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Baernstein's star-function, maximum modulus points and a problem of Erdős

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To the memory of Albert Baernstein II on his 80th birth anniversary

Abstract. The paper is devoted to the development of Baernstein's method of T^*-function. We consider the relationship between the number of separated maximum modulus points of a meromorphic function and the T^*-function. The results of Bergweiler, Bock, Edrei, Goldberg, Heins, Ostrovskii, Petrenko, Wiman are generalized. We also give examples showing that the obtained estimates are sharp.

Baernsteinin tähtifunktio, itseisarvon maksimipisteet ja Erdősin ongelma

Tiivistelmä. Tämä tutkimus keskittyy Baernsteinin T^*-funktiomenetelmän kehittämiseen. Tarkastelemme meromorfisen funktion ja T^*-funktion erillään olevien itseisarvon maksimipisteiden lukumäärän välistä suhdetta ja yleistämme Bergweilerin, Bockin, Edreinin, Goldbergin, Heinsin, Ostrovskiin, Petrenkon ja Wimanin tuloksia. Lisäksi osoitamme esimerkein, että saavutetut arviot ovat tarkkoja.

1. Introduction

We shall use standard notations of value distribution theory of meromorphic functions: $m(r, a, f)$ for the proximity function, $N(r, a, f)$ for the function counting a-points, $T(r, f)$ for Nevanlinna's characteristic, $\delta(a, f)$ for Nevanlinna's defect and λ, ρ for the lower order and order, respectively [18, 25].

Let $f \not\equiv 0$ be a meromorphic function in $\mathbb{C}$. In 1972 Albert Baernstein II in his paper 'Proof of Edrei's spread conjecture' [2] introduced for the first time the function, which is now widely referred to as *Baernstein's star-function*:

$$T^*(re^{i\theta}, f) = \sup_E \frac{1}{2\pi} \int_E \log|f(re^{i\varphi})|\, d\varphi + N(r, \infty, f),$$

where the supremum is taken over all the sets $E \subset [-\pi, \pi]$ of the Lebesgue measure $|E| = mes\ E = 2\theta$.

Theorem A. [2, 3] *Let $f(z)$ be a meromorphic function in $\mathbb{C}$. Then $T^*(z, f)$ is a subharmonic function on $\{z \in \mathbb{C}\colon \operatorname{Im} z > 0\}$ and is continuous on $\{z = re^{i\theta}\colon 0 < r < \infty,\ 0 \le \theta \le \pi\}$.*

Apart from the subharmonicity, the T^*-function also possesses the following properties:

> $T^*(re^{i\theta}, f)$ is a convex function of $\log r$ and nondecreasing on $(0, \infty)$ for $\theta \in [0, \pi]$;

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$T^*(re^{i\theta}, f) \le T(r, f)$ for $\theta \in [0, \pi]$ and $r > 0$;
$T^*(r, f) = N(r, \infty, f)$ for $r > 0$;
$\frac{dT^*}{d\theta}\Big|_{z=r} = \log \max_{|z|=r} |f(z)|$, $\frac{dT^*}{d\theta}\Big|_{z=-r} = \log \min_{|z|=r} |f(z)|$ for all $r \in (0, \infty)$ such that there are no zeros nor poles of f on the circle $\{z\colon |z| = r\}$.

As can be seen the T^*-function has various excellent properties and is applicable not only in the Nevanlinna's theory of value distribution of meromorphic functions [18, 25], but also in Petrenko's theory of growth of meromorphic functions [33]. Thus it is hardly surprising that with the help of the star-function it was possible to solve a number of previously open problems. We can list here just a few of them:

in 1972 Baernstein [2] proved Edrei's spread conjecture ([9], 1967);
in 1994 Fryntov [11] solved Weitsman's problem ([36], 1977);
in 1990 the author together with Shcherba [30] solved Petrenko's problem ([33], 1978);
in 1999 the author [29] proved the Fuchs's hypothesis ([12], 1958).

In 1974 Baernstein introduced the T^*-function for δ-subharmonic functions [4]. Let us recall that a function $u(z)$ is called a δ-subharmonic function on the domain D if $u(z) = u_1(z) - u_2(z)$, where $u_1(z)$, $u_2(z)$ are the subharmonic functions on D. Let $u(z) = u_1(z) - u_2(z)$ be a δ-subharmonic function in $\mathbb{C}$. Then

$$T^*(z, u) = \sup_{|E|=2\theta} \frac{1}{2\pi} \int_E u(re^{i\varphi})\, d\varphi + N(r, u_2),$$

where $z = re^{i\theta}$, $N(r, u_2) = \frac{1}{2\pi} \int_0^{2\pi} u_2(re^{i\varphi})\, d\varphi$. In the same paper [4] Baernstein generalized Theorem A to the case of δ-subharmonic functions.

Theorem B. *Let $u(z)$ be a δ-subharmonic function in $\mathbb{C}$. Then $T^*(z, u)$ is a subharmonic function on $\{z \in \mathbb{C}\colon \operatorname{Im} z > 0\}$ and a continuous function on $\{z = re^{i\theta}\colon 0 < r < \infty,\ 0 \le \theta \le \pi\}$.*

Let $\nu(r)$ be the number of maximum modulus points of an entire function $f(z)$ on the circle $\{z\colon |z| = r\}$. In 1964 Erdős posed the following questions ([19], Problem 2.16): Can we have a function $f(z) \ne cz^n$ such that

(a) $\limsup_{r\to\infty} \nu(r) = \infty$;
(b) $\liminf_{r\to\infty} \nu(r) = \infty$?

In 1968 Herzog and Piranian [21] found a positive solution of the Erdős's problem (a). They gave a suitable example of an entire function of infinite lower order. In the case of entire functions of finite lower order the question (a) is still open.

In 1977 Clunie stated the same question as formulated in the Erdős's problem (b) ([1], Problem 2.49): Is it true that $\liminf_{r\to\infty} \nu(r) < \infty$ for all the entire functions f? In [1] it was not mentioned that this question had been posed by Erdős first. Thus the author in [27] presented this problem as the Clunie's problem. In 2002 Piranian informed the author by letter that this problem belongs originally to Erdős and was stated in 1964.

In 1995 the author introduced the term separated maximum modulus points of meromorphic functions [27]. The number of such points can be assessed in the following manner. Let $f(z)$ be a meromorphic function in $\mathbb{C}$. For any $r \in (0, \infty)$ we denote by $p(r, \infty, f)$ the number of component intervals of the set

$$\{\theta\colon |f(re^{i\theta})| > 1\}$$

possessing at least one maximum modulus point of the function $f(z)$ on the circle $\{z\colon |z|=r\}$. We set

$$p(\infty, f) = \liminf_{r\to\infty} p(r,\infty,f).$$

Let us now come back to the star function of a δ-subharmonic function. Let then

$$\begin{aligned} u(z) &= \log^+|f(z)| = \log^+\left|\frac{g_1(z)}{g_2(z)}\right| \\ &= \max\left(\log|g_1(z)|, \log|g_2(z)|\right) - \log|g_2(z)| := u_1(z) - u_2(z) \end{aligned}$$

be a δ-subharmonic function,

$$N(r,u_2) = \frac{1}{2\pi}\int_0^{2\pi} u_2(re^{i\varphi})\,d\varphi = \frac{1}{2\pi}\int_0^{2\pi} \log|g_2(re^{i\varphi})|\,d\varphi = N(r,0,g_2) = N(r,\infty,f).$$

Hence

$$T^*(re^{i\theta},u) = \sup_{|E|=2\theta}\frac{1}{2\pi}\int_E \log^+|f(re^{i\varphi})|\,d\varphi + N(r,\infty,f).$$

For each $t\colon 0<t<\infty$, let us consider the set

$$F_t = \{re^{i\varphi}\colon u(re^{i\varphi})>t\}$$

and

$$\widetilde{u}(re^{i\varphi}) = \sup\{t\colon re^{i\varphi}\in F_t^*\},$$

where F_t^* is the symmetric rearrangement of the set F_t (see [17]).

The function $\widetilde{u}(re^{i\varphi})$ is a non-negative and non-increasing function in the interval $[0,\pi]$ and is equimeasurable with $u(re^{i\varphi})$ in φ for each fixed r. Moreover, $\widetilde{u}(re^{i\varphi})$ satisfies the relations

$$\begin{aligned} \widetilde{u}(r) &= \max_{|z|=r}\log^+|f(z)|, \\ \widetilde{u}(-r) &= \min_{|z|=r}\log^+|f(z)|, \\ T^*(z,u) &= \frac{1}{\pi}\int_0^{\theta}\widetilde{u}(re^{i\varphi})\,d\varphi + N(r,\infty,f), \qquad z=re^{i\theta}. \end{aligned}$$

Let $\alpha(r)$ be the real valued function of a real variable r and let us define

$$L\alpha(r) = \liminf_{h\to 0}\frac{\alpha(re^h)+\alpha(re^{-h})-2\alpha(r)}{h^2}.$$

When $\alpha(r)$ is twice differentiable, then

$$L\alpha(r) = r\frac{r}{dr}\left(r\frac{d}{dr}\alpha(r)\right).$$

In [27] the author proved the following estimate of LT^*.

Theorem C. *For almost all $\theta\in[0,\pi]$ and for all $r>0$ such that the function $f(z)$ has neither zeros nor poles on $\{z\colon |z|=r\}$, we have*

$$LT^*(re^{i\theta},u) \ge -\frac{p^2(r,\infty,f)}{\pi}\frac{\partial\widetilde{u}(re^{i\theta})}{\partial\theta}.$$

If $T^*(z,u)$ is twice differentiable, then Theorem A in this case can be written as

$$\Delta T^*(z,u)\ge 0,$$

$z=x+iy$, $\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ - Laplace operator.

Then

$$LT^*(z,u) = r\frac{r}{dr}\left(r\frac{d}{dr}T^*(re^{i\theta},u)\right) = r^2\Delta T^*(z,u) - \frac{\partial^2}{\partial\theta^2}T^*(re^{i\theta},u)$$
$$\geq -\frac{\partial^2}{\partial\theta^2}T^*(re^{i\theta},u) = -\frac{1}{\pi}\frac{\partial\widetilde{u}(re^{i\theta})}{\partial\theta}.$$

Therefore in the case if $T^*(z,u)$ is twice differentiable then Theorem A is equivalent to

$$LT^*(re^{i\theta},u) \geq -\frac{1}{\pi}\frac{\partial\widetilde{u}(re^{i\theta})}{\partial\theta}. \tag{1}$$

If $\delta(\infty,f) > 0$ then $p(r,\infty,f) \geq 1$ for $r > r_0$. Thus Theorem C can be considered as a generalization of Theorem A, which additionally takes into account the number of separated maximum modulus points. Inequality (1) is an inequality of the Gariepy–Lewis type. In [13] an analogue of the inequality (1) was proved in the case of δ-subharmonic functions in $\mathbb{R}^n$ $(n \geq 3)$.

Let $\mathcal{L}(r,\infty,f) = \max\limits_{|z|=r}\log^+|f(z)|$. The quantity

$$\beta(\infty,f) = \liminf_{r\to\infty}\frac{\mathcal{L}(r,\infty,f)}{T(r,f)}, \quad \beta(a,f) = \beta\left(\infty,\frac{1}{f-a}\right) \quad (a\in\mathbb{C}),$$

is called Petrenko's magnitude of deviation of a meromorphic function $f(z)$ at $a \in \overline{\mathbb{C}}$. It is clear that $\delta(a,f) \leq \beta(a,f)$, $a \in \overline{\mathbb{C}}$. In [27] the author obtained a sharp estimate of $\beta(\infty,f)$ involving $p(\infty,f)$ for meromorphic functions of finite lower order λ.

Theorem D. *For a meromorphic function $f(z)$ of finite lower order λ we have*

$$\beta(\infty,f) \leq \begin{cases} \frac{\pi\lambda}{p(\infty,f)} & \text{if } \frac{\lambda}{p(\infty,f)} \geq \frac{1}{2}, \\ \frac{\pi\lambda}{\sin\pi\lambda} & \text{if } p(\infty,f) = 1 \text{ and } \lambda < \frac{1}{2}, \\ \frac{\pi\lambda}{p(\infty,f)}\sin\frac{\pi\lambda}{p(\infty,f)} & \text{if } p(\infty,f) > 1 \text{ and } \frac{\lambda}{p(\infty,f)} < \frac{1}{2}. \end{cases}$$

It is clear that if $\beta(\infty,f) > 0$ then $p(\infty,f) \geq 1$. Then, by Theorem D we have the following estimate.

Corollary D_1. *For a meromorphic function $f(z)$ of finite lower order λ we have*

$$\beta(\infty,f) \leq \begin{cases} \pi\lambda & \text{if } \lambda \geq \frac{1}{2}, & (2) \\ \frac{\pi\lambda}{\sin\pi\lambda} & \text{if } \lambda < \frac{1}{2}. & (3) \end{cases}$$

The result in Corollary D_1 was obtained by Petrenko in 1969 [32]. In order to prove this estimate Petrenko obtained a new method, which now is called Petrenko's formula. The inequality (3) was obtained by Goldberg and Ostrovskii in 1961 [14]. It should be mentioned here that the conjecture that $\beta(\infty,f) \leq \pi\rho$ for entire functions of order ρ with $\frac{1}{2} \leq \rho < \infty$ was stated in 1932 by Paley and proved in 1969 by Govorov [16].

Corollary D_2. *For a meromorphic function $f(z)$ of the finite lower order λ and $\beta(\infty,f) > 0$ we have*

$$p(\infty,f) \leq \max\left(\left[\frac{\pi\lambda}{\beta(\infty,f)}\right],1\right) < \infty,$$

where $[x]$ means the integral part of the number x.

Corollary $\mathbf{D}_3$. *For an entire function $f(z)$ of finite lower order λ we have*

$$p(\infty, f) \le \max\left([\pi\lambda], 1\right) < \infty.$$

In [27] the author presented a sharp estimate of spread involving $\delta(\infty, f)$, $\beta(\infty, f)$ and $p(\infty, f)$.

Theorem E. *Let $f(z)$ be a meromorphic function of finite lower order λ. Then*

$$\limsup_{r\to\infty} \operatorname{mes}\{\theta \in [0, 2\pi] \colon |f(re^{i\theta})| > 1\} \ge \min\left(2\pi, \frac{4p(\infty, f)}{\lambda} \arcsin\sqrt{\frac{\delta(\infty, f)}{2}}\right).$$

Corollary $\mathbf{E}_1$. *For a meromorphic function $f(z)$ of finite lower order λ*

$$\limsup_{r\to\infty} \operatorname{mes}\{\theta \in [0, 2\pi] \colon |f(re^{i\theta})| > 1\} \ge \min\left(2\pi, \frac{4}{\lambda} \arcsin\sqrt{\frac{\delta(\infty, f)}{2}}\right).$$

The result of Corollary E_1 was obtained by Baernstein in 1972 [2].

Theorem F. *Let $f(z)$ be a meromorphic function of the finite lower order λ. Then*

$$\limsup_{r\to\infty} \operatorname{mes}\{\theta \in [0, 2\pi] \colon |f(re^{i\theta})| > 1\} \ge \min\left(2\pi, \frac{2p(\infty, f)}{\lambda} \arcsin\frac{p(\infty, f)\beta(\infty, f)}{\pi\lambda}\right).$$

Corollary $\mathbf{F}_1$. *Let $f(z)$ be a meromorphic function of the finite lower order λ. Then*

$$\limsup_{r\to\infty} \operatorname{mes}\{\theta \in [0, 2\pi] \colon |f(re^{i\theta})| > 1\} \ge \min\left(2\pi, \frac{2}{\lambda} \arcsin\frac{\beta(\infty, f)}{\pi\lambda}\right).$$

Result of Corollary F_1 was obtained by author in 1982 [26]. Let [5]

$$b(\infty, f) = \liminf_{r\to\infty} \frac{\mathcal{L}(r, \infty, f)}{rT'_{-}(r, f)},$$

where $T'_{-}(r, f)$ is the left-hand derivative of $T(r, f)$ at the point r. In 1998 the author obtained a sharp estimate of $b(\infty, f)$ in terms of $p(\infty, f)$ for meromorphic functions of the infinite lower order.

Theorem G. [28] *Let $f(z)$ be a meromorphic function of the infinite lower order. Then*

$$b(\infty, f) \le \frac{\pi}{p(\infty, f)}.$$

If $b(\infty, f) > 0$ then $p(\infty, f) \ge 1$. Therefore by Theorem G we have the following estimate.

Corollary $\mathbf{G}_1$. *For a meromorphic function $f(z)$ of the infinite lower order*

$$b(\infty, f) \le \pi.$$

The result in Corollary G_1 was obtained in 1994 by Bergweiler and Bock [5].

Corollary $\mathbf{G}_2$. *For a meromorphic function $f(z)$ of the infinite lower order and $b(\infty, f) > 0$ we have*

$$p(\infty, f) \le \left[\frac{\pi}{b(\infty, f)}\right] < \infty.$$

In 2004 the author together with Ciechanowicz [6] introduced the following generalization of the notion of separated maximum modulus points of a meromorphic function. Let $f(z)$ be a meromorphic function in $\mathbb{C}$ and let $\phi(r)$ be a positive nondecreasing convex function of $\log r$ for $r > 0$, such that $\phi(r) = o(T(r, f))$ $(r \to \infty)$. We denote by $\widehat{p}_\phi(r, \infty, f)$ the number of the component intervals of the set

$$\{\theta\colon\ \log|f(re^{i\theta})| > \phi(r)\}$$

possessing at least one maximum modulus point of the function $f(z)$ on the circle $\{z\colon |z| = r\}$. Let

$$\widehat{p}_\phi(\infty, f) = \liminf_{r\to\infty} \widehat{p}_\phi(r, \infty, f),$$
$$\widehat{p}(\infty, f) = \sup_\phi \widehat{p}_\phi(\infty, f).$$

If $\delta(\infty, f) > 0$ or $\beta(\infty, f) > 0$ or $b(\infty, f) > 0$ then $\widehat{p}(\infty, f) \geq p(\infty, f) \geq 1$. For entire functions we have $\delta(\infty, f) = 1$ and $\beta(\infty, f) \geq 1$. Thus for an entire function $f(z)$ we have $\widehat{p}(\infty, f) \geq p(\infty, f) \geq 1$.

In [6] it was possible to obtain a generalization of Theorems D, E and F involving $\widehat{p}(\infty, f)$.

Theorem H. *For a meromorphic function $f(z)$ of the finite lower order λ we have*

$$\beta(\infty, f) \leq \begin{cases} \frac{\pi\lambda}{\widehat{p}(\infty,f)} & \text{if } \frac{\lambda}{\widehat{p}(\infty,f)} \geq \frac{1}{2}, & (4)\\ \frac{\pi\lambda}{\sin\pi\lambda} & \text{if } \widehat{p}(\infty, f) = 1 \text{ and } \lambda < \frac{1}{2}, & (5)\\ \frac{\pi\lambda}{\widehat{p}(\infty,f)} \sin\frac{\pi\lambda}{\widehat{p}(\infty,f)} & \text{if } \widehat{p}(\infty, f) > 1 \text{ and } \frac{\lambda}{\widehat{p}(\infty,f)} < \frac{1}{2}. & (6)\end{cases}$$

Corollary H$_1$. *For a meromorphic function $f(z)$ of the finite lower order λ we have*

$$\widehat{p}(\infty, f) \leq \max\left(\left[\frac{\pi\lambda}{\beta(\infty, f)}\right], 1\right).$$

Corollary H$_2$. *For an entire function $f(z)$ of the finite lower order λ we have*

$$\widehat{p}(\infty, f) \leq \max\left([\pi\lambda], 1\right) < \infty.$$

Let $\Lambda(r)$ be a positive nondecreasing continuous function such that $\Lambda(r) = o(T(r, f))$. We denote

$$\sigma_\Lambda(r, \infty, f) = \mathrm{mes}\{\theta \in [0, 2\pi]\colon\ \log|f(re^{i\theta})| > \Lambda(r)\},\ \omega_\Lambda(\infty, f) = \limsup_{r\to\infty} \sigma_\Lambda(r, \infty, f).$$

The quantity

$$\omega(\infty, f) = \inf_\Lambda \omega_\Lambda(\infty, f)$$

is called the spread of the meromorphic function $f(z)$ introduced firstly by Edrei [9].

Theorem I. [6] *Let $f(z)$ be a meromorphic function $f(z)$ of the finite lower order λ. Then*

$$\omega(\infty, f) \geq \min\left(2\pi, \frac{4\widehat{p}(\infty, f)}{\lambda} \arcsin\sqrt{\frac{\delta(\infty, f)}{2}}\right).$$

Theorem J. [6] *For every meromorphic function $f(z)$ of the finite lower order λ we have*

$$\omega(\infty, f) \geq \min\left(2\pi, \frac{2\widehat{p}(\infty, f)}{\lambda} \arcsin\frac{\beta(\infty, f)\widehat{p}(\infty, f)}{\pi\lambda}\right).$$

In the section 7 we give examples showing that the estimates in the theorems H, I and J are sharp.

2. Main results

Theorem 1. *Let $f(z)$ be a meromorphic function of the finite lower order $\lambda < \frac{\widehat{p}(\infty,f)}{2}$. Then*

$$\limsup_{r\to\infty}\frac{\log^+\mu(r,f)}{T(r,f)}\geq\frac{\frac{\pi\lambda}{\widehat{p}(\infty,f)}}{\sin\frac{\pi\lambda}{\widehat{p}(\infty,f)}}\left(\delta(\infty,f)-1+\cos\frac{\pi\lambda}{\widehat{p}(\infty,f)}\right),$$

where $\mu(r,f)=\min_{|z|=r}|f(z)|$.

Corollary 1.1. *Let $f(z)$ be a meromorphic function of lower order $\lambda<\frac{1}{2}$. Then*

$$\limsup_{r\to\infty}\frac{\log^+\mu(r,f)}{T(r,f)}\geq\frac{\pi\lambda}{\sin\pi\lambda}(\delta(\infty,f)-1+\cos\pi\lambda),$$

where $\mu(r,f)=\min_{|z|=r}|f(z)|$.

Notice that Corollary 1.1 was obtained earlier by Goldberg and Ostrovskii [15, 31].

Corollary 1.2. *Suppose that $f(z)$ is a meromorphic function of finite lower order $\lambda<\frac{\widehat{p}(\infty,f)}{2}$ and $\delta(\infty,f)>1-\cos\frac{\pi\lambda}{\widehat{p}(\infty,f)}$. Then there exists a sequence of circles $\{z\colon |z|=r_k\}$, $r_k\to\infty$, on which $f(z)$ tends to ∞ uniformly with respect to $\arg z$.*

Corollary 1.3. *Suppose that $f(z)$ is a meromorphic function of finite lower order $\lambda<\frac{1}{2}$ and $\delta(\infty,f)>1-\cos\pi\lambda$. Then there is a sequence $r_n\to\infty$, such that $f(r_ne^{i\theta})$ tends uniformly to ∞ for $\theta\in[0,2\pi]$.*

It should be mentioned that the result in Corollary 1.3 was obtained earlier by Goldberg and Ostrovskii ([31], see also [15]) and Edrei [8]. It is necessary to admit that in 1939 Teichmüller [35] proved that for the meromorphic function $f(z)$ of the order $\rho<\frac{1}{2}$ such that $\delta(\infty,f)>1-\cos\pi\rho$ it holds for all $\theta\in[0,2\pi]$

$$\limsup_{r\to\infty}|f(re^{i\theta})|=\infty.$$

Therefore Teichmüller get the result of Corollary 1.3 in the case of $\delta(\infty,f)>\frac{1-\cos\pi\rho}{1-\epsilon\cos\pi\rho}$, where $\epsilon>0$, $0<\epsilon<1$.

Corollary 1.4. *Let $f(z)$ be an entire function of lower order $\lambda<\frac{\widehat{p}(\infty,f)}{2}$. Then there exists a sequence of circles $\{z\colon |z|=r_k\}$, $r_k\to\infty$, on which $f(z)$ tends to ∞ uniformly with respect to $\arg z$.*

Corollary 1.5. *Let $f(z)$ be an entire function of lower order $\lambda<\frac{1}{2}$. Then there exists a sequence of circles $\{z\colon |z|=r_k\}$, $r_k\to\infty$, on which $f(z)$ tends to ∞ uniformly with respect to $\arg z$.*

The result in Corollary 1.5 was obtained by Heins [20] in 1948 and in case when $f(z)$ is an entire function of order $\rho<\frac{1}{2}$ by Wiman [37] in 1905.

Corollary 1.6. *If $f(z)$ is an entire function of lower order $\lambda<\frac{\widehat{p}(\infty,f)}{2}$ then for all $a\in\mathbb{C}$ we have $\delta(a,f)=0$, i.e. $f(z)$ does not have finite defective values.*

Theorem 2. *Let $f(z)$ be a meromorphic function of lower order $\lambda < \frac{\widehat{p}(\infty,f)}{2}$. Then*

$$\limsup_{r\to\infty} \frac{\log^+ \mu(r,f)}{T(r,f)} \geq \frac{1}{\cos\frac{\pi\lambda}{\widehat{p}(\infty,f)}} \left(\beta(\infty,f) - \frac{\pi\lambda}{\widehat{p}(\infty,f)} \sin\frac{\pi\lambda}{\widehat{p}(\infty,f)}\right),$$

where $\mu(r,f) = \min_{|z|=r} |f(z)|$.

Corollary 2.1. *Let $f(z)$ be a meromorphic function of lower order $\lambda < \frac{1}{2}$. Then*

$$\limsup_{r\to\infty} \frac{\log^+ \mu(r,f)}{T(r,f)} \geq \frac{1}{\cos\pi\lambda}(\beta(\infty,f) - \pi\lambda\sin\pi\lambda), \tag{2.1}$$

where $\mu(r,f) = \min_{|z|=r} |f(z)|$.

It should be mentioned that the result stated in Corollary 2.1 was obtained by the author in [27].

Corollary 2.2. *Suppose that $f(z)$ is a meromorphic function of finite lower order $\lambda < \frac{\widehat{p}(\infty,f)}{2}$ and $\beta(\infty,f) > \frac{\pi\lambda}{\widehat{p}(\infty,f)} \sin\frac{\pi\lambda}{\widehat{p}(\infty,f)}$. Then there exists a sequence of circles $\{z\colon |z| = r_k\}$, $r_k \to \infty$, on which $f(z)$ tends to ∞ uniformly with respect to* $\arg z$.

Corollary 2.3. *Suppose that $f(z)$ is a meromorphic function of the finite lower order $\lambda < \frac{1}{2}$ and $\beta(a,f) > \pi\lambda\sin\pi\lambda$. Then $\beta(b,f) = 0$ for all $b \neq a$.*

Notice that the result in Corollary 2.3 was obtained earlier by Petrenko [33].

Theorem 3. *Let $f(z)$ be a meromorphic function of infinite lower order. Then*

$$b(\infty,f) \leq \frac{\pi}{\widehat{p}(\infty,f)}.$$

Corollary 3.1. *For a meromorphic function of infinite lower order and $b(\infty,f) > 0$ we have*

$$\widehat{p}(\infty,f) \leq \left[\frac{\pi}{b(\infty,f)}\right] < \infty.$$

The method of the T^*-function turns out to be very effective also in the study of other structures. In 2016 the author together with Kowalski [22] (see also [24]) introduced and investigated the notion of separated maximum points of the norm of meromorphic minimal surfaces. In 2019 we introduced the term of the separated maximum points of the entire curves [23] (see also [38]).

I would like to mention about a method of Baernstein's T^*-function. Petrenko's method is right if the extremal function has only the one maximum modulus point on the circle $\{z\colon |z| = r\}$. In the case of the single deviation $\beta(\infty,f)$ a Mittag-Leffler's function is the extremal function and has one maximum modulus point. Petrenko considered the representation of the meromorphic function on the sector in the neighborhood of the maximum modulus point (Petrenko's formula) and he get the sharp estimation of the deviation $\beta(a,f)$. In the case of the sum of deviations the extremal function is a Frithiof Nevanlinna's function (see [15], p. 317), which has n maximum modulus points ($n = 2\lambda$). In this case the Petrenko's method is not sharp. The Baernstein's T^*-function concerns all of the intervals in the neighborhood of the all maximum modulus points. Therefore by means of the T^*-function author and Shcherba get the sharp estimation of $\sum_{a\in\overline{\mathbb{C}}} \beta(a,f)$ (Petrenko's problem).

In 2021 there will be the 80th birth anniversary of Albert Baernstein II (1941–2014). I was acquainted with him personally and I have been frequently applying his brilliant function (star-function) while solving my research problems.

3. Auxiliary results

Let again $f(z)$ be a meromorphic function and $\phi(r)$ be a positive non-decreasing convex function of $\log r$ such that $\phi(r) = o(T(r, f))$, $(r \to \infty)$. We first consider the function

$$u_\phi(z) = \max(\log|f(z)|, \phi(|z|)).$$

Lemma 1. [6] *The function $u_\phi(z)$ is δ-subharmonic function in $\mathbb{C}$.*

Proof. Let $f(z) = \frac{g_1(z)}{g_2(z)}$, where $g_1(z)$ and $g_2(z)$ are entire functions without common zeros. Then it is easy to see that

$$u_\phi(z) = \max(\log|g_1(z)|, \log|g_2(z)| + \phi(|z|)) - \log|g_2(z)|.$$

The function $\phi(r)$ is a convex function of $\log r$ for $r > 0$. Therefore $\phi(|z|)$ is a subharmonic function in $\mathbb{C}$ [34]. Also

$$v_1(z) := \max(\log|g_1(z)|, \log|g_2(z)| + \phi(|z|))$$

is a subharmonic function in $\mathbb{C}$. Thus

$$u_\phi(z) = v_1(z) - \log|g_2(z)| := v_1(z) - v_2(z)$$

is a δ-subharmonic function in $\mathbb{C}$. □

Let

$$m^*(re^{i\theta}, u_\phi) = m^*(r, \theta, u_\phi) = \sup_{|E|=2\theta} \frac{1}{2\pi} \int_E u_\phi(re^{i\varphi})\, d\varphi,$$

$$T^*(re^{i\theta}, u_\phi) = T^*(r, \theta, u_\phi) = m^*(r, \theta, u_\phi) + N(r, \infty, f),$$

where $r \in (0, \infty)$, $\theta \in [0, \pi]$, E is a measurable set and $|E|$ is the Lebesgue measure of E. Now for each $t \in (0, +\infty)$, consider the set

$$F_t = \{re^{i\varphi} \colon u_\phi(re^{i\varphi}) > t\},$$

and let

$$\widetilde{u}_\phi(re^{i\varphi}) = \sup\{t \colon re^{i\varphi} \in F_t^*\},$$

where F_t^* is the symmetric rearrangement of the set F_t [17].

The function $\widetilde{u}_\phi(re^{i\varphi})$ is non-negative and non-increasing in the interval $[0, \pi]$, even with respect to ϕ and for each fixed r equimeasurable with $u_\phi(re^{i\varphi})$. Moreover, it satisfies the equalities:

$$\widetilde{u}_\phi(r) = \max(\log \max_{|z|=r} |f(z)|, \phi(r)),$$

$$\widetilde{u}_\phi(re^{i\pi}) = \max(\log \min_{|z|=r} |f(z)|, \phi(r)),$$

$$m^*(r, \theta, u_\phi) = \sup_{|E|=2\theta} \frac{1}{2\pi} \int_E u_\phi(z)\, dz = \frac{1}{\pi} \int_0^\theta \widetilde{u}_\phi(re^{i\varphi})\, d\varphi,$$

where $\widetilde{u}(r, \varphi) = \widetilde{u}(re^{i\varphi})$.

From Theorem B the function $T^*(r,\theta,u_\phi)$ is subharmonic in $D=\{re^{i\theta}\colon 0<r<\infty,\ 0<\theta<\pi\}$, continuous in $D\cup(-\infty,0)\cup(0,\infty)$ and logarithmically convex in $r>0$ for each fixed $\theta\in[0,\pi]$. Moreover,

$$T^*(r,0,u_\phi)=N(r,\infty,f),$$
$$T^*(r,\pi,u_\phi)=T(r,f)+o(T(r,f))\quad (r\to\infty),$$
$$\frac{\partial}{\partial\theta}T^*(r,\theta,u_\phi)=\frac{\widetilde{u}_\phi(re^{i\theta})}{\pi}\quad \text{for } 0<\theta<\pi.$$

Lemma 2. [6] *For almost all $\theta\in[0,\pi]$ and for all $r>0$ such that the function $f(z)$ has neither a zeros nor poles on the circle $\{z\colon |z|=r\}$ we have*

$$LT^*(re^{i\theta},u_\phi)\geq -\frac{\widehat{p}^2_\phi(r,\infty,f)}{\pi}\frac{\partial\widetilde{u}_\phi(re^{i\theta})}{\partial\theta}.$$

Lemma 2 is a generalization of Theorem C. In [6] was given sketch of the proof of Lemma 2. In this paper we give a complete proof of Lemma 2.

Proof. Let us assume that r_0 is a number satisfying the hypothesis. Since $\widetilde{u}_\phi(r_0,\theta)=\widetilde{u}_\phi(r_0e^{i\theta})$ is a non-increasing function of θ, the derivative $\frac{\partial\widetilde{u}_\phi(r_0,\theta)}{\partial\theta}$ exists for almost all $\theta\in(0,\pi)$. Let us choose $\theta\in(0,\pi)$ such that $\frac{\partial\widetilde{u}_\phi(r_0,\theta)}{\partial\theta}$ exists. If $\widetilde{u}_\phi(r_0,\theta)=\phi(r_0)$, then $\widetilde{u}_\phi(r_0,x)=\phi(r_0)$ for all $x\in[\theta,\pi)$ and so $\frac{\partial\widetilde{u}_\phi(r_0,\theta)}{\partial\theta}=0$. As $T^*(r,\theta,u_\phi)=T^*(re^{i\theta},u_\phi)$ is a convex function of $\log r$, we see that $LT^*(r,\theta,u_\phi)\geq 0$ for $r>0$. Therefore the Lemma 2 is proved in the case when $\frac{\partial\widetilde{u}_\phi(r_0,\theta)}{\partial\theta}=0$ or when $\widetilde{u}_\phi(r_0,\theta)=\phi(r_0)$.

Let us assume now that $\frac{\partial\widetilde{u}_\phi(r_0,\theta)}{\partial\theta}<0$ and $\widetilde{u}_\phi(r_0,\theta)>\phi(r_0)$. By [4] there exists a set $E(r_0,\theta)$, such that

$$\{\varphi\in[0,2\pi]\colon u_\phi(r_0,\varphi)>\widetilde{u}_\phi(r_0,\theta)\}\subset E(r_0,\theta)\subset\{\varphi\in[0,2\pi]\colon u_\phi(r_0,\varphi)\geq\widetilde{u}_\phi(r_0,\theta)\},$$

$$m^*(r_0,\theta,u_\phi)=\frac{1}{2\pi}\int_{E(r_0,\theta)}u_\phi(r_0,\varphi)\,d\varphi. \tag{3.1}$$

We will prove that the set $\{\varphi\in[0,2\pi]\colon u_\phi(r_0,\varphi)=\widetilde{u}_\phi(r_0,\theta)\}$ is finite. Let us assume that the set $\{\varphi\in[0,2\pi]\colon u_\phi(r_0,\varphi)=\widetilde{u}_\phi(r_0,\theta)\}$ is not finite. Thus there is a sequence $\{\varphi_n\}_{n=1}^\infty$, such that $\varphi_n\neq\varphi_k$ $(n\neq k)$, $\varphi_n\to\varphi_0$ $(n\to\infty)$ and $u_\phi(r_0,\varphi_n)=\widetilde{u}_\phi(r_0,\theta)$ for all $n\in\mathbb{N}$. Since $\widetilde{u}_\phi(r_0,\theta)>\phi(r_0)$ then $\log|f(r_0e^{i\varphi_n})|=\widetilde{u}_\phi(r_0,\theta)$. There is neither zeros nor poles of the function $f(z)$ on the circle $\{z\colon |z|=r_0\}$. Hence the function $F(\varphi)=\log|f(r_0e^{i\varphi})|$ is analytic for all $\varphi\in[0,2\pi]$. Moreover $F(\varphi_n)=\widetilde{u}_\phi(r_0,\theta)$ for all $n\in\mathbb{N}$ and $\varphi_n\neq\varphi_k$ $(n\neq k)$, $\varphi_n\to\varphi_0$ $(n\to\infty)$. Thus by uniqueness theorem we have $F(\varphi)=\widetilde{u}_\phi(r_0,\theta)$ for $\varphi\in[0,2\pi]$. Hence $u_\phi(r_0,\varphi)=\widetilde{u}_\phi(r_0,\theta)$ for $\varphi\in[0,2\pi]$. Thus $\widetilde{u}_\phi(r_0,\varphi)=\widetilde{u}(r_0,\theta)$ for $\varphi\in[0,2\pi]$ and $\frac{\partial\widetilde{u}_\phi(r_0,\theta)}{\partial\varphi}=0$, which is a contradiction. Hence the set $\{\varphi\in[0,2\pi]\colon u_\phi(r_0,\varphi)=\widetilde{u}_\phi(r_0,\theta)\}$ is finite.

Let $E_1(r_0,\theta)=\{\varphi\in[0,2\pi]\colon u_\phi(r_0,\varphi)>\widetilde{u}_\phi(r_0,\theta)\}$. By (3.1) we have

$$m^*(r_0,\theta,u_\phi)=\frac{1}{2\pi}\int_{E_1(r_0,\theta)}u_\phi(r_0,\varphi)\,d\varphi.$$

Let us now consider for $r>0$ the function [13]

$$\Psi(r)=\frac{1}{2\pi}\int_{E_1(r_0,\theta)}u_\phi(r,\varphi)\,d\varphi.$$

We have $m^*(r_0,\theta,u_\phi)=\Psi(r_0)$ and $m^*(r,\theta,u_\phi)\ge\Psi(r)$ for all $r>0$. Hence

$$Lm^*(r_0,\theta,u_\phi)\ge L\Psi(r_0). \tag{3.2}$$

Since the set $E_1(r_0,\theta)\subset[0,2\pi]$ is open it implies that $E_1(r_0,\theta)=\bigcup_k(\alpha_k,\beta_k)$. In the points α_k and β_k we have $F(\alpha_k)=F(\beta_k)=\widetilde{u}_\phi(r_0,\theta)$ and $F(\varphi)=\log|f(r_0e^{i\varphi})|$. It follows again from the uniqueness theorem that the family of intervals $\{(\alpha_k,\beta_k)\}$ is finite. Let $m=m(r_0)$ denote the number of those intervals, i.e.

$$E_1(r_0,\theta)=\{\varphi\in[0,2\pi]\colon u_\phi(r_0,\varphi)>\widetilde{u}_\phi(r_0,\theta)\}=\bigcup_{k=1}^{m}(\alpha_k,\beta_k),\quad m=m(r_0)<\infty.$$

The function $\log|f(z)|$ is harmonic on a certain neighborhood of the circle $\{z\colon|z|=r_0\}$ as $f(z)$ has neither zeros nor poles on this circle. Therefore

$$\begin{aligned}L\Psi(r_0)&=\frac{1}{2\pi}\sum_{k=1}^{m}\int_{\alpha_k}^{\beta_k}r_0\frac{d}{dr}r\frac{d}{dr}u_\phi(re^{i\varphi})\bigg|_{r=r_0}d\varphi\\&=\frac{1}{2\pi}\sum_{k=1}^{m}\int_{\alpha_k}^{\beta_k}r_0\frac{d}{dr}r\frac{d}{dr}\log|f(re^{i\varphi})|\bigg|_{r=r_0}d\varphi\\&=\frac{1}{2\pi}\sum_{k=1}^{m}\int_{\alpha_k}^{\beta_k}\left(-\frac{\partial^2\log|f(r_0e^{i\varphi})|}{\partial\varphi^2}\right)d\varphi\\&=-\frac{1}{2\pi}\sum_{k=1}^{m}\int_{\alpha_k}^{\beta_k}\frac{\partial^2u_\phi(r_0,\varphi)}{\partial\varphi^2}\,d\varphi=-\frac{1}{2\pi}\sum_{k=1}^{m}\left[\frac{\partial u_\phi(r_0,\varphi)}{\partial\varphi}\right]\bigg|_{\alpha_k}^{\beta_k}\end{aligned}\tag{3.3}$$

It is easy to see that there are neighborhoods of the points $\alpha_k,\ \beta_k,\ (k=1,\dots,m)$ where the function $F(\varphi)=\log|f(r_0e^{i\varphi})|$ is strictly increasing and strictly decreasing, respectively. For if not, then at least one of the numbers $\{\alpha_k,\beta_k\}$ is such that there exists a sequence $\{\varphi_n\}_{n=1}^{\infty}$ tending to this number and for which $F'(\varphi_n)=0$ for $n\in\mathbb{N}$. But the function $F'(\varphi)$ is analytic in $\varphi\in[0,2\pi]$ and hence $f'(\varphi)=0$ for all $\varphi\in[0,2\pi]$. Hence for all $\varphi\in[0,2\pi]$ we have $F(\varphi)=\log|f(r_0e^{i\varphi})|=\widetilde{u}_\phi(r_0,\theta)$. However, we have assumed that $\frac{\partial\widetilde{u}(r_0,\theta)}{\partial\theta}=0$, which is a contradiction. Thus, there exist neighborhoods of the points $\alpha_k,\ \beta_k,\ (k=1,\dots,m)$ in which the function $\log|f(r_0e^{i\varphi})|$ is strictly monotonic. Since $\widetilde{u}_\phi(r_0,\theta)>\phi(r_0)$, there exist neighborhoods of the points $\alpha_k,\ \beta_k$ where $u_\phi(r_0,\varphi)=\log|f(r_0e^{i\varphi})|$. Consequently, $u_\phi(r_0,\varphi)$ is strictly monotonic in a neighborhood of each point $\alpha_k,\ \beta_k\ (k=1,\dots,m)$.

We claim that $\frac{\partial u_\phi(r_0,\alpha_k)}{\partial\varphi}>0,\ \frac{\partial u_\phi(r_0,\beta_k)}{\partial\varphi}<0$ for all $k=1,\dots,m$. Let us choose $h>0$, such that $u_\phi(r_0,\varphi)$ is strictly increasing in the h-neighborhood of the point α_k. Then we have

$$\operatorname{mes}\{\varphi\in[0,2\pi]\colon u_\phi(r_0,\varphi)\ge u_\phi(r_0,\alpha_k+h)\}\le2\theta-h,$$

where mes is the Lebesgue measure. In view of the properties of the function $\widetilde{u}_\phi(r,\varphi)$, we have

$$\begin{aligned}\operatorname{mes}\left\{\varphi\colon u_\phi(r_0,\varphi)\ge\widetilde{u}_\phi\left(r_0,\theta-\frac{h}{2}\right)\right\}&=\operatorname{mes}\left\{\varphi\colon\widetilde{u}_\phi(r_0,\varphi)\ge\widetilde{u}_\phi\left(r_0,\theta-\frac{h}{2}\right)\right\}\\&=2\theta-h.\end{aligned}$$

Thus

$$u_\phi(r_0,\alpha_k+h)\ge\widetilde{u}_\phi\left(r_0,\theta-\frac{h}{2}\right).$$

Hence, since $u_\phi(r_0,\alpha_k)=\widetilde{u}_\phi(r_0,\theta)$,

$$\frac{\partial u_\phi(r_0,\alpha_k)}{\partial\varphi}\geq -\frac{1}{2}\frac{\partial\widetilde{u}_\phi(r_0,\theta)}{\partial\theta}>0,\quad (k=1,\dots,m).$$

Similarly,

$$\frac{\partial u_\phi(r_0,\beta_k)}{\partial\varphi}<0\quad (k=1,\dots,m).$$

For brevity of notation, in place of $u_\phi(r_0,\varphi)$ and $\widetilde{u}_\phi(r_0,\varphi)$ we shall write $u(\varphi)$ and $\widetilde{u}(\varphi)$, respectively. Thus, $u'(\alpha_k),u'(\beta_k)<0$ $(k=1,\dots,m)$. Let h_0 be a positive number such that for all $\varphi\in[-h_0,h_0]$, we have $\alpha_k+h_0<\beta_k-h_0$ and $u'(\alpha_k+\varphi)>0$, $u'(\alpha_k+\varphi)<0$ $(k=1,\dots,m)$. Further, denote by η_k the smallest value of the function $u(\varphi)$ in the interval $[\alpha_k+h_0,\beta_k-h_0]$. Set $\eta=\min_{1\leq k\leq m}\eta_k$. Clearly $u(\alpha_1+h_0)\geq\eta>u(\alpha_1)=\widetilde{u}(\theta)$. Choose a number h_1 such that $0<h_1\leq h_0$ and $u(\alpha_1+h_1)=\eta$. By the choice of h_1, the equations $u(\beta_k-x)=u(\alpha_1+h)$, $u(\alpha_k+y)=u(\alpha_1+h)$ have unique solutions for all $0<h<h_1$, which we denote respectively by x_k, y_k. Thus $x_k=x_k(h)$, $y_k=y_k(h)$ for $h\colon 0<h<h_1$.

From the continuity of the function $u(\varphi)$ and since $u(\beta_k)=u(\alpha_k)=\widetilde{u}(\theta)$ $(k=1,\dots,m)$, it follows that as $h\to+0$ the functions $x_k=x_k(h)$, $y_k=y_k(h)$ tends to zero. By the differentiability of the function $u(\varphi)$, we have

$$u(\beta_k)-u'(\beta_k)x_k+o(x_k)=u(\alpha_1)+u'(\alpha_1)h+o(h)\quad (h\to+0).$$

Hence

$$x_k=-\frac{u'(\alpha_1)}{u'(\beta_k)}h+o(h)\quad (h\to+0,\ k=1,\dots,m).$$

Similarly

$$y_k=\frac{u'(\alpha_1)}{u'(\alpha_k)}h+o(h)\quad (h\to+0,\ k=1,\dots,m).$$

From the choice of x_k, y_k it follows that

$$\begin{aligned}\operatorname{mes}\{\varphi\in[0,2\pi]\colon u(\varphi)\geq u(\alpha_1+h)\}&=2\theta-\sum_{k=1}^m(x_k+y_k)\\&=2\theta-\sum_{k=1}^m\left(\frac{u'(\alpha_1)}{u'(\alpha_k)}-\frac{u'(\alpha_1)}{u'(\beta_k)}\right)h+o(h)\\&:=2\theta-A(h).\end{aligned}$$

However, $\operatorname{mes}\{\varphi\in[0,2\pi]\colon \widetilde{u}(\varphi)\geq\widetilde{u}(\theta-\frac{1}{2}A(h))\}=2\theta-A(h)$. Thus

$$\widetilde{u}\left(\theta-\frac{1}{2}A(h)\right)=u(\alpha_1+h).$$

The function $\widetilde{u}(\varphi)$ is differentiable at the point θ, therefore

$$\widetilde{u}(\theta)-\frac{1}{2}\widetilde{u}'(\theta)A(h)+o(A(h))=u(\alpha_1)+u'(\alpha_1)h+o(h)\quad (h\to+0).$$

Hence, recalling that $u(\alpha_1)=\widetilde{u}(\theta)$, we obtain

$$-\frac{1}{2}\widetilde{u}'(\theta)\sum_{k=1}^m\left(\frac{1}{u'(\alpha_k)}-\frac{1}{u'(\beta_k)}\right)u'(\alpha_1)h=u'(\alpha_1)h+o(h)\quad (h\to+0).$$

Since $u'(\alpha_1) > 0$, it follows immediately from the previous formula that

$$-\frac{1}{2}\widetilde{u}'(\theta)\sum_{k=1}^{m}\left(\frac{1}{u'(\alpha_k)}-\frac{1}{u'(\beta_k)}\right)=1.$$

Multiplying this equality by $\sum_{j=1}^{m}(u'(\alpha_j)-u'(\beta_j))$, we have

$$\sum_{j=1}^{m}(u'(\alpha_j)-u'(\beta_j)) = -\frac{1}{2}\widetilde{u}'(\theta)\sum_{k,j=1}^{m}(u'(\alpha_j)-u'(\beta_j))\left(\frac{1}{u'(\alpha_k)}-\frac{1}{u'(\beta_k)}\right).$$

Next, we shall use a simple inequality, which can easily be proved by induction: for positive numbers $a_k > 0$, $b_k > 0$ $(k = 1,\dots,m)$ we have the inequality

$$\sum_{k,j=1}^{m}(a_j+b_j)\left(\frac{1}{a_k}+\frac{1}{b_k}\right)\geq 4m^2.$$

It follows from this and the above inequality that

$$\sum_{j=1}^{m}(u'(\alpha_j)-u'(\beta_j))\geq -2m^2\widetilde{u}'(\theta). \tag{3.4}$$

In view (3.2), (3.3) and (3.4) we have

$$Lm^*(r_0,\theta,u_\phi)\geq -\frac{m^2}{\pi}\widetilde{u}'(\theta).$$

Clearly, $m \geq \widehat{p}(r_0,\infty,f)$ and $LT^*(r_0,\theta,u_\phi)\geq Lm^*(r_0,\theta,u_\phi)$. The assertion of Lemma 2 now follows from these inequalities. □

Lemma 3. [27] *Let the function $f(x)$ be non decreasing on the interval $[a,b]$ and let $\varphi(x)$ be a non negative function having a bounded derivative of the interval $[a,b]$. Then*

$$\int_a^b f'(x)\varphi(x)\,dx \leq f(b)\varphi(b)-f(a)\varphi(a)-\int_a^b \varphi'(x)f(x)\,dx.$$

Lemma 4. [33] *Let $f(z)$ be a meromorphic function of lower order λ. Then for each $\epsilon > 0$ there exist sequences S_k, R_k tending to infinity such that $\lim_{k\to\infty}\frac{S_k}{R_k}=0$ and for $k\geq k_0(\epsilon)$*

$$\frac{T(2R_k,f)}{R_k^\lambda}+\frac{T(2S_k,f)}{S_k^\lambda}<\epsilon\int_{2S_k}^{R_k}\frac{T(r,f)}{r^{\lambda+1}}\,dr.$$

Bergweiler and Bock in [5] introduced a generalization of Pólya peaks to functions of infinite lower order. Let's remind the basic facts of this construction.

For all sequences $M_j\to\infty$, $\epsilon_j\to 0$ there exist sequences $\rho_j\to\infty$ and $\mu_j\to\infty$ such that, for all r's fulfilling the inequality $|\log\frac{r}{\rho_j}|\leq\frac{M_j}{\mu_j}$, we have

$$T(r,f)\leq(1+\epsilon_j)\left(\frac{r}{\rho_j}\right)^{\mu_j}T(\rho_j,f). \tag{3.5}$$

We can choose the sequences μ_j and M_j such that

$$\mu_j = o(\log^{\frac{3}{2}}T(\rho_j,f)),\quad M_j = o(\log T(\rho_j,f)),\quad j\to\infty.$$

Let's put

$$P_j=\rho_j e^{-\frac{M_j}{\mu_j}},\quad Q_j=\rho_j e^{\frac{M_j}{\mu_j}}.$$

Then the inequality (3.5) is true for all $r \in [P_j, Q_j]$. We shall assume that $M_j > 1$.

Let's consider the sets

$$A_j = \left\{ r \in [\rho_j, Q_j] : T(r, f) \le \frac{1}{\sqrt{\mu_j}} \left(\frac{r}{\rho_j} \right)^{\mu_j} T(\rho_j, f) \right\},$$

$$B_j = \left\{ r \in [P_j, \rho_j] : T(r, f) \le \frac{1}{\sqrt{\mu_j}} \left(\frac{r}{\rho_j} \right)^{\mu_j} T(\rho_j, f) \right\}.$$

Let's put

$$R_j = \begin{cases} \min A_j, & \text{if } A_j \ne \emptyset, \\ Q_j, & \text{if } A_j = \emptyset, \end{cases} \quad t_j = \begin{cases} \max B_j, & \text{if } B_j \ne \emptyset, \\ P_j, & \text{if } B_j = \emptyset, \end{cases}$$

$$S_j = e^{-\frac{1}{\mu_j}} R_j, \quad T_j = e^{-\frac{2}{\mu_j}} R_j.$$

Then

$$t_j < \rho_j < T_j < S_j < R_j$$

In [5] it is shown that

$$\frac{T(R_j, f)}{R_j^{\mu_j}} + \frac{T(t_j, f)}{t_j^{\mu_j}} = o\left(\mu_j \int_{t_j}^{T_j} \frac{T(r, f)}{r^{\mu_j + 1}} \, dr \right), \quad j \to \infty. \tag{3.6}$$

4. Proof of Theorem 1

If $\widehat{p}(\infty, f) = +\infty$ then by Theorem H we have $\beta(\infty, f) = 0$, Thus $\delta(\infty, f) = 0$, so the right side of inequality in the statement of the Theorem 1 is equal to zero and left side is non-negative.

Let now $\widehat{p}(\infty, f) < \infty$. If $\delta(\infty, f) \le 1 - \cos \frac{\pi\lambda}{\widehat{p}(\infty, f)}$ then the Theorem 1 is obviously. Let $\delta(\infty, f) > 1 - \cos \frac{\pi\lambda}{\widehat{p}(\infty, f)} > 0$. Then $\delta(\infty, f) > 0$ and for every $\phi(r)$ we have $\widehat{p}_\phi(\infty, f) \ge 1$. Let us consider the case $\lambda > 0$. Now we choose the number α and ψ satisfying the inequalities

$$0 < \alpha \le \min \left(\pi, \frac{\pi \widehat{p}_\phi(\infty, f)}{2\lambda} \right),$$

$$-\frac{\pi \widehat{p}_\phi(\infty, f)}{2\lambda} \le \psi \le \frac{\pi \widehat{p}_\phi(\infty, f)}{2\lambda} - \alpha.$$

We put [10, 13, 27]

$$\sigma(r) = \int_0^\alpha T^*(r, \varphi, u_\phi) \cos \frac{\lambda}{\widehat{p}_\phi(\infty, f)} (\varphi + \psi) \, d\varphi,$$

where $T^*(r, \varphi, u_\phi) = T^*(re^{i\varphi}, u_\phi)$.

Since $T^*(re^{i\varphi}, u_\phi)$ is a convex function of $\log r$, it follows that for all $r > 0$ and $h > 0$ we have

$$T^*(re^h, \varphi, u_\phi) + T^*(re^{-h}, \varphi, u_\phi) - 2T^*(r, \varphi, u_\phi) \ge 0.$$

Thus by Fatou's lemma for all $r > 0$ we have

$$L\sigma(r) \ge \int_0^\alpha LT^*(r, \theta, u_\phi) \cos \frac{\lambda}{\widehat{p}_\phi(\infty, f)} (\theta + \psi) \, d\theta \ge 0. \tag{4.1}$$

It follows from this inequality that $\sigma(r)$ is a convex function of $\log r$, and so $r\sigma_-'(r)$ is an increasing function on $(0,\infty)$. Therefore, for almost all $r>0$

$$L\sigma(r) = r\frac{d}{dr}r\sigma_-'(r).$$

It follows from (4.1)and Lemma 2 that for almost all $r>0$

$$r\frac{d}{dr}r\sigma_-'(r) \geq -\int_0^\alpha \frac{\widehat{p}_\phi^2(r,\infty,f)}{\pi}\frac{\partial\widetilde{u}_\phi(r,\theta)}{\partial\theta}\cos\frac{\lambda(\theta+\psi)}{\widehat{p}_\phi(r,\infty,f)}\,d\theta. \tag{4.2}$$

By definition $\widehat{p}_\phi(r,\infty,f)$ takes only the integral values. Thus for $r\geq r_0$ we have $\widehat{p}_\phi(\infty,f)\leq\widehat{p}_\phi(r,\infty,f)$. From this and (4.2) it follows that for almost all $r\geq r_0$

$$r\frac{d}{dr}r\sigma_-'(r) \geq -\int_0^\alpha \frac{\widehat{p}_\phi^2(\infty,f)}{\pi}\frac{\partial\widetilde{u}_\phi(r,\theta)}{\partial\theta}\cos\frac{\lambda(\theta+\psi)}{\widehat{p}_\phi(\infty,f)}\,d\theta. \tag{4.3}$$

If there are neither zeros nor poles of $f(z)$ on the circle $\{z\colon |z|=r\}$ for $r>0$, the function $u_\phi(r,\theta)=\max(\log|f(re^{i\theta})|,\phi(r))$ fulfills the Lipschitz condition in θ. Therefore $\widetilde{u}_\phi(r,\theta)$ also fulfills the Lipschitz condition on $[0,\pi]$ [17]. It implies that the function $\widetilde{u}_\phi(r,\theta)$ is absolutely continuous on $[0,\pi]$. Integrating twice by parts, we have for almost all $r\geq r_0$

$$\begin{aligned}
r\frac{d}{dr}r\sigma_-'(r) \geq &-\frac{\widehat{p}_\phi^2(\infty,f)}{\pi}\widetilde{u}_\phi(r,\alpha)\cos\frac{\lambda}{\widehat{p}_\phi(\infty,f)}(\alpha+\psi)\\
&+\frac{\widehat{p}_\phi^2(\infty,f)}{\pi}\left(\max\left(\log\max_{|z|=r}|f(z)|,\phi(r)\right)\right)\cos\frac{\lambda\psi}{\widehat{p}_\phi(\infty,f)}\\
&-\frac{\pi\lambda}{\widehat{p}_\phi(\infty,f)}T^*(r,\alpha,u_\phi)\sin\frac{\lambda(\alpha+\psi)}{\widehat{p}_\phi(\infty,f)}\\
&+\lambda\widehat{p}_\phi(\infty,f)N(r,\infty,f)\sin\frac{\lambda\psi}{\widehat{p}_\phi(\infty,f)}+\lambda^2\sigma(r) := h(r)+\lambda^2\sigma(r).
\end{aligned} \tag{4.4}$$

Dividing both sides of (4.4)by $r^{\lambda+1}$ and integrating by parts over the interval $[2S_k,R_k]$, where S_k,R_k are the sequences described in Lemma 4 we have

$$\int_{2S_k}^{R_k}\frac{h(r)}{r^{\lambda+1}}\,dr+\lambda^2\int_{2S_k}^{R_k}\frac{\sigma(r)}{r^{\lambda+1}}\,dr \leq \int_{2S_k}^{R_k}\frac{1}{r^\lambda}\frac{d}{dr}r\sigma_-'(r)\,dr = I. \tag{4.5}$$

Invoking Lemma 3 we get

$$I \leq \frac{\sigma_-'(r)}{r^{\lambda+1}}\Big|_{2S_k}^{R_k}+\lambda\int_{2S_k}^{R_k}\frac{\sigma_-'(r)}{r^\lambda}\,dr. \tag{4.6}$$

The function $\sigma(r)$ is a convex function of $\log r$ on the interval $(0,+\infty)$, i.e. $g(t)=\sigma(e^t)$ is convex on $(-\infty,\infty)$.Thus the function $g(t)$ satisfies a Lipschitz condition on each interval $[a,b]\subset(0,+\infty)$, so is also absolutely continuous on each interval. Then the function $\sigma(r)=g(\log r)$ is also absolutely continuous on the intervals $[a,b]\subset(0,+\infty)$. Integrating by parts the integral in the inequality (4.6) we have

$$\int_{2S_k}^{R_k}\frac{\sigma_-'(r)}{r^\lambda}\,dr = \int_{2S_k}^{R_k}\frac{\sigma'(r)}{r^\lambda}\,dr = \frac{\sigma(R_k)}{R_k^\lambda}-\frac{\sigma(2S_k)}{(2S_k)^\lambda}+\lambda\int_{2S_k}^{R_k}\frac{\sigma(r)}{r^{\lambda+1}}\,dr. \tag{4.7}$$

By (4.5), (4.6) and (4.7) we have

$$\int_{2S_k}^{R_k}\frac{h(r)}{r^{\lambda+1}}\,dr \leq \left(\frac{\sigma_-'(r)}{r^{\lambda-1}}+\lambda\frac{\sigma(r)}{r^\lambda}\right)\Big|_{2S_k}^{R_k}. \tag{4.8}$$

By definition of $\sigma(r)$ we get

$$0 \le \sigma(R) \le \pi(1+o(1))T(R,f) < 2\pi T(R,f) \quad (R\to\infty). \tag{4.9}$$

The function $r\sigma_-'(r)$ is non-decreasing on $(0,\infty)$, hence

$$\begin{aligned}\sigma(2R) \ge \sigma(2R)-\sigma(R) &= \int_R^{2R}\sigma'(r)\,dr = \int_R^{2R}\frac{r\sigma_-'(r)}{r}\,dr \ge R\sigma_-'(R)\int_R^{2R}\frac{dr}{r}\\ &= R\sigma_-'(R)\log 2.\end{aligned}$$

Consequently, for $R > R_0$ we have

$$R\sigma_-'(R) \le \frac{1}{\log 2}\sigma(2R) \le \frac{2\pi}{\log 2}T(2R,f). \tag{4.10}$$

Moreover, in view of the monotonicity of $R\sigma_-'(R)$ we have for $R\ge 1$

$$R\sigma_-'(R) \ge \sigma_-'(1) = C. \tag{4.11}$$

By (4.8), (4.9), (4.10) and (4.11) we have

$$\int_{2S_k}^{R_k}\frac{h(r)}{r^{\lambda+1}}\,dr \le 2\pi\left(\frac{1}{\log 2}+\lambda\right)\frac{T(2R_k,f)}{R_k^{\lambda}} - \frac{C}{(2S_k)^{\lambda}} \quad (k\to\infty).$$

It follows from the Lemma 4 that for $k \ge k_0(\epsilon)$

$$\int_{2S_k}^{R_k}\frac{h(r)}{r^{\lambda+1}}\,dr < \epsilon\int_{2S_k}^{R_k}\frac{T(r,f)}{r^{\lambda+1}}\,dr.$$

Therefore there exists a sequence $r_k\in[2S_k,R_k]$ such that $h(r_k) < \varepsilon T(r_k,f)$. Since $S_k\to\infty$ it follows that $r_k\to\infty$ as $k\to\infty$.

Recalling the definition of $h(r)$ we have for $k\ge k_0$

$$\begin{aligned}&\frac{\widehat{p}_\phi^2(\infty,f)}{\pi}\left(\max\,(\log\max_{|z|=r}|f(z)|,\phi(r))\cos\frac{\lambda\psi}{\widehat{p}_\phi(\infty,f)} - \frac{\pi\lambda}{\widehat{p}_\phi(\infty,f)}T^*(r_k,\alpha,u_\phi)\sin\frac{\lambda(\alpha+\psi)}{\widehat{p}_\phi(\infty,f)}\right)\\ &+\lambda\widehat{p}_\phi(\infty,f)N(r_k,\infty,f)\sin\frac{\lambda\psi}{\widehat{p}_\phi(\infty,f)} - \frac{\widehat{p}_\phi^2(\infty,f)}{\pi}\widetilde{u}_\phi(r_k,\alpha)\cos\frac{\lambda(\alpha+\psi)}{\widehat{p}_\phi(\infty,f)} < \epsilon T(r_k,f).\end{aligned}$$

Hence

$$\begin{aligned}&\log^+\max_{|z|=r_k}|f(z)|\cos\frac{\lambda\psi}{\widehat{p}_\phi(\infty,f)} - \frac{\pi\lambda}{\widehat{p}_\phi(\infty,f)}T^*({}_k,\alpha,u_\phi)\sin\frac{\lambda(\alpha+\psi)}{\widehat{p}_\phi(\infty,f)}\\ &+\frac{\pi\lambda}{\widehat{p}_\phi(\infty,f)}N(r_k,\infty,f)\sin\frac{\lambda\psi}{\widehat{p}_\phi(\infty,f)} - \widetilde{u}_\phi(r_k,\alpha)\cos\frac{\lambda(\alpha+\psi)}{\widehat{p}_\phi(\infty,f)}\\ &< \epsilon T(r_k,f) \quad (k>k_0).\end{aligned}\tag{4.12}$$

The quantity $\widehat{p}_\phi(\infty,f)$ is an entire non-negative number. Since $\widehat{p}(\infty,f) = \sup_\phi \widehat{p}_\phi(\infty,f)$ there is the function $\phi(r)$, such that $\widehat{p}_\phi(\infty,f) = \widehat{p}(\infty,f)$. If we apply the inequality (4.12) to the function ϕ then we have

$$\begin{aligned}&\log^+\max_{|z|=r_k}|f(z)|\cos\frac{\lambda\psi}{\widehat{p}(\infty,f)} - \frac{\pi\lambda}{\widehat{p}(\infty,f)}T^*(r_k,\alpha,u_\phi)\sin\frac{\lambda(\alpha+\psi)}{\widehat{p}(\infty,f)}\\ &+\frac{\pi\lambda}{\widehat{p}(\infty,f)}N(r_k,\infty,f)\sin\frac{\lambda\psi}{\widehat{p}(\infty,f)} - \widetilde{u}_\phi(r_k,\alpha)\cos\frac{\lambda(\alpha+\psi)}{\widehat{p}(\infty,f)}\\ &< \epsilon T(r_k,f) \quad (k>k_0).\end{aligned}\tag{4.13}$$

Let $\alpha = \pi$, $\psi = -\frac{\pi\widehat{p}(\infty,f)}{2\lambda}$. Then by (4.13) we have

$$\frac{\pi\lambda}{\widehat{p}(\infty,f)}T(r_k,f)\cos\frac{\pi\lambda}{\widehat{p}(\infty,f)} - \frac{\pi\lambda}{\widehat{p}(\infty,f)}N(r_k,\infty,f) - \widetilde{u}_\phi(r_k,\pi)\sin\frac{\pi\lambda}{\widehat{p}(\infty,f)} < \epsilon T(r_k,f).$$

Since $\delta(\infty,f) = 1 - \limsup_{r\to\infty}\frac{N(r,\infty,f)}{T(r,f)}$, then

$$N(r,\infty,f) < (1-\delta(\infty,f)+\epsilon)T(r,f) \quad (r > r_0).$$

Hence

$$\widetilde{u}_\phi(r,\pi) = \max\left(\min_{|z|=r}\log|f(z)|, \phi(r)\right) = \max\left(\min_{|z|=r}\log^+|f(z)|, \phi(r)\right) \le \min_{|z|=r}\log^+|f(z)| + \phi(r) = \log^+\mu(r,f) + o(T(r,f)) \quad (r\to\infty),$$

and

$$\frac{\pi\lambda}{\widehat{p}(\infty,f)}T(r_k,f)\cos\frac{\pi\lambda}{\widehat{p}(\infty,f)} - \frac{\pi\lambda}{\widehat{p}(\infty,f)}(1-\delta(\infty,f)+\epsilon)T(r_k,f) - \log^+\mu(r_k,f)\sin\frac{\pi\lambda}{\widehat{p}(\infty,f)} < \epsilon T(r_k,f).$$

Therefore

$$\sin\frac{\pi\lambda}{\widehat{p}(\infty,f)}\limsup_{r\to\infty}\frac{\log^+\mu(r,f)}{T(r,f)} \ge \frac{\pi\lambda}{\widehat{p}(\infty,f)}\left(\delta(\infty,f) - 1 + \cos\frac{\pi\lambda}{\widehat{p}(\infty,f)} - \epsilon\right) - \epsilon.$$

Taking $\epsilon \to 0^+$ we get statement of Theorem 1.

5. Proof of Theorem 2

In (4.13) we set $\alpha = \pi$ and $\psi = 0$. Then there exist the sequence $r_k \to \infty$, such that for $k \ge k_0$

$$\log^+\max_{|z|=r_k}|f(z)| - \frac{\pi\lambda}{\widehat{p}(\infty,f)}T^*(r_k,\alpha,u_\phi)\sin\frac{\pi\lambda}{\widehat{p}(\infty,f)} - \log^+\mu(r_k,f)\cos\frac{\pi\lambda}{\widehat{p}(\infty,f)} < \epsilon T(r_k,f).$$

Thus

$$\cos\frac{\pi\lambda}{\widehat{p}(\infty,f)}\limsup_{r\to\infty}\frac{\log^+\mu(r,f)}{T(r,f)} \ge \beta(\infty,f) - \frac{\pi\lambda}{\widehat{p}(\infty,f)}\sin\frac{\pi\lambda}{\widehat{p}(\infty,f)} - \epsilon,$$

If $\epsilon \to 0$, then we get statement of the Theorem 2.

6. Proof of Theorem 3

Let p be the number such that $1 \le p \le \max(1, \widehat{p}(\infty,f))$,

$$\sigma(r) = \int_0^{\frac{\pi p}{2\mu_j}} T^*(r,\theta,u_\phi)\cos\frac{\mu_j\theta}{p}\,d\theta,$$

where μ_j is defined in (3.5). Applying Lemma 2 and Fatou's lemma we obtain that for almost all $r \ge r_0$

$$r\frac{d}{dr}r\sigma'_-(r) \ge -\int_0^{\frac{\pi p}{2\mu_j}} \frac{p^2}{\pi}\frac{\partial\widetilde{u}_\phi(r,\theta)}{\partial\theta}\cos\frac{\mu_j\theta}{p}\,d\theta.$$

After applying integration by parts to the right side of the above inequality we have for almost all $r \geq r_0$

$$r\frac{d}{dr}r\sigma_-^{'}(r) \geq p^2 h(r,\mu_j) + \mu_j^2\sigma(r),$$

where $h(r,\mu_j) = \mathcal{L}(r,\infty,f) - \frac{\pi\mu_j}{p}T^*\left(r, \frac{\pi p}{2\mu_j}, u_\phi\right)$.
By dividing the inequality above by r^{μ_j+1} and integrating over an interval $[t_j, T_j]$ we have

$$\int_{t_j}^{T_j} \frac{1}{r^{\mu_j}}\frac{d}{dr}r\sigma_-^{'}(r)\,dr \geq p^2\int_{t_j}^{T_j}\frac{h(r,\mu_j)}{r^{\mu_j+1}}\,dr + \mu_j^2\int_{t_j}^{T_j}\frac{\sigma(r)}{r^{\mu_j+1}}\,dr, \tag{6.1}$$

where t_j, T_j are defined in (3.6).

Integrating by parts the left side of (6.1) and applying the monotonicity of $r\sigma_-^{'}(r)$, we obtain

$$p^2\int_{t_j}^{T_j}\frac{h(r,\mu_j)}{r^{\mu_j+1}}\,dr \leq \left(\frac{\sigma_-^{'}(r)}{r^{\mu_j-1}} + \mu_j\frac{\sigma_-^{'}(r)}{r\mu_j}\right)\Bigg|_{t_j}^{T_j}. \tag{6.2}$$

The definition of $\sigma(r)$ implies that

$$\sigma(r) \leq \frac{p}{\mu_j}(T(r,f) + o(T(r,f))).$$

Since $r\sigma_-^{'}(r)$ is monotonically increasing on $[t_j, T_j]$ we have

$$\begin{aligned}\sigma(s_j) - \sigma(T_j) &= \int_{T_j}^{S_j}\sigma_-^{'}(r)dr = \int_{T_j}^{S_j} r\sigma_-^{'}(r)\frac{dr}{r}\\ &\geq T_j\sigma_-^{'}(T_j)\int_{T_j}^{S_j}\frac{dr}{r} = T_j\sigma_-^{'}(T_j)\log\frac{S_j}{T_j} = \frac{1}{\mu_j}T_j\sigma_-^{'}(T_j).\end{aligned}$$

Hence

$$T_j\sigma_-^{'}(T_j) \leq \mu_j\sigma(S_j) \leq p(T(S_j,f) + o(T(S_j,f))) < 2pT(S_j,f) \quad (j\to\infty).$$

Apart from that, for all $r \geq 1$ we have

$$r\sigma_-^{'}(r) \geq \sigma_-^{'}(1).$$

Now, applying (6.2) and (3.6) we have

$$\begin{aligned}p^2\int_{t_j}^{T_j}\frac{h(r,\mu_j)}{r^{\mu_j+1}}\,dr &\leq \frac{3pT(S_j,f)}{T_j^{\mu_j}} - \frac{\sigma_1^{'}}{t_j^{\mu_j}} \leq \frac{3pe^2T(R_j,f)}{R_j^{\mu_j}} + \frac{T(t_j,f)}{t_j^{\mu_j}}\\ &< \epsilon\mu_j\int_{t_j}^{T_j}\frac{T(r,f)}{r^{\mu_j+1}}\,dr \quad (j\to\infty),\end{aligned} \tag{6.3}$$

$$\begin{aligned}T^*(r,\alpha,u_\phi) &\leq T(r,f) + o(T(r,f)) \quad (r\to\infty).\\ h(r,\mu_j) &= \mathcal{L}(r,\infty,f) - \frac{\pi\mu_j}{p}T^*(r,\frac{\pi p}{2\mu_j},u_\phi)\\ &\geq \mathcal{L}(r,\infty,f) - \frac{\pi\mu_j}{p}(T(r,f) + o(T(r,f))).\end{aligned}$$

Therefore

$$\int_{t_j}^{T_j}\frac{\mathcal{L}(r,\infty,f)}{r^{\mu_j+1}}\,dr < (\frac{\pi}{p} + \epsilon)\mu_j\int_{t_j}^{T_j}\frac{T(r,f)}{r^{\mu_j+1}}dr. \tag{6.4}$$

Using integrating by parts and applying (3.6), we obtain

$$\mu_j \int_{t_j}^{T_{j+1}} \frac{T(r,f)}{r^{\mu_j+1}}\,dr = \frac{T(t_j,f)}{t_j^{\mu_j}} - \frac{T(T_j,f)}{T_j^{\mu_j}} + \int_{t_j}^{T_{j+1}} \frac{rT'_-(r,f)}{r^{\mu_j+1}}\,dr <$$

$$< (1+\epsilon)\int_{t_j}^{T_{j+1}} \frac{rT'_-(r,f)}{r^{\mu_j+1}}\,dr \quad (j\to\infty).$$

Hence

$$\int_{t_j}^{T_j} \frac{\mathcal{L}(r,\infty,f)}{r^{\mu_j+1}}\,dr < \left(\frac{\pi}{p}+\epsilon\right)(1+\epsilon)\int_{t_j}^{T_{j+1}} \frac{rT'_-(r,f)}{r^{\mu_j+1}}\,dr.$$

Therefore there is a sequence $r_j \in [t_j, T_j]$ such that

$$\mathcal{L}(r_j,\infty,f) < \left(\frac{\pi}{p}+\epsilon\right)(1+\epsilon) r_j T'_-(r_j,f).$$

The definition of sequences $\{t_j\}$ implies that $t_j \geq P_j = p_j e^{-\frac{M_j}{\mu_j}}$, where $p_j \to \infty$, $\frac{M_j}{\mu_j} \to 0$. The sequence $p_j \to \infty$ as $j \to \infty$. Thus $t_j \to \infty$ and $r_j \to \infty$ as $j \to \infty$. From the definition of $b(\infty, f)$ and from (6.4) we get

$$b(\infty,f) \leq \left(\frac{\pi}{p}+\epsilon\right)(1+\epsilon).$$

As it is true for any $\epsilon > 0$ then for all number $1 \leq p \leq \widehat{p}(\infty, f)$ we have

$$b(\infty,f) \leq \frac{\pi}{p} \tag{6.5}$$

If $\widehat{p}(\infty, f) < \infty$ then putting in (6.5) $p = \widehat{p}(\infty, f)$ we obtain the statement. If, on the other hand, $\widehat{p}(\infty, f) = \infty$ then the inequality (6.5) is true for all numbers $p \geq 1$. Hence in this case $b(\infty, f) = 0$. This completes the proof of Theorem 3.

7. Examples

Let $E_\rho(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma\left(1+\frac{k}{\rho}\right)}$ be the Mittag-Leffer function of order $\rho > 0$ ($\lambda = \rho$). Let $n \in \mathbb{N}$ and λ: $\frac{\lambda}{n} \geq \frac{1}{2}$. Let us consider the entire function

$$F_1(z) = E_{\frac{\lambda}{n}}(z^n).$$

It is clear that $F_1(z)$ is an entire function of lower order λ. From the asymptotics of the function $E_\rho(z)$ ([15, p. 114]) it follows easily that $\widehat{p}(\infty, F_1) = n$, $\beta(\infty, F_1) = \frac{\pi\lambda}{n}$. Thus the estimate (4) of Theorem H is attained for the function $F_1(z)$. The estimate (5) is attained for the function $E_\lambda(z)$ for $0 < \lambda < \frac{1}{2}$.

To prove the sharpness of the estimate (6) for $\lambda > 0$ we consider for $n \in \mathbb{N}$ the meromorphic function $F_2(z) = f_{\frac{\lambda}{n}}(z^n)$, where $f_\rho(z)$ is the function given by Teichmüller [35] (see also [15, p. 282]). The function $f_\rho(z)$ is of the order ρ: $0 < \rho < \frac{1}{2}$, $\delta(\infty, f_\rho) = 1 - \cos\pi\rho$, $\beta(\infty, f_\rho) = \pi\rho\sin\pi\rho$ and $|f(-r)| \leq 2$ for $r \in [0, \infty)$. Clearly, $F_2(z) = f_{\frac{\lambda}{n}}(z^n)$ is of the finite lower order λ: $0 < \frac{\lambda}{n} < \frac{1}{2}$ ($\lambda = \rho$), $\widehat{p}(\infty, F_2) = n$, $\beta(\infty, F_2) = \frac{\pi\lambda}{n}\sin\frac{\pi\lambda}{n}$, $\delta(\infty, F_2) = 1 - \cos\frac{\pi\lambda}{n}$. Consequently, the estimate (6) is attained for the function $F_2(z)$ for $\lambda > 0$.

If $\lambda = 0$ and $\beta(\infty, f) > 0$ then by Corollary 3.2 we have $\widehat{p}(\infty, f) = 1$. The function $f_0(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{e^n}\right)$ is of the order 0 and $\beta(\infty, f_0) = 1$. Therefore in the case of $\lambda = 0$ the estimate (5) of Theorem H are attained for the function $f_0(z)$.

Let notice that the estimates in Theorem I and Theorem J are attained for the functions $F_1(z)$ and $F_2(z)$ introduced earlier in this section.

The example of the function $F_2(z)$ proves that if $\delta(\infty, f) = 1 - \cos \frac{\pi\lambda}{\widehat{p}(\infty,f)}$, then the statement of Corollary 1.2 not always is true, i.e. the condition $\delta(\infty, f) > 1 - \cos \frac{\pi\lambda}{\widehat{p}(\infty,f)}$ can not be replaced by $\delta(\infty, f) \geq 1 - \cos \frac{\pi\lambda}{\widehat{p}(\infty,f)}$. The function $F_2(z)$ also shows that the condition $\beta(\infty, f) > \frac{\pi\lambda}{\widehat{p}(\infty,f)} \sin \frac{\pi\lambda}{\widehat{p}(\infty,f)}$ in Corollary 2.2 can not be changed to $\beta(\infty, f) \geq \frac{\pi\lambda}{\widehat{p}(\infty,f)} \sin \frac{\pi\lambda}{\widehat{p}(\infty,f)}$.

Let us consider the function $F_3(z) = \cos(z^{\frac{n}{2}}) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{nk}}{(2k)!}$, $(n \in \mathbb{N})$. Then

$$\widehat{p}(\infty, F_3) = n, \quad \rho(F_3) = \frac{n}{2} \quad \text{and} \quad |F_3(x)| = |\cos x^{\frac{n}{2}}| \leq 1 \quad \text{for } x > 0.$$

The example of the function $F_3(z)$ shows that if $\lambda = \frac{\widehat{p}(\infty,f)}{2}$, then the Corollary 1.4 may not hold.

For the function from the Example 1 ([15, p. 277]) we have $f_{uv}(z) = \frac{f(z,u)}{f(-z,v)}$, where $f(z,u)$ is a Weierstrass product with positive zeros and $n(r,0) \sim ur^{\lambda}$ $(r \to \infty)$, $0 < \lambda < 1$. It is easy to check (see [15], p. 277–278) that the function $f_{uv}(z)$ is a meromorphic function of lower order λ $(\lambda = \rho)$ and for $0 < \lambda < \frac{1}{2}$ we have

$$\beta(\infty, f_{uv}) = \frac{\pi\lambda}{\sin \pi\lambda} \frac{(u - v\cos \pi\lambda)}{u},$$

$$\limsup_{r\to\infty} \frac{\log^+ \mu(r, f_{uv})}{T(r, f_{uv})} = \frac{\pi\lambda}{\sin \pi\lambda} \frac{(u\cos \pi\lambda - v)}{u} = \frac{1}{\cos \pi\lambda}(\beta(\infty, f_{uv}) - \pi\lambda \sin \pi\lambda).$$

Hence for all $\lambda\colon 0 < \lambda < \frac{1}{2}$ there exists a meromorphic function of lower order λ such that in the estimate (2.1) we have an equality.

To prove the sharpness of the Theorem 3 we consider the entire function $E_0(z)$ of infinite order [18, p. 126]. For $E_0(z)$ we have (see [18, p. 128])

$$E_0(z) = \begin{cases} \exp(e^z + z) + O(\frac{1}{|z|^2}), & \text{if } z \in A_0, \\ O(\frac{1}{|z|^2}), & \text{if } z \notin A_0, \end{cases}$$

where $A_0 = \{z = x + iy \in \mathbb{C}\colon x > 0,\ -\pi \leq y \leq \pi\}$ and

$$\mathcal{L}(r, \infty, E_0) \sim e^r, \quad T(r, E_0) \sim \frac{e^r}{\pi r} \quad (r \to \infty).$$

For each $n \in \mathbb{N}$ we consider the function

$$F_4(z) = E_0(z^n).$$

It is easy to see that for this function we have $\widehat{p}(\infty, F_4) = n$, $\mathcal{L}(r, \infty, F_4) \sim e^{r^n}$, $T(r, F_4) \sim \frac{e^{r^n}}{\pi r^n}$, $rT'_-(r, F_4) \sim \frac{ne^{r^n}}{\pi}$ $(r \to \infty)$ and $b(\infty, F_4) = \frac{\pi}{n}$. Thus estimate of theorem 3 is attained fot function $F_4(z)$.

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