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Uniformization of metric surfaces using isothermal coordinates

Toni Ikonen

Abstract. We establish a uniformization result for metric surfaces—metric spaces that are topological surfaces with locally finite Hausdorff 2-measure. Using the geometric definition of quasiconformality, we show that a metric surface that can be covered by quasiconformal images of Euclidean domains is quasiconformally equivalent to a Riemannian surface. To prove this, we construct an atlas of suitable isothermal coordinates.

Metristen pintojen uniformisaatio isotermisillä koordinaateilla

Tiivistelmä. Todistamme metristen pintojen uniformisaatiolauseen. Metrinen pinta on topologinen pinta varustettuna etäisyysfunktiolla, jonka kaksiulotteinen Hausdorffin mitta on lokaalisti äärellinen. Tutkimme milloin metrinen pinta on riemannilaisen pinnan geometrisesti kvasikonformaalinen kuva. Osoitamme riittäväksi ehdoksi, että metrinen pinta voidaan peittää Eukleideen avaruuden alueiden kvasikonformaalisilla kuvilla. Konstruoimme todistusta varten kartaston isotermisiä koordinaatteja.

1. Introduction

1.1. Overview. The Riemann mapping theorem states that given a simply connected proper subdomain U of $\mathbb{R}^2$, there exists a conformal map $\phi\colon \mathbb{D} \to U$, where $\mathbb{D}$ is the Euclidean disk. Recall that conformal maps preserve angles but they do not necessarily preserve lengths of paths or areas. We say that domains U and V are *conformally equivalent* if there exists a conformal map from U to V.

When the topological type of U is more complicated, so is the classification result. For example, if $U = A(1, r) \subset \mathbb{R}^2$ in an Euclidean annulus of inner radius 1 and outer radius $r > 1$, two such annuli $A(1, r)$ and $A(1, r')$ are conformally equivalent if and only if $r = r'$.

If we relax the definition of conformal map to allow for distortion of infinitesimal balls in a uniformly controlled manner, we obtain the class of quasiconformal maps. With this relaxation, it turns out that for every pair of outer radii $1 < r$ and $1 < r'$, there exists a quasiconformal map from $A(1, r)$ onto $A(1, r')$. Such a map takes the infinitesimal Euclidean balls in $A(1, r)$ to infinitesimal ellipses in $A(1, r')$, and the distortion is determined from the eccentricity of the ellipses.

Similar questions can be considered when the topology type of the surface is more complicated. This is the domain of Teichmüller theory of surfaces; see for example [Leh87, IT92, Hub06]. Roughly speaking, the Teichmüller theory classifies Riemann

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surfaces up to conformal maps, and quasiconformal maps measure how far apart two Riemann surfaces are from one another.

Quasiconformal maps also arise when we try to find isothermal coordinates in a given Riemannian surface, that is, a smooth surface with a smooth Riemannian metric. Indeed, given a Riemannian surface (Y, g) and a smooth chart $f\colon V \to U \subset \mathbb{R}^2$, by considering a smaller open set $V' \subset V$, we may assume without loss of generality that f is quasiconformal. We interpret the Riemannian metric g on V as a particular choice of an ellipse at each point of V. Then the chart f maps these ellipses to ellipses in U. We ask whether it is possible to find a diffeomorphism $\eta\colon U \to W \subset \mathbb{R}^2$ such that the particular ellipses in U are mapped to Euclidean balls by η. The existence of such a diffeomorphism η is guaranteed by the measurable Riemann mapping theorem; see, for example, [AB60, AIM09]. When we apply this theorem to the ellipse field of f, the composition $\eta \circ f$ maps the ellipses in V to Euclidean balls. Classically, the coordinates $\eta \circ f$ are called *isothermal coordinates.*

We are interested in two questions. Given a metric space (Y, d_Y) homeomorphic to a surface, what conditions guarantee that there exists a Riemannian surface Z and a quasiconformal map $f\colon Y \to Z$? Moreover, is it possible to find a good notion of isothermal coordinates on Y?

We use an approach based on [Raj17]. Let Y be a metric surface and $V \subset Y$ homeomorphic to $\mathbb{R}^2$. We say that V is a *reciprocal disk* if there exists a quasiconformal homeomorphism $f\colon V \to U \subset \mathbb{R}^2$. Given such an f, the inverse f^{-1} has an *approximate metric differential*, which defines a field of convex bodies on U. We obtain a field of ellipses on U by associating to each of the convex bodies its *distance ellipse* (see for example [Rom19, Section 2], [TJ89, Chapter 37] or Section 4). As before, there exists a quasiconformal homeomorphism $\eta\colon U \to W \subset \mathbb{R}^2$ mapping the field of distance ellipses to Euclidean balls. We call $(V, \eta \circ f)$ an *isothermal chart* of Y. The reason we define the charts in this manner is that every isothermal chart is $(\pi/2)$-quasiconformal; see [Rom19] or Section 4. We prove that whenever Y can be covered by reciprocal disks, the isothermal charts form an atlas $\mathcal{C}$ on Y with transition maps holomorphic or antiholomorphic. Using the atlas $\mathcal{C}$, we prove that Y is quasiconformally equivalent to a Riemannian surface.

Given a metric surface, a cover by reciprocal disks can be found if the 2-dimensional Hausdorff measure of any ball is bounded from above by a constant multiple of the radius squared [Raj17, Theorem 1.6]. In fact, it suffices to require a (locally) uniform upper bound for the 2-dimensional Hausdorff upper density [RRR21, Proposition 3.9]. Next, we give an example for which such a cover does not exist. To this end, we consider a Cantor set $E \subset \mathbb{R}^2$ of positive Lebesgue measure and any continuous function $\omega\colon \mathbb{R}^2 \to [0, \infty)$ with $E = \{x\colon \omega(x) = 0\}$. We define a distance d_ω by setting $d_\omega(x, y) = \inf \int_\gamma \omega\, ds$, the infimum taken over absolutely continuous paths joining x to y. The metric space $(\mathbb{R}^2, d_\omega)$ is homeomorphic to the plane but no Lebesgue density point of E can be covered by a reciprocal disk $V \subset (\mathbb{R}^2, d_\omega)$ [Raj17, Example 2.1].

1.2. Main results. A metric space (Y, d_Y) with a locally finite Hausdorff 2-measure is a *metric surface* if it is homeomorphic to a connected 2-manifold without boundary.

Definition 1.1. A metric surface (Y, d_Y) is a *quasiconformal surface* if every point of (Y, d_Y) is contained in a quasiconformal image of an open set $U \subset \mathbb{R}^2$.

A necessary and sufficient condition for Y to be a quasiconformal surface is given by [Raj17, Theorem 1.4]. Note that every Riemannian surface is a quasiconformal surface and being a quasiconformal surface is a quasiconformal invariant.

We now state the first of our main results.

Theorem 1.2. *Every quasiconformal surface is quasiconformally equivalent to a Riemannian surface.*

To prove Theorem 1.2 for a given quasiconformal surface (Y, d_Y), we construct in Section 4 an atlas of isothermal charts for (Y, d_Y). The atlas defines a conformal structure $\mathcal{C}$ on (Y, d_Y), uniquely determined from the distance d_Y. The classical uniformization theorem implies the existence of a Riemannian norm field G on $(Y, \mathcal{C})$ of Gaussian curvature -1, 0, or 1 in such a way that the associated length distance d_G on Y is complete and that every element of $\mathcal{C}$ is an isothermal chart for the Riemannian surface. The norm field G is not uniquely determined by $\mathcal{C}$ but different choices of G are conformally equivalent. Having fixed such a G, the identity map from (Y, d_G) to (Y, d_Y) is called the *uniformization map* and denoted by u. Theorem 1.2 follows from our next theorem.

Theorem 1.3. *For every quasiconformal surface (Y, d_Y), the uniformization map $u\colon (Y, d_G) \to (Y, d_Y)$ is $(\pi/2)$-quasiconformal. More precisely, it satisfies*

$$\frac{\pi}{4} \operatorname{mod} \Gamma \le \operatorname{mod} u\Gamma \le \frac{\pi}{2} \operatorname{mod} \Gamma \tag{1}$$

for all path families Γ in (Y, d_G).

In this generality, both the lower and upper bounds in (1) are best possible for any quasiconformal map from a Riemannian surface onto (Y, d_Y) [Raj17, Example 2.2].

As a particular application of Theorem 1.3, we consider a quasiconformal surface (Y, d_Y) homeomorphic to a domain in the sphere $\mathbb{S}^2$. Using the notation from Theorem 1.3, we recall the existence of a 1-quasiconformal embedding $\psi\colon (Y, d_G) \to \mathbb{S}^2$ [AS60, Section III.4]. Then the composition $f = \psi \circ u^{-1}$ is a $(\pi/2)$-quasiconformal embedding of (Y, d_Y) into the sphere $\mathbb{S}^2$, satisfying the bounds $(2/\pi) \operatorname{mod} \Gamma \le \operatorname{mod} f\Gamma \le (4/\pi) \operatorname{mod} \Gamma$ for all path families in (Y, d_Y). Romney proved in [Rom19] the existence of such an embedding for reciprocal disks.

Next, we refer the reader to Section 6.2 for the definitions of Ahlfors 2-regularity, linear local contractibility, and quasisymmetries.

Theorem 1.4. *If (Y, d_Y) is a compact, linearly locally contractible, and Ahlfors 2-regular metric surface, then (Y, d_Y) is a quasiconformal surface. Furthermore, a uniformization map $u\colon (Y, d_G) \to (Y, d_Y)$ is η-quasisymmetric with η depending only on the data of (Y, d_Y).*

In the statement, *the data* of (Y, d_Y) refers to the constants appearing in the definitions of linear local contractibility and Ahlfors 2-regularity. When (Y, d_Y) is homeomorphic to $\mathbb{S}^2$, we need to choose the uniformization map with care.

The main theorem from [BK02] proves that if (Y, d_Y) is as in the statement of Theorem 1.4 and homeomorphic to $\mathbb{S}^2$, then there exists an η'-quasisymmetry $\psi\colon \mathbb{S}^2 \to (Y, d_Y)$. We recover this result from Theorem 1.4, since (Y, d_G) is isometric to $\mathbb{S}^2$.

Theorem 1.2 of [GW18] proves that if (Y, d_Y) is as in the statement of Theorem 1.4, orientable and not homeomorphic to $\mathbb{S}^2$, there exists a complete Riemannian surface Z of constant curvature and an η'-quasisymmetric homeomorphism $\phi\colon Z \to (Y, d_Y)$ with η' depending only on the data of (Y, d_Y). Using Theorem 1.3, our isothermal coordinates, and a modified version of their proof, we prove that the uniformization map is η-quasisymmetric with η depending only on the data of (Y, d_Y). The modified proof also works for the non-orientable case.

We refer the interested reader to [BK02, Raj17, GW18], and references therein, for further reading about the quasisymmetric uniformization problem.

2. Outline of the paper

In Section 3, we introduce our notations and recall some prerequisite knowledge. In Section 4, we prove the existence of isothermal charts and the uniformization mapping. In Section 5, we analyze quasiconformal homeomorphisms between quasiconformal surfaces. These results are applied in Section 6, where we introduce isothermal parametrizations of quasiconformal surfaces by Riemannian surfaces. We prove that up to a conformal diffeomorphism, the isothermal parametrizations are uniquely determined by the uniformization mapping. We also prove Theorem 1.4 in this section. In Section 7, we have some concluding remarks.

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3. Preliminaries

Let (Y, d_Y) be a metric space. We drop the subscript from d_Y when convenient. We recall the definition of Hausdorff measure. For all $Q \geq 0$, the *Q-dimensional Hausdorff measure* is defined by

$$\mathcal{H}^Q_Y(B) = \frac{\alpha(Q)}{2^Q} \sup_{\delta > 0} \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} B_i)^Q \colon B \subset \bigcup_{i=1}^{\infty} B_i, \operatorname{diam} B_i < \delta \right\}$$

for all sets $B \subset Y$, where the normalization constant is chosen in such a way that $\mathcal{H}^n_{\mathbb{R}^n}$ coincides with the Lebesgue measure $\mathcal{L}^n$ for all positive integers n.

A *path* is a continuous function from a compact interval into a metric space. A path in Y will typically be denoted by γ. The *length* of the path $\gamma\colon [a,b] \to Y$ is defined as

$$\ell_d(\gamma) = \sup \sum_{j=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)),$$

where the supremum is taken over all finite sequences $a = t_0 \leq t_1 \leq \cdots \leq t_n = b$. A path is *rectifiable* if it has finite length.

The *metric speed* of a path $\gamma\colon [a,b] \to Y$ at the point $t \in [a,b]$ is defined as

$$v_\gamma(t) = \lim_{t \neq s \to t} \frac{d(\gamma(s), \gamma(t))}{|t-s|}$$

whenever this limit exists. If γ is rectifiable, its metric speed exists at $\mathcal{L}^1$-almost every $t \in [a,b]$ [Dud07, Theorem 2.1].

A rectifiable path $\gamma\colon [a,b] \to Y$ is *absolutely continuous* if for all $a \leq s \leq t \leq b$,

$$d(\gamma(t), \gamma(s)) \leq \int_s^t v_\gamma(u)\, d\mathcal{L}^1(u)$$

with $v_\gamma \in L^1([a,b])$ where $\mathcal{L}^1$ is the Lebesgue measure on the real line. Equivalently, γ is absolutely continuous if it maps sets of $\mathcal{L}^1$-measure zero to sets of $\mathcal{H}^1_Y$-measure zero in its image [Dud07, Section 3]. We refer to Chapter 5 of [HKST15] for further details about rectifiable paths.

If $\gamma\colon [a,b] \to Y$ is rectifiable, then there exist a 1-Lipschitz path $\widetilde{\gamma}\colon [0,\ell(\gamma)] \to Y$ whose metric speed equals one $\mathcal{L}^1$-almost everywhere on $[0,\ell(\gamma)]$, and for which there exists a non-decreasing surjective map $\psi\colon [a,b] \to [0,\ell(\gamma)]$ with $\widetilde{\gamma} \circ \psi = \gamma$.

Let $\rho\colon Y \to [0,\infty]$ be a Borel function. The *(path) integral* of ρ over γ is defined by

$$\int_\gamma \rho\, ds = \int_0^{\ell(\gamma)} \rho \circ \widetilde{\gamma}\, d\mathcal{L}^1. \tag{2}$$

A Borel function ρ is *integrable over* γ if (2) is finite. If γ is an absolutely continuous path, then

$$\int_\gamma \rho\, ds = \int_a^b (\rho \circ \gamma)\, v_\gamma\, d\mathcal{L}^1;$$

see [Dud07]. A path is *non-constant* if $\ell(\gamma) > 0$.

Let Γ be a family of paths in Y. A Borel function $\rho\colon Y \to [0,\infty]$ is *admissible* for Γ if for every rectifiable path in Γ,

$$\int_\gamma \rho\, ds \geq 1. \tag{3}$$

The *(conformal) modulus* of Γ is

$$\operatorname{mod} \Gamma = \inf \int_Y \rho^2\, d\mathcal{H}^2_Y, \tag{4}$$

where the infimum is taken over all admissible functions ρ. A Borel function $\rho\colon Y \to [0,\infty]$ is *weakly admissible* for Γ if there exists a path family $\Gamma' \subset \Gamma$ such that $\operatorname{mod} \Gamma' = 0$ and for every $\gamma \in \Gamma \setminus \Gamma'$ (3) holds. We refer to [HKST15, Section 5.2] and [Wil12, Lemma 2.2] for basic properties of modulus. We recall that $\Gamma \mapsto \operatorname{mod} \Gamma$ is an outer measure on the collection of path families.

We say that a path family Γ is *negligible* if $\operatorname{mod} \Gamma = 0$. A property holds for *almost every* path if the path family along which it fails is negligible. We recall that a family Γ of non-constant paths is negligible if and only if there exists $\rho \in L^2(Y)$ such that the integral of ρ over every rectifiable $\gamma \in \Gamma$ is infinite [HKST15, Lemma 5.2.8]. The equivalence also holds for $\rho \in L^2_{\mathrm{loc}}(Y)$ by the countably subadditivity of modulus.

Let $\phi\colon (Y,d_Y) \to (Z,d_Z)$ be a homeomorphism between metric surfaces. The map ϕ is an element of the Sobolev space $N^{1,2}_{\mathrm{loc}}(Y,Z)$ if there exists a non-negative Borel function $\rho \in L^2_{\mathrm{loc}}(Y)$ such that for all non-constant rectifiable paths $\gamma\colon [a,b] \to Y$,

$$d_Z(\phi(\gamma(a)), \phi(\gamma(b))) \leq \int_\gamma \rho\, ds. \tag{5}$$

Such a function ρ is called an *upper gradient* of ϕ. A Borel function is a *weak upper gradient* of ϕ if (5) holds for almost all non-constant paths. A weak upper gradient ρ of $\phi \in N^{1,2}_{\mathrm{loc}}(Y,Z)$ is *minimal* if for every other weak upper gradient $\widetilde{\rho} \in L^2_{\mathrm{loc}}(Y)$, $\rho \leq \widetilde{\rho}$ $\mathcal{H}^2_Y$-almost everywhere. Every $\phi \in N^{1,2}_{\mathrm{loc}}(Y,Z)$ has a minimal weak upper gradient, uniquely defined $\mathcal{H}^2_Y$-almost everywhere, which we denote by ρ_ϕ. We refer the reader to [HKST15] and [Wil12] for details.

Let $C \subset Y$ be a Borel set. The length of γ in C, denoted by $\ell(\gamma \cap C)$, is the integral of χ_C over γ. Then Γ_C^+ denotes those rectifiable paths that have positive length in C.

Observe that if $\mathcal{H}_Y^2(C) = 0$, then Γ_C^+ is negligible; consider the admissible function $\infty \cdot \chi_C$. We prove in Lemma 3.2 a partial converse of this fact. We use the converse later on, since quasiconformal surfaces can have Borel subsets $C \subset Y$ of positive measure for which $\operatorname{mod} \Gamma_C^+ = 0$. See Remark 3.4 for further discussion.

Definition 3.1. For a metric surface (Y, d_Y) and for each Borel set $C \subset Y$, we denote $\nu_Y(C) = \int_C \rho_{\mathrm{id}_Y} \, d\mathcal{H}_Y^2$.

Lemma 3.2. *Let (Y, d_Y) be a metric surface. Then there exists a Borel set $C_0 \subset Y$ such that $\rho_{\mathrm{id}_Y} = \chi_{Y \setminus C_0}$. Moreover, for each Borel set $C \subset Y$, $\operatorname{mod} \Gamma_C^+ = 0$ if and only if $\nu_Y(C) = 0$.*

Proof. Fix a Borel representative ρ of the minimal weak upper gradient ρ_{id_Y}. Since ρ and χ_Y are weak upper gradients of id_Y, so is their pointwise minimum. Therefore, we may assume without loss of generality that $\rho \leq \chi_Y$ everywhere.

For $A = \{\rho < 1\}$, we have that $\operatorname{mod} \Gamma_A^+ = 0$, since otherwise ρ cannot be a weak upper gradient of id_Y [HKST15, Proposition 6.3.3]. Therefore, $\rho_0 = \rho \chi_{Y \setminus A} = \chi_{Y \setminus A}$ is a weak upper gradient of id_Y, and $\rho_0 \leq \rho$ implies that ρ_0 is a representative of ρ_{id_Y}. We denote $C_0 := A$.

Consider $\rho_0 = \chi_{Y \setminus C_0}$ as above. If $C \subset Y$ is a Borel set with $\operatorname{mod} \Gamma_C^+ = 0$, then $\rho_0 \chi_{Y \setminus C}$ is a representative of ρ_{id_Y}, so $0 = \mathcal{H}_Y^2(C \setminus C_0) = \nu_Y(C)$. Conversely, if $0 = \nu_Y(C) = \mathcal{H}_Y^2(C \setminus C_0)$, then $\operatorname{mod} \Gamma_{C \setminus C_0}^+ = 0$. Also, $\operatorname{mod} \Gamma_{C \cap C_0}^+ \leq \operatorname{mod} \Gamma_{C_0}^+ = 0$. These facts imply that $\operatorname{mod} \Gamma_C^+ = 0$. The set C_0 has the claimed properties. □

Consider a homeomorphism $\phi\colon (Y, d_Y) \to (Z, d_Z)$ between metric surfaces. We denote $\phi^* \mathcal{H}_Z^2(A) = \mathcal{H}_Z^2(\phi(A))$ for all sets $A \subset Y$. Then there exists a decomposition $\phi^* \mathcal{H}_Z^2 = J_\phi \mathcal{H}_Y^2 + \mu^\perp$ with $\mathcal{H}_Y^2$ and $\mu^\perp$ singular [Bog07, Sections 3.1–3.2, Volume I]. We refer to the density J_ϕ as the *Jacobian* of ϕ.

We say that ϕ satisfies Lusin's Condition (N) if $\phi^* \mathcal{H}_Z^2$ is absolutely continuous with respect to $\mathcal{H}_Y^2$. It satisfies Lusin's Condition (N^{-1}) if $\mathcal{H}_Y^2$ is absolutely continuous with respect to $\phi^* \mathcal{H}_Z^2$.

A homeomorphism $\phi\colon (Y, d_Y) \to (Z, d_Z)$ between metric surfaces is *quasiconformal* if there exist constants $K_O, K_I \geq 1$ such that $K_O^{-1} \operatorname{mod} \Gamma \leq \operatorname{mod} \phi\Gamma \leq K_I \operatorname{mod} \Gamma$ for every path family Γ in (Y, d_Y). Recalling [Wil12, Theorem 1.1], an equivalent definition is obtained by requiring

$$\phi \in N^{1,2}_{\mathrm{loc}}(Y, Z) \quad \text{and} \quad \rho_\phi^2 \leq K_O J_\phi \quad \mathcal{H}_Y^2\text{-a.e.} \quad \text{and} \tag{6}$$

$$\phi^{-1} \in N^{1,2}_{\mathrm{loc}}(Z, Y) \quad \text{and} \quad \rho_{\phi^{-1}}^2 \leq K_I J_{\phi^{-1}} \quad \mathcal{H}_Z^2\text{-a.e.} \tag{7}$$

with the same constants K_O and K_I. The smallest constant K_O (resp. K_I) for which (6) (resp. (7)) holds is called the *outer dilatation* of ϕ (resp. *inner dilatation*) and denoted by $K_O(\phi)$ (resp. $K_I(\phi)$). We say that a quasiconformal mapping is K-quasiconformal if $K_O(\phi) \leq K$ and $K_I(\phi) \leq K$. The smallest $K \geq 1$ for which ϕ is K-quasiconformal is called the *maximal dilatation* of ϕ.

Having defined quasiconformal mappings, we prove the following.

Lemma 3.3. *Let $\phi\colon (Y, d_Y) \to (Z, d_Z)$ be a quasiconformal homeomorphism between metric surfaces. Then for each Borel sets $C \subset Y$, the following four conditions* *are equivalent:*

$$\nu_Y(C)=0, \quad \operatorname{mod}\Gamma_C^+=0, \quad \operatorname{mod}\Gamma_{\phi(C)}^+=0 \quad \textit{and} \quad \nu_Z(\phi(C))=0.$$

Proof. Let K denote the maximal dilatation of ϕ. Fix Borel representatives of ρ_ϕ and $\rho_{\phi^{-1}}\circ\phi$. We denote $\rho=\rho_\phi\,(\rho_{\phi^{-1}}\circ\phi)$. We recall from (6) and (7) that

$$\rho_\phi^2 \le KJ_\phi \in L^1_{\mathrm{loc}}(Y) \quad \text{and} \quad \rho_{\phi^{-1}}^2 \le KJ_{\phi^{-1}} \in L^1_{\mathrm{loc}}(Z) \tag{8}$$

hold $\mathcal{H}^2_Y$- and $\mathcal{H}^2_Z$-almost everywhere, respectively.

Proposition 6.3.3 of [HKST15] implies that for almost every non-constant absolutely continuous path $\gamma\colon [0,1]\to Y$, the path $\phi\circ\gamma$ is absolutely continuous and for $\mathcal{L}^1$-almost every $0\le t\le 1$,

$$v_{\phi\circ\gamma}(t) \le (\rho_\phi\circ\gamma(t))\,v_\gamma(t) \in L^1([0,1]). \tag{9}$$

The right-hand side is interpreted to be zero in the set $\{v_\gamma\equiv 0\}$. Let Γ_1 denote the collection of those non-constant paths for which (9) fails.

As above, for almost every non-constant absolutely continuous path $\theta\colon [0,1]\to V$, the path $\phi^{-1}\circ\theta$ is absolutely continuous and for $\mathcal{L}^1$-almost every $0\le t\le 1$,

$$v_{\phi^{-1}\circ\theta}(t) \le (\rho_{\phi^{-1}}\circ\theta(t))\,v_\theta(t) \in L^1([0,1]). \tag{10}$$

Let Γ_2 denote the collection of those paths γ in Y for which $\theta=\phi\circ\gamma$ fails (10).

Since ϕ is quasiconformal, $\operatorname{mod}(\Gamma_1\cup\Gamma_2)=0$. Therefore, for almost every absolutely continuous $\gamma\colon [0,1]\to Y$ and $\theta=\phi\circ\gamma$ both (9) and (10) hold $\mathcal{L}^1$-almost everywhere. For such γ,

$$v_\gamma(t)\le (\rho\circ\gamma(t))\,v_\gamma(t)$$

for $\mathcal{L}^1$-almost every $0\le t\le 1$. This implies that ρ is a weak upper gradient of the identity map $\mathrm{id}_Y\colon Y\to Y$, and we conclude from (8) that

$$\rho\in L^2_{\mathrm{loc}}(Y). \tag{11}$$

Similar reasoning as above yields that

$$\rho\circ\phi^{-1}\in L^2_{\mathrm{loc}}(Z) \tag{12}$$

is a weak upper gradient of id_Z.

Let Γ_3 denote the collection of those absolutely continuous paths in U along which ρ fails to be integrable or those γ for which $\rho\circ\phi^{-1}$ fails to be integrable along $\phi\circ\gamma$. Then (11) and (12) imply that $\operatorname{mod}\Gamma_3=0$ as well.

Consider $\Gamma_0=\Gamma_1\cup\Gamma_2\cup\Gamma_3$. Observe that given a Borel set $C\subset U$, an absolutely continuous path $\gamma\colon [0,1]\to U\not\in\Gamma_0$ has positive length in C, i.e.,

$$\int_0^1 (\chi_C\circ\gamma)\,v_\gamma\,d\mathcal{L}^1>0$$

if and only if the absolutely continuous path $\phi\circ\gamma$ has positive length in $\phi(C)$. Since Γ_0 and $\phi\Gamma_0$ are negligible, we deduce from this that $\operatorname{mod}\Gamma_C^+=0$ if and only if $\operatorname{mod}\Gamma_{\phi(C)}^+=0$. Then Lemma 3.2 proves the claim. □

Remark 3.4. As a consequence of Lemma 3.2, a quasiconformal homeomorphism ϕ from (Y,d_Y) into (Z,d_Z) satisfies Lusin's Conditions (N) and (N^{-1}) with respect to the measures ν_Y and ν_Z. That is, for all Borel subsets $B\subset Y$, $\nu_Y(B)=0$ if and only if $\nu_Z(\phi(B))=0$. We use this fact in Section 5.

As an application of Lemma 3.2, we fix a Borel set $B_0\subset Y$ such that $\nu_Y=\chi_{Y\setminus B_0}\mathcal{H}^2_Y$ and $\nu_Z=\chi_{Z\setminus\phi(B_0)}\mathcal{H}^2_Z$. The product $\rho_\phi(\rho_{\phi^{-1}}\circ\phi)$ is uniquely defined ν_Y-almost everywhere, since every representative of ρ_ϕ is zero $\mathcal{H}^2_Y$-almost everywhere in B_0 and $\rho_{\phi^{-1}}$ zero $\mathcal{H}^2_Z$-almost everywhere in $\phi(B_0)$. We apply this fact already in Section 4.

If Z is an open subset of $\mathbb{R}^2$ or a Riemannian surface, we have $\nu_Z \equiv \mathcal{H}^2_Z$ in Lemma 3.3. Therefore, for such Z, any quasiconformal mapping ϕ as above satisfies Lusin's Condition (N). For such a Z, if we have $\rho_{\mathrm{id}_Y} = \chi_{Y \setminus B_0}$ with $\mathcal{H}^2_Y(B_0) > 0$, ϕ fails Lusin's Condition (N^{-1}), with respect to the Hausdorff 2-measures, exactly at Borel subsets of B_0 of positive measure. We note that there are quasiconformal surfaces for which $\mathcal{H}^2_Y(B_0) > 0$; see [Raj17, Proposition 17.1]. Due to this fact, many results in Section 5 are only phrased in terms of ν_Y.

We sometimes write $TU = U \times \mathbb{R}^2$ when $U \subset \mathbb{R}^2$ is an open set. We refer to TU as the *tangent bundle* of U. For each $x \in U$, we refer to $\{x\} \times \mathbb{R}^2$ as a *fiber* of TU and denote it by T_xU.

At times, we consider quasiconformal maps $\psi\colon U \to \widetilde{U}$ between open subsets of $\mathbb{R}^2$. Such maps have a *differential* $D\psi$ $\mathcal{L}^2$-almost everywhere, which just means its classical derivative. The differential defines a map

$$D\psi\colon TU \to T\widetilde{U},$$

where the fiber T_xU is taken to $T_{\psi(x)}\widetilde{U}$ by the linear map $D_x\psi$.

Next, we consider a measurable seminorm field $\mathrm{N}\colon TU \to [0,\infty]$. This means that we have a measurable map from TU into $[0,\infty]$ such that for $\mathcal{L}^2$-almost every $x \in U$, the restriction of N to T_xU is a seminorm. If the restriction of N to $\mathcal{L}^2$-almost every fiber is a norm, we say that N is a *norm field*. In this case, the pair (TU, N) is called a *normed bundle*, where the fibers refer to $(TU, \mathrm{N})_x := (T_xU, \mathrm{N}\,|_{T_xU})$.

We sometimes consider the differential $D\psi$ between two normed bundles, i.e., the map

$$D\psi\colon (TU, \mathrm{N}) \to (T\widetilde{U}, \widetilde{\mathrm{N}}). \tag{13}$$

The *operator norm* $\|D\psi\|$ of (13) at $x \in U$ refers to the operator norm of the linear map $D_x\psi\colon (TU, \mathrm{N})_x \to (TU, \mathrm{N})_{\psi(x)}$. We denote the Jacobian of $D\psi$ at x by $J_2(D\psi)(x)$. The *outer dilatation* $K_O(D\psi)$ at $x \in U$ is defined as

$$K_O(D\psi)(x) = \frac{\|D\psi\|^2(x)}{J_2(D\psi)(x)}. \tag{14}$$

The *inner dilatation* $K_I(D\psi)$ at $x \in U$ is defined by the formula

$$K_I(D\psi)(x) = K_O(D(\psi^{-1}))(\psi(x)). \tag{15}$$

The *maximal dilatation* $K(D\psi)$ of $D\psi$ at $x \in U$ is the maximum of (14) and (15).

The objects (13), (14) and (15) are well-defined even if we consider norms $\{N_x\}_{x\in U}$ and $\{\widetilde{N}_x\}_{x\in U}$ together with linear maps $L_x\colon (TU, N)|_x \to (TU, \widetilde{N})|_x$. The objects above are defined similarly when U is an open subset of a smooth surface.

4. Proof of Theorem 1.3

We define isothermal parametrizations in Section 4.1 and state some of their properties. In Section 4.2, we analyze general quasiconformal maps from planar domains into metric surfaces. Using results from that subsection, we prove the claims from Section 4.1 in Section 4.3.

We construct the atlas of isothermal coordinates for (Y, d_Y) in Section 4.4. We define the uniformization map in Section 4.5 and prove Theorem 1.3 there.

4.1. Isothermal parametrizations.

Definition 4.1. A quasiconformal homeomorphism $\phi\colon U \to V \subset Y$, with $U \subset \mathbb{R}^2$ open, is an *isothermal parametrization* of V if for every other quasiconformal homeomorphism $\widetilde{\phi}\colon \widetilde{U} \to V$ with $\widetilde{U} \subset \mathbb{R}^2$,

$$\rho_\phi(x)\,(\rho_{\phi^{-1}} \circ \phi(x)) \leq \rho_{\widetilde{\phi}}(\widetilde{x})\left(\rho_{\widetilde{\phi}^{-1}} \circ \widetilde{\phi}(\widetilde{x})\right) \tag{16}$$

for $\widetilde{x} = (\widetilde{\phi}^{-1} \circ \phi)(x)$ and $\mathcal{L}^2$-almost every $x \in U$. If the image of ϕ is clear, we say that ϕ is *isothermal*.

Here ρ_ϕ denotes a minimal weak upper gradient of ϕ and $\rho_{\phi^{-1}}$ a minimal weak upper gradient of ϕ^{-1}. Lemma 3.3 implies that both sides of (16) are independent of the representatives we use.

It turns out that the left-hand side of (16) is the geometric mean of the pointwise versions of the dilatations $K_O(\phi)$ and $K_I(\phi)$; this is made precise in (19) and the discussion following (19). This observation implies that isothermal parametrizations minimize the geometric mean of the pointwise dilatations; see Theorem 4.12 for the precise statement. We highlight two consequences of Theorem 4.12.

Proposition 4.2. *Let $\phi\colon U \to V$ be K-quasiconformal, $U \subset \mathbb{R}^2$ and $V \subset Y$ open. Then there exist a set $\widetilde{U} \subset \mathbb{R}^2$ and a $(4K/\pi)$-quasiconformal homeomorphism $\psi\colon \widetilde{U} \to U$ such that $\widetilde{\phi} = \phi \circ \psi$ is isothermal.*

Proposition 4.3. *Every isothermal parametrization $\phi\colon U \to V$ satisfies*

$$\frac{\pi}{4}\operatorname{mod}\Gamma \leq \operatorname{mod}\phi\Gamma \leq \frac{\pi}{2}\operatorname{mod}\Gamma \tag{17}$$

for all path families $\Gamma \subset U$. Moreover, if $V' \subset V$ is open and $\phi'\colon U' \to V'$ is quasiconformal with $U' \subset \mathbb{R}^2$, ϕ' is an isothermal parametrization of V' if and only if $\phi^{-1} \circ \phi'$ is holomorphic or antiholomorphic.

We see from Proposition 4.3 that isothermal parametrizations satisfy the same dilatation bounds as the parametrizations constructed in [Rom19]. In fact, our isothermal parametrizations coincide with the parametrizations considered by Romney for simply connected domains. This observation is not immediately apparent from our definition, but is a corollary of Theorem 4.12.

4.2. Quasiconformal parametrizations. Before proving the existence of isothermal parametrizations, we first analyze a given quasiconformal map $\phi\colon U \to V \subset Y$ with open $U \subset \mathbb{R}^2$ and Y a metric surface. Since $\phi \in N^{1,2}_{\mathrm{loc}}(U, V)$, there exists a measurable seminorm field

$$\mathrm{N}_\phi\colon TU \to \mathbb{R}$$

that encodes the following geometric properties of ϕ.

Lemma 4.4. *The following properties hold.*

(a) *The maximal stretching of N_ϕ,*

$$L(\mathrm{N}_\phi)(x) := \sup_{\|v\|_2 \leq 1} \mathrm{N}_\phi(x, v) \qquad \text{for } x \in U,$$

defines a representative of the minimal weak upper gradient ρ_ϕ;

(b) *The Jacobian function*

$$x \mapsto J_2(\mathrm{N}_\phi)(x) := \frac{\pi}{\mathcal{L}^2\left(\{v \in \mathbb{R}^2\colon\ \mathrm{N}_\phi(x, v) \leq 1\}\right)}$$

is a representative of the Jacobian J_ϕ *of* ϕ*;*

(c) *For almost every non-constant absolutely continuous path* $\gamma\colon [a,b] \to U$,

$$v_{\phi\circ\gamma}(t) = \mathrm{N}_\phi \circ D\gamma(t)$$

for $\mathcal{L}^1$*-almost every* t*, where* $D\gamma(t) = (\gamma(t), \gamma'(t))$ *is the derivative of* γ *at* t*.*

See [LW18, Sections 3.3-3.4 and 3.6] for the proof of Lemma 4.4. The seminorm field N_ϕ is refered to as the *approximate metric differential* of ϕ. Lemma 3.3 implies that ϕ satisfies Lusin's Condition (N^{-1}) (see also Remark 3.4). Then the Sobolev regularity of ϕ implies the following; see, for example, [Raj17, Lemma 14.1].

Lemma 4.5. *The homeomorphism* ϕ *satisfies Lusin's Condition* (N^{-1}) *and there exists a Borel set* $B_0 \subset U$ *with* $\mathcal{L}^2(B_0) = 0$ *such that* $\phi|_{U\setminus B_0}$ *satisfies Lusin's Condition* (N)*.*

Lemma 4.5 implies the following.

Corollary 4.6. *If* B_0 *is as in Lemma 4.5, then the Jacobian of* ϕ^{-1} *equals* $1/(J_\phi \circ \phi^{-1})$ $\mathcal{H}^2_Y$*-almost everywhere in* $V \setminus \phi(B_0)$*.*

Rajala's example [Raj17, Proposition 17.1] illustrates that the set $\phi(B_0)$ can have positive $\mathcal{H}^2_Y$-measure, so ϕ does not necessarily satisfy Lusin's Condition (N).

Since ϕ satisfies Lusin's Condition (N^{-1}), the Jacobian of ϕ is non-zero $\mathcal{L}^2$-almost everywhere in U. In other words, the approximate metric differential N_ϕ is a norm $\mathcal{L}^2$-almost everywhere in U. Consequently, $\omega(\mathrm{N}_\phi)(x) := \inf_{\|v\|_2 \geq 1} \mathrm{N}_\phi(x, v)$ is an element in $(0, \infty)$ for $\mathcal{L}^2$-almost every $x \in U$.

Lemma 4.7. *Let* B_0 *be as in Lemma 4.5 and*

$$\widetilde{\rho}(y) = \left(\frac{1}{\omega(\mathrm{N}_\phi)} \circ \phi^{-1}(y) \right) \chi_{V \setminus \phi(B_0)}(y) \quad \textit{for each } y \in V.$$

Here $\widetilde{\rho} \equiv 0$ *in* $\phi(B_0)$*. Then* $\widetilde{\rho}$ *is a representative of the minimal upper gradient* $\rho_{\phi^{-1}}$*.*

Proof. The L^2_{loc}-integrability of $\widetilde{\rho}$ follows from the change of variables formula for ϕ. Lemma 4.5 and Lemma 3.3 imply that $\operatorname{mod} \Gamma^+_{\phi(B_0)} = 0$.

We conclude that almost every non-constant path has zero length in $\phi(B_0)$ and that $\widetilde{\rho}$ is integrable over the path. We may also assume that the image path γ in U is absolutely continuous and satisfies Lemma 4.4 (c). These facts imply that $\widetilde{\rho}$ is a weak upper gradient of ϕ^{-1}.

To see that $\widetilde{\rho}$ is a minimal upper gradient, it suffices to fix a upper gradient $\rho \in L^2_{\mathrm{loc}}(Y)$ of ϕ^{-1} and to prove $\widetilde{\rho} \leq \rho$ $\mathcal{H}^2_Y$-almost everywhere. This is clear everywhere in $\phi(B_0)$. Since $\phi|_{U\setminus B_0}$ satisfies Lusin's Condition (N) and (N^{-1}), it suffices to verify $\widetilde{\rho}(y_0) \leq \rho(y_0)$ for $y_0 = \phi(x_0)$ for $\mathcal{L}^2$-almost every $x_0 \in U \setminus B_0$. We fix $v_0, w_0 \in \mathbb{S}^1$ perpendicular to one another.

Consider now a rectangle $R \subset U$ with a foliation $\gamma_t(s) = x_0 + tv + sw$, for $-1 \leq s, t \leq 1$, $r = \|v\|_2 = \|w\|_2$ with $v = rv_0$ and $w = rw_0$. For $\mathcal{L}^1$-almost every t, Lemma 4.4 (c) holds for γ_t, and $\theta_t := \phi \circ \gamma_t$ is absolutely continuous. Then the upper gradient inequality and Fubini's theorem imply

$$\rho(\phi(x))\, \mathrm{N}_\phi((x, w)) \geq \|w\|_2 \quad \text{for } \mathcal{L}^2\text{-almost every } x \in R \setminus B_0.$$

Covering U by such rectangles implies

$$\rho(\phi(x)) \geq \frac{1}{\mathrm{N}_\phi((x, w_0))} \quad \text{for } \mathcal{L}^2\text{-almost every } x \in U \setminus B_0. \tag{18}$$

Since the inequality (18) holds for a countable dense set $\{w_i\}_{i=1}^{\infty} \subset \mathbb{S}^1$ for $\mathcal{L}^2$-almost every $x_0 \in U \setminus B_0$, taking the supremum over i yields $\rho(\phi(x)) \geq \widetilde{\rho}(\phi(x))$ for $\mathcal{L}^2$-almost every $x \in U \setminus B_0$. This was sufficient for the claim. $\square$

Definition 4.8. Let $\phi\colon U \to V$ be quasiconformal. The *pointwise outer dilatation* of ϕ at $x \in U$ is

$$K_O(\phi)(x) = \frac{\rho_\phi^2(x)}{J_\phi(x)}$$

and the *pointwise inner dilatation* of ϕ at $x \in U$ is

$$K_I(\phi)(x) = \left(\rho_{\phi^{-1}}^2(\phi(x))\right) J_\phi(x)\chi_{U\setminus B_0}(x).$$

The *pointwise maximum dilatation* of ϕ at $x \in U$ is the maximum of the corresponding outer and inner dilatations.

We consider the differential

$$D\mathrm{id}\colon (TU, \|\cdot\|_2) \to (TU, \mathrm{N}_\phi) \tag{19}$$

as defined in (13). Then Lemma 4.4 (a) implies that the operator norm of Did from (19) is a representative of ρ_ϕ. Similarly, Lemma 4.4 (b) implies that the Jacobian $J_2(D\mathrm{id})$ is a representative of the Jacobian of ϕ. Lemma 4.7 and Corollary 4.6 yield similar identities for the inverse of the map in (19). Consequently, the pointwise outer (resp. inner) dilatation of ϕ and the differential in (19) coincide. These facts imply that the left-hand side of (16) equals $\sqrt{K_O(D\mathrm{id})K_I(D\mathrm{id})}$ $\mathcal{L}^2$-almost everywhere. Therefore, the left-hand side in (16) is the geometric mean of the outer and inner dilatations of the differential (19). This fact connects the definition of isothermal parametrizations to convex analysis.

4.3. Banach–Mazur distance and isothermal parametrizations. In this section, we associate a Beltrami differential to the approximate metric differential of any given quasiconformal parametrization. For this purpose, we introduce Banach–Mazur distance from convex analysis.

Definition 4.9. Let M and N be norms on $\mathbb{R}^2$. Then $GL_2[M,N]$ is the collection of all invertible linear maps $S\colon (\mathbb{R}^2, M) \to (\mathbb{R}^2, N)$. An invertible linear map $S \in GL_2[M,N]$ is a *Banach–Mazur minimizer from M to N* if S attains the infimum

$$\rho(M,N) = \inf_{T \in GL[M,N]} \sqrt{K_O(T)K_I(T)}.$$

If the domain and codomain of the linear map S are clear from the context, we say that S is a *Banach–Mazur minimizer.* The number $\rho(M,N)$ is the *Banach–Mazur distance* from M to N.

If N is induced by an inner product, $\rho(M,N) \leq \sqrt{2}$ [TJ89, Proposition 9.12], with $\rho(M,N) = \sqrt{2}$ if M is the supremum norm [TJ89, Proposition 37.6]. Therefore, $\rho(M,N) \leq 2$ for every pair of norms. Then a compactness argument implies that Banach–Mazur minimizers exist for each pair of norms, see e.g. [TJ89, Section 37].

We recall some notations. The group O_2 is the group of linear isometries of $\mathbb{R}^2$ and $\mathbb{R}_+ \cdot O_2$ denotes the group of invertible linear maps $L = \lambda \cdot S$, where $\lambda > 0$ and $S \in O_2$. The group SO_2 consists of the elements of O_2 with determinant equal to 1. The group $\mathbb{R}_+ \cdot O_2$ are the linear conformal automorphisms of $\mathbb{R}^2$, and $\mathbb{R}_+ \cdot SO_2$ the subgroup of $\mathbb{R}_+ \cdot O_2$ whose elements have positive determinant.

Lemma 4.10. *Let M be a norm on $\mathbb{R}^2$ and $L\colon (\mathbb{R}^2, M) \to (\mathbb{R}^2, \|\cdot\|_2)$ a Banach–Mazur minimizer. Then*

$$\frac{\pi}{4}\rho^2(M, \|\cdot\|_2) \leq K_O\,(L) \leq \frac{\pi}{2} \quad \text{and} \tag{20}$$

$$\frac{2}{\pi}\rho^2(M, \|\cdot\|_2) \leq K_I\,(L) \leq \frac{4}{\pi}. \tag{21}$$

Moreover, $L' \in GL_2[M, \|\cdot\|_2]$ is a Banach–Mazur minimizer if and only if $L' \circ L^{-1} \in \mathbb{R}_+ \cdot O_2$.

Proof. The inequalities (20) and (21) are slight reformulations of Lemma 2.1 of [Rom19]. Lemma 2.2 of [Rom19] proves that if L' is a Banach–Mazur minimizer, then $L' \circ L^{-1} \in \mathbb{R}_+ \cdot O_2$. Conversely, if $L' = S \circ L$ for some $S \in \mathbb{R}_+ \cdot O_2$, the outer and inner dilatations of L' and L coincide. Therefore, L' is a Banach–Mazur minimizer. ☐

If M is the supremum norm, we have that $\rho^2(M, \|\cdot\|_2) = 2$. Thus (20) and (21) are equalities in this case. In fact, $K_O\,(L) = \pi/2$ and $K_I\,(L) = 4/\pi$ for a Banach–Mazur minimizer from M to $\|\cdot\|_2$ if and only if M is isometric to the supremum norm [TJ89, Proposition 37.4].

We identify $\mathbb{R}^2$ with the complex plane in the following statement.

Corollary 4.11. *Suppose that M is a norm on $\mathbb{R}^2$. Then there exists a unique complex number μ_M in the Euclidean ball $\mathbb{D}$ such that*

$$T_M = \mathrm{id} + \mu_M \cdot \overline{\mathrm{id}}\colon \big(\mathbb{R}^2, M\big) \to \big(\mathbb{R}^2, \|\cdot\|_2\big)$$

is a Banach–Mazur minimizer from M to $\|\cdot\|_2$. Moreover, μ_M and T_M depend continuously on the norm M.

Here $\mu_M \cdot \overline{\mathrm{id}}$ refers to the complex multiplication and $\overline{\mathrm{id}}(w) = \overline{w_1 + iw_2} = w_1 - iw_2$ denotes the complex conjugation map.

Proof. Consider an orientation-preserving Banach–Mazur minimizer $L\colon (\mathbb{R}^2, M) \to (\mathbb{R}^2, \|\cdot\|_2)$, the existence of which follows from Lemma 4.10.

Fix an orientation-preserving $L' \in GL_2[M, \|\cdot\|_2]$. Lemma 4.10 implies that L' is a Banach–Mazur minimizer if and only if $L' = S \circ L$ for some $S \in \mathbb{R}_+ \cdot SO_2$. Such an S exists if and only if L' and L have the same Beltrami differential [AIM09, Section 2.4]. Moreover, for a given L, there exists $S \in \mathbb{R}_+ \cdot SO_2$ such that $L = S \circ T$ for some $T = \mathrm{id} + \mu \cdot \overline{\mathrm{id}}$ with $\mu \in \mathbb{D}$. So $T = T_M$ and $\mu = \mu_M$ are uniquely defined.

Next, we establish the continuity of $M \mapsto \mu_M$ and $M \mapsto T_M$. To this end, given a sequence of norms $(M_j)_{j=1}^{\infty}$ and a norm M, with $M_j \to M$ uniformly in compact subsets of $\mathbb{R}^2$, we claim that $T_{M_j} \to T_M$. First, we note that Banach–Mazur distances $\rho(M_j, \|\cdot\|_2)$ converge to $\rho(M, \|\cdot\|_2)$. Indeed, for every $\epsilon > 0$, there exists j_0 such that for every $j \geq j_0$, the identity mapping from $(\mathbb{R}^2, M_j)$ to $(\mathbb{R}^2, M)$ is $(1+\epsilon)$-bi-Lipschitz. This implies the claimed convergence. This convergence implies that $(T_{M_j})_{j=1}^{\infty}$ is a normal family.

Consider a convergent subsequence with $T_{M_{j_i}} \to T$. Then the sequence of outer (resp. inner) dilatations of $T_{M_{j_i}}\colon (\mathbb{R}^2, M_{j_i}) \to (\mathbb{R}^2, \|\cdot\|_2)$ converge to the outer (resp. inner) dilatation of $T\colon (\mathbb{R}^2, M) \to (\mathbb{R}^2, \|\cdot\|_2)$. Therefore,

$$\rho(M, \|\cdot\|_2) \leq \sqrt{K_O(T)K_I(T)} = \lim_{i\to\infty} \sqrt{K_O(T_{j_i})K_I(T_{j_i})} = \lim_{i\to\infty} \rho(M_{j_i}, \|\cdot\|_2).$$

The right-hand side equals $\rho(M, \|\cdot\|_2)$, so T must be a Banach–Mazur minimizer. Since every accumulation point of $(T_{M_j})_{j=1}^{\infty}$ is of the form $T = \mathrm{id} + \mu \cdot \overline{\mathrm{id}}$, we conclude that $T = T_M$ by the uniqueness of T_M. This implies $\mu = \mu_M$. Since $(T_{M_j})_{j=1}^{\infty}$ has a unique accumulation point, the sequence itself converges to T_M. This also implies $\mu_{M_j} \to \mu_M$. □

Let M and T_M be as in Corollary 4.11. We call the ellipse

$$\mathcal{E}_M := \left\{ v \in \mathbb{R}^2 \colon \left\| T_M^{-1} \right\| \left\| T_M(v) \right\|_2 \leq 1 \right\}$$

the *distance ellipse* of $\{M \leq 1\}$. We note that for every ellipse $\mathcal{E} \subset \{M \leq 1\}$ and every $\lambda > 0$ with $\{M \leq 1\} \subset \lambda\mathcal{E}$, we have $\lambda \geq \rho(M, \|\cdot\|_2)$. The equality $\lambda = \rho(M, \|\cdot\|_2)$ holds if and only if $\mathcal{E}$ is the distance ellipse. Observe that $\mathcal{E}_M$ is an Euclidean ball if and only if $\mu_M = 0$.

In the following statement, $\phi\colon U \to V \subset Y$ is a quasiconformal homeomorphism with $U \subset \mathbb{R}^2$ open. Furthermore, we denote $\mu_\phi := \mu_{\mathrm{N}_\phi}$ for the approximate metric differential N_ϕ. We refer to μ_ϕ as the Beltrami differential of ϕ.

Theorem 4.12. *Let $W \subset \mathbb{R}^2$ be open and $\psi\colon W \to U$ be a quasiconformal map, possibly orientation-reversing. Then the following are equivalent:*

(a) *The composition $\phi \circ \psi$ is isothermal;*
(b) *The equality $\mu_{\phi\circ\psi} = 0$ holds $\mathcal{L}^2$-almost everywhere.*

If either one of the conditions hold and ϕ is K-quasiconformal, then ψ is $(4K/\pi)$-quasiconformal. Moreover, the above conditions are equivalent to any one of the following.

(c) *Either ψ^{-1} or $\overline{\psi^{-1}}$ is an orientation-preserving solution of the Beltrami equation $\partial_{\overline{z}} f = \mu_\phi \partial_z f$;*
(d) *The map $D\mathrm{id}_W\colon (TW, \mathrm{N}_{\phi\circ\psi}) \to (TW, \|\cdot\|_2)$ is a Banach–Mazur minimizer pointwise $\mathcal{L}^2$-almost everywhere;*
(e) *The pointwise dilatations satisfy the equality*

$$K_O(\phi\circ\psi)K_I(\phi\circ\psi) = \rho^2(\|\cdot\|_2, \mathrm{N}_{\phi\circ\psi})$$

$\mathcal{L}^2$-almost everywhere in W.

We discussed normed bundles (TU, N_ϕ) in Section 3. We refer the reader to [AIM09, Chapter 5] for the basics of Beltrami equations and the measurable Riemann mapping theorem.

Proof of Theorem 4.12. Lemma 4.10 yields that

$$D(\psi^{-1})\colon (TU, \mathrm{N}_\phi) \to (TW, \|\cdot\|_2)$$

is a Banach–Mazur minimizer $\mathcal{L}^2$-almost everywhere if and only if there exists a measurable map $x \mapsto S(x) \in \mathbb{R}_+ \cdot O_2$ such that $D(\psi^{-1}) = S \circ T_{\mathrm{N}_\phi}$ pointwise $\mathcal{L}^2$-almost everywhere. The map ψ is orientation-preserving if and only if S is orientation-preserving $\mathcal{L}^2$-almost everywhere. In that case $\mu_{\psi^{-1}} = \mu_\phi$ holds $\mathcal{L}^2$-almost everywhere. Otherwise, $\overline{\psi^{-1}}$ is orientation-preserving and $\mu_{\overline{\psi^{-1}}} = \mu_\phi$ holds $\mathcal{L}^2$-almost everywhere. These facts and the chain rule $\mathrm{N}_{\phi\circ\psi} = \mathrm{N}_\phi \circ D\psi$ now imply that Properties (c) and (d) are equivalent.

We recall from (19) and the following discussion that the pointwise dilatations satisfy $K_O(\phi\circ\psi) = K_O(D\mathrm{id}_W)$ and $K_I(\phi\circ\psi) = K_I(D\mathrm{id}_W)$ $\mathcal{L}^2$-almost everywhere. Therefore, the dilatations also satisfy

$$K_O(\phi\circ\psi)K_I(\phi\circ\psi) \geq \rho^2(\|\cdot\|_2, \mathrm{N}_{\phi\circ\psi}) \quad \mathcal{L}^2\text{-almost everywhere.} \tag{22}$$

Moreover, the equality (22) holds $\mathcal{L}^2$-almost everywhere if and only if Property (d) holds.

Also, if ψ_1 and ψ_2 are two maps for which $\phi \circ \psi_1$ and $\phi \circ \psi_2$ are isothermal parametrizations, the pointwise dilatations satisfy

$$K_O(\phi \circ \psi_1)K_I(\phi \circ \psi_1) = [K_O(\phi \circ \psi_2)K_I(\phi \circ \psi_2)] \circ (\psi_2^{-1} \circ \psi_1) \tag{23}$$

$\mathcal{L}^2$-almost everywhere.

By applying (22) and (23), the equivalence of Properties (c) to (e) and Property (a) follows if it can be shown that there exists a quasiconformal map ψ such that the equality in (22) holds $\mathcal{L}^2$-almost everywhere. By Property (c), it suffices to solve the Beltrami equation $\mu_f = \mu_\phi$ induced by ϕ.

Suppose that we know that the L^∞-norm of μ_ϕ is bounded from above by some constant $C < 1$. Then we extend μ_ϕ as zero to the Euclidean plane and let f be the normalized solution to the corresponding Beltrami equation. The existence of f is guaranteed by the measurable Riemann mapping theorem; see for example [AIM09]. The restriction of f^{-1} to the appropriate open set is the desired map ψ.

Lemma 4.10 implies that

$$\|\mu_\phi\|_{L^\infty(U)} \leq \frac{\frac{4}{\pi}K - 1}{\frac{4}{\pi}K + 1} =: C,$$

where we use the fact that ϕ is K-quasiconformal. This inequality also implies that the maximal dilatation of ψ is bounded from above by $(4K/\pi)$.

By expressing $\phi \circ \psi$ as $(\phi \circ \psi) \circ \mathrm{id}_W$, we see that Property (b) is equivalent to the other properties. □

Proof of Proposition 4.2. Let ψ^{-1} solve the Beltrami equation $\partial_{\overline{z}} f = \mu_\phi \partial_z f$ induced by ϕ. Then Theorem 4.12 proves that ψ is $(4K/\pi)$-quasiconformal and $\widetilde{\phi} = \phi \circ \psi$ isothermal. □

Proof of Proposition 4.3. The outer and inner dilatation bounds follow from Theorem 4.12 (d) and the dilatation bounds in Lemma 4.10.

Next, consider an open set $U' \subset \mathbb{R}^2$, $V' \subset V$ and a quasiconformal homeomorphism $\phi' \colon U' \to V'$. Here $\phi' = \phi \circ \psi$ for $\psi = \phi^{-1} \circ \phi'$. Theorem 4.12 (d) proves that ϕ' is isothermal if and only if ψ^{-1}, or $\overline{\psi^{-1}}$, is orientation-preserving and its Beltrami differential equals $\mu_\phi = 0$. Thus, [AIM09, Weyl's lemma] yields that ϕ' is isothermal if and only if ψ is holomorphic or antiholomorphic. □

4.4. Conformal surfaces.

We fix a quasiconformal surface (Y, d) for this section. Given an open set $V \subset (Y, d)$ and a quasiconformal homeomorphism $\phi' \colon U' \to V$ with $U' \subset \mathbb{R}^2$, Proposition 4.2 yields the existence of an isothermal parametrization $\phi \colon U \to V$ of V. Given such a ϕ, we denote $f := \phi^{-1}$ and call the pair (V, f) an *isothermal chart* of (Y, d).

Let $\mathcal{I}_d = \{(V_i, f_i)\}_{i \in I}$ denote the collection of all isothermal charts of (Y, d). Since a quasiconformal surface (Y, d) can be covered by quasiconformal images of planar domains, we conclude that $\bigcup_{i \in I} V_i = Y$. The subscript d refers to the dependence of the collection on the distance of Y.

Definition 4.13. A *conformal atlas* $\mathcal{D}$ is an atlas whose transition maps are holomorphic or antiholomorphic maps. A conformal atlas $\mathcal{D}$ is *maximal* if for every other conformal atlas $\mathcal{D}'$ with $\mathcal{D} \cap \mathcal{D}' \neq \emptyset$, we have $\mathcal{D}' \subset \mathcal{D}$. If $\mathcal{D}$ is a maximal conformal atlas, the pair $(Y, \mathcal{D})$ is a *conformal surface*. A smooth surface is defined analogously.

Proposition 4.14. *The pair $(Y, \mathcal{I}_d)$ is a conformal surface.*

Proof. Proposition 4.3 implies that restrictions of isothermal charts to open subsets of their domains are isothermal charts, and that the transition maps between isothermal charts are holomorphic or antiholomorphic. Consequently, $\mathcal{I}_d$ is a conformal atlas. The maximality of $\mathcal{I}_d$ also follows from Proposition 4.3. □

We define and recall some terminology from Riemannian geometry. A *Riemannian norm (field)* G on a conformal (or a smooth) surface $(Y, \mathcal{A})$ is a map $G\colon TY \to \mathbb{R}$ for which there exists a smooth Riemannian metric g such that $G(v) = [g(v,v)]^{1/2}$ for $v \in TY$. Here TY is the tangent bundle of Y.

The length distance induced by g is denoted by d_G. We say that d_G is the *Riemannian distance* induced by G. The metric space (Y, d_G) has *constant curvature* k if the corresponding Riemannian metric g has constant curvature k. The curvature refers to Gaussian curvature.

A *Riemannian surface* is a conformal (or smooth) surface with a Riemannian norm field. A map $\psi\colon (Y_1, G_1) \to (Y_2, G_2)$ between Riemannian surfaces is *conformal in the Riemannian sense* if ψ is a diffeomorphism and there exists a positive smooth function $h\colon Y_2 \to (0, \infty)$ such that the pushforward Riemannian norm field $\psi_* G_1$ equals $h \cdot G_2$. A Riemannian norm G is *compatible* with a conformal atlas $\mathcal{I}$ if every chart $(V, f) \in \mathcal{I}$ is conformal in the Riemannian sense.

Proposition 4.15. *The conformal surface $(Y, \mathcal{I}_d)$ has a Riemannian distance d_G such that G is compatible with the isothermal charts $\mathcal{I}_d$ of Y and (Y, d_G) is complete and has constant curvature -1, 0 or 1. Additionally, $\mathcal{I}_d = \mathcal{I}_{d_G}$ and the charts $(V, f) \in \mathcal{I}_{d_G}$ are conformal in the Riemannian sense.*

Proof. The existence of G follows from the classical uniformization theorem. Theorem 4.12 Property (e) and [AIM09, Weyl's lemma] imply that the elements of $\mathcal{I}_{d_G}$ are conformal in the Riemannian sense. The construction of G implies that when the elements of $\mathcal{I}_d$ are considered as maps from Euclidean domains into (Y, d_G), then they are conformal in the Riemannian sense. Thus $\mathcal{I}_d = \mathcal{I}_{d_G}$. □

4.5. Uniformization map. Let d_G denote the Riemannian distance obtained from Proposition 4.15. We define $Y_G = (Y, d_G)$ and let $Y = (Y, d)$. We denote the Hausdorff 2-measure of Y_G by $\mathcal{H}^2_G$.

We call the map $u = \mathrm{id}_Y\colon Y_G \to Y$ the *uniformization map.* Proposition 4.15 implies that every isothermal parametrization of $V \subset Y$ can be written in the form $u \circ \phi$ for an isothermal parametrization $\phi\colon U \to u^{-1}(V)$.

Let Y be a quasiconformal surface. If $u \circ \phi_1$ and $u \circ \phi_2$ are isothermal charts and $\psi = \phi_2^{-1} \circ \phi_1$, then $\mathrm{N}_{u\circ\phi_2} \circ D\psi = \mathrm{N}_{u\circ\phi_1}$ by the chain rule. Since $D\psi$ is a diffeomorphism, the equality actually holds everywhere whenever the left-hand side or the right-hand side are defined.

Remark 4.16. For a given quasiconformal surface Y, there is a norm field N on Y_G such that for every isothermal parametrization $u \circ \phi\colon U \to V$, its approximate metric differential $\mathrm{N}_{u\circ\phi}$ satisfies $\mathrm{N}_{u\circ\phi} = \mathrm{N} \circ D\phi$ everywhere.

Corollary 4.17. *Let u be the uniformization map. Then the pointwise dilatations of u satisfy*

$$\rho^2(G, \mathrm{N}) = K_O(u) K_I(u) \quad \mathcal{H}^2_G\text{-almost everywhere,} \tag{24}$$

where $\rho(G, \mathrm{N})$ is the Banach–Mazur distance between (TY, G) and (TY, N). In par-

ticular,

$$\frac{\pi}{4}\operatorname{mod}\Gamma \le \operatorname{mod} u\Gamma \le \frac{\pi}{2}\operatorname{mod}\Gamma \tag{25}$$

for all path families Γ *in* Y_G*.*

Proof. It suffices to verify (24) and (25) in any given planar domain $V \subset Y_G$. Consider an isothermal parametrization $\phi\colon U \to V \subset Y_G$. Remark 4.16 yields $\mathrm{N} \circ D\phi = \mathrm{N}_{u\circ\phi}$ and Proposition 4.15 implies $G \circ D\phi = \omega \left\|\cdot\right\|_2$ for some smooth function ω. The equality (24) follows from the corresponding claim about $u \circ \phi$, see Theorem 4.12. The inequalities (25) follow the corresponding property of $u \circ \phi$, see Proposition 4.3. □

Proof of Theorem 1.3. The claim was that the uniformization map satisfies $K_O(u) \le 4/\pi$ and $K_I(u) \le \pi/2$. These inequalities follow from (25). □

Lemma 4.18. *The map* $u\colon (Y, \mathcal{H}^2_G) \to (Y, \nu_Y)$ *satisfies Lusin's Conditions (N) and* (N^{-1})*.*

Proof. This follows from Lemma 3.3 since $\nu_{Y_G} \equiv \mathcal{H}^2_G$. □

We sometimes consider the differential

$$Du\colon (TY, G) \to (TY, \mathrm{N}),$$

where the norm field N is understood to be well-defined ν_Y-almost everywhere in Y. This makes sense due to Lemma 4.18.

5. Quasiconformal maps between quasiconformal surfaces

Given two quasiconformal surfaces $Y_1 = (Y_1, d_1)$ and $Y_2 = (Y_2, d_2)$, we let $Y_{G_i} = (Y_i, d_{G_i})$ and $u_i\colon Y_{G_i} \to Y_i$ be as in Section 4.5 for $i = 1, 2$. For $i = 1, 2$, we denote $\nu_i = \nu_{Y_i}$ for the measures from Definition 3.1.

Our goal is to understand an analog of Corollary 4.17 for the quasiconformal surfaces Y_1 and Y_2 and for an arbitrary quasiconformal map

$$\Psi\colon Y_1 \to Y_2. \tag{26}$$

A technical difficulty is posed by the fact that Ψ can fail to satisfy Lusin's Condition (N) and (N^{-1}) with respect to Hausdorff measures. As a consequence, the pointwise results we prove hold only ν_1-almost everywhere.

We observe that the mapping

$$\widetilde{\Psi} = u_2^{-1} \circ \Psi \circ u_1\colon Y_{G_1} \to Y_{G_2}$$

is quasiconformal as a map between two Riemannian surfaces, it is classically differentiable $\mathcal{H}^2_{G_1}$-almost everywhere and it satisfies Lusin's Conditions (N) and (N^{-1}). Then Lemma 4.18 implies the following.

Lemma 5.1. *The differential*

$$D\Psi\colon (TY_1, \mathrm{N}_1) \to (TY_2, \mathrm{N}_2) \tag{27}$$

is well-defined ν_1*-almost everywhere. Moreover,*

$$D(\Psi^{-1}) \circ D\Psi = D\mathrm{id}_{Y_1}\colon (TY_1, \mathrm{N}_1) \to (TY_1, \mathrm{N}_1)$$

ν_1*-almost everywhere.*

Lemma 5.1 implies that we can compute the operator norm and the Jacobian of (27) ν_1-almost everywhere. These objects are defined as in Section 3. The chain rule implies that the inverse of (27) is well-defined ν_2-almost everywhere. We define pointwise outer and inner dilatations $K_O(\Psi) = \rho_\Psi^2/J_\Psi$ and $K_I(\Psi) = \rho_{\Psi^{-1}}^2 J_\Psi$, which are uniquely defined ν_1-almost everywhere.

Theorem 5.2. *The equalities $K_O(\Psi) = K_O(D\Psi)$ and $K_I(\Psi) = K_I(D\Psi)$ hold ν_1-almost everywhere. In particular, the pointwise dilatations satisfy*

$$K_O(D\Psi)K_I(D\Psi) \geq \rho^2(\mathrm{N}_1, \mathrm{N}_2 \circ D\Psi) \tag{28}$$

ν_1-almost everywhere. The equality (28) *holds ν_1-almost everywhere if and only if the differential*

$$D\Psi\colon (TY_1, \mathrm{N}_1) \to (TY_2, \mathrm{N}_2)$$

is a Banach–Mazur minimizer ν_1-almost everywhere.

Since Ψ is 1-quasiconformal if and only if the pointwise dilatations satisfy $K_O(\Psi) = \chi_{Y_1}$ and $K_I(\Psi) = \chi_{Y_1}$ ν_1-almost everywhere, Theorem 5.2 implies the following.

Corollary 5.3. *A quasiconformal homeomorphism $\Psi\colon Y_1 \to Y_2$ is* 1*-quasiconformal if and only if there exists a Borel function $\omega\colon Y_1 \to (0, \infty)$ such that $\mathrm{N}_2 \circ D\psi = \omega\, \mathrm{N}_1$ ν_1-almost everywhere.*

The rest of the section is spent on proving Theorem 5.2. To this end, let $B_0 \subset Y_{G_1}$ be a Borel set of $\mathcal{H}^2_{G_1}$-measure zero such that the restrictions of u_1 and $u_2 \circ \widetilde{\Psi}$ to $Y_{G_1} \setminus B_0$ satisfy Conditions (N) and (N^{-1}). The existence of such a set is guaranteed by Lemma 4.18 and by the fact that $\widetilde{\Psi}$ satisfies Conditions (N) and (N^{-1}). We fix such a set for the rest of this section.

Lemma 5.4. *The Jacobian J_Ψ of Ψ equals $J_2(D\Psi)$ $\mathcal{H}^2_{Y_1}$-almost everywhere in $Y_1 \setminus u_1(B_0)$. In particular, this identity holds ν_1-almost everywhere.*

Proof. The claim is local, so it suffices to consider the claim using isothermal charts of Y_1 and Y_2. The isothermal charts satisfy Conditions (N) and (N^{-1}) when restricted to the complement of $u_1(B_0)$ and $\Psi \circ u_1(B_0)$, respectively. Then the claim follows from the chain rule of Jacobians of linear maps between Banach spaces [AK00, Lemma 4.2] and the corresponding Euclidean results formulated in Lemma 4.4 and Corollary 4.6. $\square$

We fix a Borel set $B_1 \supset B_0$ of zero $\mathcal{H}^2_{G_1}$-measure for which the following properties hold:

(a) The maps $Y_1 \setminus u_1(B_1) \ni y \mapsto \mathrm{N}_1(y)$ and $Y_2 \setminus \Psi(u_1(B_1)) \ni y \mapsto \mathrm{N}_2(y)$ are norms everywhere and also Borel measurable;

(b) The maps $Y_1 \setminus u_1(B_1) \ni y \mapsto D\Psi(y)$ and $Y_2 \setminus \Psi(u_1(B_1)) \ni y \mapsto D(\Psi^{-1})(y)$ are Borel measurable and the chain rule $D(\Psi^{-1}) \circ D\Psi = D\mathrm{id}_{Y_1}$ holds everywhere in $Y_1 \setminus u_1(B_1)$.

The set B_1 is defined to guarantee that the operator norms of $D\Psi$ and its inverse $D(\Psi^{-1})$ are well-defined everywhere in the complement of $u_1(B_1)$ and $\Psi(u_1(B_1))$, respectively. Also, the restriction of Ψ to the complement of $u_1(B_1)$ satisfies Conditions (N) and (N^{-1}).

Proposition 5.5. *The Borel functions $x \mapsto \|D\Psi\|(x)\big(\chi_{Y_1 \setminus u_1(B_1)}(x)\big) =: I_\Psi(x)$ and $x \mapsto \|D(\Psi^{-1})\|(x)\big(\chi_{Y_2 \setminus \Psi(u_1(B_1))}(x)\big) =: I_{\Psi^{-1}}(x)$ are minimal weak upper gradients of Ψ and Ψ^{-1}, respectively.*

Proof. First, for almost every non-constant absolutely continuous path $\theta\colon [0,1] \to Y_1$, the paths $u_1^{-1}\circ\theta$, $\Psi\circ\theta$, and $u_2^{-1}\circ\Psi\circ\theta$ are absolutely continuous, and the measures on $[0,1]$ induced by their metric speeds are absolutely continuous with respect to one another.

Second, Lemma 3.3 and Lemma 4.18 imply that the path families $\Gamma^{+}_{u_1(B_1)}$ and $\Gamma^{+}_{\Psi\circ u_1(B_1)}$ are negligible. The fact that I_Ψ is a minimal weak upper gradient of Ψ is a local property. For this reason, we fix isothermal parametrizations $u_i \circ \phi_i\colon U_i \to V_i$ for $i = 1, 2$ with $\Phi(V_1) = V_2$. Now for almost every path $\theta\colon [0,1] \to V_1$, Lemma 4.4 (c) holds for $(u_1 \circ \phi_1)^{-1} \circ \theta$ and for $(u_2 \circ \phi_2)^{-1} \circ (\Psi \circ \theta)$.

The previous two paragraphs imply that I_Ψ is a weak upper gradient of Ψ. To see the minimality of I_Ψ, we fix an upper gradient $\rho \in L^2_{\mathrm{loc}}(V_1)$ of Ψ. Fix a rectangle $R \subset U_1$ with a foliation $\gamma_t(s) = x_0 + tv + sw$, for $-1 \leq s,t \leq 1$, with v and w orthogonal. By arguing as in the proof of Lemma 4.7, Fubini's theorem implies for $\mathcal{L}^2$-almost every $x \in R \setminus \phi_1^{-1}(B_1)$ and $y = u_1(\phi_1(x))$,

$$\rho(y)\,\mathrm{N}_{u_1\circ\phi_1}((x,w)) \geq \mathrm{N}_{\Psi\circ(u_1\circ\phi_1)}((x,w)). \tag{29}$$

By slightly modifying the corresponding argument from the proof of Lemma 4.7, the inequality (29) implies $\rho(y) \geq I_\Psi(y)$ $\mathcal{H}^2_{Y_1}$-almost everywhere in $V_1 \setminus u_1(B_1)$. Therefore, I_Ψ is a representative of ρ_Ψ. The claim for $I_{\Psi^{-1}}$ follows from symmetry, given the fact from Lemma 3.3 that $\nu_1(B) = 0$ if and only if $\nu_2(\Psi(B)) = 0$. ☐

Proof of Theorem 5.2. Lemma 5.4 proves that the Jacobian of Ψ and the Jacobian $J_2(D\Psi)$ coincide ν_1-almost everywhere. Proposition 5.5 implies that the operator norm of $D\Psi$ determines the minimal weak upper gradient of Ψ ν_1-almost everywhere. This implies that the pointwise outer dilatation of Ψ is determined by the outer dilatation of $D\Psi$. Similar reasoning holds for the inner dilatation.

The inequality (28) follows from the fact that $D\Psi$ is a linear map between Banach spaces. The defining property of a Banach–Mazur minimizer yields that $D\Psi$ is a Banach–Mazur minimizer ν_1-almost everywhere if and only if the inequality (28) is an equality ν_1-almost everywhere. ☐

6. Applications

In Section 6.1, we establish the uniqueness of the uniformization map up to conformal diffeomorphisms. We prove Theorem 1.4 in Section 6.2.

6.1. Isothermal parametrizations using Riemannian surfaces. We start this section by considering global isothermal parametrizations of quasiconformal surfaces.

Definition 6.1. (Isothermal parametrizations) Let Z be a Riemannian surface and $\Psi\colon Z \to Y$ a quasiconformal map. The pair (Z, Ψ) is an *isothermal parametrization of Y* if for every other Riemannian surface $\widetilde{Z}$ and quasiconformal map $\widetilde{\Psi}\colon \widetilde{Z} \to Y$ we have that

$$(K_O(\Psi)K_I(\Psi))(z) \leq (K_O(\widetilde{\Psi})K_I(\widetilde{\Psi}))(\widetilde{z}) \tag{30}$$

for $\widetilde{z} = (\widetilde{\Psi}^{-1} \circ \Psi)(z)$ at $\mathcal{H}^2_Z$-almost every $z \in Z$. If the image of the map (Z, Ψ) is clear from the context, we say that (Z, Ψ) is *isothermal.* If also the domain is clear, we simply say that Ψ is *isothermal.*

The following theorem is a global version of Theorem 4.12.

Theorem 6.2. *The uniformization map u is isothermal. Moreover, the following are equivalent for every Riemannian surface Z and a quasiconformal homeomorphism $\Psi\colon Z \to Y$:*

(a) *The map Ψ is isothermal;*
(b) *The composition $u^{-1} \circ \Psi$ is conformal in the Riemannian sense;*
(c) *The pointwise dilatations satisfy*

$$(K_O(\Psi)K_I(\Psi)) \circ (\Psi^{-1} \circ u) = K_O(u)K_I(u) \tag{31}$$

$\mathcal{H}^2_G$-almost everywhere in Y_G.

(d) *The differential $D\Psi\colon (TZ, G) \to (TY, \mathrm{N})$ is a Banach–Mazur minimizer at $\mathcal{H}^2_Z$-almost every point $z \in Z$.*

Proof. Since $\Psi\colon Z \to Y$ is quasiconformal, Theorem 5.2 shows that

$$\begin{aligned} K_O(\Psi)K_I(\Psi) &\geq \rho^2(G_Z, \mathrm{N} \circ D\Psi) \\ &= \rho^2(G_Z \circ D(\Psi^{-1}) \circ Du, \mathrm{N} \circ Du) \circ (u^{-1} \circ \Psi) \end{aligned} \tag{32}$$

$\mathcal{H}^2_Z$-almost everywhere in Z. The composition $G_Z \circ D(\Psi^{-1}) \circ Du$ is a norm induced by a Riemannian norm $\mathcal{H}^2_G$-almost everywhere in Y. Therefore the identity

$$\rho^2(G_Z \circ D(\Psi^{-1}) \circ Du, \mathrm{N} \circ Du) \circ (u^{-1} \circ \Psi) = \rho^2(G, \mathrm{N} \circ Du) \circ (u^{-1} \circ \Psi)$$

holds $\mathcal{H}^2_Z$-almost everywhere in Z. Applying Corollary 4.17 to the latter term shows that

$$\rho^2(G_Z \circ D(\Psi^{-1}) \circ Du, \mathrm{N} \circ Du) \circ (u^{-1} \circ \Psi) = (K_O(u)K_I(u)) \circ (u^{-1} \circ \Psi) \tag{33}$$

$\mathcal{H}^2_Z$-almost everywhere in Z. Now (32) and (33) show that

$$(K_O(\Psi)K_I(\Psi)) \circ (\Psi^{-1} \circ u) \geq K_O(u)K_I(u) \tag{34}$$

$\mathcal{H}^2_G$-almost everywhere in Y_G. We deduce from (34) that u is isothermal.

The map Ψ is isothermal if and only if the inequality in (34) is an equality $\mathcal{H}^2_G$-almost everywhere, and, by (32) and (33), this happens if and only if

$$D\Psi\colon (TZ, G_Z) \to (TY, \mathrm{N}) \tag{35}$$

is a Banach–Mazur minimizer $\mathcal{H}^2_Z$-almost everywhere. Hence, Properties (a), (c), and (d) are equivalent.

Having verified that Properties (a) and (d) are equivalent, we see that the property of being isothermal is a local property. Hence, the equivalence of Properties (a) and (b) follow after we verify the equivalence in the domain of an arbitrary isothermal chart of Z.

Let $\phi_1\colon U_1 \to V_1 \subset Z$ be an isothermal parametrization of a domain $V_1 \subset Z$. Then $\mathrm{N}_{\phi_1} = G_Z \circ D\phi_1 = \omega \left\|\cdot\right\|_2$ for some smooth function $\omega > 0$. Observe that $\Psi|_{V_1}$ is isothermal if and only if $\Psi \circ \phi_1$ is isothermal. Proposition 4.15 implies that the latter property holds if and only if $u^{-1} \circ (\Psi \circ \phi_1)$ is conformal in the Riemannian sense if and only if $u^{-1} \circ \Psi|_{V_1}$ is conformal in the Riemannian sense. This establishes the claim. □

Theorem 6.2 can be applied, for example, in the following manner. Given an isothermal map $\Phi\colon Z \to Y$ and a 1-quasiconformal homeomorphism $f\colon Y \to Y$, the mapping $\Phi^{-1} \circ f \circ \Phi\colon Z \to Z$ is conformal in the Riemannian sense. To see why, we first apply Corollary 5.3 to show that $f \circ \Phi$ is isothermal. Then Theorem 6.2 implies that $\Phi^{-1} \circ (f \circ \Phi)$ is conformal in the Riemannian sense. This fact imposes a structure and size restriction on the group generated by such f. A similar reasoning implies that for any given 1-quasiconformal homeomorphism $f\colon Y_1 \to Y_2$ and isothermal $\Phi_i\colon Z_i \to Y_i$, the homeomorphism $\Phi_2^{-1} \circ f \circ \Phi_1\colon Z_1 \to Z_2$ is conformal in the Riemannian sense.

6.2. Quasisymmetries. In this section, we investigate properties of isothermal charts of (Y, d_Y) under the assumption that (Y, d_Y) is compact, Ahlfors 2-regular, and linearly locally contractible.

6.2.1. Basic definitions. Let Y and Z be metric spaces. For a homeomorphism $\phi\colon Y \to Z$, $y \in Y$ and $r > 0$, let

$$L_\phi(y,r) = \sup\left\{d_Z(\phi(y), \phi(w)) \mid d_Y(y,w) \leq r\right\} \quad \text{and}$$
$$l_\phi(y,r) = \inf\left\{d_Z(\phi(y), \phi(w)) \mid d_Y(y,w) \geq r\right\}.$$

The map ϕ is *quasisymmetric* if there exists a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ for which for every $y \in Y$ and $0 < r_1, r_2 < \operatorname{diam} Y$,

$$L_\phi(y, r_1) \leq \eta\left(\frac{r_1}{r_2}\right) l_\phi(y, r_2). \tag{36}$$

Such a homeomorphism η is called a *(quasisymmetric) distortion function* of ϕ and we say that ϕ is η-quasisymmetric.

A metric surface Y is *Ahlfors* 2*-regular* if there exists a constant $C_A \geq 1$ such that for every $y \in Y$ and $\operatorname{diam} Y > r > 0$,

$$C_A^{-1} r^2 \leq \mathcal{H}^2_Y(\overline{B}(y,r)) \leq C_A r^2. \tag{37}$$

Here $\overline{B}(y,r) \subset Y$ is the closed ball of radius r centered at y.

Let $\lambda \geq 1$. A metric surface Y is λ*-linearly locally contractible* if for every $y \in Y$ and $0 < r < \frac{\operatorname{diam} Y}{\lambda}$, the metric ball $B(y,r)$ is contractible inside the ball $B(y, \lambda r)$. That is, there exists $y_0 \in B(y, \lambda r)$ and a continuous map $H\colon B(y,r) \times [0,1] \to B(y,\lambda r)$ such that $H(z,0) = z$ and $H(z,1) = y_0$ for every $z \in B(y,r)$.

6.2.2. Global parametrizations of compact surfaces. When we say that something in this section depends only on the *data of* Y, we mean that it depends only on C_A and λ, defined as above. Theorem 1.4 is an immediate consequence of Theorem 6.3 and Theorem 6.4.

Theorem 6.3. *Suppose that Y is an Ahlfors 2-regular metric surface that is linearly locally contractible and homeomorphic to $\mathbb{S}^2$. Then there exists a Riemannian distance $d_{G'}$ on Y of constant curvature 1 for which*

$$u' = \operatorname{id}_Y\colon Y_{G'} \to Y$$

is isothermal and η-quasisymmetric with η depending only on the data of Y.

Proof. Let $(Y, d_G) = Y_G$ denote the Riemannian surface obtained from Proposition 4.15. The surface has curvature equal to one. The uniformization map $u = \operatorname{id}_Y\colon Y_G \to Y$ is isothermal, and therefore $\frac{\pi}{2}$-quasiconformal.

We fix an isometry $I\colon \mathbb{S}^2 \to Y_G$, and choose three points $p_1, p_2, p_3 \in Y$ such that $d_Y(p_i, p_j) \geq \operatorname{diam} Y/2$ for each $i \neq j$. There exists a Möbius transformation $M\colon \mathbb{S}^2 \to \mathbb{S}^2$ so that $v' = u \circ I \circ M$ takes the north pole to p_1, the south pole to p_3, and a point from the equator to p_2. Since v' is $(\pi/2)$-quasiconformal, v' is η-quasisymmetric with η depending only on the data of Y; see [BK02, Proposition 9.1 and Section 3]. We denote $d_{G'}(x,y) := d_{\mathbb{S}^2}((I \circ M)^{-1}(x), (I \circ M)^{-1}(y))$ for all $x, y \in Y_G$ and set $Y_{G'} := (Y, d_{G'})$. Then the identity mapping $u'\colon (Y, d_{G'}) \to (Y, d_Y)$ is isothermal and η-quasisymmetric. □

Theorem 6.4. *Suppose that Y is a compact Ahlfors 2-regular and linearly locally contractible metric surface that is not homeomorphic to $\mathbb{S}^2$. Then the uniformization map*

$$u = \mathrm{id}_Y \colon Y_G \to Y \tag{38}$$

is η-quasisymmetric, where η depends only on the data of Y.

We postpone the proof of Theorem 6.4 until the end of this section.

Lemma 6.5. *Let Y be a quasiconformal surface and suppose that $\phi\colon \mathbb{D} \to V \subset Y$ is an η-quasisymmetric homeomorphism. Then ϕ is K-quasiconformal with K depending only on η. Moreover, there exists a $(4K/\pi)$-quasiconformal homeomorphism $\psi\colon \mathbb{D} \to \mathbb{D}$ such that $\psi(0) = 0$ and $\phi \circ \psi$ is an isothermal η'-quasisymmetric map with η' depending only on η.*

Proof. It follows from [Tys00, Theorem 3.13] that the outer dilatation of ϕ is bounded by some constant K_O depending only on η. Since V has a $(\pi/2)$-quasiconformal chart, the inner dilatation bound $(\pi/2)^2 K_O$ of ϕ follows from Euclidean regularity results [AIM09, Definition 3.1.1 and Theorem 3.7.7]. Therefore, ϕ is K-quasiconformal with $K = (\pi/2)^2 K_O$.

Proposition 4.2 and the Riemann mapping theorem, together with Proposition 4.3, imply the existence of a $(4K/\pi)$-quasiconformal mapping $\psi\colon \mathbb{D} \to \mathbb{D}$ with $\psi(0) = 0$ such that $\phi \circ \psi$ is isothermal. Corollary 3.10.4 of [AIM09] implies that ψ is $\widetilde{\eta}$-quasisymmetric with $\widetilde{\eta}$ depending only on the maximal dilatation of ψ. Hence, $\phi \circ \psi$ is $\eta \circ \widetilde{\eta}$-quasisymmetric. Since K and $\widetilde{\eta}$ depend only on η, the claim follows. □

Proposition 6.6. *Let Y_G be a complete Riemannian surface of curvature -1, 0, or 1 and*

$$\phi\colon \mathbb{D} \to Y_G$$

a conformal embedding. Suppose that Y_G is not homeomorphic to the sphere $\mathbb{S}^2$ or that

$$2 \operatorname{diam} \phi(\mathbb{D}) \le \operatorname{diam} Y_G.$$

Then there is a constant $2^{-1} > \beta > 0$ and a distortion function $\widetilde{\eta}$ for which

$$\phi(\beta\mathbb{D}) \subset B_G\left(\phi(0), \frac{l_\phi(0, \frac{1}{2})}{6}\right) \tag{39}$$

and the restriction of ϕ to $\beta\mathbb{D}$ is $\widetilde{\eta}$-quasisymmetric. The constant β and distortion function $\widetilde{\eta}$ are independent of ϕ and the surface Y_G.

Proof. First, suppose that Y_G is not homeomorphic to the sphere $\mathbb{S}^2$. The surface Y_G has a universal cover $\pi\colon \Omega \to Y_G$, where π is a local isometry and where Ω is either the hyperbolic disk $\mathbb{D}_{\mathrm{hyp}}$, the Euclidean plane $\mathbb{R}^2$, or the Riemann sphere $\mathbb{S}^2$. If $\Omega = \mathbb{S}^2$, the covering group of π is generated by the antipodal map.

Suppose that $\phi\colon \mathbb{D} \to Y_G$ is as in the claim. Then there exists a conformal embedding $\psi\colon \mathbb{D} \to \Omega$ for which $\phi = \pi \circ \psi$. Since ϕ is an embedding, so are ψ and the restriction of π to the image of ψ.

Claim (1): There exists a $2^{-1} > \beta' > 0$ and a distortion function η for which the restriction of ψ to $\beta'\mathbb{D}$ is η-quasisymmetric.

Proof of Claim (1): If Ω is the hyperbolic disk or the Euclidean plane, the existence of β' and η follows from Propositions 5 and 7 of [GW18] (which are stated for the case when ψ is orientable. However, the non-orientable case follows from the orientable one by applying the conjugate map $z \mapsto \overline{z}$ in the Euclidean unit disk $\mathbb{D}$).

Consider the case $\Omega = \mathbb{S}^2$. We rotate the sphere $\mathbb{S}^2$ in such a way that $\psi(0) = (0,0,-1)$. Moreover, we identify $\mathbb{S}^2$ with the extended plane $\mathbb{R}^2 \cup \{\infty\}$ using the stereographic projection $\tau\colon \mathbb{S}^2 \to \mathbb{R}^2 \cup \{\infty\}$ which fixes the *equator* $\mathbb{S}^1 = \mathbb{S}^1 \times \{0\} \subset \mathbb{R}^3$ and maps the *south pole* $(0,0,-1)$ to 0. With this identification, τ maps the southern hemisphere to the unit disk $\mathbb{D}$. Recall that τ is a conformal map.

By construction, the restriction of π to the image of ψ is injective. We claim that $\psi(10^{-1}\mathbb{D})$ is contained in the southern hemisphere. We prove this by employing the following growth estimate for conformal embeddings [Dur83, Theorem 2.6]: If $0 < r < 1$ and $\|x\|_2 = r$, then

$$\|D(\tau \circ \psi)\|(0)\frac{r}{(1+r)^2} \leq \|\tau \circ \psi\|_2(x) \leq \|D(\tau \circ \psi)\|(0)\frac{r}{(1-r)^2}. \tag{40}$$

If $\psi(10^{-1}\mathbb{D})$ is not contained in the southern hemisphere, then (40) implies that

$$\frac{81}{10} \leq \|D(\tau \circ \psi)\|(0). \tag{41}$$

Then (40) and (41) imply that $\tau \circ \psi(2^{-1}\mathbb{D})$ contains the closed unit disk $\overline{\mathbb{D}}$. This is a contradiction with the injectivity of π in the image of ψ.

The restriction of the stereographic projection τ to the southern hemisphere is a biLipschitz map. Also, the restriction of $\tau \circ \psi$ to the disk $10^{-1}\mathbb{D}$ is η'-quasisymmetric with η' independent of ψ [AIM09, Theorem 3.6.2]. The existence of β' and η follows.

Claim (2): Let $\beta' > 0$ be as in Claim (1). There exists a constant $\beta' > \beta'' > 0$ such that

$$\psi(\beta''\mathbb{D}) \subset B_{d_\Omega}\left(\psi(0), \frac{l_\psi(0,\frac{1}{2})}{6}\right). \tag{42}$$

Proof of Claim (2): Suppose that $\beta' > 0$ and η are as in Claim (1) and consider $\beta' > \beta'' > 0$. Since the restriction of ψ to the disk $\beta'\mathbb{D}$ is η-quasisymmetric,

$$L_\psi(0,\beta'') \leq \eta\left(\frac{\beta''}{\beta'}\right) l_\psi(0,\beta') \leq \eta\left(\frac{\beta''}{\beta'}\right) l_\psi\left(0, \frac{1}{2}\right).$$

Therefore, it suffices to pick $\beta'' > 0$ so small that $\eta\left(\frac{\beta''}{\beta'}\right) < \frac{1}{6}$. Claim (2) follows.

We complete the proof of the claim using Claims (1) and (2) (when Y_G is not homeomorphic to $\mathbb{S}^2$). Recall that the restriction of π to $\psi(\mathbb{D})$ is injective. Let $\beta'' > 0$ be as in Claim (2). Since

$$B_{d_\Omega}\left(\psi(0), l_\psi\left(0, \frac{1}{2}\right)\right) \subset \psi\left(2^{-1}\mathbb{D}\right),$$

the restriction of π to $B_{d_\Omega}\left(\psi(0), 6^{-1} l_\psi\left(0, \frac{1}{2}\right)\right)$ is an isometry onto its image. This is an immediate consequence of the fact that

$$d_G(x,y) = \inf\left\{ d_\Omega(x',y') \mid x' \in \pi^{-1}(x) \text{ and } y' \in \pi^{-1}(y) \right\}.$$

In conclusion, the map ψ can be replaced with ϕ and Ω with Y_G everywhere in Claims (1) and (2). We define $\beta = \beta''$ as in Claim (2) and $\widetilde{\eta} = \eta$ as in Claim (1) to conclude the proof of Proposition 6.6 when Y_G is not homeomorphic to $\mathbb{S}^2$.

We are left to consider the case when Y_G is homeomorphic to $\mathbb{S}^2$. Then there exists an isometry $\pi\colon \mathbb{S}^2 \to Y_G$. Therefore, there exists a conformal embedding $\psi\colon \mathbb{D} \to \mathbb{S}^2$ for which $\phi = \pi \circ \psi$. By rotating the sphere, we can assume that $\psi(0)$ is the south pole. The diameter bound on the image of ϕ implies that $\psi(10^{-1}\mathbb{D})$ is contained in the southern hemisphere. The rest of the proof is argued as above. □

For the rest of the section, we assume that $\operatorname{diam} Y = 1$. This can be done without loss of generality since the properties we study are left unchanged by rescaling. The diameter normalization is needed for the results we use from [GW18]. We formulate the following corollary of [GW18, Theorem 9] and Lemma 6.5.

Proposition 6.7. *There is a quantity $A_0 \geq 1$ and a distortion function η, each depending only on the data of Y, such that for every $0 < R \leq \frac{1}{A_0}$ and $y \in Y$, there is a neighbourhood U of y for which*

(a) $B(y, \frac{R}{A_0}) \subset U \subset B(y, A_0 R)$;
(b) *there exists an η-quasisymmetric homeomorphism $f \colon U \to \mathbb{D}$ that is an isothermal chart of Y with $f(y) = 0$.*

The only difference between [GW18, Theorem 9] and Proposition 6.7 is the condition that f is an isothermal chart. We state next a modified version of [GW18, Lemma 10].

Lemma 6.8. *Suppose that $2^{-1} > \beta > 0$ is the constant in Proposition 6.6 and η is as in Proposition 6.7. Then there exist radii α and $r_0 > 0$ and a positive integer n such that the following statements hold.*

(a) *There exists an atlas $\mathcal{A}_\beta = \{(U_j, f_j)\}_{j=1}^n$, where $f_j(U_j) = \mathbb{D}$ and each f_j is an η-quasisymmetric isothermal chart of Y.*
(b) *Let $x_j = f_j^{-1}(0)$. The collection $\{B(x_j, r_0)\}_{j=1}^n$ is pairwise disjoint.*
(c) *The collection $\{B(x_j, 2r_0)\}_{j=1}^n$ covers Y.*
(d) *For each $j = 1, \ldots, n$, we have $B(x_j, 10r_0) \subset U_j$ and*

$$\alpha\mathbb{D} \subset f_j(B(x_j, r_0)) \subset f_j(B(x_j, 10r_0)) \subset \beta\mathbb{D}.$$

The radii α and r_0, and the integer n depend only on the data of Y and β.

Lemma 6.8 is proved exactly as [GW18, Lemma 10], but instead of applying [GW18, Theorem 9] as in the proof of [GW18, Lemma 10], we apply Proposition 6.7.

Proof of Theorem 6.4. Let $(Y, d_G) = Y_G$ denote the Riemannian surface obtained from Proposition 4.15. The surface Y_G has curvature equal to 1, 0, or -1 and is not homeomorphic to $\mathbb{S}^2$. Let $u = \mathrm{id}_Y \colon Y_G \to Y$ denote the uniformization map.

Recall that the claim is that u is quasisymmetric with distortion depending only on the data of Y. It suffices to prove that $v = u^{-1} = \mathrm{id}_Y \colon Y \to Y_G$ is quasisymmetric with quasisymmetric distortion function depending only on the data of Y.

For the duration of the proof, we use the notations introduced in Lemma 6.8, and denote $\psi_j = v \circ f_j^{-1} \colon \mathbb{D} \to Y_G$. We first observe that for each $j = 1, 2, \ldots, n$,

$$v\big|_{B(x_j,10r_0)} = \psi_j \circ f_j\big|_{B(x_j,10r_0)} \quad \text{is } \eta_1\text{-quasisymmetric} \tag{43}$$

with $\eta_1 = \widetilde{\eta} \circ \eta$, where η is from Lemma 6.8 and $\widetilde{\eta}$ from Proposition 6.6. Recall that $\widetilde{\eta}$ is independent of Y and the η depends only on the data of Y.

Next, we claim that for each $x, x' \in Y$ with $d_Y(x, x') = 4r_0$,

$$d_G(v(x), v(x')) \geq \delta = C^{-1} \operatorname{diam} Y_G, \tag{44}$$

where C depends only on the data of Y. To this end, since $\{B(x_j, 2r_0)\}_{j=1}^n$ covers Y, the union $\bigcup_{j=1}^n \psi_j(\beta\mathbb{D})$ covers Y_G. As Y_G is connected, we conclude

$$\max_j \{\operatorname{diam} \psi_j(\beta\mathbb{D})\} \geq \frac{\operatorname{diam} Y_G}{n}. \tag{45}$$

Consider a pair of indices $i, k = 1, 2, \ldots, n$ with $d_Y(x_i, x_k) < 4r_0$. Then $x_k \in B(x_i, 10r_0)$, so Lemma 6.8 implies $d_G(v(x_i), v(x_k)) \leq L_{\psi_i}(0, \beta)$. If i and k are distinct, $d_Y(x_i, x_k) > r_0$, so the same lemma implies $d_G(v(x_i), v(x_k)) \geq \ell_{\psi_i}(0, \alpha)$. Observe that

$$\ell_{\psi_i}(0, \alpha) \geq \frac{L_{\psi_i}(0, \beta)}{\widetilde{\eta}\left(\frac{\beta}{\alpha}\right)} \geq \frac{\operatorname{diam} \psi_i(\beta \mathbb{D})}{2\widetilde{\eta}\left(\frac{\beta}{\alpha}\right)}.$$

We have now verified that the quantities

$$\ell_{\psi_i}(0, \alpha), L_{\psi_i}(0, \beta), \operatorname{diam} \psi_i(\beta \mathbb{D}), d_G(v(x_i), v(x_k)) \tag{46}$$

are comparable with constants depending only on the data of Y.

Observe that for every pair $i, j = 1, 2, \ldots, n$ with $i \neq j$, there exists $m \leq n$ and a chain $\{x_{i_k}\}_{k=1}^m$ with $x_{i_1} = x_i$ and $x_{i_m} = x_j$, and $4r_0 > d_Y(x_{i_k}, x_{i_{k+1}}) > r_0$ for each $k = 1, 2, \ldots m-1$. Recall from Lemma 6.8 that n depends only on the data of Y. This fact and (46) imply that there exists $C_0 > 0$, depending only on the data of Y, such that for every pair $i, j = 1, 2, \ldots, n$,

$$\ell_{\psi_i}(0, \alpha) \geq \frac{\operatorname{diam} \psi_j(\beta \mathbb{D})}{C_0}. \tag{47}$$

Given the inequalities (45) and (47), we have

$$\ell_{\psi_i}(0, \alpha) \geq \frac{\operatorname{diam} Y_G}{nC_0} \quad \text{for every } i. \tag{48}$$

Suppose that $x, x' \in Y$ with $d_Y(x, x') = 4r_0$. Then there exist i and k such that $d_Y(x, x_i) < 2r_0$ and $d_Y(x', x_k) < 2r_0$. As $2r_0 \leq d_Y(x', x_i) \leq 6r_0$, we have $x, x', x_k \in B(x_i, 10r_0)$. Then (43) implies

$$d_G(v(x'), v(x)) \geq \frac{d_G(v(x'), v(x_i))}{\eta_1(3/2)}. \tag{49}$$

Since $x' \in Y \setminus B_Y(x_i, r_0)$, the inequality (48) yields that

$$d_G(v(x'), v(x_i)) \geq \ell_{\psi_i}(0, \alpha) \geq \frac{\operatorname{diam} Y_G}{nC_0}. \tag{50}$$

The inequality (44) follows from the inequalities (49) and (50).

Lastly, Lemma 6.8 implies that $L = 8r_0$ is a Lebesgue number of $\{B(x_j, 10r_0)\}_{j=1}^n$. Then a theorem by Tukia and Väisälä, as formulated in [GW18, Theorem 4], states that v is η_2-quasisymmetric, where η_2 depends only on η_1 from (43) and the ratios $\frac{\operatorname{diam} Y}{L} = \frac{1}{L}$ and $\frac{\operatorname{diam} Y_G}{\delta}$, where δ is from (44). Hence η_2 depends only on the data of Y. This implied the claim. □

7. Concluding remarks

The classical uniformization theorem states that every smooth Riemannian surface Y is 1-quasiconformally equivalent to a complete Riemannian surface of curvature -1, 0, or 1. For such Y, our uniformization map $u \colon Y_G \to Y$ is 1-quasiconformal. Given this observation, we pose the following question.

Open Problem A. Let Y be a quasiconformal surface. Is Y 1-quasiconformally equivalent to a metric surface Z with desirable geometric properties?

One might ask if Open Problem A holds in such a way that Z is bi-Lipschitz equivalent to the space Y_G obtained from Proposition 4.15, or even if the space is $\sqrt{2}$-bi-Lipschitz equivalent to Y_G.

When (Y, d_Y) is constructed from a sufficiently regular norm field on a smooth surface, such a Z can be constructed using John's theorem and regularity results for Beltrami differential equations. However, we cannot always take in Open Problem A the surface Z to be bi-Lipschitz equivalent to Y_G, or to any other Riemannian surface.

Theorem 7.1. [IRar, Theorem 1.6] *There exists a distance d on $\mathbb{R}^2$ such that the identity map $\iota\colon (\mathbb{R}^2, \|\cdot\|_2) \to (\mathbb{R}^2, d)$ is an isothermal parametrization, but ι does not factor as $\iota = \widehat{\iota} \circ P$, where (Z, d_Z) is a metric surface, $\widehat{\iota}\colon (Z, d_Z) \to (\mathbb{R}^2, d)$ is quasiconformal with distortion $H(\widehat{\iota}) < \sqrt{2}$ and $P\colon (\mathbb{R}^2, \|\cdot\|_2) \to (Z, d_Z)$ is bi-Lipschitz.*

Here $H(\widehat{\iota}) = \operatorname{ess\,sup} \sqrt{K_O(\widehat{\iota})(x) K_I(\widehat{\iota})(x)}$ for the pointwise dilatations of $\widehat{\iota}$. Theorem 7.1 shows that we cannot require in Open Problem A Z to be bi-Lipschitz equivalent to Y_G, even if we allow for non-conformal distortion $1 < H(\widehat{\iota}) < \sqrt{2}$. We note that the isothermal parametrization ι in Theorem 7.1 has distortion exactly $H(\iota) = \sqrt{2}$.

It is not clear whether Z in Open Problem A can be chosen in such a way that Z is locally quasisymmetrically equivalent to some Riemannian surface, or even what is the answer to the following problem.

Open Problem B. Is every quasiconformal surface 1-quasiconformally equivalent to a metric surface Z that is locally Ahlfors 2-regular and locally linearly locally contractible?

We note that Open Problem A is trivially true for each quasiconformal surface for which the uniformization map is 1-quasiconformal. This holds, for example, when (Y, d_Y) has bounded integral curvature [Res01] and [BL03], or $(Y, d_Y) \subset \mathbb{R}^N$ for $N \geq 2$.

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Toni Ikonen
University of Jyväskylä
Department of Mathematics and Statistics
P. O. Box 35 (MaD)
FI-40014 University of Jyväskylä, Finland
toni.m.h.ikonen@jyu.fi