb{D}$. There has been significant and extensive research carried out on Improved-Bohr inequality and Bohr–Rogosinski radius (see [14, 29, 30, 34, 31, 32, 33, 35, 36, 38]).

Lemma 1.4. [40] *Let $a \in \mathbb{D}$ and $f \in \mathcal{B}(\mathbb{D})$ with*

$$f(z) = \sum_{n=0}^{\infty} \alpha_n (z-a)^n, \quad |z-a| \leq 1 - |a|.$$

Then,

$$|\alpha_n| \leq (1+|a|)^{n-1} \frac{1-|\alpha_0|^2}{(1-|a|^2)^n}, \ n \geq 1.$$

Recently, Evdoridis et al. [27] obtained the following coefficient bounds for functions defined in Ω_γ.

Lemma 1.5. [27] *For $\gamma \in [0,1)$, let*

$$\Omega_\gamma = \left\{ z \in \mathbb{C} \colon \left| z + \frac{\gamma}{1-\gamma} \right| < \frac{1}{1-\gamma} \right\},$$

and let f be an analytic function in Ω_γ, bounded by 1, with the series representation $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in the unit disk $\mathbb{D}$. Then

$$|a_n| \leq \frac{1-|a_0|^2}{1+\gamma} \quad \text{for} \ \ n \geq 1.$$

2. Main results

Before we state an improved version of inequality of Theorem 1.2, we prove the following lemma.

Lemma 2.1. *Let $g \colon \mathbb{D} \to \overline{\mathbb{D}}$ be an analytic function, $m(\geq 2)$ be an integer, and let $\gamma \in \mathbb{D}$ be such that $g(z) = \sum_{n=0}^{\infty} \alpha_n (z-\gamma)^n$ for $|z-\gamma| \leq 1-|\gamma|$. Then*

$$|\alpha_0| + \sum_{n=1}^{\infty} \left(|\alpha_n| + \beta |\alpha_n|^m \right) \rho^n \leq 1 \quad \text{for} \ \ \rho \leq \rho_0 := (1-\gamma^2)/(3+\gamma), \tag{2.2}$$

where

$$\beta = \frac{(1-\gamma)^m (3+\gamma) - (1-\gamma^2)}{8(m-1)} \quad \text{for} \ \ 0 \leq \gamma \leq \gamma_* < 1,$$

where γ_* is the smallest root of the equation $(1-\gamma)^m(3+\gamma) + \gamma^2 - 1 = 0$.

Using Lemma 2.1, we obtain the following improved version of Theorem 1.2 for the class $\mathcal{B}(\Omega_\gamma)$.

Theorem 2.3. *For $0 \leq \gamma < 1$, and integer m (≥ 2), let $f \in \mathcal{B}(\Omega_\gamma)$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $z \in \mathbb{D}$, then we have*

$$|a_0| + \sum_{n=1}^{\infty} \left(|a_n| + \beta \frac{|a_n|^m}{(1-\gamma)^{(m-1)n}} \right) r^n \leq 1 \quad \text{for} \ \ r \leq r_0 = \frac{1+\gamma}{3+\gamma},$$

where β as in Lemma 2.1. Furthermore, the quantities β and $(1+\gamma)/(3+\gamma)$ cannot be improved.

Figure 1 demonstrates values of γ_* in $[0,1)$ for which $\beta(\gamma) > 0$ with $0 \leq \gamma \leq \gamma_* < 1$. The values of γ_* are $\gamma_*(10) = 0.1083$, $\gamma_*(21) = 0.0519$, $\gamma_*(50) = 0.0219$ and $\gamma_*(100) = 0.011$.

Lemma 2.4. *Let $g \colon \mathbb{D} \to \overline{\mathbb{D}}$ be an analytic function, $\lambda \in [0, 512/243]$ and let $\gamma \in \mathbb{D}$ be such that $g(z) = \sum_{n=0}^{\infty} \alpha_n (z-\gamma)^n$ for $|z-\gamma| < 1-|\gamma|$. Then*

$$\sum_{n=0}^{\infty} |\alpha_n| \rho^n + \left(\frac{8}{9} - \frac{27}{64}\lambda \right) \left(\frac{S_\rho^\gamma}{\pi} \right) + \lambda \left(\frac{S_\rho^\gamma}{\pi} \right)^2 \leq 1 \quad \text{for} \ \ \rho \leq \rho_0 = \frac{1-|\gamma|^2}{3+|\gamma|},$$

where S_ρ^γ denotes the area of the image of the disk $\mathbb{D}(\gamma; r(1-|\gamma|))$ under the mapping g.

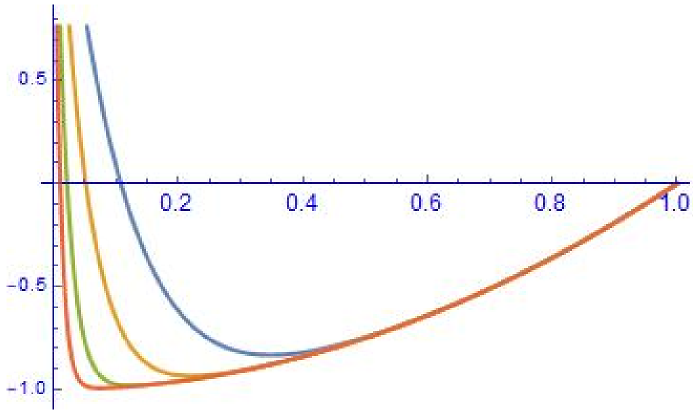

Figure 1. The roots $\gamma_*(m)$ of the equation $(1-\gamma)^m(3+\gamma)+\gamma^2-1=0$.

By applying Lemma 2.4, we obtain the following improved version of Theorem 1.2.

Theorem 2.5. *For $0 \leq \gamma < 1$ and $0 \leq \lambda \leq 512/243$, let $f \in \mathcal{B}(\Omega_\gamma)$ with $f(z)=\sum_{n=0}^{\infty} a_n z^n$ for $z \in \mathbb{D}$, then we have*

$$\sum_{n=0}^{\infty}|a_n|r^n + \left(\frac{8}{9}-\frac{27}{64}\lambda\right)\left(\frac{S_{r(1-\gamma)}}{\pi}\right)+\lambda\left(\frac{S_{r(1-\gamma)}}{\pi}\right)^2 \leq 1 \quad \text{for } r \leq r_0 = \frac{1+\gamma}{3+\gamma}.$$

Furthermore, the radius r_0 is sharp, and the bounds of λ and $8/9-27\lambda/64$ cannot be improved.

Lemma 2.6. *For $\gamma \in \mathbb{D}$, let $g \in \mathcal{B}(\mathbb{D})$ with $g(z)=\sum_{n=0}^{\infty}\alpha_n(z-\gamma)^n$, for $|z-\gamma| \leq 1-|\gamma|$, then*

$$|g(z)| + \sum_{n=N}^{\infty}|\alpha_n|\rho^n \leq 1, \quad \text{for } \rho \leq \rho_N,$$

where ρ_N is the root of

$$2(1+\gamma)\rho^N + (1+\gamma)(1-\gamma)^{N-1}(\rho-1)(1-\gamma-\rho)=0$$

in $(0,1)$.

Using Lemma 2.6, we obtain the following Bohr–Rogosinski radius for the class $\mathcal{B}(\Omega_\gamma)$.

Theorem 2.7. *For $0 \leq \gamma < 1$ and integer N (≥ 1), let $f \in \mathcal{B}(\Omega_\gamma)$ with $f(z)=\sum_{n=0}^{\infty}a_n z^n$ for $z \in \mathbb{D}$. Then, we have*

$$\left|f\left(\frac{z-\gamma}{1-\gamma}\right)\right| + \sum_{n=N}^{\infty}|a_n|r^n \leq 1 \quad \text{for } r \leq r_0 = \frac{\rho_N}{1-\gamma},$$

where ρ_N is the root of the equation

$$2(1+\rho)\rho^N + (1+\gamma)(1-\gamma)^{N-1}(\rho-1)(1-\gamma-\rho)=0. \tag{2.8}$$

Furthermore, the constant $\rho_N/(1-\gamma)$ cannot be improved.

Using Lemma 1.5, we establish the following refined Bohr inequality for the class $\mathcal{B}(\Omega_\gamma)$.

Theorem 2.9. *For $0 \leq \gamma < 1$, let $f \in \mathcal{B}(\Omega_\gamma)$ with $f(z) = \sum_{n=1}^{\infty} a_n z^n$ for $z \in \mathbb{D}$. Then we have*

$$\sum_{n=1}^{\infty} |a_n| r^n + \left(\frac{1}{1+|a_1|} + \frac{r}{1-r}\right) \sum_{n=2}^{\infty} |a_n|^2 r^{2(n-1)} \leq 1 \quad \textit{for } r \leq r_0 = \frac{1+\gamma}{3+\gamma}.$$

The constant r_0 cannot be improved.

3. Proofs of the main results

Proof of the Lemma 2.1. Without loss of generality, we may assume that $\gamma \in [0,1)$. Using Lemma 1.4, we obtain

$$\sum_{n=1}^{\infty} |\alpha_n| \rho^n \leq \frac{1-|\alpha_0|^2}{1+\gamma} \sum_{n=1}^{\infty} \left(\frac{\rho}{1-\gamma}\right)^n = \frac{(1-|\alpha_0|^2)\,\rho}{(1+\gamma)(1-\gamma-\rho)}. \tag{3.1}$$

Further, we have

$$\sum_{n=1}^{\infty} |\alpha_n|^m \rho^n \leq \frac{(1-|\alpha_0|^2)^m}{(1+\gamma)^m} \sum_{n=1}^{\infty} \left(\frac{\rho}{(1-\gamma)^m}\right)^n = \frac{(1-|\alpha_0|^2)^m\,\rho}{(1+\gamma)^m((1-\gamma)^m-\rho)}. \tag{3.2}$$

The series in (2.2) contains positive terms for $\beta \geq 0$. Our aim is to find the smallest value of γ in $[0,1)$ for which $\beta \geq 0$. That is

$$\beta = \frac{(1-\gamma)^m(3+\gamma) - (1-\gamma^2)}{8(m-1)} := \frac{Q(\gamma)}{8(m-1)} \geq 0,$$

where $Q(\gamma) = (1-\gamma)^m(3+\gamma) - (1-\gamma^2)$. Clearly, $\gamma = 1$ is a root of $Q(\gamma)$. Since $Q(\gamma)$ is a polynomial such that $Q(0) = 2 > 0$ and $m \geq 2$, we have

$$Q\left(\frac{9}{10}\right) = \frac{3.9}{10^m} + \frac{81}{100} - 1 \leq \frac{84.9}{100} - 1 = -\frac{15.1}{100} < 0.$$

Therefore, there exists at least one root of $Q(\gamma)$ in $(0,1)$. Let γ_* be the smallest root of $Q(\gamma)$. Then, it is easy to see that $Q(\gamma) \geq 0$, and hence $\beta \geq 0$ for all $\gamma \in [0, \gamma_*]$. A simple computation using (3.1) and (3.2) shows that

$$\begin{aligned} & |\alpha_0| + \sum_{n=1}^{\infty} |\alpha_n| \rho^n + \beta \sum_{n=1}^{\infty} |\alpha_n|^m \rho^n \\ & \leq |\alpha_0| + \frac{(1-|\alpha_0|^2)\,\rho}{(1+\gamma)(1-\gamma-\rho)} + \beta \frac{(1-|\alpha_0|^2)^m\,\rho}{(1+\gamma)^m((1-\gamma)^m-\rho)} \\ & = 1 + \Psi_\gamma(\rho) \leq 1 \end{aligned} \tag{3.3}$$

provided $\Psi_\gamma(\rho) \leq 0$, where

$$\Psi_\gamma(\rho) = \frac{1-|\alpha_0|^2}{1+\gamma}\left(\frac{\rho}{1-\gamma-\rho}\right) + \beta\left(\frac{1-|\alpha_0|^2}{1+\gamma}\right)^m \left(\frac{\rho}{(1-\gamma)^m-\rho}\right) - (1-|\alpha_0|).$$

Since $(1-\gamma)-\rho > (1-\gamma)^m - \rho$, it is easy to see that $\Psi_\gamma(\rho)$ is an increasing function of r for $r < (1-\gamma)^m$. A simplification shows that

$$\begin{aligned} & \Psi_\gamma(\rho) \\ & = K\left(1 + (1-|\alpha_0|^2)^{m-1}\left(\frac{2\beta\rho}{(1+\gamma)[(1-\gamma)^m-\rho]} + \frac{\phi_\gamma(\rho)}{(1-|\alpha_0|^2)^{m-1}}\right) - \frac{2}{1+|\alpha_0|}\right), \end{aligned}$$

where

$$K = \frac{1-|\alpha_0|^2}{2} \quad \text{and} \quad \phi_\gamma(\rho) = \frac{2r}{(1+\gamma)(1-\gamma-\rho)} - 1.$$

Let $\rho \leq \rho_0$ be such that $\Psi_\gamma(\rho) \leq \Psi_\gamma(\rho_0)$, and $\phi_\gamma(\rho_0) = 0$. Then, it is easy to see that $\phi_\gamma(\rho_0) = 0$ if, and only if, $\rho_0 = (1-\gamma^2)/(3+\gamma)$. Therefore, it is enough to prove that $\Psi_\gamma(\rho_0) \leq 0$ for $|\alpha_0| \leq 1$. Let $\beta = \eta\left((1-\gamma)^m(3+\gamma) - (1-\gamma^2)\right)$, then it is easy to see

$$\Psi_\gamma(\rho_0) = K\left(1 + 2\eta(1-|\alpha_0|^2)^{m-1}\frac{1-\gamma^2}{(1+\gamma)^m} - \frac{2}{1+|\alpha_0|}\right) := KG_\gamma(|\alpha_0|),$$

where

$$G_\gamma(x) = 1 + 2\eta A(\gamma)(1-x^2)^{m-1} - \frac{2}{x+1} \tag{3.4}$$

and

$$A(\gamma) = \frac{1-\gamma^2}{(1+\gamma)^m} > 0 \quad \text{for } \gamma \in [0,1).$$

It now remains to show that $G_\gamma(x) \leq 0$ for $\gamma \in [0,1)$ and $x \in [0,1]$. Since

$$A'(\gamma) = -\frac{2(1+\gamma)\gamma + m(1-\gamma^2)}{(1+\gamma)^{m+1}} \leq 0, \quad \text{for } \gamma \in [0,1)$$

and $A(0) = 1$, $A(1) = 0$, it follows that $A(\gamma)$ is a decreasing function and hence $A(\gamma) \leq A(0) = 1$. Since $x \leq 1$ and $0 < A(\gamma) \leq 1$, we have

$$-A(\gamma)x(1+x)^2(1-x^2)^{m-2} > -4.$$

From (3.4), we have

$$\begin{aligned}(G_\gamma(x))' &= \frac{2}{(1+x)^2}\left(1 - 2\eta A(\gamma)(m-1)x(1+x)^2(1-x^2)^{m-2}\right)\\ &\geq \frac{2\left(1-8(m-1)\eta\right)}{(1+x)^2}.\end{aligned}$$

Clearly, $(G_\gamma(x))' > 0$ for $x \in (0,1)$ whenever $\eta \leq 1/(8(m-1))$. Therefore, $G_\gamma(x)$ is an increasing function on $[0,1]$ for $\eta \leq 1/(8(m-1))$. Equivalently,

$$\beta \leq \frac{(1-\gamma)^m(3+\gamma) - (1-\gamma^2)}{8(m-1)}.$$

In particular, $G_\gamma(x) \leq 0$ for $\gamma \in [0, \gamma_*]$ and $x \in [0,1]$, where γ_* is the smallest root of the equation $(1-\gamma)^m(3+\gamma) - (1-\gamma^2) = 0$. This completes the proof. □

Proof of Theorem 2.3. For $0 \leq \gamma < 1$, let

$$\Omega_\gamma = \left\{z \in \mathbb{C}\colon \left|z + \frac{\gamma}{1-\gamma}\right| < \frac{1}{1-\gamma}\right\}$$

and the function $f\colon \Omega_\gamma \to \mathbb{D}$ be given by $f(z) = \sum_{n=0}^\infty a_n z_n$. Then the function g defined by

$$g(z) = f\left(\frac{z-\gamma}{1-\gamma}\right) = \sum_{n=0}^\infty \frac{a_n}{(1-\gamma)^n}(z-\gamma)^n \quad \text{for } |z-\gamma| < 1-\gamma$$

belongs to $\mathcal{B}(\mathbb{D})$. Applying Lemma 2.1 to the function g, we obtain

$$|a_0| + \sum_{n=1}^\infty \left(\frac{|a_n|}{(1-\gamma)^n} + \beta\left(\frac{|a_n|}{(1-\gamma)^n}\right)^m\right)\rho^n \leq 1 \quad \text{for } \rho \leq \rho_0 = \frac{1-\gamma^2}{3+\gamma}.$$

That is

$$|a_0| + \sum_{n=1}^{\infty} \left(|a_n| + \beta \frac{|a_n|^m}{(1-\gamma)^{(m-1)n}} \right) \left(\frac{\rho}{1-\gamma} \right)^n \leq 1 \quad \text{for } \rho \leq \rho_0 = \frac{1-\gamma^2}{3+\gamma}$$

which is equivalent to

$$|a_0| + \sum_{n=1}^{\infty} \left(|a_n| + \beta \frac{|a_n|^m}{(1-\gamma)^{(m-1)n}} \right) r^n \leq 1 \quad \text{for } r \leq r_0 = \frac{1+\gamma}{3+\gamma},$$

where $\rho = r(1-\gamma)$ and

$$\beta = \frac{(1-\gamma)^m(3+\gamma) - (1-\gamma^2)}{8(m-1)} \quad \text{for } 0 \leq \gamma \leq \gamma_* < 1.$$

Here γ_* is the smallest root of the equation $(1-\gamma)^m(3+\gamma) + \gamma^2 - 1 = 0$.

In order to prove the sharpness of the radius, we consider the composition function $f_a = h \circ H$ which maps Ω_γ univalently onto $\mathbb{D}$, where $H\colon \Omega_\gamma \to \mathbb{D}$ defined by $H(z) = (1-\gamma)z + \gamma$ and $h\colon \mathbb{D} \to \mathbb{D}$ with $h(z) = (a-z)/(1-az)$, for $a \in (0,1)$. A simple computation shows that

$$f_a(z) = \frac{a - \gamma - (1-\gamma)z}{1 - a\gamma - a(1-\gamma)z} = C_0 - \sum_{n=1}^{\infty} C_n z^n \quad \text{for } z \in \mathbb{D},$$

where $a \in (0,1)$ and

$$C_0 = \frac{a-\gamma}{1-a\gamma} \quad \text{and} \quad C_n = \frac{1-a^2}{a(1-a\gamma)} \left(\frac{a(1-\gamma)}{1-a\gamma} \right)^n.$$

A simple computation shows that

$$\begin{aligned}
|C_0| &+ \sum_{n=1}^{\infty} \left(|C_n| + \beta \frac{|C_n|^m}{(1-\gamma)^{(m-1)n}} \right) r^n = \frac{a-\gamma}{1-a\gamma} + \sum_{n=1}^{\infty} \left(\frac{1-a^2}{a(1-a\gamma)} \left(\frac{a(1-\gamma)}{1-a\gamma} \right)^n \right.\\
&\left. + \frac{\beta}{(1-\gamma)^{(m-1)n}} \frac{(1-a^2)^m}{a^m(1-a\gamma)^m} \left(\frac{a(1-\gamma)}{1-a\gamma} \right)^{mn} \right) r^n \\
&= \frac{a-\gamma}{1-a\gamma} + \frac{(1+a)(1-a)(1-\gamma)r}{(1-a\gamma)\left((1-a\gamma) - ar(1-\gamma)\right)} + \frac{\beta(1-a)^m(1+a)^m(1-\gamma)r}{(1-a\gamma)^m\left((1-a\gamma)^m - a^m(1-\gamma)r\right)} \\
&= 1 - (1-a)\Phi_\gamma(r),
\end{aligned}$$

where

$$\begin{aligned}
\Phi_\gamma(r) &= -\frac{(1+a)(1-\gamma)r}{(1-a\gamma)((1-a\gamma) - ar(1-\gamma))} - \frac{\beta(1-a)^{m-1}(1+a)^m(1-\gamma)r}{(1-a\gamma)^m\left((1-a\gamma)^m - a^m(1-\gamma)r\right)} \\
&\quad - \frac{1}{1-a}\left(\frac{a-\gamma}{1-a\gamma} - 1 \right) \\
&= -\frac{(1+a)(1-\gamma)r}{(1-a\gamma)((1-a\gamma) - ar(1-\gamma))} - \frac{\beta(1-a)^{m-1}(1+a)^m(1-\gamma)r}{(1-a\gamma)^m\left((1-a\gamma)^m - a^m(1-\gamma)r\right)} \\
&\quad + \frac{1+\gamma}{1-a\gamma}.
\end{aligned}$$

We note that for $r \in (0,1)$,

$$\Phi'_\gamma(r) = -\frac{(1+a)(1-\gamma)}{\left((1-a\gamma) - ar(1-\gamma)\right)^2} - \frac{\beta(1-a)^{m-1}(1+a)^m(1-\gamma)}{\left((1-a\gamma)^m - a^m(1-\gamma)r\right)^2} < 0.$$

Therefore, $\Phi_\gamma(r)$ is strictly decreasing function of r in $(0,1)$. Hence, for $r > r_0 = (1+\gamma)/(3+\gamma)$, we have $\Phi_\gamma(r) < \Phi_\gamma(r_0)$. A simple computation shows that

$$\lim_{a\to 1} \Phi_\gamma(r_0) = -\frac{2r_0}{(1-\gamma)(1-r_0)} + \frac{1+\gamma}{1-\gamma} = 0.$$

Thus, $\Phi_\gamma(r) < 0$ for $r > r_0$. Hence, $1-(1-a)\Phi_\gamma(r) > 1$ for $r > r_0$, which shows that r_0 is the best possible. This completes the proof. □

Proof of Lemma 2.4. Without loss of generality, we assume that $\gamma \in [0,1)$. Also let $z \in \mathbb{D}_\gamma := \mathbb{D}(\gamma; 1-\gamma)$ if, and only if, $\phi = (z-\gamma)/(1-\gamma) \in \mathbb{D}$. Then we have

$$g(z) = \sum_{n=0}^{\infty} \alpha_n (1-\gamma)^n \phi^n(z) = \sum_{n=0}^{\infty} b_n \phi^n(z) := G(\phi(z))$$

for $z \in \mathbb{D}_\gamma$, where $b_n = \alpha_n(1-\gamma)^n$. It is known that for arbitrary function $H(z) = \sum_{n=0}^{\infty} h_n z^n$, $z \in \mathbb{D}$, the area functional

$$\frac{S_r}{\pi} = \text{Area}\Big(H(\mathbb{D}(0,r))\Big) = \frac{1}{\pi}\iint_{|z|<r} |\, H'(z) \,|^2 \; dx\, dy \leq \sum_{n=1}^{\infty} n|h_n|^2 r^{2n}.$$

A simple computation shows that

$$\frac{S_\rho^\gamma}{\pi} = \frac{1}{\pi}\text{Area}\Big(G(\mathbb{D}(0,\rho))\Big) \leq (1-|b_0|^2)^2 \frac{\rho^2}{(1-\rho^2)^2} = (1-|\alpha_0|^2)^2 \frac{\rho^2}{(1-\rho^2)^2}. \tag{3.5}$$

Using Lemma 1.4, we deduce that

$$\sum_{n=1}^{\infty} |\alpha_n|\rho^n \leq \frac{1-|\alpha_0|^2}{1+\gamma} \sum_{n=1}^{\infty} \left(\frac{\rho}{1-\gamma}\right)^n = \frac{1-|\alpha_0|^2}{1+\gamma} \frac{\rho}{1-\gamma-\rho}. \tag{3.6}$$

In view of (3.5) and (3.6), we obtain

$$\begin{aligned}
&|\alpha_0| + \sum_{n=1}^{\infty} |\alpha_n|\rho^n + k\left(\frac{S_\rho^\gamma}{\pi}\right) + \lambda\left(\frac{S_\rho^\gamma}{\pi}\right)^2 \\
&= |\alpha_0| + \frac{(1-|\alpha_0|^2)\rho}{(1+\gamma)(1-\gamma-\rho)} + k\frac{(1-|\alpha_0|^2)^2\rho^2}{(1-\rho^2)^2} + \lambda\frac{(1-|\alpha_0|^2)^4\rho^4}{(1-\rho^2)^4} \\
&= 1 + \Psi_1^\gamma(\rho),
\end{aligned}$$

where

$$\Psi_1^\gamma(\rho) = \frac{(1-|\alpha_0|^2)\rho}{(1+\gamma)(1-\gamma-\rho)} + k\frac{(1-|\alpha_0|^2)^2\rho^2}{(1-\rho^2)^2} + \lambda\frac{(1-|\alpha_0|^2)^4\rho^4}{(1-\rho^2)^4} - (1-|\alpha_0|)$$

which can be written as

$$\begin{aligned}
\Psi_1^\gamma(\rho) = \frac{1-|\alpha_0|^2}{2}\Bigg(&1 + 2\lambda(1-|\alpha_0|^2)^3\Bigg(\left(\frac{\rho^4}{(1-\rho^2)^4} + \frac{k}{\lambda}\frac{\rho^2}{(1-\rho^2)^2(1-|\alpha_0|^2)^2}\right) \\
&+ \frac{1}{2\lambda(1-|\alpha_0|^2)^3}\left(\frac{2\rho}{(1+\gamma)(1-\gamma-\rho)} - 1\right)\Bigg) - \frac{2}{1+|\alpha_0|}\Bigg).
\end{aligned}$$

We note that

$$\begin{aligned}
(\Psi_1^\gamma)'(\rho) = &\frac{(1-|\alpha_0|^2)}{(1+\gamma)} \frac{1-\gamma}{((1-\gamma-\rho))^2} + k\frac{(1-|\alpha_0|^2)^2\left(2\rho(1-\rho^2)^2 + 4\rho^3(1-\rho^2)\right)}{(1-\rho^2)^4} \\
&+ \lambda\frac{(1-|\alpha_0|^2)^4 4\rho^3(1-\rho^2)^4 + 8\rho^5(1-\rho^2)^3}{(1-\rho^2)^8} > 0.
\end{aligned}$$

Therefore, $\Psi_1^\gamma(\rho)$ is an increasing function and hence $\Psi_1^\gamma(\rho) \le \Psi_1^\gamma(\rho_0)$ for $\rho \le \rho_0$, where

$$\frac{2\rho_0}{(1+\gamma)(1-\gamma-\rho_0)} = 1, \quad \text{i.e.,}\ \rho_0 = \frac{1-\gamma^2}{3+\gamma}.$$

A simple computation shows that

$$\begin{aligned}\Psi_1^\gamma(\rho_0) &= \frac{1-|\alpha_0|^2}{2}\left(1+2\lambda(1-|\alpha_0|^2)^3A^4(\gamma)+2k(1-|\alpha_0|^2)A^2(\gamma)-\frac{2}{1+|\alpha_0|}\right)\\ &= \frac{1-|\alpha_0|^2}{2}J(|\alpha_0|),\end{aligned}$$

where

$$J(x) = 1+2\lambda(1-x^2)^3A^4(\gamma)+2k(1-x^2)A^2(\gamma)-\frac{2}{1+x} \quad \text{for}\ x\in[0,1]$$

and

$$A(\gamma) = \frac{(3+\gamma)(1-\gamma^2)}{(3+\gamma)^2-(1-\gamma^2)^2}.$$

It is enough to show that $J(x) \le 0$ for $x \in [0,1]$ and $\gamma \in [0,1)$ so that $\Psi_1^\gamma(\rho_0) \le 0$. We note that $A(\gamma) > 0$ for $\gamma \in [0,1)$. Further,

$$J(0) = 2\lambda A^4(\gamma)+2kA^2(\gamma)-1, \quad \text{and} \quad \lim_{x\to 1^-} J(x) = 0.$$

It can be seen that $A(\gamma) = (f_1 \circ f_2)(\gamma)$, where $f_1(\rho) = \rho/(1-\rho^2)$ and $f_2(\gamma) = (1-\gamma^2)/(3+\gamma)$. Since $A'(\gamma) = f_1'(f_2(\gamma))f_2'(\gamma)$, where

$$f_2'(\gamma) = -\left(\frac{\gamma^2+6\gamma+1}{(3+\gamma)^2}\right) < 0 \tag{3.7}$$

which implies that $f_1(\rho)$ is an increasing function of ρ in $(0,1)$, and f_2 is a decreasing function of γ in $[0,1)$. Hence, it follows that $A(\gamma)$ is a decreasing function of γ in $[0,1)$, with $A(0) = 3/8$ and $A(1) = 0$. It can be seen that $A^2(\gamma)$ and $A^4(\gamma)$ are decreasing functions on $[0,1)$. Therefore, we have

$$A^2(\gamma) \le A^2(0) = \frac{9}{64} \quad \text{and} \quad A^4(\gamma) \le A^4(0) = \frac{81}{4096}.$$

Since $x \in [0,1]$, we have

$$x(1+x)^2A^2(\gamma) \le \frac{9}{16} \quad \text{and} \quad x(1+x)^2(1-x^2)^2A^4(\gamma) \le \frac{81}{1024}.$$

As a consequence, we obtain

$$\begin{aligned}J'(x) &= \frac{2}{(1+x)^2}\left(1-2kx(1+x)^2A^2(\gamma)-6\lambda x(1+x)^2(1-x^2)^2A^4(\gamma)\right)\\ &\ge \frac{2}{(1+x)^2}\left(1-\left(\frac{9k}{8}+\frac{243\lambda}{512}\right)\right)\\ &\ge 0, \quad \text{if}\ 9k/8+243\lambda/512 \le 1.\end{aligned}$$

Since $0 \le \lambda \le 512/243$ and $k \ge 0$, therefore, $J(x)$ is an increasing function in $[0,1]$ for $k+27\lambda/64 \le 8/9$. Hence, $J(x) \le 0$ for all $x \in [0,1]$ and $\gamma \in [0,1)$. This completes the proof. □

Proof of Theorem 2.5. Let $f \in \mathcal{B}(\Omega_\gamma)$ and $g(z) = f((z-\gamma)/(1-\gamma))$. Then, it is easy to see that $g \in \mathcal{B}(\mathbb{D})$ and

$$g(z) = \sum_{n=0}^{\infty} \frac{a_n}{(1-\gamma)^n}(z-\gamma)^n.$$

Using Lemma 2.4, we obtain

$$\sum_{n=0}^{\infty} \frac{|a_n|}{(1-\gamma)^n}\rho^n + \left(\frac{8}{9} - \frac{27}{64}\lambda\right)\left(\frac{S_\rho^\gamma}{\pi}\right) + \lambda\left(\frac{S_\rho^\gamma}{\pi}\right)^2 \leq 1 \quad \text{for } \rho \leq \frac{1-\gamma^2}{3+\gamma}$$

which is equivalent to

$$\sum_{n=0}^{\infty} |a_n|\left(\frac{\rho}{(1-\gamma)}\right)^n + \left(\frac{8}{9} - \frac{27}{64}\lambda\right)\left(\frac{S_\rho^\gamma}{\pi}\right) + \lambda\left(\frac{S_\rho^\gamma}{\pi}\right)^2 \leq 1 \quad \text{for } \rho \leq \frac{1-\gamma^2}{3+\gamma}. \tag{3.8}$$

Set $\rho = r(1-\gamma)$, then in view of (3.8), we obtain

$$\sum_{n=0}^{\infty} |a_n|r^n + \left(\frac{8}{9} - \frac{27}{64}\lambda\right)\left(\frac{S_{r(1-\gamma)}^\gamma}{\pi}\right) + \lambda\left(\frac{S_{r(1-\gamma)}^\gamma}{\pi}\right)^2 \leq 1 \quad \text{for } r \leq \frac{1+\gamma}{3+\gamma}. \tag{3.9}$$

To show the sharpness of the result, we consider the following function

$$f_a(z) = \frac{a-\gamma-(1-\gamma)z}{1-a\gamma-a(1-\gamma)z} \quad \text{for } z \in \Omega_\gamma \text{ and } a \in (0,1).$$

Define $\phi\colon \mathbb{D} \to \mathbb{D}$ by $\phi(z) = (a-z)/(1-az)$ and $H\colon \Omega_\gamma \to \mathbb{D}$ by $H(z) = (1-\gamma)z+\gamma$. Then, the function $f_a = \phi \circ H$ maps Ω_γ, univalently onto $\mathbb{D}$. A simple computation shows that

$$f_a(z) = \frac{a-\gamma-(1-\gamma)z}{1-a\gamma-a(1-\gamma)z} = C_0 - \sum_{n=1}^{\infty} C_n z^n \quad \text{for } z \in \mathbb{D},$$

where $a \in (0,1)$ and

$$C_0 = \frac{a-\gamma}{1-a\gamma} \quad \text{and} \quad C_n = \frac{1-a^2}{a(1-a\gamma)}\left(\frac{a(1-\gamma)}{1-a\gamma}\right)^n.$$

A simple computation using (3.9) shows that

$$\begin{aligned}
&\sum_{n=0}^{\infty} |C_n|r^n + \left(\frac{8}{9} - \frac{27}{64}\lambda\right)\left(\frac{S_{r(1-\gamma)}^\gamma}{\pi}\right) + \lambda\left(\frac{S_{r(1-\gamma)}^\gamma}{\pi}\right)^2 \\
&= \frac{a-\gamma}{1-a\gamma} + \left(\frac{1-a^2}{1-a\gamma}\right)\frac{(1-\gamma)r}{1-a\gamma-ar(1-\gamma)} \\
&\quad + \left(\frac{8}{9} - \frac{27}{64}\lambda\right)\frac{r^2(1-a^2)^2(1-\gamma)^4}{((1-a\gamma)^2-a^2r^2(1-\gamma)^4)^2} + \lambda\frac{r^4(1-a^2)^4(1-\gamma)^8}{((1-a\gamma)^2-a^2r^2(1-\gamma)^4)^4} \\
&:= 1-(1-a)\Phi_1^\gamma(r),
\end{aligned}$$

where

$$\begin{aligned}
\Phi_1^\gamma(r) = &-\frac{(1+a)(1-\gamma)r}{(1-a\gamma-ar(1-\gamma))(1-a\gamma)} - \left(\frac{8}{9} - \frac{27}{64}\lambda\right)\frac{r^2(1-a)(1+a)^2(1-\gamma)^4}{((1-a\gamma)^2-a^2r^2(1-\gamma)^4)^2} \\
&- \lambda\frac{r^4(1-a)^3(1+a)^4(1-\gamma)^8}{((1-a\gamma)^2-a^2r^2(1-\gamma)^4)^4} - \frac{1}{1-a}\left(\frac{a-\gamma}{1-a\gamma} - 1\right).
\end{aligned}$$

A simple computation shows that

$$\left(\Phi_1^{\gamma}\right)'(r) = -\frac{(1+a)(1-\gamma)}{(1-a\gamma-ar(1-\gamma))^2} - \left(\frac{8}{9}-\frac{27}{64}\lambda\right)\frac{2r((1-a\gamma)^2+a^2r^2(1-\gamma)^4)}{((1-a\gamma)^2-a^2r^2(1-\gamma)^4)^3}$$
$$-\lambda\frac{4r^3(1-a)^3(1+a)^4(1-\gamma)^8((1-a\gamma)^2+a^2r^2(1-\gamma)^4)}{((1-a\gamma)^2-a^2r^2(1-\gamma)^4)^5} < 0$$

for r in $(0,1)$ and hence $\Phi_1^{\gamma}(r)$ is strictly decreasing function of r. Therefore, for $r > r_0 = (1+\gamma)/(3+\gamma)$, we have $\Phi_1^{\gamma}(r) < \Phi_1^{\gamma}(r_0)$. An elementary calculation shows that

$$\lim_{a\to 1}\Phi_1^{\gamma}(r_0) = -\frac{2r_0}{(1-\gamma)(1-r_0)} + \frac{1+\gamma}{1-\gamma} = 0.$$

Therefore, $\Phi_1^{\gamma}(r) < 0$ for $r > r_0$. Hence, $1-(1-a)\Phi_{\gamma}(r) > 1$ for $r > r_0$, which shows that r_0 is the best possible. □

Proof of Lemma 2.6. Let $g \in \mathcal{B}(\mathbb{D})$. Then by the Schwarz–Pick lemma, for any $g \in \mathcal{B}(\mathbb{D})$, we have

$$|g(z)| \le \frac{\rho+|g(\gamma)|}{1+\rho|g(\gamma)|} = \frac{\rho+|\alpha_0|}{1+\rho|\alpha_0|} \quad \text{for } z\in\mathbb{D}. \tag{3.10}$$

For functions $g \in \mathcal{B}(\mathbb{D})$, from Lemma 1.4, we have

$$|\alpha_n| \le (1+|\gamma|)^{n-1}\frac{1-|\alpha_0|^2}{(1-|\gamma|^2)^n} \quad \text{for } n\ge 1. \tag{3.11}$$

A simple computation using (3.11) gives

$$\sum_{n=N}^{\infty}|\alpha_n|\rho^n \le \frac{1-|\alpha_0|^2}{1+\gamma}\sum_{n=N}^{\infty}\left(\frac{\rho}{1-\gamma}\right)^n = \frac{(1-|\alpha_0|^2)}{(1+\gamma)(1-\gamma)^{N-1}}\left(\frac{\rho^N}{1-\gamma-\rho}\right). \tag{3.12}$$

From (3.10) and (3.12) we obtain

$$\begin{aligned}|g(z)| + \sum_{n=N}^{\infty}|\alpha_n|\rho^n &\le \frac{\rho+|\alpha_0|}{1+\rho|\alpha_0|} + \frac{(1-|\alpha_0|^2)}{(1+\gamma)(1-\gamma)^{N-1}}\left(\frac{\rho^N}{1-\gamma-\rho}\right)\\ &= 1 + \frac{\Phi_N^{\gamma}(\rho)}{(1+\rho|\alpha_0|)(1+\gamma)(1-\gamma)^{N-1}(1-\gamma-\rho)},\end{aligned}$$

where

$$\begin{aligned}\Phi_N^{\gamma}(\rho) &= (\rho+|\alpha_0|)A(\gamma)(1-\gamma-\rho) + (1+|\alpha_0|)(1-|\alpha_0|)(1+\rho|\alpha_0|)\rho^N\\ &\quad -(1+\rho|\alpha_0|)A(\gamma)(1-\gamma-\rho)\\ &= (1-|\alpha_0|)\Big((1+|\alpha_0|)(1+\rho|\alpha_0|)\rho^N + A(\gamma)(\rho-1)(1-\gamma-\rho)\Big)\\ &\le (1-|\alpha_0|)\Big(2(1+\gamma)\rho^N + A(\gamma)(\rho-1)(1-\gamma-\rho)\Big),\end{aligned}$$

where $A(\gamma) = (1+\gamma)(1-\gamma)^{N-1}$ and $|\alpha_0| \le 1$. An observation shows that $\Phi_N^{\gamma}(\rho) \le 0$ if $2(1+\gamma)\rho^N + A(\gamma)(\rho-1)(1-\gamma-\rho) \le 0$, and this holds for $\rho \le \rho_N$, where ρ_N is the root of

$$F_N(\gamma,\rho) = 2(1+\gamma)\rho^N + A(\gamma)(\rho-1)(1-\gamma-\rho) = 0. \tag{3.13}$$

The existence of the root ρ_N in $(0,1)$ follows from the fact that $F_N(\gamma,\rho)$ is continuous and $F_N(\gamma,0)F_N(\gamma,1) < 0$. □

Proof of Theorem 2.7. For $0 \leq \gamma < 1$, let $f \in \mathcal{B}(\Omega_\gamma)$ such that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $z \in \mathbb{D}$. Then, it is easy to see that

$$g(z) = f\left(\frac{z-\gamma}{1-\gamma}\right) \in \mathcal{B}(\mathbb{D}) \quad \text{for } |z-\gamma| < 1 - |\gamma|.$$

Further,

$$g(z) = f\left(\frac{z-\gamma}{1-\gamma}\right) = \sum_{n=0}^{\infty} \frac{a_n}{(1-\gamma)^n}(z-\gamma)^n.$$

An application of Lemma 2.6 shows that

$$\left| f\left(\frac{z-\gamma}{1-\gamma}\right)\right| + \sum_{n=N}^{\infty} \frac{|a_n|}{(1-\gamma)^n}\rho^n \leq 1 \quad \text{for } \rho \leq \rho_N. \tag{3.14}$$

Since $|z-\gamma| < 1-\gamma$, we set $z-\gamma = w(1-\gamma)$ for some $w \in \mathbb{D}$ and $\rho = r(1-\gamma)$. Then, from (3.14), we obtain

$$|f(w)| + \sum_{n=N}^{\infty} |a_n| r^n \leq 1 \quad \text{for } r \leq \frac{\rho_N}{1-\gamma},$$

where ρ_N as in Lemma 2.6. That is, ρ_N is the smallest root of the equation $2(1+\rho)\rho^N + A(\gamma)(\rho-1)(1-\gamma-\rho) = 0$.

In order to show the sharpness of the result, we consider the following function f_a defined by

$$f_a(z) = \frac{1-\gamma-(1-\gamma)z}{(1-a\gamma)-(1-\gamma)z} = C_0 - \sum_{n=1}^{\infty} C_n z^n \quad \text{for } z \in \mathbb{D}.$$

For $\gamma \in [0,1)$, $a > \gamma$ and $\rho = r(1-\gamma)$, we obtain

(3.15)

$$\begin{aligned}
M &:= |f_a(-\rho)| + \sum_{n=N}^{\infty} |C_n|\rho^n \\
&= \frac{(a-\gamma)+(1-\gamma)\rho}{(1-a\gamma)+a(1-\gamma)\rho} + \frac{1-a^2}{a(1-a\gamma)}\left(\frac{a(1-\gamma)}{1-a\gamma}\right)^N \rho^N \left(\frac{1-a\gamma}{(1-a\gamma)-a(1-\gamma)\rho}\right) \\
&= \frac{(a-\gamma)+(1-\gamma)\rho}{(1-a\gamma)+a(1-\gamma)\rho} + \frac{(1-a^2)B^N\rho^N}{a\left((1-a\gamma)-a(1-\gamma)\rho\right)}, \quad \text{where } B = \frac{a(1-\gamma)}{1-a\gamma} \\
&= \frac{\left((a-\gamma)+(1-\gamma)\rho\right)\left((1-a\gamma)-a(1-\gamma)\rho\right) + d\rho^N(1-a^2)\left((1-a\gamma)+a(1-\gamma)\rho\right)}{\left((1-a\gamma)+a(1-\gamma)\rho\right)\left((1-a\gamma)-a(1-\gamma)\rho\right)} \\
&= 1 + \frac{V(\rho)}{\left((1-a\gamma)+a(1-\gamma)\rho\right)\left((1-a\gamma)-a(1-\gamma)\rho\right)},
\end{aligned}$$

where $d = B^N/a$ and

$$\begin{aligned}
V(\rho) &:= \left((a-\gamma)+(1-\gamma)\rho\right)\left((1-a\gamma)-a(1-\gamma)\rho\right) \\
&\quad + d\rho^N(1-a^2)((1-a\gamma)+a(1-\gamma)\rho) \\
&\quad - \left((1-a\gamma)+a(1-\gamma)\rho\right)\left((1-a\gamma)-a(1-\gamma)\rho\right) \\
&= (1-a)((1+a)\left((1-a\gamma)+a(1-\gamma)\rho\right) d\rho^N \\
&\quad + ((1-a\gamma)-a(1-\gamma)\rho)(\rho(1-\gamma)-(1+\gamma))).
\end{aligned}$$

From (3.15), it is easy to see that $M > 1$ if $V(\rho) > 0$. Note that $V(\rho) > 0$ if

$$W_a(\rho) := (1+a)\left((1-a\gamma) + a(1-\gamma)\rho\right) d\rho^N + ((1-a\gamma) - a(1-\gamma)\rho)(\rho(1-\gamma) - (1+\gamma)). \tag{3.16}$$

Allowing $a \to 1$, from the inequality (3.16), it can be seen that

$$W_1(\rho) = 2(1-\gamma)(1+\rho)\rho^N + (1-\gamma)(1-\rho)(\rho(1-\gamma) - (1+\gamma)).$$

Now for $\rho > \rho_N$ and $0 < \gamma < 1$, we obtian

$$W_1(\rho) > (1-\gamma)\left(2(1+\rho_N)\rho_N^N + (1+\gamma)(1-\gamma)^{N-1}(\rho_N - 1)(1-\gamma-\rho_N)\right) \tag{3.17}$$

Since ρ_N is a root of (3.13), we have

$$2(1+\gamma)\rho_N^N + A(\gamma)(\rho_N - 1)(1-\gamma-\rho_N) = 0. \tag{3.18}$$

In view of (3.17) and (3.18), it is easy to see that $W_1(\rho) > 0$ for $\rho > \rho_N$. Thus, $M > 1$ if $r > \rho_N/(1-\gamma)$. This proves the sharpness. ☐

Proof of Theorem 2.9. Let $f \in \mathcal{B}(\Omega_\gamma)$ be given by $f(z) = \sum_{n=1}^{\infty} a_n z^n$ for $z \in \mathbb{D}$. Then, f can be expressed as $f(z) = zh(z)$, where $h \in B(\Omega_\gamma)$ with $h(z) = \sum_{n=0}^{\infty} b_n z^n$ and $b_n = a_{n+1}$. Let $|b_0| = |a_1| = a$, and $h_0(z) = g(z) - b_0$. Using Lemma 1.5, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} |b_n| r^n &+ \left(\frac{1}{1+|b_0|} + \frac{r}{1+r}\right) \sum_{n=1}^{\infty} |b_n|^2 r^{2n} \\ &\leq a + \frac{1-a^2}{1+\gamma}\frac{r}{1-r} + \left(\frac{1}{1+a} + \frac{r}{1-r}\right)\left(\frac{1-a^2}{1+\gamma}\right)^2 \frac{r^2}{1-r^2}.\end{aligned} \tag{3.19}$$

That is,

$$\begin{aligned}\sum_{n=0}^{\infty} |b_n| r^n \leq a &+ \frac{1-a^2}{1+\gamma}\frac{r}{1-r} + \left(\frac{1}{1+a} + \frac{r}{1-r}\right)\left(\frac{1-a^2}{1+\gamma}\right)^2 \frac{r^2}{1-r^2} \\ &- \left(\frac{1}{1+|b_0|} + \frac{r}{1+r}\right) \sum_{n=1}^{\infty} |b_n|^2 r^{2n}.\end{aligned} \tag{3.20}$$

Since

$$\sum_{n=1}^{\infty} |a_n| r^n = \sum_{n=0}^{\infty} |b_n| r^{n+1} = r \sum_{n=0}^{\infty} |b_n| r^n, \tag{3.21}$$

in view of (3.20) and (3.21), we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} |a_n| r^n &\leq r\left(a + \frac{1-a^2}{1+\gamma}\frac{r}{1-r}\right) + \left(\frac{1}{1+a} + \frac{r}{1-r}\right)\left(\frac{1-a^2}{1+\gamma}\right)^2 \frac{r^3}{1-r^2} \\ &\quad - \left(\frac{1}{1+a} + \frac{r}{1-r}\right) \sum_{n=1}^{\infty} |a_{n+1}|^2 r^{2n+1} \\ &= ra + \left(\frac{1-a^2}{1+\gamma}\right)\frac{r^2}{1-r} + \left(\frac{1}{1+a} + \frac{r}{1-r}\right)\left(\frac{1-a^2}{1+\gamma}\right)^2 \frac{r^3}{1-r^2} \\ &\quad - \left(\frac{1}{1+a} + \frac{r}{1-r}\right) \sum_{n=2}^{\infty} |a_n|^2 r^{2n-1}.\end{aligned}$$

Further simplification shows that

$$\begin{aligned}
&\sum_{n=1}^{\infty}|a_n|r^n+\left(\frac{1}{1+a}+\frac{r}{1-r}\right)\sum_{n=2}^{\infty}|a_n|^2r^{2n-1}\\
&\leq ra+\left(\frac{1-a^2}{1+\gamma}\right)\frac{r^2}{1-r}+\left(\frac{1}{1+a}+\frac{r}{1-r}\right)\left(\frac{1-a^2}{1+\gamma}\right)^2\frac{r^3}{1-r^2}\\
&:=\mathcal{T}(a).
\end{aligned}$$

It is easy to see that $\mathcal{T}$ can be represented as

$$\mathcal{T}(a)=ar+A(1-a^2)+B(1-a)(1-a^2)+C(1-a^2)^2,$$

where

$$\begin{aligned}
A=A(r)&=\frac{r^2}{(1+\gamma)(1-r)},\\
B=B(r)&=\frac{r^3}{(1+\gamma)^2(1-r^2)}\quad\text{and}\\
C=C(r)&=\frac{r^4}{(1+\gamma)^2(1-r)(1-r^2)}.
\end{aligned}$$

Clearly, B and C are positive. We note that,

$$\begin{aligned}
\mathcal{T}'(a)&=r-2Aa+B(3a^2-2a-1)+4C(a^3-a),\\
\mathcal{T}''(a)&=-2A+2B(3a-1)+4C(3a^2-1)\quad\text{and}\\
\mathcal{T}'''(a)&=6B+24Ca.
\end{aligned}$$

Since B and C are positive, it follows that $\mathcal{T}'''(a)>0$ for $a\in[0,1]$. In other words, $\mathcal{T}''$ is an increasing function of a in $[0,1]$. Therefore,

$$\mathcal{T}''(a)\leq\mathcal{T}''(1)=-2A+4B+8C=\frac{2r^2}{(1+\gamma)^2(1-r)(1-r^2)}L(r),$$

where

$$L(r)=4r^2+2r(1-r)-(1+\gamma)(1-r^2)=(1+r)(r(3+\gamma)-(1+\gamma)).$$

It is easy to see that $L(r)\leq 0$ for $r\leq r_0=(1+\gamma)/(3+\gamma)$. Hence, $\mathcal{T}''(a)\leq 0$ for $a\in[0,1]$ which implies that $\mathcal{T}'$ is decreasing in $[0,1]$. Therefore, for $r\leq r_0=(1+\gamma)/(3+\gamma)$, we obtain

$$\mathcal{T}'(a)>\mathcal{T}'(1)=1-2A=r\frac{1+\gamma-r(3+\gamma)}{(1+\gamma)(1-r)}.$$

Clearly, for $r\leq r_0$, we have $\mathcal{T}'(1)\geq 0$ for all $a\in[0,1]$. Since $\mathcal{T}'(a)\geq 0$ in $[0,1]$, $\mathcal{T}$ is an increasing function in $[0,1]$, and hence, we have $\mathcal{T}(a)\leq\mathcal{T}(1)=r$. A simple computation shows that

$$\sum_{n=1}^{\infty}|a_n|r^n+\left(\frac{1}{1+|a_1|}+\frac{r}{1-r}\right)\sum_{n=2}^{\infty}|a_n|^2r^{2(n-1)}\leq 1\quad\text{for}\ \ r\leq r_0=\frac{1+\gamma}{3+\gamma}.$$

To show that the sharpness of the radius we consider the function f_a^* by

$$f_a^*(z):=zf_a(z)=z\left(\frac{a-\gamma-(1-\gamma)z}{1-a\gamma-a(1-\gamma)z}\right)=C_0z-\sum_{n=1}^{\infty}C_nz^{n+1}\quad\text{for}\ \ z\in\mathbb{D},$$

where

$$C_0 = \frac{a-\gamma}{1-a\gamma} \quad \text{and} \quad C_n = \frac{(1-a^2)}{a(1-a\gamma)}\left(\frac{a(1-\gamma)}{1-a\gamma}\right)^n.$$

For $n \geq 2$, it is easy to see that $a_1(f_a) = C_0$ and $a_n(f_a) = -C_{n-1}$. For $\gamma \in [0,1]$, and $a > \gamma$, a simple calculation shows that

$$\begin{aligned}
D(r) &:= \sum_{n=1}^{\infty} |C_n| r^n + \left(\frac{1}{1+|C_0|} + \frac{r}{1-r}\right) \sum_{n=2}^{\infty} |C_n|^2 r^{2n-1} \\
&= \left(\frac{a-\gamma}{1-a\gamma}\right) r + \sum_{n=2}^{\infty} \frac{(1-a^2)}{a(1-a\gamma)} \left(\frac{a(1-\gamma)}{1-a\gamma}\right)^{n-1} r^n \\
&\quad + \left(\frac{1}{1+|C_0|} + \frac{r}{1-r}\right) \sum_{n=2}^{\infty} \frac{(1-a^2)^2}{a^2(1-a\gamma)^2} \left(\frac{a(1-\gamma)}{1-a\gamma}\right)^{2(n-1)} r^{2n-1} \\
&= \left(1 - \frac{1-a}{1-a\gamma}\chi(r)\right) r,
\end{aligned}$$

where

$$\begin{aligned}
\chi(r) &:= 1 + \gamma - \frac{(1+a)(1-\gamma)r}{1-a\gamma-a(1-\gamma)r} \\
&\quad - \left(\frac{1-a\gamma}{(1+a)(1-\gamma)} + \frac{r}{1-r}\right) \frac{(1+a)(1-a^2)}{1-a\gamma} \frac{(1-\gamma)^2 r^2}{(1-a\gamma)^2 - a^2(1-\gamma)^2 r^2}.
\end{aligned}$$

Note that

$$\begin{aligned}
\chi'(r) &= -\frac{(1+a)(1-\gamma)(1-a\gamma)}{(1-a\gamma-a(1-\gamma)r)^2} - \frac{(1+a)(1-a^2)}{(1-a\gamma)(1-r)^2} \frac{(1-\gamma)^2 r^2}{(1-a\gamma)^2 - a^2(1-\gamma)^2 r^2} \\
&\quad - \left(\frac{1-a\gamma}{(1+a)(1-\gamma)} + \frac{r}{1-r}\right) \frac{2(1-a\gamma)(1+a)(1-a^2)(1-\gamma)^2 r}{((1-a\gamma)^2 - a^2(1-\gamma)^2 r^2)^2} < 0.
\end{aligned}$$

Therefore, χ is strictly decreasing function in $r \in (0,1)$. Hence, for $r > r_0$, we have $\chi(r) < \chi(x_0)$. It is worth to point out that

$$\lim_{a\to 1} \chi(r_0) = 1 + \gamma - \frac{2(1-\gamma)r_0}{1-\gamma-(1-\gamma)r_0} = 1 + \gamma - \frac{2r_0}{1-r_0} = 0.$$

This shows that $\chi(r) \leq 0$ for $r > r_0$ as $a \to 1$, and hence $D(r) > r$ for $r > r_0$. Therefore,

$$\sum_{n=1}^{\infty} |a_n| r^n + \left(\frac{1}{1+|a_1|} + \frac{r}{1-r}\right) \sum_{n=2}^{\infty} |a_n|^2 r^{2(n-1)} > 1$$

and hence r_0 is the best possible. This completes the proof. □

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