Annales Fennici Mathematici
Volumen 47, 2022, 283–303

Weak quasicircles have Lipschitz dimension 1

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Abstract. We prove that the Lipschitz dimension of any bounded turning Jordan circle or arc is equal to 1. Equivalently, the Lipschitz dimension of any weak quasicircle or arc is equal to 1.

Heikkojen kvasiympyröiden Lipschitzin ulottuvuus on 1

Tiivistelmä. Todistamme, että jokaisen rajoitetusti kiertävän Jordanin ympyrän tai kaaren Lipschitzin ulottuvuus on 1. Yhtäpitävästi jokaisen heikon kvasiympyrän tai -kaaren Lipschitzin ulottuvuus on 1.

1. Background

In [CK13], Cheeger and Kleiner introduced the concept of Lipschitz dimension and proved deep results about metric spaces of Lipschitz dimension at most 1. In [Dav21], David further developed various dimension-theoretic properties of Lipschitz dimension and posed several questions to prompt additional study. In this paper we answer one of these questions. To this end, we now provide a few core definitions and briefly survey the theoretical context of our main result.

We write $\mathbb{N}$ to denote the set $\{0, 1, 2, \dots\}$ consisting of non-negative integers, and $\mathbb{R}$ to denote the Euclidean line. Given metric spaces X and Y and a constant $C \geq 1$, a map $f\colon X \to Y$ is said to be *C-Lipschitz* provided that, for all points $a, b \in X$, we have $d_Y(f(a), f(b)) \leq C\, d_X(a, b)$. Furthermore, an embedding is *C-bi-Lipschitz* if it is also true that $d_X(a, b) \leq C\, d_Y(f(a), f(b))$.

In order to specialize from Lipschitz maps to Lipschitz light maps (as defined below), we must analyze the diameters of pre-image components. However, these components are not necessarily connected. Instead, they are defined in terms of the following metric condition. Given a metric space (X, d) and $\delta > 0$, a *δ-sequence* in X is a finite sequence of points $\{x_i\}_{i=0}^n$ such that, for each $0 \leq i \leq n-1$, we have $d(x_i, x_{i+1}) \leq \delta$. A subset $U \subset X$ is said to be *δ-connected* if every pair of points in U is contained in a δ-sequence in U. A *δ-component* of X is a maximal δ-connected subset of X.

We say that a map $F\colon X \to Y$ is *Lipschitz light* provided there exists $C \geq 1$ such that F is C-Lipschitz, and, for every $r > 0$ and every subset $E \subset Y$ with $\operatorname{Diam}(E) \leq r$, the r-components of $F^{-1}(E)$ have diameter bounded above by $C\, r$. Here we employ the definition of Lipschitz light used in [Dav21, Definition 1.2]. As shown by David in [Dav21, Section 1.4], this definitions is equivalent to [CK13, Definition 1.14] for maps into $\mathbb{R}^n$ (for $n \geq 1$).

A metric space X has *Lipschitz dimension* $\dim_L(X) = n \in \mathbb{N}$ if n is minimal such that there exists a Lipschitz light map $F\colon X \to \mathbb{R}^n$. In [CK13], Cheeger and Kleiner prove that any metric space admitting a Lipschitz light mapping into the real line

https://doi.org/10.54330/afm.113453
2020 Mathematics Subject Classification: Primary 30L05; Secondary 30C65, 54F45.
Key words: Lipschitz dimension, Lipschitz light mappings, quasisymmetric mappings, bi-Lipschitz embeddings.

can be mapped into L_1 via a bi-Lipschitz embedding [CK13, Theorem 1.7]. That is, if $\dim_L(X) \leq 1$, then there exists a measure space (Z, μ) such that X admits a bi-Lipschitz embedding into $L_1(Z, \mu)$.

Furthermore, in [CK13], Cheeger and Kleiner study metric spaces appearing as the inverse limits of certain admissible inverse systems of graphs. On one hand, they prove that the inverse limit of such a system has Lipschitz dimension at most 1 [CK13, Theorem 1.10]. On the other hand, they prove that any metric space of Lipschitz dimension at most 1 can be realized (up to a bi-Lipschitz homeomorphism) as the inverse limit of an admissible inverse system of graphs [CK13, Theorem 1.11]. We refer the reader to [CK13] for definitions and more precise statements.

In [Dav21], David studies Lipschitz dimension in a somewhat broader context. For example, he explores various dimension-theoretic properties of Lipschitz dimension, clarifies its relationship to other notions of dimension, and reveals the behavior of Lipschitz dimension under the action of various classes of mappings. In particular, David proves that, in general, Lipschitz dimension is not invariant under quasisymmetric mappings [Dav21, Corollary 8.4]. However, the examples David uses to demonstrate such non-invariance all have Hausdorff dimension at least 4. In light of this, David comments on the possibility that Lipschitz dimension is in fact invariant under quasisymmetric deformations of lower-dimensional Euclidean spaces. More specifically, he poses the following question.

Question 1.1. [Dav21, Question 8.7] Does every quasisymmetric image of the unit interval $[0,1] \subset \mathbb{R}$ have Lipschitz dimension 1?

In what follows we provide a positive answer to Question 1.1.

2. Main result

We begin the presentation of our main result by introducing additional terminology. An embedding $f\colon X \to Y$ is *quasisymmetric* provided there exists a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that, for all points $x, y, z \in X$ and $t \in [0,\infty)$,

$$d(x,y) \leq t\, d(x,z) \quad \text{implies} \quad d(f(x), f(y)) \leq \eta(t) d(f(x), f(z)).$$

An embedding $f\colon X \to Y$ is *weakly quasisymmetric* provided there exists a constant $H \geq 1$ such that, for all $x, y, z \in X$,

$$d(x,y) \leq d(x,z) \quad \text{implies} \quad d(f(x), f(y)) \leq H d(f(x), f(z)).$$

While all quasisymmetries are weak quasisymmetries (with $H := \eta(1)$), in general, a weak quasisymmetry need not be a quasisymmetry. We refer to the (weak) quasisymmetric image of the unit circle as a *(weak) quasicircle*, and such an image of the closed unit interval as a *(weak) quasiarc*. Thus every quasicircle/arc is a weak quasicircle/arc. Conversely, by [TV80, Theorem 4.9], every weak quasicircle/arc that is *doubling* is a quasicircle/arc. Here we say that a space X is doubling if there exists $D \geq 1$ such that any open ball of radius $2r > 0$ can be covered by D open metric balls of radius r. For additional information about weak quasicircles and weak quasiarcs, we refer the reader to [Mey11] and references therein.

A *Jordan circle* Γ is a homeomorphic image of the unit circle. Given two points $x, y \in \Gamma$, we write $\Gamma(x,y)$ to denote a component of $\Gamma \backslash \{x,y\}$ of minimal diameter. We write $\Gamma[x,y]$ to denote the topological closure of $\Gamma(x,y)$; thus $\Gamma[x,y] = \Gamma(x,y) \cup \{x,y\}$. Analogously, a *Jordan arc* Γ is a homeomorphic image of the closed unit interval. In this setting, given two points $x, y \in \Gamma$, we write $\Gamma(x,y)$ to denote the connected component of $\Gamma \setminus \{x, y\}$ whose closure contains $\{x, y\}$. Again, $\Gamma[x, y] = \Gamma(x, y) \cup \{x, y\}$.

A Jordan circle or arc is said to be *bounded turning* provided that there exists a constant $C \geq 1$ such that, for all pairs of points $x, y \in \Gamma$, we have $\mathrm{Diam}(\Gamma[x, y]) \leq C\, d(x, y)$. In this case we say that Γ is C-bounded turning. This property is at times referred to in the literature as *linear connectivity*.

While it is straightforward to verify that weak quasicircles and arcs are bounded turning, in [Mey11] Meyer proves that bounded turning quasicircles/arcs are weakly quasisymmetric images of the unit circle/interval. We record this result as follows.

Theorem 2.1. [Mey11, Theorem 1.1] *A Jordan circle/arc Γ is a weak quasicircle/arc if and only if Γ is bounded turning.*

In light of Meyer's characterization and the fact that every quasicircle is a weak quasicircle, Question 1.1 is (more than) answered by the following result.

Theorem 2.2. *Bounded turning Jordan circles/arcs have Lipschitz dimension* 1.

We point out that the Lipschitz light maps we will construct in order to prove Theorem 2.2 are *not* injective (in general). Indeed, if Γ is not locally rectifiable, an injective Lipschitz light map $F\colon \Gamma \to \mathbb{R}$ does not exist. This is because an injective Lipschitz light map $F\colon \Gamma \to \mathbb{R}$ must be bi-Lipschitz. To see this, let a, b denote any pair of points in $F(\Gamma) \subset \mathbb{R}$. Then $F^{-1}(\{a, b\}) = \{F^{-1}(a), F^{-1}(b)\}$ and so

$$d(F^{-1}(a), F^{-1}(b)) = \mathrm{Diam}(F^{-1}(\{a, b\})) \leq C\, \mathrm{Diam}(\{a, b\}) = C|a - b|.$$

Therefore, the maps we construct in order to prove Theorem 2.2 are quite different from the homeomorphisms constructed in [Mey11].

Via [CK13, Theorem 1.7 and Theorem 1.11] (as mentioned in Section 1), we also have the following two corollaries.

Corollary 2.3. *If Γ is a bounded turning Jordan circle or arc, then there exists a measure space (Z, μ) such that Γ admits a bi-Lipschitz embedding into $L_1(Z, \mu)$.*

Corollary 2.4. *If Γ is a bounded turning Jordan circle or arc, then Γ is bi-Lipschitz homeomorphic to an inverse limit of an admissible inverse system of graphs.*

Indeed, given a metric space X and a Lipschitz light map $F\colon X \to \mathbb{R}$, Cheeger and Kleiner explicitly construct an admissible inverse system of graphs whose inverse limit is bi-Lipschitz homeomorphic to X [CK13, Section 4]. In Appendix A, we provide an alternate means of viewing a given bounded turning Jordan circle as an inverse limit.

Theorem 2.2 is also relevant to the following result of David from [Dav21].

Theorem 2.5. [Dav21, Theorem 5.9] *Let $\mathbb{G}$ denote a non-abelian Carnot group, and let $K \subset \mathbb{G}$ denote a compact subset of positive measure. Then the Lipschitz dimension of K is equal to ∞.*

Since the Lipschitz dimension of a compact space is bounded from below by its topological dimension [Dav21, Observation 1.4], given an integer $n \geq 1$, we note that the Lipschitz dimension of the product of n bounded turning Jordan arcs is at least n. Furthermore, the Lipschitz dimension of the product of n bounded turning Jordan arcs is bounded from above by n (here we use Theorem 2.2 and [Dav21, Proposition 3.1]). Therefore, the Lipschitz dimension of the product of n bounded turning Jordan arcs is equal to n. Since Lipschitz dimension is invariant under bi-Lipschitz homeomorphisms, we arrive at the following corollary.

Corollary 2.6. *Let $\mathbb{G}$ denote a non-abelian Carnot group. If $K \subset \mathbb{G}$ is a compact subset of positive measure, then K does not admit a bi-Lipschitz embedding into a product of finitely many bounded turning Jordan arcs.*

Before proceeding, we briefly sketch the proof of Theorem 2.2. The starting point is provided by the following result of Herron and Meyer.

Theorem 2.7. [HM12] *If Γ is a bounded turning Jordan circle, then Γ is bi-Lipschitz homeomorphic to some Jordan circle in $\mathcal{S}_1$.*

Here $\mathcal{S}_1$ is the collection of all Jordan circles given by dyadic diameter functions as constructed in Section 3 below. This result allows us to distort any given bounded turning Jordan circle into the limit of piecewise linear Jordan circles. This structure is amenable to the construction of a Lipschitz light map into $\mathbb{R}$.

The organization of this paper is as follows. In Section 3, we define and analyze the catalogue $\mathcal{S}_1$. In Section 4, we construct a 1-Lipschitz mapping from any Jordan circle $\Gamma \in \mathcal{S}_1$ onto the unit circle $\mathbb{S}$. Finally, in Section 5 we prove that this mapping is Lipschitz light via a series of technical lemmas. An appendix is also provided, in which we explain how our Lipschitz light map from Γ onto $\mathbb{S}$ can be understood as an inverse limit.

3. Dyadic intervals and dyadic diameter functions

Following [HM12], we view the unit circle $\mathbb{S}$ as $[0,1]/\{0,1\}$, the closed unit interval whose endpoints are identified. We equip $\mathbb{S}$ with the arc-length metric λ. That is, for two points $s, t \in \mathbb{S}$ such that $0 \le s \le t \le 1$, we have

$$\lambda(s,t) := \min\{t-s, 1-(t-s)\}.$$

The space $\mathbb{S}$ is endowed with a positive orientation via the usual left-to-right orientation on $[0,1]$. For $a, b \in \mathbb{S}$, the interval $[a,b] \subset \mathbb{S}$ consists of $\{a,b\}$ and points $c \in \mathbb{S}$ such that the orientation given by progressing from a to c to b along $[a,b]$ agrees with the positive orientation on $\mathbb{S}$.

Given $n \in \mathbb{N}$, we write $\mathcal{I}_n$ to denote the collection of 2^n closed dyadic intervals in $\mathbb{S}$, each of length 2^{-n}. For example, $\mathcal{I}_1 = \{[0,1/2],[1/2,1]\}$. We write $\hat{\mathcal{I}}_n$ to denote the collection $\bigcup_{m=0}^{n} \mathcal{I}_m$. Furthermore, we write $\hat{\mathcal{I}}$ to denote the collection $\bigcup_{n=0}^{\infty} \mathcal{I}_n$. Given an interval $I \in \hat{\mathcal{I}}$, we write $l(I)$ to denote the unique index $n \in \mathbb{N}$ such that $I \in \mathcal{I}_n$. For convenience, we use the language of a dyadic tree to describe intervals in $\hat{\mathcal{I}}$. In particular, given any $I \in \hat{\mathcal{I}}$, there are exactly two dyadic *children* contained in I, and I is contained in its unique dyadic *parent* interval. Two children with the same parent are called *siblings*, and if a dyadic interval I is strictly contained in a dyadic interval J, we say that I is a *descendent* of J.

Similarly, we write $\mathcal{D}_n$ to denote the collection of 2^n dyadic endpoints of intervals in $\mathcal{I}_n$. For example, $\mathcal{D}_0 = \{0\} = \{1\}$, $\mathcal{D}_1 = \{0,1/2\} = \{1,1/2\}$, etc. Note that, for each $n \in \mathbb{N}$, we have $\mathcal{D}_n \subset \mathcal{D}_{n+1}$. We write $\mathcal{D}$ to denote $\bigcup_{n=0}^{\infty} \mathcal{D}_n$.

We call a function $\Delta\colon \hat{\mathcal{I}} \to (0,1]$ a *dyadic diameter function* provided that $\Delta(\mathbb{S}) = 1$ and, for any $I \in \hat{\mathcal{I}}$, either

$$\Delta(I') = \Delta(I'') = \frac{1}{2}\Delta(I) \quad \text{or} \quad \Delta(I') = \Delta(I'') = \Delta(I).$$

Here I' and I'' denote the two dyadic children of I. We also require that

$$\lim_{n\to+\infty} \max\{\Delta(I) \mid I \in \mathcal{I}_n\} = 0. \tag{3.1}$$

Note that, contrary to [HM12], we omit the parameter σ from the definition of Δ. This is because $\sigma = 1$ for all dyadic diameter functions utilized in what follows. For use below, we denote the collection of all such dyadic diameter functions as $\mathfrak{D}$.

For completeness, we also provide the definition of the catalogue $\mathcal{S}_1$. Indeed, for every $\Delta \in \mathfrak{D}$, the function d_Δ on $\mathbb{S} \times \mathbb{S}$ is defined as

$$d_\Delta(x, y) := \inf \sum_{k=1}^{N} \Delta(J_k),$$

where the infimum is taken over all chains $J_1, \ldots, J_N$ of intervals from $\hat{\mathcal{I}}$ joining x to y. That is, $\{x, y\}$ is contained in the connected set $J_1 \cup \cdots \cup J_N$. By [HM12, Lemma 3.1], the function d_Δ is a distance, and the metric space $(\mathbb{S}, d_\Delta)$ is a 1-bounded turning Jordan curve. We write $\mathcal{S}_1$ to denote the collection of all such curves, each given by some dyadic diameter function Δ. That is,

$$\mathcal{S}_1 := \{(\mathbb{S}, d_\Delta) \mid \Delta \in \mathfrak{D}\}$$

Given a fixed $\Delta \in \mathfrak{D}$, for each $n \in \mathbb{N}$ we define a distance d_n on $\mathbb{S}$ using the *truncated* dyadic diameter function Δ_n. For $m \leq n$ and $I \in \mathcal{I}_m$, we define $\Delta_n(I) := \Delta(I)$. For every $m > n$ and $I \in \mathcal{I}_m$, we inductively define $\Delta_n(I) = \frac{1}{2}\Delta_n(\tilde{I})$, where $\tilde{I} \in \mathcal{I}_{m-1}$ denotes the dyadic parent of I. We then define

$$d_n(x, y) := d_{\Delta_n}(x, y).$$

We write Γ_n to denote the metric space $(\mathbb{S}, d_n)$, and Γ to denote $(\mathbb{S}, d_\Delta)$. For $n \in \mathbb{N}$, we write $\mathrm{Diam}_n(E)$ to denote the d_n-diameter of a set $E \subset \mathbb{S}$. Furthermore, we write $\mathrm{Diam}_\Delta(E)$ to denote the d_Δ-diameter of E. For notational consistency, we will write d_0 for λ. Thus, given any $\Gamma \in \mathcal{S}_1$, the space Γ_0 denotes $(\mathbb{S}, \lambda)$.

We say that a chain of dyadic intervals $\{I_i\}_{i=1}^N$ is *minimal* provided that it consists of intervals with pairwise disjoint interiors and that no union of at least two distinct intervals from the chain forms an interval in $\hat{\mathcal{I}}$. In particular, if the union of intervals $\cup_{k=1}^M I_{i_k}$ from a minimal chain $\{I_i\}_{i=1}^N$ is equal to some interval $J \in \hat{\mathcal{I}}$, then $M = 1$ and $J = I_{i_1} \in \{I_i\}_{i=1}^N$. In graph-theoretic language, a chain of dyadic intervals is minimal if no two siblings are contained in the chain and no interval is a descendent of another in the chain.

Lemma 3.1. *Given $\Delta \in \mathfrak{D}$, the definition of d_Δ is unchanged by the assumption that the chains of dyadic intervals utilized in this definition are minimal.*

Proof. Suppose that $\{I_i\}_{i=1}^N$ is a chain of dyadic intervals joining x and y in $\mathbb{S}$. Suppose I_j and I_k have non-disjoint interiors. Since both I_j and I_k are dyadic, one must be a subset of the other. Without loss of generality, $I_j \subset I_k$. Therefore, the sum $\sum_{i=1}^N \Delta(I_i)$ can be decreased by eliminating the interval I_j from $\{I_i\}_{i=1}^N$. It follows that d_Δ can be defined using only chains consisting of intervals with pairwise disjoint interiors.

Next, suppose there exists $M \geq 2$ and a subcollection $\{I_{i_k}\}_{k=1}^M \subset \{I_i\}_{i=1}^N$ such that $J := \cup_{k=1}^M I_{i_k} \in \hat{\mathcal{I}}$. Since $\Delta(J) \leq \sum_{k=1}^M \Delta(I_{i_k})$, the sum $\sum_{i=1}^N \Delta(I_i)$ will not increase when replacing the intervals $\{I_{i_k}\}_{k=1}^M$ in $\{I_i\}_{i=1}^N$ with the single interval J. Since $N < +\infty$, such a replacement can happen at most finitely many times. It follows that d_Δ can be defined using only minimal chains. $\square$

For use below, we record the following technical lemma.

Lemma 3.2. *Assume $\{I_i\}_{i=1}^N$ is a minimal chain of dyadic intervals in $\mathbb{S}$ indexed such that, for $1 \le i \le N-1$, the right endpoint of I_i is the left endpoint of I_{i+1}. Under this assumption, there exists either a unique interval or a unique pair of adjacent intervals from $\{I_i\}_{i=1}^N$ of maximal λ-diameter. Write i_* to denote the index of such a maximal interval. If $i_* > 1$, then $l(I_i)$ is strictly decreasing for $i = 1, \dots, i_* - 1$. If $i_* < N$, then $l(I_i)$ is strictly increasing for $i = i_* + 1, \dots, N$.*

Proof. We may assume that $\sigma := \bigcup_{i=1}^N I_i \neq \mathbb{S}$, else $N = 1$ and $I_1 = \mathbb{S}$. Suppose there are two distinct intervals I_j and I_k in $\{I_i\}_{i=1}^N$ of maximal λ-diameter, where $j < k$ and $n := l(I_j) = l(I_k)$. If these intervals are not adjacent, then the chain $\{I_i\}_{i=j+1}^{k-1}$ consisting of intervals from $\{I_i\}_{i=1}^N$ joins the right endpoint of I_j to the left endpoint of I_k. Since I_j and I_k are not adjacent, the union $\bigcup_{i=j}^k I_i \subset \sigma$ contains at least three consecutive intervals from $\mathcal{I}_n$. Such a union must contain some interval J from $\mathcal{I}_{n-1}$. It follows from the minimality of $\{I_i\}_{i=1}^N$ that the interval J must be an element of $\{I_i\}_{i=1}^N$. However, this violates the assumption that I_j and I_k are of maximal λ-diameter. Therefore, I_j and I_k must be adjacent.

To verify the second part of the lemma, suppose $i_* > 1$. If $i_* = 2$ then the desired conclusion is trivial, so we may assume that $i_* \ge 3$. Let $1 \le k \le i_* - 2$, and write $m := l(I_k)$. Since $k \le i_* - 2$ and $l(I_{i_*}) < m$, the interval J of $\mathcal{I}_m$ immediately to the right of I_k is contained in σ. By minimality, no union of at least two intervals from $\{I_i\}_{i=1}^N$ is equal to J. Therefore, either $J = I_{k+1}$ or J is strictly contained in I_{k+1}. If $J = I_{k+1}$, then minimality implies that the intervals I_k and I_{k+1} are not siblings. In particular, I_{k+1} is the left child of its parent interval $\tilde{J} \in \mathcal{I}_{m-1}$. Since $\tilde{J} \in \mathcal{I}_{m-1}$ strictly contains I_{k+1} and $\tilde{J} \subset \sigma$, we contradict minimality. In conclusion, $J \neq I_{k+1}$, and so J is strictly contained in I_{k+1}. This implies that $l(I_{k+1}) < l(I_k)$, as desired.

An analogous argument verifies the final assertion of the lemma. □

We emphasize that the conclusion of Lemma 3.2 applies to any minimal chain of dyadic intervals in $\mathbb{S}$, up to a possible rearrangement of indices.

Since, for any $I \in \hat{\mathcal{I}}$, we have $\Delta_n(I) \le \Delta(I)$, it follows that, for any $x, y \in \mathbb{S}$,

$$d_n(x,y) \le d_\Delta(x,y). \tag{3.2}$$

Therefore, for any set $E \subset \mathbb{S}$, we have $\mathrm{Diam}_n(E) \le \mathrm{Diam}_\Delta(E)$. Furthermore, given $n \in \mathbb{N}$ such that $n \ge 1$ and $[a,b] = I \in \hat{\mathcal{I}}_n$, via [HM12, Lemma 3.1], we have

$$d_n(a,b) = \mathrm{Diam}_n(I) = \Delta_n(I) = \Delta(I) = \mathrm{Diam}_\Delta(I) = d_\Delta(a,b). \tag{3.3}$$

Lemma 3.3. *If $x, y \in \mathbb{S}$ and $n \in \mathbb{N}$, then*

$$d_\Delta(x,y) \le d_n(x,y) + 2\max\{\Delta(I) \,|\, I \in \mathcal{I}_n\}.$$

In particular, $d_n(x,y) \to d_\Delta(x,y)$ as $n \to +\infty$.

Proof. Let $M(n) := \max\{\Delta(I) \,|\, I \in \mathcal{I}_n\}$. Fix $x, y \in \mathbb{S}$, $n \in \mathbb{N}$, and let $0 < \varepsilon < M(n)$ be given. Let $\{I_i\}_{i=1}^N$ be a minimal chain of dyadic intervals joining x and y, indexed as in the assumptions of Lemma 3.2, such that $\sum_{i=1}^N \Delta_n(I_i) < d_n(x,y) + \varepsilon$. If $\{I_i\}_{i=1}^N \subset \hat{\mathcal{I}}_n$, then we are done, because $\Delta = \Delta_n$ on $\hat{\mathcal{I}}_n$. If not, then let I_{i_*} denote an interval from $\{I_i\}_{i=1}^N$ such that $m := l(I_{i_*})$ is minimal. If x and y are contained in adjacent intervals $J, K \in \mathcal{I}_n$, then

$$d_\Delta(x,y) \le \Delta(J) + \Delta(K) \le 2M(n) \le d_n(x,y) + 2M(n).$$

Therefore, we can assume that x and y are contained in non-adjacent intervals from $\mathcal{I}_n$. It follows from the minimality of $\{I_i\}_{i=1}^N$ that $m \le n$. Therefore, either $l(I_1) \le n$, or, by Lemma 3.2, there exists a maximal index i_1 such that $1 \leq i_1 < i_*$ and, if $1 \leq i \leq i_1$, then $l(I_i) > n$. Similarly, either $l(I_N) \leq n$, or there exists a minimal index i_2 such that $i_* < i_2 \leq N$ and, if $i_2 \leq i \leq N$, then $l(I_i) > n$. Assume the existence of such i_1 and i_2 (else the following argument simplifies). Via Lemma 3.2, one can verify that the interval $\sigma_1 := \bigcup_{i=1}^{i_1} I_i$ is contained in some interval $J_1 \in \mathcal{I}_n$ adjacent (on the left) to I_{i_1+1}. Similarly, $\sigma_2 := \bigcup_{i=i_2}^{N} I_i$ is contained in some interval $J_2 \in \mathcal{I}_n$ adjacent (on the right) to I_{i_2-1}. Thus we have

$$d_\Delta(x,y) \leq \Delta(J_1) + \sum_{i=i_1+1}^{i_2-1} \Delta(I_i) + \Delta(J_2) = \Delta(J_1) + \sum_{i=i_1+1}^{i_2-1} \Delta_n(I_i) + \Delta(J_2)$$
$$\leq \sum_{i=1}^{N} \Delta_n(I_i) + 2M(n) < d_n(x,y) + \varepsilon + 2M(n).$$

Since $\varepsilon > 0$ was arbitrary, we are done. □

4. Constructing a 1-Lipschitz map $F_0 \colon \Gamma \to \mathbb{S}$

Let Γ denote a bounded turning Jordan circle or arc. Our first step towards the construction of a Lipschitz light map $F \colon \Gamma \to \mathbb{R}$ is to realize that it is sufficient to find a Lipschitz light map $F \colon \Gamma \to \mathbb{S}$. This is because $\mathbb{S}$ is easily seen to admit a Lipschitz light map into $\mathbb{R}$, and one can verify that the composition of a Lipschitz light map from Γ to $\mathbb{S}$ with a Lipschitz light map from $\mathbb{S}$ into $\mathbb{R}$ is Lipschitz light (as noted in [Dav21, Section 5]). See also our comments at the outset of Section 5.

Next, we again recall the following result of Herron and Meyer.

Theorem 4.1. [HM12] *If Γ is a bounded turning Jordan circle (or arc), then Γ is bi-Lipschitz homeomorphic to some Jordan circle in $\mathcal{S}_1$ (or $\mathcal{S}_1'$).*

Here $\mathcal{S}_1$ is defined as in Section 3. The collection $\mathcal{S}_1'$ can be analogously defined using dyadic diameter functions on the unit interval $[0,1]$. We remark that the validity of this extension of the main result of [HM12] to Jordan arcs is pointed out by Herron and Meyer on page 605 of [HM12].

Since Lipschitz dimension is invariant under bi-Lipschitz homeomorphisms, we may work exclusively with Jordan circles in $\mathcal{S}_1$ (or arcs in $\mathcal{S}_1'$). We will only present the details for weak quasicircles; the details for weak quasiarcs are analogous. Thus, given a curve $\Gamma = (\mathbb{S}, d_\Delta) \in \mathcal{S}_1$, we construct a Lipschitz light map $F \colon \Gamma \to \mathbb{S}$.

We will need the following map f in order to achieve this goal, which we will refer to as a *folding map*. First, we divide $[0,1]$ into two dyadic subintervals, and denote these two subintervals by I^0 and I^1, respectively. We also divide $[0,1]$ into four consecutive dyadic subintervals of equal length with disjoint interiors, and denote these four subintervals by I^{00}, I^{01}, I^{10}, and I^{11}, respectively. Thus $I^0 = I^{00} \cup I^{01}$ and $I^1 = I^{10} \cup I^{11}$. Assume these intervals are indexed (in binary) such that adjacent intervals proceed consecutively from left to right along $[0,1]$. Finally, divide each of I^{01} and I^{10} into two dyadic subintervals of equal length with disjoint interiors, and denote these subintervals by I^{010}, I^{011}, I^{100}, and I^{101}, respectively. Thus $I^{01} = I^{010} \cup I^{011}$ and $I^{10} = I^{100} \cup I^{101}$. Again we index these intervals such that their order reflects the positive orientation of $[0,1]$. The map $f \colon [0,1] \to [0,1]$ is defined by its action on these subintervals (see Figure 1). It maps

- I^{00} linearly onto I^0 in an orientation preserving manner,
- I^{010} linearly onto I^{10} in an orientation preserving manner,

- I^{011} linearly onto I^{10} in an orientation reversing manner,
- I^{100} linearly onto I^{01} in an orientation reversing manner,
- I^{101} linearly onto I^{01} in an orientation preserving manner, and
- I^{11} linearly onto I^{1} in an orientation preserving manner.

We note that this definition can be scaled linearly and applied to any interval $I \subset \mathbb{S}$. Moreover, by identifying the endpoints of $[0,1]$, the map f can be applied to $\mathbb{S}$. Thus, for any $n \in \mathbb{N}$, let $I \in \mathcal{I}_n$. If the two dyadic children I' and I'' of I satisfy $\Delta(I') = \Delta(I'') = \frac{1}{2}\Delta(I)$, then we define the map $f_n \colon I \to I$ to be the identity map. Thus f_n is an isometry from $(I, d_{n+1}) \to (I, d_n)$. Indeed, in the case that f_n is defined as the identity map, the distances d_{n+1} and d_n agree when restricted to I. If $\Delta(I') = \Delta(I'') = \Delta(I)$, then we define the map $f_n \colon I \to I$ to be a folding map. The map $f_n \colon \Gamma_{n+1} \to \Gamma_n$ is defined in this manner on each interval $I \in \mathcal{I}_n$.

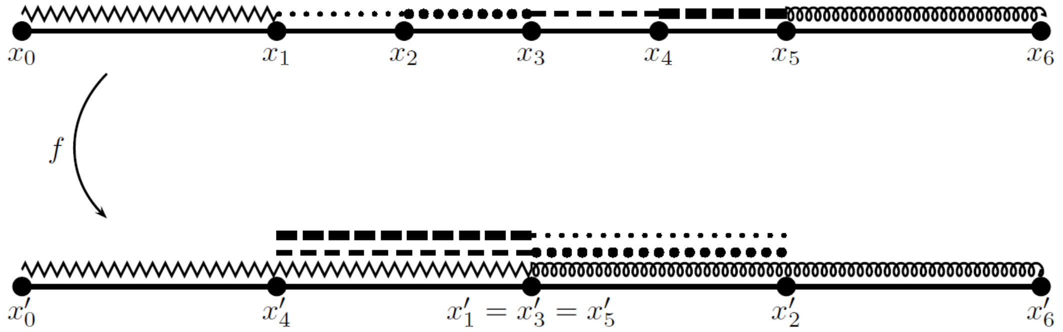


Figure 1. The action of the folding map f. Here x'_i denotes $f(x_i)$, and, for $1 \leq i \leq 6$, the map f is linear on $[x_{i-1}, x_i]$.

Lemma 4.2. *For each $n \in \mathbb{N}$, the map $f_n \colon \Gamma_{n+1} \to \Gamma_n$ is 1-Lipschitz.*

Before beginning the proof of Lemma 4.2, we remark that this lemma is clearly false when the folding map as seen in Figure 1 is understood with respect to Euclidean distance on the intervals $I \in \mathcal{I}_n$. Indeed, with respect to the Euclidean distance the folding map f has Lipschitz constant 2. We obtain a 1-Lipschitz map $f_n \colon \Gamma_{n+1} \to \Gamma_n$ only by virtue of the fact that Γ_{n+1} is equipped with the distance d_{n+1} and Γ_n is equipped with the distance d_n.

Proof of Lemma 4.2. We examine the image of an interval $I \in \hat{\mathcal{I}}$ under the map f_n. If $I \in \hat{\mathcal{I}}_n$, then $f_n(I) = I \in \hat{\mathcal{I}}$ and $\mathrm{Diam}_n(f_n(I)) = \mathrm{Diam}_n(I) = \mathrm{Diam}_{n+1}(I)$.

If $I \in \hat{\mathcal{I}} \setminus \hat{\mathcal{I}}_{n+1}$, then $f_n(I) \in \hat{\mathcal{I}}$ and $\mathrm{Diam}_n(f_n(I)) \leq \mathrm{Diam}_{n+1}(I)$. Indeed, let $J \in \mathcal{I}_n$ denote the unique dyadic interval such that $I \subset J$. If f_n is defined as a folding map on J, then

$$\mathrm{Diam}_n(f_n(I)) = \frac{1}{2}\,\mathrm{Diam}_{n+1}(f_n(I)) \leq \mathrm{Diam}_{n+1}(I).$$

If f_n is defined as the identity map on J, then

$$\mathrm{Diam}_n(f_n(I)) = \mathrm{Diam}_n(I) = \mathrm{Diam}_{n+1}(I).$$

If $I \in \mathcal{I}_{n+1}$ (the only remaining possibility), then f_n fixes the endpoints of I, and $I \subset f_n(I)$. In particular, if f_n is the identity on $\tilde{I}$ (the dyadic parent of I), then $f_n(I) = I$ and $\mathrm{Diam}_n(f_n(I)) = \mathrm{Diam}_n(I) = \mathrm{Diam}_{n+1}(I)$. If f_n is a folding map on $\tilde{I}$, then $f_n(I)$ is the union of I with an interval $J \in \mathcal{I}_{n+2}$ that is adjacent to I.

Moreover, it is straightforward to verify that

$$\begin{aligned}\operatorname{Diam}_n(f_n(I)) &= \operatorname{Diam}_n(I\cup J) = \operatorname{Diam}_n(I) + \operatorname{Diam}_n(J)\\ &= \frac{3}{4}\operatorname{Diam}_n(\tilde I) = \frac{3}{4}\operatorname{Diam}_{n+1}(I) < \operatorname{Diam}_{n+1}(I).\end{aligned}$$

Therefore, given a chain $\{I_i\}_{i=1}^N$ of dyadic intervals, $\{f_n(I_i)\}_{i=1}^N$ can be written as a chain of dyadic intervals $\{I'_j\}_{j=1}^{N'}$ such that $\bigcup_{i=1}^N f_n(I_i) = \bigcup_{j=1}^{N'} I'_j$, and

$$\sum_{i=1}^N \operatorname{Diam}_n(f_n(I_i)) = \sum_{j=1}^{N'} \operatorname{Diam}_n(I'_j) = \sum_{j=1}^{N'} \Delta_n(I'_j).$$

Thus, if $\{I_i\}_{i=1}^N$ joins x and y, then $\{I'_j\}_{j=1}^{N'}$ joins $f_n(x)$ and $f_n(y)$, and

$$d_n(f_n(x), f_n(y)) \le \sum_{j=1}^{N'} \Delta_n(I'_j) = \sum_{i=1}^N \operatorname{Diam}_n(f_n(I_i)) \le \sum_{i=1}^N \Delta_{n+1}(I_i).$$

It follows from the definition of $d_{n+1}(x,y)$ that $d_n(f_n(x), f_n(y)) \le d_{n+1}(x,y)$. □

Given any $m \le n \in \mathbb{N}$, we define $F_{m,n} := f_m \circ f_{m+1} \circ \cdots \circ f_n \colon \Gamma_{n+1} \to \Gamma_m$. If $m = n$, then we understand $F_{m,m} = F_{n,n}$ to denote f_n. As a composition of 1-Lipschitz maps (Lemma 4.2), each map $F_{m,n} \colon \Gamma_{n+1} \to \Gamma_m$ is 1-Lipschitz. Furthermore, this sequence of maps induces a 1-Lipschitz map $F_m \colon (\mathcal{D}, d_\Delta) \to (\Gamma_m, d_m)$. To see this, let $x \in \mathcal{D}$, and let $k \in \mathbb{N}$ be the smallest integer such that $x \in \mathcal{D}_k$. For any $n \ge k$, the map f_n fixes the set $\mathcal{D}_k$. Therefore, if $n \ge k$, then $f_n(x) = x$. If $n \ge k > m$, we then observe that $\lim_{n\to+\infty} F_{m,n}(x) = F_{m,k}(x)$. In this case, define $F_m(x) := F_{m,k}(x)$. If $n > m \ge k$, then define $F_m(x) := x$.

To see that F_m thus defined is 1-Lipschitz on $\mathcal{D}$, let x, y denote any two points in $\mathcal{D}$. Choose $k \in \mathbb{N}$ such that $x, y \in \mathcal{D}_k$ and $k > m$. Via (3.2) and the fact that, for all $n \ge m$ the map $F_{m,n}$ is 1-Lipschitz, we have

$$d_m(F_m(x), F_m(y)) = d_m(F_{m,k}(x), F_{m,k}(y)) \le d_{k+1}(x,y) \le d_\Delta(x,y).$$

Since Γ_m is complete, $\mathcal{D}$ is dense in Γ (cf. (3.1)), and F_m is Lipschitz on $\mathcal{D}$, it is then straightforward to extend F_m such that $F_m \colon \Gamma \to \Gamma_m$ is 1-Lipschitz.

We note that we may also view the maps $F_{m,n}$ as acting on Γ. Moreover, it follows from (3.2) that $F_{m,n} \colon \Gamma \to \Gamma_m$ is 1-Lipschitz. With this in mind, we prove the following lemma.

Lemma 4.3. *For each $m \in \mathbb{N}$, the maps $F_{m,n} \colon \Gamma \to \Gamma_m$ uniformly converge to $F_m \colon \Gamma \to \Gamma_m$ as $n \to +\infty$.*

Proof. Fix $m \in \mathbb{N}$ and $\varepsilon > 0$. Choose $M \in \mathbb{N}$ such that

$$n \ge M \quad \text{implies} \quad \max\{\Delta(I) \mid I \in \mathcal{I}_n\} < \varepsilon/2.$$

For any $x \in \Gamma$, there exists a nested sequence of dyadic intervals $I_n \in \mathcal{I}_n$ such that, for every $n \in \mathbb{N}$, we have $I_n \subset I_{n-1}$ and $x \in I_n$. Furthermore, there exists $x_n \in \mathcal{D}_n$ such that $x_n \in I_n$, and so $d_\Delta(x_n, x) \to 0$. For $n \ge m$ and $j \in \mathbb{N}$, we have $F_m(x_{n+j}) = F_{m,n+j}(x_{n+j})$. If $j = 0$, write $w_{n,0} := x_n$. If $j \ge 1$, write $w_{n,j} := f_{n+1} \circ \cdots \circ f_{n+j}(x_{n+j})$. In either case, we note that $F_m(x_{n+j}) = F_{m,n+j}(x_{n+j}) = F_{m,n}(w_{n,j})$. Since, for all $k \in \mathbb{N}$, we have $f_{n+k}(I_n) = I_n$, it follows that $w_{n,j} \in I_n$. Therefore,

$$F_m(x) = \lim_{j\to+\infty} F_m(x_{n+j}) = \lim_{j\to+\infty} F_{m,n}(w_{n,j}) \in F_{m,n}(I_n).$$

Combining these observations, we find that, for $n \geq \max\{m, M\}$, we have

$$\begin{aligned} d_m(F_{m,n}(x), F_m(x)) &\leq d_m(F_{m,n}(x), F_{m,n}(x_n)) + d_m(F_{m,n}(x_n), F_m(x)) \\ &\leq d_\Delta(x, x_n) + \mathrm{Diam}_m(F_{m,n}(I_n)) \\ &\leq 2\,\mathrm{Diam}_\Delta(I_n) \leq 2\max\{\Delta(I) \mid I \in \mathcal{I}_n\} < \varepsilon. \end{aligned}$$

It follows that $F_{m,n}\colon \Gamma \to \Gamma_m$ is uniformly convergent to $F_m\colon \Gamma \to \Gamma_m$. □

At this point the informed reader may notice that $\{F_{m,n}\colon \Gamma_{n+1} \to \Gamma_m\}$ and $\{F_m\colon \Gamma \to \Gamma_m\}$ bear some resemblance to an inverse system and an inverse limit, respectively. Indeed, in Appendix A, we re-frame our construction in the language of inverse systems and inverse limits.

5. Proving that $F_0\colon \Gamma \to \mathbb{S}$ is Lipschitz light

Our goal in this section is to verify the existence of a constant $C \geq 1$ such that, for any subset $E \subset \mathbb{S}$, the $\mathrm{Diam}_0(E)$-components of $F_0^{-1}(E)$ have d_Δ-diameter bounded above by $C\,\mathrm{Diam}_0(E)$. Here we remind the reader that Diam_0 denotes λ-diameter in $\mathbb{S}$. Via the following lemma, this will be sufficient to prove that $F_0\colon \Gamma \to \mathbb{S}$ is Lipschitz light, and thus (via the comments at the outset of Section 4) that Γ has Lipschitz dimension equal to 1.

Lemma 5.1. *Suppose there exists a constant $C \geq 1$ such that $F\colon \Gamma \to \mathbb{S}$ is C-Lipschitz, and, for any subset $E \subset \mathbb{S}$ such that $\mathrm{Diam}_0(E) > 0$, the $\mathrm{Diam}_0(E)$-components of $F^{-1}(E)$ have d_Δ-diameter bounded by $C\,\mathrm{Diam}_0(E)$. This implies that, for any $r > 0$ and any subset $E \subset \mathbb{S}$ satisfying $\mathrm{Diam}_0(E) \leq r$, the r-components of $F^{-1}(E)$ have d_Δ-diameter bounded by $C'r$, for $C' := \max\{C, 8\}$.*

Proof. Let $r > 0$, and let $E \subset \mathbb{S}$ be such that $\mathrm{Diam}_0(E) \leq r$. We may assume that E is compact. If $\mathrm{Diam}_0(E) = r$, then (by assumption) the r-components of $F^{-1}(E)$ have d_Δ-diameter bounded by Cr. Therefore, we may assume that $\mathrm{Diam}_0(E) < r$. If $r \geq 1/8$, then we note that $\mathrm{Diam}_0(F^{-1}(E)) \leq \mathrm{Diam}_\Delta(\Gamma) \leq 1 \leq 8r$. Thus, we may assume that $r < 1/8$.

We claim that E is contained in a subset $E' \subset \mathbb{S}$ such that $\mathrm{Diam}_0(E') = r$. To see this, we modify the argument employed in [Dav21, Remark 1.9]. Given $x \in \mathbb{S}$, the subset $I_x := \{y \in \mathbb{S} \mid \lambda(x, y) \leq 1/8\}$ is isometric to an interval in $\mathbb{R}$ of length $1/4$. If $x \in E$, then $E \subset I_x$. Let a and b denote the first and last points in E along the interval I_x. Thus $\lambda(a, b) = \mathrm{Diam}_0(E) < r$. Let $c \in I_x$ be such that $\lambda(a, c) = r < 1/8$ and $\mathbb{S}[a, b] \subset \mathbb{S}[a, c] =: E'$. Then $E \subset E'$ and $\mathrm{Diam}_0(E') = r$. By assumption, the r-components of $F^{-1}(E')$ have d_Δ-diameter bounded by Cr. Since the r-components of $F^{-1}(E)$ are contained in the r-components of $F^{-1}(E')$, we arrive at the desired conclusion. □

With the above lemma in hand, we begin our proof that $F_0\colon \Gamma \to \mathbb{S}$ is Lipschitz light. Fix $E \subset \mathbb{S}$ such that $\mathrm{Diam}_0(E) > 0$, and let $M^* \in \mathbb{N}$ be such that

$$2^{-M^*-1} \leq \mathrm{Diam}_0(E) < 2^{-M^*}. \tag{5.1}$$

We may assume that $M^* \geq 3$, else $\mathrm{Diam}_\Delta(F_0^{-1}(E)) \leq \mathrm{Diam}_\Delta(\Gamma) \leq 8\,\mathrm{Diam}_0(E)$. By definition of M^*, there exist two adjacent dyadic subintervals $I, J \in \mathcal{I}_{M^*}$ such that $E \subset I \cup J$. In fact, E may be contained in a single element of $\mathcal{I}_{M^*}$, but it will do no harm to assume E is contained in the union of two such intervals.

We claim that it is sufficient to examine pre-images of $H := I \cup J$. Indeed, given any $\delta > 0$, the δ-components of $F_0^{-1}(E)$ are contained in δ-components of $F_0^{-1}(H)$.

For the remainder of this section, we set $\delta := \mathrm{Diam}_0(E)$.

Lemma 5.2. *Given $n \in \mathbb{N}$ and $U \subset \mathbb{S}$, we have $F_{n+1}(F_0^{-1}(U)) = F_{0,n}^{-1}(U)$.*

Proof. Suppose $x \in F_{n+1}(F_0^{-1}(U))$, so $x = F_{n+1}(w)$ for some $w \in F_0^{-1}(U)$. Then

$$F_{0,n}(x) = F_{0,n}(F_{n+1}(w)) = \lim_{m\to\infty} F_{0,n}(F_{n+1,m}(w)) = \lim_{m\to\infty} F_{0,m}(w) = F_0(w).$$

Since $F_0(w) \in U$, it follows that $F_{n+1}(F_0^{-1}(U)) \subset F_{0,n}^{-1}(U)$.

Next, let $x \in F_{0,n}^{-1}(U)$. Write $z_{n+1} := x$, and choose a point $z_{n+2} \in f_{n+1}^{-1}(z_{n+1})$ so that $f_{n+1}(z_{n+2}) = z_{n+1}$. Inductively, for each $k \geq 2$, define z_{n+k} such that

$$f_{n+k-1}(z_{n+k}) = z_{n+k-1}.$$

We claim that the sequence $\{z_{n+k}\}_{k=1}^{\infty}$ is Cauchy with respect to d_Δ, and thus convergent to some point $z \in \Gamma$. Indeed, for any $1 \leq i < j$, we note that

$$z_{n+i} = f_{n+i} \circ \cdots \circ f_{n+j-1}(z_{n+j}).$$

Let $I \in \mathcal{I}_{n+i}$ denote an interval containing z_{n+j}. For all $k \in \mathbb{N}$, we have $f_{n+i+k}(I) = I$. Therefore, $z_{n+i} \in I$, and so

$$d_\Delta(z_{n+i}, z_{n+j}) \leq \mathrm{Diam}_\Delta(I) \leq \max\{\Delta(J) \mid J \in \mathcal{I}_{n+i}\}.$$

Since $\max\{\Delta(J) \mid J \in \mathcal{I}_{n+i}\} \to 0$ as $i \to \infty$, our claim follows.

Next, we claim that $z = \lim_{k\to+\infty} z_{n+k} \in F_0^{-1}(U)$. Via Lemma 4.3, we have

$$\begin{aligned} F_0(z) &= \lim_{m\to+\infty} F_{0,n+m-1}(z_{n+m}) = \lim_{m\to+\infty} F_{0,n}(F_{n+1,n+m-1}(z_{n+m})) \\ &= \lim_{m\to+\infty} F_{0,n}(z_{n+1}) = F_{0,n}(x) \in U. \end{aligned}$$

Finally, we claim $F_{n+1}(z) = x$. Again via Lemma 4.3, we note that

$$F_{n+1}(z) = \lim_{m\to\infty} F_{n+1,m}(z_{m+1}) = \lim_{m\to\infty} f_{n+1} \circ \cdots \circ f_m(z_{m+1}) = z_{n+1} = x.$$

Therefore, $x \in F_{n+1}(F_0^{-1}(U))$, and so $F_{0,n}^{-1}(U) \subset F_{n+1}(F_0^{-1}(U))$. □

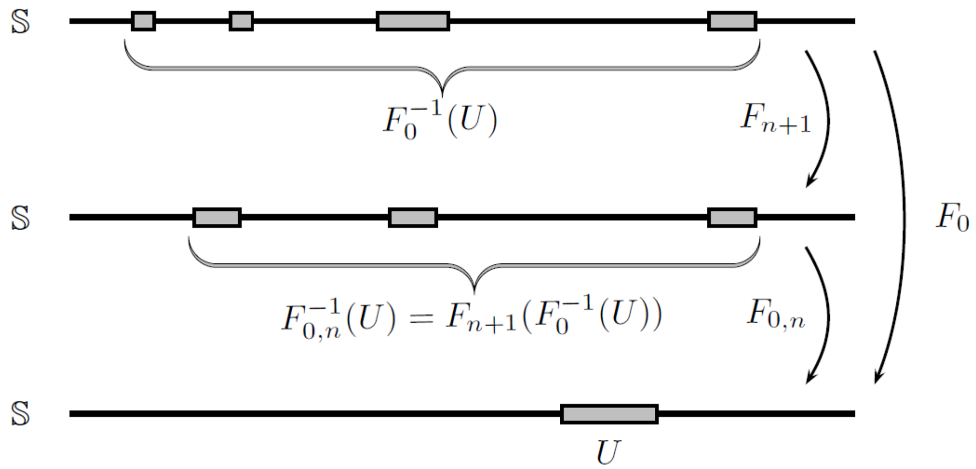


Figure 2. Interaction between the maps F_0, $F_{0.n}$, and F_{n+1}, as described in Lemma 5.2.

Lemma 5.3. *Given $m \leq n \in \mathbb{N}$ and $x \in \Gamma$, we have $F_{m,n}(F_{n+1}(x)) = F_m(x)$.*

Proof. $F_{m,n}(F_{n+1}(w)) = \lim_{k\to\infty} F_{m,n}(F_{n+1,k}(w)) = \lim_{k\to\infty} F_{m,k}(w)$. □

Let W denote any fixed δ-component of $F_0^{-1}(H)$. By Lemma 5.2, we have $F_{n+1}(W) \subset F_{0,n}^{-1}(H)$. Given any $n \in \mathbb{N}$, via Lemma 4.3, the set $F_{n+1}(W)$ is δ-connected in Γ_{n+1}. In particular, it is contained in a single δ-component of $F_{0,n}^{-1}(H)$

in Γ_{n+1}. We denote this δ-component by V_{n+1}. Thus, for every $n \geq 1$, write V_n to denote the δ-component of $F^{-1}_{0,n-1}(H)$ containing $F_n(W)$. We also write $V_0 := H$.

Lemma 5.4. *For $n \in \mathbb{N}$, we have $f_n(V_{n+1}) \subset V_n$. Furthermore, the set V_{n+1} is a δ-component of $f_n^{-1}(V_n)$.*

Proof. If $n = 0$, then $V_1 \subset F_0^{-1}(H) = f_0^{-1}(H)$, and so $f_0(V_1) \subset H = V_0$. We assume $n \geq 1$. Via Lemma 5.3, we have $F_n(W) = f_n(F_{n+1}(W)) \subset f_n(V_{n+1}) \subset F^{-1}_{0,n-1}(H)$. By definition, $F_n(W) \subset V_n \subset F^{-1}_{0,n-1}(H)$. Therefore, the sets V_n and $f_n(V_{n+1})$ are both subsets of $F^{-1}_{0,n-1}(H)$ and have non-trivial intersection. Since $f_n \colon \Gamma_{n+1} \to \Gamma_n$ is 1-Lipschitz, the set $f_n(V_{n+1})$ is δ-connected. Since V_n is a maximal δ-connected subset of $F^{-1}_{0,n-1}(H)$, we must have $f_n(V_{n+1}) \subset V_n$.

Since $V_{n+1} \subset f_n^{-1}(V_n)$ and V_{n+1} is δ-connected, V_{n+1} is contained in a single δ-component of $f_n^{-1}(V_n) \subset F^{-1}_{0,n}(H)$. Since V_{n+1} is a maximal δ-connected subset of $F^{-1}_{0,n}(H)$, the set V_{n+1} is equal to a single δ-component of $f_n^{-1}(V_n)$. □

Lemma 5.5. *There exists $N^* \in \mathbb{N}$ such that, if $n \geq N^*$, then*

$$\mathrm{Diam}_\Delta(W) \leq \mathrm{Diam}_n(V_n) + \delta.$$

Proof. Choose $N^* \in \mathbb{N}$ such that, for $n \geq N^*$, we have $\max\{\Delta(I) \mid I \in \mathcal{I}_n\} < \delta/4$. Let $x, y \in W$. Since, for $n \in \mathbb{N}$, the map F_n fixes elements of $\mathcal{I}_n$, we note that

$$d_\Delta(x, y) \leq d_\Delta(F_n(x), F_n(y)) + 2\max\{\Delta(I) \mid I \in \mathcal{I}_n\} < d_\Delta(F_n(x), F_n(y)) + \delta/2.$$

Furthermore, via Lemma 3.3, we also have (for $n \geq N^*$)

$$d_\Delta(F_n(x), F_n(y)) \leq \mathrm{Diam}_\Delta(F_n(W)) \leq \mathrm{Diam}_n(F_n(W)) + \delta/2.$$

Since $F_n(W) \subset V_n$, we conclude that, for any $n \geq N^*$ and any $x, y \in W$, we have

$$d_\Delta(x, y) \leq \mathrm{Diam}_n(V_n) + \delta.$$

It follows that $\mathrm{Diam}_\Delta(W) \leq \mathrm{Diam}_n(V_n) + \delta$. □

Lemma 5.6. *If, for some $n, m \in \mathbb{N}$, the set V_n is contained in an interval $I_n \in \mathcal{I}_n$ and $\mathrm{Diam}_n(V_n) \geq 2^{-m}\,\mathrm{Diam}_n(I_n)$, then, for any $k \in \mathbb{N}$, we have $\mathrm{Diam}_{n+k}(V_{n+k}) \leq 2^m\,\mathrm{Diam}_n(V_n)$.*

Proof. We first note that $V_{n+1} \subset I_n$, since, by Lemma 5.4, $f_n(V_{n+1}) \subset V_n \subset I_n$, and $f_n(I_n) = I_n$. Via induction, for all $k \in \mathbb{N}$, we have $V_{n+k} \subset I_n$. Therefore, via (3.3), we have $\mathrm{Diam}_{n+k}(V_{n+k}) \leq \mathrm{Diam}_{n+k}(I_n) = \mathrm{Diam}_n(I_n) \leq 2^m\,\mathrm{Diam}_n(V_n)$. □

Lemma 5.7. *Suppose that, for some $n \in \mathbb{N}$, we have*

1. $\mathrm{Diam}_n(V_n) = \mathrm{Diam}_0(V_0)$,
2. *V_n is the union of two adjacent intervals from $\mathcal{I}_m$, for some $m \geq M^*$,*
3. *V_n is not symmetric about a point in $\mathcal{D}_n$,*
4. *V_n is contained in a single interval $I_n \in \mathcal{I}_n$, and*
5. $\mathrm{Diam}_n(V_n) \leq \frac{1}{4}\,\mathrm{Diam}_n(I_n)$.

Under these assumptions, $\mathrm{Diam}_{n+1}(V_{n+1}) \leq 2\,\mathrm{Diam}_n(V_n)$. *If* $\mathrm{Diam}_{n+1}(V_{n+1}) > \mathrm{Diam}_n(V_n)$, *then* $\mathrm{Diam}_{n+1}(V_{n+1}) = 2\,\mathrm{Diam}_n(V_n)$ *and V_{n+1} is symmetric about a point in $\mathcal{D}_{n+3} \setminus \mathcal{D}_{n+1}$. If, on the other hand,* $\mathrm{Diam}_{n+1}(V_{n+1}) < \mathrm{Diam}_n(V_n)$, *then* $\mathrm{Diam}_{n+1}(V_{n+1}) = 0$ *and V_{n+1} is a point in $\mathcal{D}_{n+3} \setminus \mathcal{D}_{n+2}$.*

Proof. Note that Assumption (3) follows from Assumption (4) when $n \geq 1$; we list Assumption (3) to address the case that $n = 0$.

We use binary superscripts to index the four second-generation dyadic subintervals I_n^{00}, I_n^{01}, I_n^{10}, and I_n^{11} in I_n such that they proceed consecutively along the positive orientation in I_n.

If f_n is the identity on I_n, then the lemma is trivial. Therefore, we assume that f_n is a folding map on I_n, and we consider the cases below. We preface this case analysis with the reminder that

$$\delta < \frac{1}{2^{M^*}} = \frac{1}{2}\operatorname{Diam}_0(V_0) = \frac{1}{2}\operatorname{Diam}_n(V_n) \leq \frac{1}{8}\operatorname{Diam}_n(I_n).$$

Case 1: $V_n \subset I_n^{00}$. In this case, $f_n^{-1}(V_n)$ consists of either one δ-component or (if V_n contains the right endpoint of I_n^{00}) it consists of two (see Figure 3). If one, then, via Lemma 5.4, we have $V_{n+1} = f_n^{-1}(V_n) \subset I_n^{00}$ and $\operatorname{Diam}_{n+1}(V_{n+1}) = \operatorname{Diam}_n(V_n)$. If two, then one δ-component is contained in I_n^{00} and satisfies $\operatorname{Diam}_{n+1}(V_{n+1}) = \operatorname{Diam}_n(V_n)$ while the other is a single point located at the midpoint of I_n^{10}.

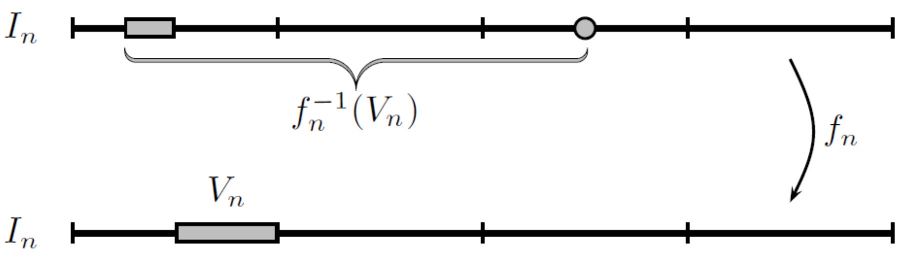


Figure 3. An example of Case 1 (from the proof of Lemma 5.7) in which $f_n^{-1}(V_n)$ consists of two δ-components.

Case 2: $V_n \subset I_n^{01}$. In this case, there are at most three δ-components of $f_n^{-1}(V_n)$: one in I_n^{00} and either one or two in I_n^{10} (see Figure 4). The component in I_n^{00} has d_{n+1}-diameter equal to $\operatorname{Diam}_n(V_n)$. If there are two components in I_n^{10}, then they each have d_{n+1}-diameter equal to $\operatorname{Diam}_n(V_n)$. If there is one component in I_n^{10}, then it has d_{n+1}-diameter equal to $2\operatorname{Diam}_n(V_n)$, and it is symmetric about the midpoint of I_n^{10}. That is, if $\operatorname{Diam}_{n+1}(V_{n+1}) > \operatorname{Diam}_n(V_n)$, then V_{n+1} is symmetric about a point in $\mathcal{D}_{n+3} \setminus \mathcal{D}_{n+1}$ and $\operatorname{Diam}_{n+1}(V_{n+1}) = 2\operatorname{Diam}_n(V_n)$.

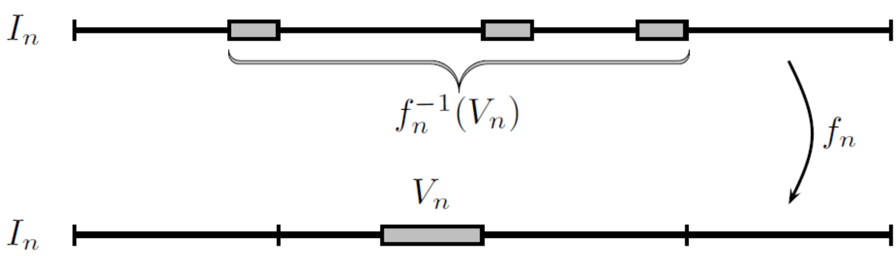


Figure 4. An example of Case 2 (from the proof of Lemma 5.7) in which $f_n^{-1}(V_n)$ consists of three δ-components.

Case 3: $V_n \subset I_n^{10}$. By symmetry, we can apply an argument parallel to that used in Case 2 to conclude that $\operatorname{Diam}_{n+1}(V_{n+1}) \leq 2\operatorname{Diam}_n(V_n)$. Furthermore, if $\operatorname{Diam}_{n+1}(V_{n+1}) > \operatorname{Diam}_n(V_n)$, then V_{n+1} is symmetric about a point in $\mathcal{D}_{n+3} \setminus \mathcal{D}_{n+1}$ and $\operatorname{Diam}_{n+1}(V_{n+1}) = 2\operatorname{Diam}_n(V_n)$.

Case 4. $V_n \subset I_n^{11}$. By symmetry, we can apply an argument parallel to that used in Case 1 to conclude that, either $V_{n+1} \subset I_n^{11}$ has diameter equal to $\operatorname{Diam}_n(V_n)$, or V_{n+1} is a single point at the midpoint of I_n^{01}.

Case 5: V_n is symmetric about a point in $\mathcal{D}_{n+2} \setminus \mathcal{D}_{n+1}$. In this case, $f_n^{-1}(V_n)$ consists of two δ-components. One is contained in I_n^{00} (or I_n^{11}), and the other is contained in I_n^{10} (or I_n^{01}) (see Figure 5). Each component has d_{n+1}-diameter equal to $\mathrm{Diam}_n(V_n)$. Via Lemma 5.4, $\mathrm{Diam}_{n+1}(V_{n+1}) = \mathrm{Diam}_n(V_n)$.

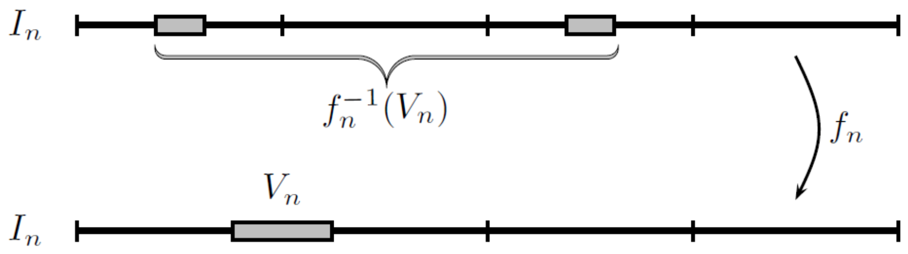


Figure 5. An example of Case 5 (from the proof of Lemma 5.7) in which $f_n^{-1}(V_n)$ consists of two δ-components.

Case 6: V_n is symmetric about a point in $\mathcal{D}_{n+1} \setminus \mathcal{D}_n$. In this case, there are three δ-components of $f_n^{-1}(V_n)$, and each has d_{n+1}-diameter equal to $\mathrm{Diam}_n(V_n)$ (see Figure 6). We note that one of these δ-components is symmetric about a point in $\mathcal{D}_{n+1} \setminus \mathcal{D}_n$. In particular, this is the only case in which V_{n+1} might not be contained in a single interval from $\mathcal{I}_{n+1}$. Via Lemma 5.4, $\mathrm{Diam}_{n+1}(V_{n+1}) = \mathrm{Diam}_n(V_n)$.

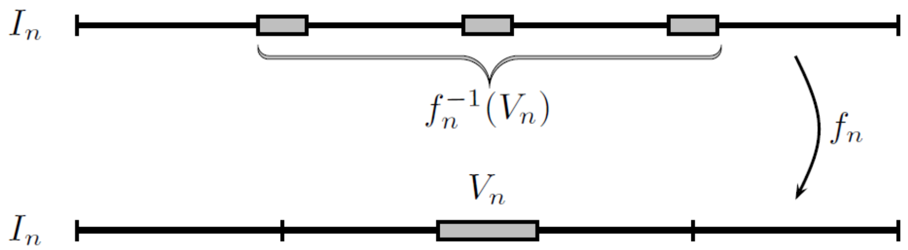


Figure 6. Case 6 (from the proof of Lemma 5.7) in which $f_n^{-1}(V_n)$ consists of three δ-components.

Having exhausted the possible cases, we conclude the proof of the lemma. □

Lemma 5.8. *If there exists $K \in \mathbb{N}$ such that, for all $k \le K$, the set V_k is not symmetric about a point in $\mathcal{D}_{k+2}$, then, either there exists $n \ge N^*$ (for N^* as in Lemma 5.5) such that $\mathrm{Diam}_n(V_n) \le 16\delta$, or, for all $k \le K$,*

(k.1) $\mathrm{Diam}_k(V_k) = \mathrm{Diam}_0(V_0)$,
(k.2) *V_k is the union of two adjacent intervals from $\mathcal{I}_m$ for some $m \ge M^*$,*
(k.3) *V_k is contained in a single interval $I_k \in \mathcal{I}_k$, and*
(k.4) $\mathrm{Diam}_k(V_k) \le \frac{1}{4}\mathrm{Diam}_k(I_k)$.

Proof. Suppose $K \in \mathbb{N}$ is such that, for all $k \le K$, no set V_k is symmetric about a point in $\mathcal{D}_{k+2}$. In preparation for an inductive argument, we affirm the base case $k = 0 \le K$. Indeed, for $V_0 = H$, we have

(0.1) $\mathrm{Diam}_0(V_0) = \mathrm{Diam}_0(V_0)$,
(0.2) V_0 is the union of two adjacent intervals from $\mathcal{I}_{M^*}$, and
(0.3) V_0 is contained in a single interval $I_0 \in \mathcal{I}_0$.
(0.4) $\mathrm{Diam}_0(V_0) \le \frac{1}{4}\mathrm{Diam}_0(I_0)$.

Here (0.4) follows from the fact that $M^* \ge 3$. We next assume that, either there exists $n \ge N^*$ such that $\mathrm{Diam}_n(V_n) \le 16\delta$, or, for all $n \le k-1 \le K-1$, we have

(n.1) $\mathrm{Diam}_n(V_n) = \mathrm{Diam}_0(V_0)$,
(n.2) V_n is the union of two adjacent intervals from $\mathcal{I}_m$, for some $m \ge M^*$,

$(n.3)$ V_n is contained in a single interval $I_n \in \mathcal{I}_n$, and
$(n.4)$ $\mathrm{Diam}_n(V_n) \leq \frac{1}{4}\mathrm{Diam}_n(I_n)$.

Therefore, either there exists $n \geq N^*$ such that $\mathrm{Diam}_n(V_n) \leq 16\delta$, or we satisfy the assumptions of Lemma 5.7 for V_{k-1}. Since V_k is not symmetric about $\mathcal{D}_{k+2}$, the conclusion of Lemma 5.7 tells us that

$(k.1)$ $\mathrm{Diam}_k(V_k) = \mathrm{Diam}_{k-1}(V_{k-1})$.

We note that, if f_{k-1} is the identity on I_{k-1}, then $V_k = V_{k-1}$. If f_{k-1} is a folding map on I_{k-1}, then V_k is the union of two adjacent intervals in $\mathcal{I}_{m+1}$ (here we are using (k.1)). In either case,

$(k.2)$ V_k is the union of two adjacent intervals in $\mathcal{I}_m$, for some $m \geq M^*$.

Furthermore, since V_{k-1} is not symmetric about a point in $\mathcal{D}_k$, Case 6 (in the proof of Lemma 5.7) cannot occur. It follows that

$(k.3)$ V_k is contained in a single interval $I_k \in \mathcal{I}_k$.

Furthermore, if $\mathrm{Diam}_k(V_k) > \frac{1}{4}\mathrm{Diam}_k(I_k)$, then, (since V_k is the union of two adjacent dyadic intervals) we must have either $\mathrm{Diam}_k(V_k) = \frac{1}{2}\mathrm{Diam}_k(I_k)$ or $\mathrm{Diam}_k(V_k) = \mathrm{Diam}_k(I_k)$. Since, by assumption, V_k is not symmetric about a point in $\mathcal{D}_{k+2}$, neither case can occur. Therefore,

$(k.4)$ $\mathrm{Diam}_k(V_k) \leq \frac{1}{4}\mathrm{Diam}_k(I_k)$.

Thus we conclude our inductive argument, and the proof of the lemma. □

Lemma 5.9. *Suppose there exist $n, K \in \mathbb{N}$ such that $n \leq K$, and, for all $n \leq k \leq K$, the set V_k is not symmetric about a point in $\mathcal{D}_{k+1} \setminus \mathcal{D}_k$. Furthermore, suppose n is such that*

$(n.1)$ *Either* $\mathrm{Diam}_n(V_n) = 0$ *or* $\mathrm{Diam}_0(V_0) \leq \mathrm{Diam}_n(V_n) \leq 2\,\mathrm{Diam}_0(V_0)$,
$(n.2)$ *V_n is symmetric about a point in $\mathcal{D}_{n+2} \setminus \mathcal{D}_{n+1}$,*
$(n.3)$ *V_n is contained in a single interval $I_n \in \mathcal{I}_n$, and*
$(n.4)$ $\mathrm{Diam}_n(V_n) \leq \frac{1}{4}\mathrm{Diam}_n(I_n)$.

Under these assumptions on n and K, for all $n \leq k \leq K$, it is true that

$(k.1)$ $\mathrm{Diam}_k(V_k) = \mathrm{Diam}_n(V_n)$,
$(k.2)$ *V_k is symmetric about a point in $\mathcal{D}_{k+2} \setminus \mathcal{D}_{k+1}$,*
$(k.3)$ *V_k is contained in a single interval $I_k \in \mathcal{I}_k$, and*
$(k.4)$ $\mathrm{Diam}_k(V_k) \leq \frac{1}{4}\mathrm{Diam}_n(I_k)$.

Proof. By way of induction, we first note that the base case $k = n$ is included in our assumptions. Thus, we assume that $K > n$ and, for all $n \leq j \leq k-1 \leq K-1$,

$(j.1)$ $\mathrm{Diam}_j(V_j) = \mathrm{Diam}_n(V_n)$,
$(j.2)$ V_j is symmetric about a point in $\mathcal{D}_{j+2} \setminus \mathcal{D}_{j+1}$,
$(j.3)$ V_j is contained in a single interval $I_j \in \mathcal{I}_j$, and
$(j.4)$ $\mathrm{Diam}_j(V_j) \leq \frac{1}{4}\mathrm{Diam}_j(I_j)$.

We prove that the analogous conclusions hold for V_k. Indeed, if f_{k-1} is the identity on I_{k-1}, then $V_k = V_{k-1}$ is symmetric about a point in $\mathcal{D}_{k+1} \setminus \mathcal{D}_k$. Since, by assumption, this cannot occur, we only need to consider the case that f_{k-1} is a folding map on I_{k-1}. Thus, we find ourselves in a situation analogous to Case 5 in the proof of Lemma 5.7 (see Figure 5). It follows that

$(k.1)$ $\mathrm{Diam}_k(V_k) = \mathrm{Diam}_{k-1}(V_{k-1})$,
$(k.2)$ V_k is symmetric about a point in $\mathcal{D}_{k+2} \setminus \mathcal{D}_{k+1}$,
$(k.3)$ V_k is contained in a single interval $I_k \in \mathcal{I}_k$, and

(k.4) $\mathrm{Diam}_k(V_k) \le \frac{1}{4}\,\mathrm{Diam}_k(I_k)$.

This completes the inductive argument, and the proof of the lemma. □

Lemma 5.10. *Suppose there exist $n, K \in \mathbb{N}$ such that $n \le K$, and, for all $n \le k \le K$, the set V_k is not symmetric about a point in $\mathcal{D}_k$. Furthermore, suppose n is such that*

(n.1) *Either* $\mathrm{Diam}_n(V_n) = 0$ *or* $\mathrm{Diam}_0(V_0) \le \mathrm{Diam}_n(V_n) \le 2\,\mathrm{Diam}_0(V_0)$,
(n.2) *V_n is symmetric about a point in $\mathcal{D}_{n+1} \setminus \mathcal{D}_n$,*
(n.3) *V_n is contained in a single interval $I_n \in \mathcal{I}_n$, and*
(n.4) $\mathrm{Diam}_n(V_n) \le \frac{1}{4}\,\mathrm{Diam}_n(I_n)$.

Under these assumptions, for all $n \le k \le K$,

(k.1) $\mathrm{Diam}_k(V_k) = \mathrm{Diam}_n(V_n)$,
(k.2) *V_k is symmetric about a point in $\mathcal{D}_{k+1}$,*
(k.3) *V_k is contained in a single interval $I_k \in \mathcal{I}_k$, and*
(k.4) $\mathrm{Diam}_k(V_k) \le \frac{1}{4}\,\mathrm{Diam}_n(I_k)$.

Proof. The proof consists of a straightforward inductive argument similar to that used to prove Lemma 5.9. Indeed, we consider a situation analogous to Case 6 in the proof of Lemma 5.7 (see Figure 6). We omit the details. □

We are now ready to prove the following, which, via Lemmas 5.5 and 5.1, will be sufficient to prove that $F_0\colon \Gamma \to \mathbb{S}$ is Lipschitz light.

Lemma 5.11. *There exists $n \in \mathbb{N}$ such that $n \ge N^*$ (for N^* as in Lemma 5.5) and* $\mathrm{Diam}_n(V_n) \le 128\delta$.

Proof. If there is no index n_1 for which V_{n_1} is symmetric about a point in $\mathcal{D}_{n_1+2}$, then, by Lemma 5.8, we conclude that $\mathrm{Diam}_{N^*}(V_{N^*}) \le 16\delta$. Therefore, we may assume n_1 is the minimal such index. We first consider the case that $n_1 \ge 1$. By the definition of n_1 and Lemma 5.8, we may assume that

- $\mathrm{Diam}_{n_1-1}(V_{n_1-1}) = \mathrm{Diam}_0(V_0)$,
- V_{n_1-1} is the union of two adjacent intervals from $\mathcal{I}_m$, for some $m \ge M^*$,
- V_{n_1-1} is contained in a single interval $I_{n_1-1} \in \mathcal{I}_{n_1-1}$, and
- $\mathrm{Diam}_{n_1-1}(V_{n_1-1}) \le \frac{1}{4}\,\mathrm{Diam}_{n_1-1}(I_{n_1-1})$.

Via Lemma 5.4, it follows from the definition of f_{n_1-1} and the minimality of n_1 that V_{n_1} is symmetric about a point in $\mathcal{D}_{n_1+2} \setminus \mathcal{D}_{n_1+1}$, and, either $\mathrm{Diam}_{n_1}(V_{n_1}) = 0$, or

$$\mathrm{Diam}_0(V_0) \le \mathrm{Diam}_{n_1}(V_{n_1}) \le 2\,\mathrm{Diam}_{n_1-1}(V_{n_1-1}) = 2\,\mathrm{Diam}_0(V_0).$$

Furthermore, V_{n_1} is contained in a single interval $I_{n_1} \in \mathcal{I}_{n_1}$. If $\mathrm{Diam}_{n_1}(V_{n_1}) > \frac{1}{4}\,\mathrm{Diam}_{n_1}(I_{n_1})$, then, via Lemma 5.6 and (5.1), there exists $n \ge N^*$ such that

$$\mathrm{Diam}_n(V_n) \le 4\,\mathrm{Diam}_{n_1}(V_{n_1}) \le 8\,\mathrm{Diam}_0(V_0) \le 32\delta.$$

Therefore, we assume that $\mathrm{Diam}_{n_1}(V_{n_1}) \le \frac{1}{4}\,\mathrm{Diam}_{n_1}(I_{n_1})$.

If there is no index $n > n_1$ such that V_n is symmetric about a point in $\mathcal{D}_{n+1} \setminus \mathcal{D}_n$, then, by Lemma 5.9, we conclude that there exists $n \ge N^*$ such that

$$\mathrm{Diam}_n(V_n) = \mathrm{Diam}_{n_1}(V_{n_1}) \le 2\,\mathrm{Diam}_0(V_0) \le 8\delta.$$

Therefore, we may assume that there exists $n_2 > n_1$ minimal such that V_{n_2} is symmetric about a point in $\mathcal{D}_{n_2+1} \setminus \mathcal{D}_{n_2}$. By Lemma 5.9,

- $\mathrm{Diam}_{n_2-1}(V_{n_2-1}) = \mathrm{Diam}_{n_1}(V_{n_1})$,
- V_{n_2-1} is symmetric about a point in $\mathcal{D}_{n_2+1} \setminus \mathcal{D}_{n_2}$,
- V_{n_2-1} is contained in a single interval $I_{n_2-1} \in \mathcal{I}_{n_2-1}$, and

- $\mathrm{Diam}_{n_2-1}(V_{n_2-1}) \le \frac{1}{4}\,\mathrm{Diam}_{n_1}(I_{n_2-1})$.

In particular, $\mathrm{Diam}_{n_2-1}(V_{n_2-1}) \le 2\,\mathrm{Diam}_0(V_0)$.

Since V_{n_2-1} is symmetric about a point in $\mathcal{D}_{n_2+1} \setminus \mathcal{D}_{n_2}$, it follows that f_{n_2-1} is the identity on I_{n_2-1}. Therefore, $V_{n_2} = V_{n_2-1}$ and V_{n_2} is contained in an interval $I_{n_2} \in \mathcal{I}_{n_2}$. If $\mathrm{Diam}_{n_2}(V_{n_2}) > \frac{1}{4}\,\mathrm{Diam}_{n_2}(I_{n_2})$, then, via Lemma 5.6 and (5.1), there exists $n \ge N^*$ such that

$$\mathrm{Diam}_n(V_n) \le 4\,\mathrm{Diam}_{n_2}(V_{n_2}) = 4\,\mathrm{Diam}_{n_2-1}(V_{n_2-1}) \le 8\,\mathrm{Diam}_0(V_0) \le 32\delta.$$

Therefore, we may assume that $\mathrm{Diam}_{n_2}(V_{n_2}) \le \frac{1}{4}\,\mathrm{Diam}_{n_2}(I_{n_2})$,

If there is no index $n > n_2$ such that V_n is symmetric about a point in $\mathcal{D}_n$, then, via Lemma 5.10, we conclude that there exists $n \ge N^*$ such that

$$\mathrm{Diam}_n(V_n) = \mathrm{Diam}_{n_2}(V_{n_2}) \le 8\delta.$$

Thus we may assume that there exists $n_3 > n_2$ minimal such that V_{n_3} is symmetric about a point in $\mathcal{D}_{n_3}$.

If $\mathrm{Diam}_{n_3}(V_{n_3}) = 0$, then, for all $n \ge n_3$, we have $f_n^{-1}(V_{n_3}) = V_{n_3}$ (since f_n fixes points in $\mathcal{D}_{n_3}$). Therefore, there exists $n \ge N^*$ such that $\mathrm{Diam}_n(V_n) = 0 < \delta$. Thus we may assume that $\mathrm{Diam}_{n_3}(V_{n_3}) > 0$. In this case, we note that n_3 is minimal such that V_{n_3} is not contained in a single interval from $\mathcal{I}_{n_3}$. Indeed, V_{n_3} is contained in the interior of the union of two adjacent intervals from $\mathcal{I}_{n_3}$ whose union forms $I_{n_3-1} \in \mathcal{I}_{n_3-1}$. It is also easy to verify (via Lemma 5.10) that $\mathrm{Diam}_{n_3}(V_{n_3}) = \mathrm{Diam}_{n_2}(V_{n_2})$.

Write I'_{n_3} to denote the left dyadic child of I_{n_3-1}, and write $V'_{n_3} := V_{n_3} \cap I'_{n_3}$. Inductively, for each $k \ge 1$, write I'_{n_3+k} to denote the right dyadic child of I'_{n_3+k-1}. Since $\mathrm{Diam}_{n_3}(V'_{n_3}) \le \frac{1}{4}\,\mathrm{Diam}_{n_3}(I'_{n_3})$, we have $V'_{n_3} \subset I'_{n_3+2}$. If $\mathrm{Diam}_{n_3}(V'_{n_3}) = \frac{1}{4}\,\mathrm{Diam}_{n_3}(I'_{n_3})$, then

$$\mathrm{Diam}_{n_3-1}(V_{n_3-1}) \ge \mathrm{Diam}_{n_3}(V'_{n_3}) = \frac{1}{4}\,\mathrm{Diam}_{n_3}(I'_{n_3}) \ge \frac{1}{8}\,\mathrm{Diam}_{n_3-1}(I_{n_3-1}).$$

Therefore, by Lemma 5.6 and (5.1), there exists $n \ge N^*$ such that

$$\mathrm{Diam}_n(V_n) \le 8\,\mathrm{Diam}_{n_3-1}(V_{n_3-1}) \le 16\,\mathrm{Diam}_0(V_0) \le 64\delta.$$

Therefore, we may assume that $\mathrm{Diam}_{n_3}(V'_{n_3}) < \frac{1}{4}\,\mathrm{Diam}_{n_3}(I'_{n_3})$.

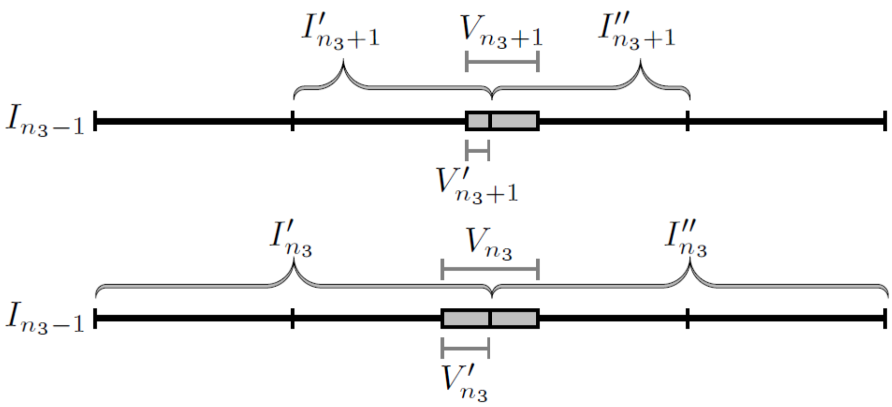


Figure 7. An example of how V'_{n_3} and V'_{n_3+1} can situated within I'_{n_3}. Note that in this example f_{n_3} is the identity map on I''_{n_3} and a folding map on I'_{n_3}.

For each $n \ge n_3$, we define $V'_n := V_n \cap I'_{n_3}$, and we define the ratio

$$R(n) := \frac{\mathrm{Diam}_n(V'_n)}{\mathrm{Diam}_n(I'_n)}.$$

Since $R(n_3) < \frac{1}{4}$, we have $\mathrm{Diam}_{n_3+1}(V'_{n_3+1}) = \mathrm{Diam}_{n_3}(V'_{n_3})$. If f_{n_3} is a folding map on I'_{n_3}, then $V'_{n_3+1} \subset I'_{n_3+3}$ and $R(n_3+1) = R(n_3) < \frac{1}{4}$. If f_{n_3} is the identity on I_{n_3}, then $R(n_3+1) = 2R(n_3)$. If $R(n_3+1) \geq \frac{1}{4}$, then define $n_4 := n_3 + 1$. If not, then we proceed inductively, and assume that, for all $n_3 + 1 \leq j \leq k-1$, we have $V'_{n_3+j} \subset I'_{n_3+j+2}$, $\mathrm{Diam}_{n_3+j}(V'_{n_3+j}) = \mathrm{Diam}_{n_3}(V'_{n_3})$, and $R(n_3+j) < \frac{1}{4}$.

Under this inductive hypothesis, we examine V'_{n_3+k}. Either f_{n_3+k-1} is a folding map on I'_{n_3+k-1}, and $R(n_3+k) = R(n_3+k-1) < \frac{1}{4}$, or f_{n_3+k-1} is the identity on I'_{n_3+k-1}, and $R(n_3+k) = 2R(n_3+k-1)$. If $R(n_3+k) \geq \frac{1}{4}$, then write $n_4 := n_3 + k$.

Via induction, we are faced with two possibilities: either there exists $n_4 > n_3$ minimal such that $V'_{n_4} \subset I'_{n_4}$ and $R(n_4) \geq \frac{1}{4}$, or, for all $n > n_3$, we have $V'_n \subset I'_{n+2}$ and $R(n) < \frac{1}{4}$. We claim this latter case cannot occur. Indeed, we note that, for all $n > n_3$, we have $R(n+1) \geq R(n)$. Moreover, $R(n+1) > R(n)$ if and only if f_n is the identity on I'_n and $R(n+1) = 2R(n)$. Since $\mathrm{Diam}_n(I'_n) \to 0$, the map f_n must be the identity on I'_n infinitely often, and thus $R(n+1) = 2R(n)$ infinitely often. This would imply that $R(n) \to +\infty$, and this contradiction proves our claim.

Thus we have $V'_{n_4} = V_{n_4} \cap I'_{n_3} \subset I'_{n_4}$ such that $\mathrm{Diam}_{n_4}(V'_{n_4}) \geq \frac{1}{4}\,\mathrm{Diam}_{n_4}(I'_{n_4})$. Recall that, for any $n \geq n_4$, the map f_n fixes elements of $\mathcal{I}_{n_4}$ and $\mathcal{I}_{n_3}$. Therefore, for any $n \geq n_4$, we have $V'_n \subset I'_{n_4}$, and so

$$\begin{aligned}\mathrm{Diam}_n(V'_n) &\leq \mathrm{Diam}_n(I'_{n_4}) = \mathrm{Diam}_{n_4}(I'_{n_4})\\ &\leq 4\,\mathrm{Diam}_{n_4}(V'_{n_4}) = 4\,\mathrm{Diam}_{n_3}(V'_{n_3})\\ &\leq 8\,\mathrm{Diam}_{n_3}(V_{n_3}) = 8\,\mathrm{Diam}_{n_2}(V_{n_2}) \leq 64\delta\end{aligned}$$

An analogous argument applies to the set $V''_{n_3} := V_{n_3} \cap I''_{n_3}$, where I''_{n_3} denotes the right dyadic child of I_{n_3-1} (see Figure 7). In particular, there exists $n_5 > n_3$ such that, if $n \geq n_5$, then $\mathrm{Diam}_n(V''_n) \leq 64\delta$. Therefore, there exists $n \geq N^*$ such that

$$\begin{aligned}\mathrm{Diam}_n(V_n) &\leq \mathrm{Diam}_n(V_n \cap I'_{n_3}) + \mathrm{Diam}_n(V_n \cap I''_{n_3})\\ &= \mathrm{Diam}_n(V'_n) + \mathrm{Diam}_n(V''_n) \leq 128\delta.\end{aligned}$$

We finish by briefly considering the case that $n_1 = 0$. If V_0 is symmetric about a point in $\mathcal{D}_2 \setminus \mathcal{D}_1$, then we argue as in the case that $n_1 \geq 1$. If V_0 is symmetric about a point in $\mathcal{D}_1 \setminus \mathcal{D}_0$, then we apply the argument utilized in our above analysis of V_{n_2}. If V_0 is symmetric about the point in $\mathcal{D}_0$, then we apply (a simple modification of) the argument used in our above analysis of V_{n_3}. □

Appendix A. Lipschitz light maps as inverse limits

Here we re-frame our construction of the Lipschitz light map $F_0\colon \Gamma \to \mathbb{S}$ using the language of inverse systems and inverse limits. We generally follow the notation of [RZ10, Section 1.1]. In particular, an *inverse system* (in our case indexed by $\mathbb{N}$) consists of a collection of topological spaces $\{X_i\}$ along with a collection of continuous mappings $\varphi_{ij}\colon X_i \to X_j$ (defined for $i \geq j$) such that, for $i \geq j \geq k$, we have $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$. Here we assume that φ_{ii} is the identity map. We denote such an inverse system by $\{X_i, \varphi_{ij}\}$.

Given any topological space Y and continuous mappings $\{\psi_i\colon Y \to X_i\}$, we say that the mappings $\{\psi_i\}$ are *compatible* with the inverse system $\{X_i, \varphi_{ij}\}$ provided that, for any $i \geq j$, we have $\varphi_{ij} \circ \psi_i = \psi_j$.

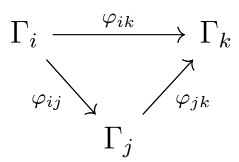
A topological space X along with compatible mappings $\{\varphi_i\colon X \to X_i\}$ constitute an *inverse limit* of $\{X_i, \varphi_{ij}\}$ provided that, for any topological space Y and compatible continuous mappings $\{\psi_i\colon Y \to X_i\}$, there exists a unique continuous mapping $\psi\colon Y \to X$ such that, for all $i \in \mathbb{N}$, we have $\varphi_i \circ \psi = \psi_i$.

Given any bounded turning Jordan circle Γ, we assume (up to a bi-Lipschitz homeomorphism) that Γ is an element of $\mathcal{S}_1$ (as described in Theorem 4.1). Thus $\Gamma = (\mathbb{S}, d_\Delta)$. We then obtain the corresponding sequence $\Gamma_n = (\mathbb{S}, d_n)$ such that $d_n \to d_\Delta$ as $n \to +\infty$ (see Lemma 3.3).

We re-index the maps $F_{m,n} : \Gamma_{n+1} \to \Gamma_m$ as defined in Section 4 in order to define $\varphi_{ij}\colon \Gamma_i \to \Gamma_j$ as

$$\varphi_{ij} := \begin{cases} f_j \circ \cdots \circ f_{i-1} & \text{if } i > j, \\ \mathrm{id} & \text{if } i = j. \end{cases}$$

It is then clear that (for $i \geq j \geq k$) the following diagram commutes:



In particular, $\{\Gamma_i, \varphi_{ij}\}$ is an inverse system.

It is important to note that $\Gamma = (\mathbb{S}, d_\Delta)$ and the spaces $\{\Gamma_i\} = \{(\mathbb{S}, d_i)\}$ are pairwise homeomorphic and share the underlying set $\mathbb{S}$ in common. Therefore, we may view the maps φ_{ij} as self-maps of $\mathbb{S}$. Of course, the metric behavior of φ_{ij} will depend on the metrics with which its domain and range are equipped. With this in mind, for a fixed $j \in \mathbb{N}$, Lemma 4.3 implies that the maps $\varphi_{ij}\colon \Gamma \to \Gamma_j$ uniformly converge to $\varphi_j\colon \Gamma \to \Gamma_j$ as $i \to \infty$. Therefore, given any $x \in \mathbb{S}$ and $i \geq j$, we have

$$\varphi_{ij} \circ \varphi_i(x) = \lim_{k\to\infty} \varphi_{ij} \circ \varphi_{ki}(x) = \lim_{k\to\infty} \varphi_{kj}(x) = \varphi_j(x).$$

In particular, $\{\varphi_i\}$ is compatible with $\{\Gamma_i, \varphi_{ij}\}$.

In what follows, we say that a sequence $\{x_i\} \in \mathbb{S}^{\mathbb{N}}$ is *compatible* with the maps $\{\varphi_{ij}\}$ provided that, for each $i \geq j$, we have $\varphi_{ij}(x_i) = x_j$.

Lemma A.1. *Given $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that, for any sequence $\{x_i\} \in \mathbb{S}^{\mathbb{N}}$ that is compatible with $\{\varphi_{ij}\}$, if $i, j \geq N_\varepsilon$, then $d_\Delta(x_i, x_j) < \varepsilon$.*

Proof. Fix $\varepsilon > 0$, and choose $N_\varepsilon \in \mathbb{N}$ such that $\max\{\Delta(I) \mid I \in \mathcal{I}_{N_\varepsilon}\} < \varepsilon$. Given any $i \geq j \geq N_\varepsilon$, let $J \in \mathcal{I}_j$ denote an interval containing x_i. For all $k \geq j$, we have $\varphi_{kj}(J) = J$. In particular, $x_j = \varphi_{ij}(x_i) \in J$, and so

$$d_\Delta(x_i, x_j) \leq \mathrm{Diam}_\Delta(J) \leq \max\{\Delta(I) \mid I \in \mathcal{I}_{N_\varepsilon}\} < \varepsilon. \qquad \square$$

By (the proof of) [RZ10, Proposition 1.1.1], the inverse limit to $\{\Gamma_i, \varphi_{ij}\}$ is given by the subset $S \subset \mathbb{S}^{\mathbb{N}}$ consisting of sequences compatible with $\{\varphi_{ij}\}$. Here S is equipped with the topology inherited as a subspace of $\mathbb{S}^{\mathbb{N}}$ equipped with the product topology.

Lemma A.2. *The space S is homeomorphic to $\mathbb{S}$.*

Proof. Let $\varphi\colon \mathbb{S} \to S$ be defined component-wise as $\varphi(x) = \{\varphi_i(x)\}$. To see that φ is surjective, fix any $\{x_i\} \in S$. By Lemma A.1, we obtain the point $x := \lim_{i\to\infty} x_i$. Here the limit is taken in $\Gamma = (\mathbb{S}, d_\Delta)$. For any $i \geq j$, we have $x_j = \varphi_{ij}(x_i)$. Therefore, taking a limit as $i \to \infty$ (and with reference to Lemma 4.3), we obtain

$$\varphi_j(x) = \lim_{i\to\infty} \varphi_{ij}(x_i) = x_j. \tag{A.1}$$

To see that φ is injective, we note that, for any $x \in \mathbb{S}$, we have

$$x = \lim_{i\to\infty} \varphi_i(x). \tag{A.2}$$

Here the limit is taken in $\Gamma = (\mathbb{S}, d_\Delta)$. Indeed, this easily follows from the facts that φ_i fixes intervals in $\mathcal{I}_i$ and $\max\{\Delta(I) \mid I \in \mathcal{I}_i\} \to 0$. Therefore, if $\varphi(x) = \varphi(y)$ for points $x, y \in \mathbb{S}$, we have

$$x = \lim_{i\to\infty} \varphi_i(x) = \lim_{i\to\infty} \varphi_i(y) = y.$$

Since φ is continuous in each component, it is continuous (by virtue of the product topology). Since $\mathbb{S}$ is compact and S is Hausdorff (by [IM12, Theorem 172], where we note that [HM12] *defines* the inverse limit of $\{\Gamma_i, \varphi_{ij}\}$ to be the space S), the continuous bijection $\varphi\colon \mathbb{S} \to S$ is a homeomorphism. □

Theorem A.3. *$\{\Gamma, \varphi_i\}$ is the inverse limit of $\{\Gamma_i, \varphi_{ij}\}$.*

Proof. Let Y denote any topological space and $\{\psi_i\}$ a collection of compatible continuous mappings $\psi_i\colon Y \to \Gamma_i$. Given $y \in Y$, define $\hat\psi(y) = \{\psi_i(y)\} \in S$. Since $\hat\psi$ is continuous in each component, it is continuous (by virtue of the product topology). Via (the proof of) Lemma A.2, we obtain a continuous map $\psi\colon Y \to \Gamma$ given by $\psi(y) := \varphi^{-1} \circ \hat\psi(y) = \lim_{i\to\infty} \psi_i(y)$. Moreover, given $y \in Y$ and $j \in \mathbb{N}$, as in (A.1), we have

$$\varphi_j(\psi(y)) = \lim_{i\to\infty} \varphi_{ij}(\psi_i(y)) = \psi_j(y).$$

Suppose $\tau\colon Y \to \Gamma$ is some other continuous map such that, for all $i \in \mathbb{N}$, we have $\varphi_i \circ \tau = \psi_i$. Then, for any $y \in Y$, we observe (as in (A.2)) that

$$\psi(y) = \lim_{i\to\infty} \varphi_i(\psi(y)) = \lim_{i\to\infty} \psi_i(y) = \lim_{i\to\infty} \varphi_i(\tau(y)) = \tau(y).$$

Therefore, ψ as defined above is the unique map such that, for $i \in \mathbb{N}$, we have $\varphi_i \circ \psi = \psi_i$. Moreover, we confirm that $\{\Gamma, \varphi_i\}$ is the inverse limit of $\{\Gamma_i, \varphi_{ij}\}$. □

We conclude by remarking that our inverse limit construction is somewhat different from that of [CK13, Theorem 1.11]. Nevertheless, we note that the distance d_Δ on our inverse limit space is similar in spirit to the distance $\hat d_\infty$ on the inverse limit of [CK13] (as defined in [CK13, Lemma 2.4]).

Acknowledgments. The author thanks Guy C. David for enlightening dialogue regarding the results and theoretical context of both this paper and [Dav21]. In particular, the author is grateful for David's suggestion of Corollary 2.6. The author also thanks Daniel Meyer for comments on an earlier draft of this paper, and the anonymous referee for numerous helpful suggestions. For example, Appendix A was included in response to a suggestion of the referee.

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Received 12 August 2020 • Accepted 12 March 2021 • Published online 10 January 2022

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