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Weighted norm inequalities for the maximal operator on $L^{p(\cdot)}$ over spaces of homogeneous type

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Abstract. Given a space of homogeneous type (X, d, μ), we prove strong-type weighted norm inequalities for the Hardy–Littlewood maximal operator over the variable exponent Lebesgue spaces $L^{p(\cdot)}$. We prove that the variable Muckenhoupt condition $A_{p(\cdot)}$ is necessary and sufficient for the strong type inequality if $p(\cdot)$ satisfies log-Hölder continuity conditions and $1 < p_- \le p_+ < \infty$. Our results generalize to spaces of homogeneous type the analogous results in Euclidean space proved by Cruz-Uribe, Fiorenza and Neugebauer (2012).

Maksimaalioperaattorin painotetut normiepäyhtälöt homogeenisen tyypin avaruuden funktioluokissa $L^{p(\cdot)}$

Tiivistelmä. Annetussa homogeenisen tyypin avaruudessa (X, d, μ) todistamme Hardyn–Littlewoodin maksimaalioperaattorille vahvan tyypin painotettuja normiepäyhtälöitä muuttuvaeksponenttisissa Lebesguen avaruuksissa $L^{p(\cdot)}$. Osoitamme, että muuttuvaeksponenttinen Muckenhouptin ehto $A_{p(\cdot)}$ on riittävä ja välttämätön vahvan tyypin epäyhtälölle, mikäli $p(\cdot)$ toteuttaa logaritmisen Hölderin jatkuvuusehdon ja $1 < p_- \le p_+ < \infty$. Tuloksemme yleistävät homogeenisen tyypin avaruuksiin Cruz-Uriben, Fiorenzan ja Neugebauerin (2012) todistamia vastaavia euklidisen avaruuden tuloksia.

1. Introduction

This paper is concerned with extending established results in the theory of variable exponent Lebesgue spaces to the setting of spaces of homogeneous type. In recent decades—largely as a result of [29]—interest has arisen over the natural extension of the classical Lebesgue spaces L^p in which the exponent p is itself a function of the underlying space; see [10, 19] for extensive discussions on such spaces. In particular, the development of a variable exponent Calderón–Zygmund theory has been the subject of much research, especially since Cruz-Uribe, Fiorenza, and Neugebauer [12], building on the work of Diening [18], proved that the Hardy–Littlewood maximal operator is bounded on $L^{p(\cdot)}$ for $p(\cdot)$ satisfying a continuity condition weaker than Hölder continuity.

Just as with the development of classical Calderón–Zygmund theory, many results in the theory of variable exponent spaces have only been proved to hold over $\mathbb{R}^n$ or in metric spaces (see [23] for the latter). In the 1970s, this restriction was removed by Coifman and Weiss, who in [7] introduced spaces of homogeneous type, which they later developed in [8] as the natural spaces onto which Calderón–Zygmund theory could be generalized. A logical step for variable exponent theory, then, is

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to perform the same generalization for $L^{p(\cdot)}$. Such a program has been underway since the maximal operator was shown to be bounded over $\mathbb{R}^n$. Early results include [20, 21, 26, 31]; for a more detailed history, see [1].

Spaces of homogeneous type have a topological structure weaker than metric spaces: namely, that of a quasi-metric space.

Definition 1.1. Given a set X and a function $d\colon X\times X\to[0,\infty)$, we say (X,d) is a *quasi-metric space* if

(1) $d(x,y)=0$ if and only if $x=y$,
(2) $d(x,y)=d(y,x)$ for all $x,y\in X$,
(3) there exists a constant $A_0\geq 1$ for which $d(x,y)\leq A_0(d(x,z)+d(z,y))$ for all $x,y,z\in X$.

The constant A_0 is referred to as the *quasi-metric constant*. Some authors (e.g. [27]) also loosen condition (2) to symmetry up to a constant, $d(x,y)\leq Kd(y,x)$. An important property of quasi-metric spaces is that quasi-metric balls need not be open; however, Macías and Segovia [30] showed that there is always an equivalent quasi-metric whose balls are all open. Analysis can be done on quasi-metric spaces without additional structure—see [2]—but typically measures on quasi-metric measure spaces are taken to be at least doubling.

Definition 1.2. A measure μ on a space X is said to be *doubling* if there exists a constant $C_\mu\geq 1$ such that, for any $x\in X$ and $r>0$,

$$0<\mu(B(x,2r))\leq C_\mu\mu(B(x,r))<\infty,$$

where $B(x,r)=\{y\in X\colon d(x,y)<r\}$ is the quasi-metric ball of radius r centered at x. The constant C_μ is called the *doubling constant*.

The assumption that balls have positive, finite measure avoids trivial measures, and also ensures that μ is σ-finite. We are now led naturally to the well-known setting of spaces of homogeneous type.

Definition 1.3. A *space of homogeneous type* is a triple (X,d,μ) where X is a non-empty set, d is a quasi-metric on X, and μ is a doubling regular measure on the σ-algebra generated by quasi-metric balls and open sets.

Hereafter, we will let (X,d,μ) be a fixed space of homogeneous type, and often denote it simply by X. The assumption that μ is regular is used only to apply the Lebesgue Differentiation Theorem in Section 5; see [2] for the possibility of weakening this hypothesis.

We now introduce some basic notions of the variable exponent spaces $L^{p(\cdot)}(X)$.

Definition 1.4. Define $\mathcal{P}(X)$ to be the set of measurable functions $p(\cdot)\colon X\to[1,\infty]$. The elements of $\mathcal{P}(X)$ are called *exponent functions*. Given an exponent function and a set $E\subseteq X$, we define

$$p_-(E)=\operatorname*{ess\,inf}_{x\in E}p(x),\quad p_+(E)=\operatorname*{ess\,sup}_{x\in E}p(x).$$

In particular we denote $p_-(X)=p_-$ and $p_+(X)=p_+$.

When considering the conjugate exponent function $p'(\cdot)$ defined by $p'(x)=\frac{p(x)}{p(x)-1}$ (with the convention that $1/0=\infty$ and $1/\infty=0$), to avoid the ambiguity inherent to notation like "p'_+" we will always write $(p')_+$ to denote the essential supremum of $p'(\cdot)$, etc.

Definition 1.5. Given an exponent $p(\cdot) \in \mathcal{P}(X)$, define

- $X_\infty = \{x \in X : p(x) = \infty\}$,
- $X_1 = \{x \in X : p(x) = 1\}$,
- $X_* = \{x \in X : 1 < p(x) < \infty\}$.

Intuitively, given an exponent $p(\cdot) \in \mathcal{P}(X)$, we would like to define $L^{p(\cdot)}(X)$ as the collection of all functions on X satisfying

$$\int_X |f|^{p(x)}\, d\mu < \infty.$$

To properly formulate this, we require an analog to the constant exponent p-norm. It is well-known that the following modular function provides such an analog.

Definition 1.6. The space $L^{p(\cdot)}(X)$ is the set of measurable functions f on X for which the modular

$$\rho_{p(\cdot)}(f) = \int_{X\setminus X_\infty} |f(x)|^{p(x)}\, d\mu + \|f\|_{L^\infty(X_\infty)}$$

satisfies $\rho_{p(\cdot)}(f/\lambda) < \infty$ for some $\lambda > 0$.

If $p_+ < \infty$, we say f is locally $p(\cdot)$-integrable if $\rho_{p(\cdot)}(f\chi_B) < \infty$ for every ball $B \subset X$. Often we write $L^{p(\cdot)}$ for $L^{p(\cdot)}(X)$; similarly, when the exponent is clear from context, we will simply write $\rho_{p(\cdot)} = \rho$. It is shown in [10, 19] that this modular induces the following Luxembourg norm on $L^{p(\cdot)}$.

Proposition 1.7. *The function $\|\cdot\| : L^{p(\cdot)}(X) \to \mathbb{R}$ given by*

$$\|f\|_{L^{p(\cdot)}(X)} = \inf\{\lambda > 0 : \rho(f/\lambda) \le 1\}$$

is a norm on $L^{p(\cdot)}(X)$, which is a Banach space with respect to $\|\cdot\|_{L^{p(\cdot)}(X)}$.

When the underlying space is clear from context, we write $\|\cdot\|_{L^{p(\cdot)}} = \|\cdot\|_{p(\cdot)}$. In the case that $p(\cdot)$ is constant, $p(\cdot) = p$, it is easy to show that $\|\cdot\|_{p(\cdot)}$ reduces to the usual norm in L^p.

For most purposes, the set $\mathcal{P}(X)$ of possible exponent functions is far too broad to prove meaningful results. Indeed, even piecewise-constant exponents lose many of the properties of classical L^p spaces (see [10]), such as the boundedness of the maximal operator. In the study of variable exponent theory, it has become clear that in many cases a sufficient condition on the exponent is log-Hölder continuity.

Definition 1.8. We say that an exponent $p(\cdot) \in \mathcal{P}(X)$ is locally Log-Hölder continuous, $p(\cdot) \in \mathrm{LH}_0$, if there exists a constant C_0 such that for any $x, y \in X$ with $d(x, y) < 1/2$,

$$|p(x) - p(y)| < \frac{-C_0}{\log(d(x, y))}.$$

We say that $p(\cdot)$ is Log-Hölder continuous at infinity, $p(\cdot) \in \mathrm{LH}_\infty$, with respect to a base point $x_0 \in X$, if there exist constants C_∞ and p_∞ such that for every $x \in X$,

$$|p(x) - p_\infty| < \frac{C_\infty}{\log(e + d(x, x_0))}.$$

We call C_0 the $\mathrm{LH}_{\mathbf{0}}$ constant of $p(\cdot)$ and C_∞ the LH_∞ constant of $p(\cdot)$. If $p(\cdot) \in \mathrm{LH} = \mathrm{LH}_0 \cap \mathrm{LH}_\infty$, we say that $p(\cdot)$ is globally Log-Hölder continuous.

Note that $p(\cdot) \in \mathrm{LH}$ implies that $p_+ < \infty$, a condition that is crucial to most of the results in this paper. Note also that the above definition appears to depend on the choice of base point x_0. In fact, such a choice is irrelevant, as shown by the following lemma, which was proved in [1].

Lemma 1.9. *Choose $x_0, y_0 \in X$. If $p(\cdot) \in LH_\infty$ with respect to x_0, then $p(\cdot) \in LH_\infty$ with respect to y_0.*

Whenever x_0 is not chosen explicitly, we assume that X has a fixed, arbitrarily chosen base point x_0.

We are interested in weighted norm inequalities on $L^{p(\cdot)}$. For classical Lebesgue spaces, much of the theory of such inequalities is due to Muckenhoupt (see e.g. [32]). The following definition clarifies some standard notation.

Definition 1.10. A weight is a locally integrable function $w\colon X \to [0, \infty]$ with $0 < w(x) < \infty$ almost everywhere. Given a weight w, we define its associated measure by $dw(x) = w(x)\, d\mu(x)$. The weighted average integral of a function f over a set $E \subset X$ with $w(E) > 0$ is denoted

$$\fint_E f(x)\, dw = \frac{1}{w(E)} \int_E f(x)\, w(x)\, d\mu.$$

If $w = 1$, we replace dw with $d\mu$.

We denote by M the uncentered Hardy–Littlewood maximal operator; that is,

$$Mf(x) = \sup_{B \ni x} \fint |f(y)|\, d\mu.$$

For classical Lebesgue spaces $L^p(\mathbb{R}^n)$, Muckenhoupt proved in [32] that a necessary and sufficient condition for strong-type (p, p) weighted norm inequalities, $p > 1$, is that for every ball B,

$$\fint_B w(x)\, dx \left(\fint_B w(x)^{1-p'}\, dx \right)^{p-1} \le C < \infty.$$

This is the famous Muckenhoupt A_p condition. In [14] the A_p condition is recast into an equivalent form which may be generalized to variable exponent spaces.

Definition 1.11. Given an exponent $p(\cdot) \in \mathcal{P}(X)$, we say $w \in A_{p(\cdot)}$ if there exists a constant K such that for any ball B,

$$\|w\chi_B\|_{p(\cdot)} \|w^{-1}\chi_B\|_{p'(\cdot)} \le K\mu(B).$$

The infimum over all such K is called the $A_{p(\cdot)}$ constant and is denoted $[w]_{A_{p(\cdot)}}$.

Remark 1.12. If we adopt the usual convention that

$$c \cdot \infty = \begin{cases} 0, & c = 0, \\ \infty, & c > 0, \end{cases}$$

if $w \in A_{p(\cdot)}$, then $\|w\chi_B\|_{p(\cdot)} = \infty$ implies that $\|w^{-1}\chi_B\|_{p'(\cdot)} = 0$, and thus that w^{-1} is the zero element in $L^{p'(\cdot)}$, contrary to w being finite almost everywhere. Thus $w \in L^{p(\cdot)}$ and if $p_+ < \infty$ we can say that w is locally $p(\cdot)$-integrable.

In the case that $p(\cdot)$ is constant, $p(\cdot) = p \in (1, \infty)$, the $A_{p(\cdot)}$ condition for w is equivalent to Mucknhoupt's A_p condition for w^p. The necessity and sufficiency of the $A_{p(\cdot)}$ condition for strong-type weighted norm inequalities of the maximal operator in $\mathbb{R}^n$ was first proved in [9] and simultaneously [14]. The following theorem, which is our main result, generalizes this to the case of spaces of homogeneous type.

Theorem 1.13. *Given $p(\cdot) \in LH$ with $1 < p_- \le p_+ < \infty$ and a weight w,*

$$\|(Mf)w\|_{p(\cdot)} \le C\|fw\|_{p(\cdot)}$$

if and only if $w \in A_{p(\cdot)}$.

By analogy with weak- and strong-type inequalities in classical L^p spaces, Theorem 1.13 naturally suggests the following weak-type inequality, which remains an open problem.

Conjecture 1.14. *Given $p(\cdot) \in LH$ with $p_+ < \infty$ and a weight w,*

$$\|t\chi_{\{x\in X\,:\,Mf(x)>t\}}w\|_{p(\cdot)} \le C\|fw\|_{p(\cdot)}$$

if and only if $w \in A_{p(\cdot)}$.

We will prove the necessity of the $A_{p(\cdot)}$ condition for the weak-type inequality in Section 5. Moreover, if $p_- > 1$ then the conjecture follows from Theorem 1.13. Conjecture 1.14 is claimed to be true in $\mathbb{R}^n$ in [14], but the proof contains a gap: if $p_- = 1$, then $(p')_+ = \infty$, and so Lemmas 3.3-3.6 (which are analogous to Lemmas 3.3 and 3.4 in this paper) may not be applied to $w^{-1} \in A_{p'(\cdot)}$, as is done several times throughout the proof.

The remainder of this paper is devoted to proving Theorem 1.13. In Sections 2, 3, and 4, we will collect several elementary results about $L^{p(\cdot)}$ spaces, the $A_{p(\cdot)}$ condition, and dyadic grids, respectively, on spaces of homogeneous type. In Section 5 we will prove the necessity of the $A_{p(\cdot)}$ condition, and in Section 6 we will prove its sufficiency.

We adopt the convention throughout that C denotes a large constant dependent only on fixed quantities (usually X, $p(\cdot)$, w, and the dyadic grid $\mathcal{D}$, unless otherwise stated or obvious from context). Multiples of balls are written as $CB(x,r) = B(x,Cr)$. By $A \approx B$, we mean that there are constants c, C with $cB \le A \le CB$. Finally for a weight w and a set E we write $w(E) = \int_E w(x)\,d\mu$.

2. Variable Lebesgue spaces

This section is a collection of elementary results regarding variable Lebesgue spaces on spaces of homogeneous type. We begin with two lemmas concerning spaces of homogeneous type which will be used to prove many of the results in this paper. The first is well-known, and we omit the proof. The second characterizes finite spaces of homogeneous type and is proved in [4, Lemma 1.9].

Lemma 2.1. (Lower Mass Bound) *There exists a positive constant $C = C(X)$ such that for all $x \in X$, $0 < r < R$, and $y \in B(x,R)$,*

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \ge C\left(\frac{r}{R}\right)^{\log_2 C_\mu}.$$

Lemma 2.2. *A space of homogeneous type X has $\mu(X) < \infty$ if and only if $\mu(X) = B(x,r)$ for some $x \in X$ and $r > 0$.*

The remainder of the lemmas in this section are facts which are well-known in $\mathbb{R}^n$. We omit proofs that are unchanged from their Euclidean case, which may be found in [10, 19]. We do, however, reproduce the proof of Lemma 2.3, as we later make reference to the constants implicit in the proof.

Lemma 2.3. *Given $p(\cdot) \in \mathcal{P}(X)$ with $p_+ < \infty$, $\|f\|_{p(\cdot)} \le C_1$ if and only if*

$$\int_X |f(x)|^{p(x)}\, d\mu \le C_2. \tag{2.1}$$

Moreover, if one constant is equal to 1, we may take both to be.

Proof. Assume first that (2.1) holds. Since $p_+ < \infty$, we have that $\|f\|_{L^\infty(X_\infty)} = 0$. Given $C_2 \le 1$, we have $\rho(f/1) \le 1$ and hence we may take $C_1 = 1$. If $C_2 \ge 1$, then we may divide to obtain

$$\int_X \left| \frac{f(x)}{C_2^{1/p(x)}} \right|^{p(x)} d\mu \le 1.$$

Now $C_2^{1/p(x)}$ is bounded by $C_1 = C_2^{1/p_-}$, for which $\rho(f/C_1) \le 1$ and so $\|f\|_{p(\cdot)} \le C_1$.

Conversely, given $\|f\|_{p(\cdot)} \le C_1$, then by the definition of the norm we get that

$$1 \ge \int_X \left| \frac{f(x)}{C_1 + 1} \right|^{p(x)} d\mu \ge \frac{1}{(C_1+1)^{p_+}} \int_X |f(x)|^{p(x)}\, d\mu,$$

and so that (2.1) holds. If $C_1 = 1$, then for any $\epsilon > 0$, there exists $\lambda_\epsilon \in [1, 1+\epsilon)$ such that

$$\rho(f/\lambda_\epsilon) = \int_X \left| \frac{f(x)}{\lambda_\epsilon} \right|^{p(x)} d\mu \le 1.$$

Since the integrand is dominated by $|f(x)|^{p(x)}$, taking $\epsilon = 1/n$ and applying the Dominated Convergence Theorem, we get that

$$\int_X |f(x)|\, d\mu \le 1,$$

and so C_2 may be taken to be 1. □

Lemma 2.4. *Given $p(\cdot) \in \mathcal{P}(X)$ with $p_+ < \infty$,*

$$\int_X \left(\frac{|f(x)|}{\|f\|_{p(\cdot)}} \right)^{p(x)} d\mu = 1.$$

In particular, if $\|f\|_{p(\cdot)} = 1$, then

$$\int_X |f(x)|^{p(x)}\, d\mu = 1.$$

Lemma 2.5. *Let $p(\cdot) \in \mathcal{P}(X)$ be such that $p_+ < \infty$. If $\|f\|_{p(\cdot)} \le 1$, then*

$$\|f\|_{p(\cdot)}^{p_+} \le \int_X |f(x)|^{p(x)}\, d\mu \le \|f\|_{p(\cdot)}^{p_-}.$$

On the other hand, if $\|f\|_{p(\cdot)} \ge 1$, then

$$\|f\|_{p(\cdot)}^{p_-} \le \int_X |f(x)|^{p(x)}\, d\mu \le \|f\|_{p(\cdot)}^{p_+}.$$

Lemma 2.6. *If $p(\cdot) \in \mathcal{P}(X)$ is such that $p_+ < \infty$, bounded functions with support contained in $B_r(x_0)$ for some r and x_0 (bounded support) are dense in $L^{p(\cdot)}$. Moreover, any nonnegative $f \in L^{p(\cdot)}$ is the limit of an increasing sequence of such functions.*

Proof. All bounded functions of bounded support are in $L^{p(\cdot)}$, because they are bounded by constant functions on finite-measure domains and $p_+ < \infty$. To prove that such functions are dense, choose $f \in L^{p(\cdot)}$ and let $\epsilon > 0$. By decomposing f as

$$f(x) = f_+(x) - f_-(x),$$

with both $f_+, f_- \geq 0$, it suffices to consider the case $f(x) \geq 0$. Since $\|f\|_{p(\cdot)} < \infty$, there exists $\Lambda > 0$ for which $\rho(f/\Lambda) \leq 1$. If $\Lambda/\epsilon = \lambda \geq 1$, then

$$\int_X \left|\frac{f(x)}{\epsilon}\right|^{p(x)} d\mu = \int_X \left|\frac{\lambda f(x)}{\Lambda}\right|^{p(x)} d\mu \leq \lambda^{p_+} \int_X \left|\frac{f(x)}{\Lambda}\right|^{p(x)} d\mu \leq \lambda^{p_+} < \infty.$$

If $\lambda < 1$, the same argument holds with p_- replacing p_+. Thus $\rho(f/\epsilon) < \infty$. Now define (choosing some base point $x_0 \in X$)

$$f_n(x) = \min\{f(x), n\}\chi_{B(x_0,n)}.$$

It is clear that $f_n \to f$ pointwise. Then by the dominated convergence theorem,

$$\int_X \left|\frac{f(x) - f_n(x)}{\epsilon}\right|^{p(x)} d\mu \to 0,$$

since $\rho(f/\epsilon)$ is finite and $|\frac{f(x)}{\epsilon}|^{p(x)}$ dominates the above integrand. But then for n large enough that the above integral is less than one, $\|f - f_n\|_{p(\cdot)} \leq \epsilon$. It follows that the f_n converge to f in $L^{p(\cdot)}$ and consequently that bounded functions of bounded support are dense. Finally, we have that $f_{n+1}(x) \geq f_n(x)$, so the sequence increases to f. □

Lemma 2.7. (Monotone Convergence Theorem) *Given an exponent $p(\cdot) \in \mathcal{P}(X)$, let $\{f_k\}_{k=1}^{\infty}$ be a sequence of non-negative measureable functions that increase pointwise almost everywhere to a function $f \in L^{p(\cdot)}$. Then $\|f_k\|_{p(\cdot)} \to \|f\|_{p(\cdot)}$.*

Lemma 2.8. (Hölder's Inequality) *Given an exponent $p(\cdot) \in \mathcal{P}(X)$,*

$$\int_X |f(x)g(x)|\, d\mu \leq 4\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}$$

for any f, g.

The following two lemmas are stated erroneously in [14, Lemmas 2.7, 2.8]. For clarity, we provide the correct proofs here.

Lemma 2.9. *Given a set $G \subseteq X$ and two exponents $s(\cdot)$ and $r(\cdot)$ such that*

$$0 \leq r(y) - s(y) \leq \frac{C_0}{\log(e + d(x_0, y))}$$

for each $y \in G$, then for every $t \geq 1$ there exists a constant $C = C(t, C_0)$ such that for all functions f,

$$\int_G |f(y)|^{s(y)}\, dy \leq C \int_G |f(y)|^{r(y)}\, dy + \int_G \frac{1}{(e + d(x_0, y))^{ts_-(G)}}\, dy. \tag{2.2}$$

Proof. Let $G' = \{y \in G\colon |f(y)| \geq (e + d(x_0, y))^{-t}\}$. Then decomposing the domain of the left integral in the inequality into G' and $G \setminus G'$, we see that since $(e + d(x_0, y))^{-t} \leq 1$,

$$\int_{G\setminus G'} |f(y)|^{s(y)}\, dy \leq \int_{G\setminus G'} (e + d(x_0, y))^{-ts(y)}\, dy \leq \int_G \frac{1}{(e + d(x_0, y))^{ts_-(G)}}\, dy.$$

If $y \in G'$, then

$$\begin{aligned}|f(y)|^{s(y)} &= |f(y)|^{r(y)}|f(y)|^{s(y)-r(y)} \le |f(y)|^{r(y)}(e+d(x_0,y))^{t(r(y)-s(y))}\\ &\le |f(y)|^{r(y)}(e+d(x_0,y))^{C_0 t/\log(e+d(x_0,y))} \le C|f(y)|^{r(y)}.\end{aligned}$$

The desired inequality follows. □

Lemma 2.10. *Given a set $G \subseteq X$ and two exponents $s(\cdot)$ and $r(\cdot)$ such that*

$$|r(y)-s(y)| \le \frac{C_0}{\log(e+d(x_0,y))}$$

for each $y \in G$, then for every $t \ge 1$ there exists a constant $C = C(t, C_0)$ such that (2.2) *holds for all functions f with $|f(y)| \le 1$.*

Proof. Define the two sets $A = \{y \in G : r(y) \ge s(y)\}$ and $B = G \setminus A$. Lemma 2.9 takes care of A. For B, construct $B' = \{y \in B : |f(y)| \ge (e+d(x_0,y))^{-t}\}$ and observe that the $B \setminus B'$ component holds as in the previous proof. But since $|f(y)| \le 1$,

$$|f(y)|^{s(y)} = |f(y)|^{r(y)}|f(y)|^{s(y)-r(y)} \le |f(y)|^{r(y)}.$$

Since $C \ge 1$, this proves the inequality. □

Our proof of the final lemma in this section is based on the proof of [1, Lemma 3.1].

Lemma 2.11. *Given an exponent $p(\cdot) \in LH$, for all balls $B \subseteq X$,*

$$\mu(B)^{p_-(B)-p_+(B)} \le C.$$

Proof. Fix $B = B(y_0, r)$ and define $B_0 = B(x_0, 1)$, where x_0 is the LH_∞ condition base point. Also let $k = 2\lceil \log_2 4A_0 \rceil + 2$ and $C_1 = \mu(B_0)/C_\mu^k$. We will show that for any $x, y \in B$,

$$\mu(B)^{-|p(x)-p(y)|} \le C;$$

a simple limiting argument shows that this is equivalent to the stated form, by the continuity of $p(\cdot)$. If $\mu(B) \ge \min\{1, C_1\}$, then

$$\mu(B)^{-|p(x)-p(y)|} \le \min\{1, C_1\}^{-|p(x)-p(y)|} \le \min\{1, C_1\}^{-p_+},$$

since $p(\cdot) \in \mathrm{LH}$ implies $|p(x) - p(y_0)| \le p_+ < \infty$. Thus we may assume $\mu(B) \le \min\{1, C_1\}$.

We begin by asserting that we need only prove the inequality when one of the points is the center point y_0 of B. If this is not the case, then

$$\mu(B)^{-|p(x)-p(y)|} = \mu(B)^{-|p(x)-p(y_0)+p(y_0)-p(y)|} \le \mu(B)^{-|p(x)-p(y_0)|-|p(y)-p(y_0)|},$$

and so it suffices to prove that

$$\mu(B)^{-|p(x)-p(y_0)|} \le C$$

for any $x \in B$.

We consider first the case where $r \ge 1$. For any $y \in r^{-1}B$, we have that

$$d(x_0, y) \le A_0(d(x_0, y_0) + d(y_0, y)) \le A_0(1 + d(x_0, y_0)).$$

Consequently $r^{-1}B \subseteq A_0(1+d(x_0,y_0))B_0$, and so by the lower mass bound (Lemma 2.1),

$$\begin{aligned}\mu(B_0) &\le \mu(A_0(1+d(x_0,y_0))B_0) \le C(1+d(x_0,y_0))^{\log_2 C_\mu}\mu(r^{-1}B)\\ &\le C(1+d(x_0,y_0))^{\log_2 C_\mu}\mu(B).\end{aligned}$$

Dividing by $\mu(B)$ and raising to the power of $|p(x) - p(y_0)|$, we get

$$\mu(B)^{-|p(x)-p(y_0)|} \le C(1+d(x_0,y_0))^{\log_2 C_\mu |p(x)-p(y_0)|}. \tag{2.3}$$

To estimate the quantity on the right of (2.3), we argue that $B_0 \cap 2A_0B = \varnothing$. If to the contrary there exists $y \in B_0 \cap 2A_0B$, then for any $z \in B_0$ we have

$$d(y_0, z) \le A_0(d(y_0, y) + d(y, z)) \le 2A_0(1 + A_0r) \le 4A_0^2 r,$$

since $A_0, r \ge 1$. Consequently $B_0 \subseteq 4A_0^2 B$. From the doubling condition,

$$\mu(B) \ge \mu(4A_0^2B)/C_\mu^k \ge \mu(B_0)/C_\mu^k = C_1,$$

contrary to assumption. Hence the claim is true; in particular $x_0 \notin 2A_0B$, so $d(x_0, y_0) > 2A_0r$. By the quasi-triangle inequality,

$$d(x_0, y_0) \le A_0(d(x_0, x) + d(x, y_0)) \le A_0(r + d(x_0, x)).$$

Since $d(x_0, y_0) > 2A_0r$ we have that $d(x_0, x) > r$ and so $d(x_0, y_0) \le 2A_0 d(x_0, x)$. Thus

$$(1 + d(x_0, y_0))^{|p(x)-p(y_0)|} \le (1 + d(x_0, y_0))^{|p(y_0)-p_\infty|}(1 + 2A_0d(x_0, x))^{|p(x)-p_\infty|}.$$

That this is bounded by a constant is implied by the LH_∞ condition and the fact that the function

$$f(x) = \frac{\log(e + ax)}{\log(e + bx)} \tag{2.4}$$

is bounded on $x > 0$ by $\frac{a}{b}$ when $a > b > 0$. This completes the case when $r \ge 1$.

If $r < 1$, we argue much as before with the lower mass bound to obtain

$$\mu(B)^{-|p(x)-p(y_0)|} \le Cr^{-|p(x)-p(y_0)|}\mu(r^{-1}B)^{-|p(x)-p(y_0)|};$$

The $r^{-|p(x)-p(y_0)|}$ term is bounded by the LH_0 condition. □

3. The $A_{p(\cdot)}$ Condition

In this section we develop the $A_{p(\cdot)}$ condition in spaces of homogeneous type. Our first lemma characterizes various properties of A_∞ weights. For a proof, see [33, Chapter I, Theorem 15].

Lemma 3.1. *Given a weight W, the following are equivalent.*

- $W \in A_\infty = \bigcup_{p\ge1} A_p$.
- *There exist constants $\epsilon > 0$ and $C_2 > 1$ such that given any ball B and any measurable set $E \subseteq B$,*

$$\frac{\mu(E)}{\mu(B)} \le C_2 \left(\frac{W(E)}{W(B)}\right)^\epsilon.$$

- *W is doubling (in the sense that the measure ν given by $d\nu(x) = W(x)\,d\mu(x)$ is doubling) and there exist constants $\delta > 0$ and $C_1 > 1$ such that given any ball B and any measurable set $E \subseteq B$,*

$$\frac{W(E)}{W(B)} \le C_1 \left(\frac{\mu(E)}{\mu(B)}\right)^\delta.$$

To utilize the properties described in Lemma 3.1, we will use the $A_{p(\cdot)}$ condition to construct a weight W in A_∞. To do so, we require the following lemmas.

Lemma 3.2. *Given an exponent $p(\cdot) \in \mathcal{P}(X)$, if $w \in A_{p(\cdot)}$, then there exists a constant C depending on $p(\cdot)$ and w such that given any ball B and any measurable set $E \subset B$,*

$$\frac{\mu(E)}{\mu(B)} \le C\frac{\|w\chi_E\|_{p(\cdot)}}{\|w\chi_B\|_{p(\cdot)}}.$$

Proof. Fix B and $E \subset B$. By Hölder's inequality and the $A_{p(\cdot)}$ condition (Definition 1.11),

$$\begin{aligned}\mu(E) = \int_X w(x)\chi_E w(x)^{-1}\chi_B \, d\mu &\leq C\|w\chi_E\|_{p(\cdot)}\|w^{-1}\chi_B\|_{p'(\cdot)} \\ &\leq C\|w\chi_E\|_{p(\cdot)}\|w\chi_B\|_{p(\cdot)}^{-1}\mu(B).\end{aligned}$$

□

Lemma 3.3. *Given an exponent $p(\cdot) \in LH$ and a weight $w \in A_{p(\cdot)}$, there exists a constant C_0 depending on $p(\cdot)$, w, and X such that for all balls B,*

$$\|w\chi_B\|_{p(\cdot)}^{p_-(B)-p_+(B)} \leq C_0.$$

Proof. Our proof is reminiscent of the proof of Lemma 2.11. Fix $B = B(y_0, r)$ and define $B_0 = B(x_0, 1)$. If $\|w\chi_B\|_{p(\cdot)} \geq 1$, then $\|w\chi_B\|_{p(\cdot)}^{p_-(B)-p_+(B)} \leq 1$, so we may assume that $\|w\chi_B\|_{p(\cdot)} < 1$. We consider three cases; first, suppose $r \leq 1$ and $d(x_0, y_0) \leq 2A_0$. By the quasi-triangle inequality (Definition 1.1), for any point $y \in B$, we have that

$$d(x_0, y) \leq A_0(d(y_0, y) + d(x_0, y_0)) \leq A_0(r + 2A_0) \leq A_0(1 + 2A_0),$$

and so

$$B \subseteq A_0(1 + 2A_0)B_0 = B_1.$$

If we apply Hölder's inequality, the lower mass bound on B_0 and B_1, and the $A_{p(\cdot)}$ condition, we get

$$\begin{aligned}\mu(B) &= \int_B w(x)w(x)^{-1}\, d\mu \leq C\|w\chi_B\|_{p(\cdot)}\|w^{-1}\chi_B\|_{p'(\cdot)} \\ &\leq C\|w\chi_B\|_{p(\cdot)}\|w^{-1}\chi_{B_1}\|_{p'(\cdot)}(2A_0)^{\log_2 C_\mu}\frac{\mu(B_0)}{\mu(B_1)} \leq C\|w\chi_B\|_{p(\cdot)}\|w\chi_{B_1}\|_{p(\cdot)}^{-1}.\end{aligned} \tag{3.1}$$

Here the constant depends on both X and x_0. After rearranging, raising to the power $p_-(B) - p_+(B)$, and applying Lemma 2.11, we obtain

$$\|w\chi_B\|_{p(\cdot)}^{p_-(B)-p_+(B)} \leq C\mu(B)^{p_-(B)-p_+(B)}\|w\chi_{B_q}\|^{p_-(B)-p_+(B)} \leq C(1 + \|w\chi_{B_1}\|_{p(\cdot)}^{-1})^{p_+-p_-},$$

which is a bound independent of B.

Consider now the case where $r > 1$ and $d(x_0, y_0) \leq 2A_0 r$. Applying the quasi-triangle inequality as before,

$$B_0 \subseteq A_0(1 + 2A_0 r)B = B_2.$$

Using Hölder's inequality and the $A_{p(\cdot)}$ condition as in the previous case,

$$\begin{aligned}\mu(B) &\leq C\|w\chi_B\|_{p(\cdot)}\|w^{-1}\chi_B\|_{p'(\cdot)} \leq C\|w\chi_B\|_{p(\cdot)}\|w^{-1}\chi_{B_2}\|_{p'(\cdot)} \\ &\leq C\|w\chi_B\|_{p(\cdot)}\mu(B_2)\|w\chi_{B_2}\|_{p(\cdot)}^{-1} \leq C\|w\chi_B\|_{p(\cdot)}\mu(B)\|w\chi_{B_0}\|_{p(\cdot)}^{-1}.\end{aligned} \tag{3.2}$$

Thus

$$\|w\chi_B\|_{p(\cdot)}^{p_-(B)-p_+(B)} \leq C(1 + \|w\chi_{B_0}\|_{p(\cdot)}^{-1})^{p_+-p_-}.$$

Consider now the remaining case, namely when $d(x_0, y_0) > 2A_0 \max\{1, r\}$. Let $d = 2A_0 d(x_0, y_0)$ so that $B, B_0 \subseteq B(x_0, d) = B_3$. Arguing as we did in inequality (3.1) (if $1 \geq r$) or (3.2) (if $r > 1$) with B_3 in place of B_1 or B_2, we get

$$\mu(B) \leq C\mu(B_3)\|w\chi_B\|_{p(\cdot)}\|w\chi_{B_3}\|_{p(\cdot)}^{-1}.$$

In order to bring $\mu(B_3)$ into the constant as in the previous cases and obtain the corresponding inequality, we need

$$\mu(B_3)^{p_+(B)-p_-(B)} \leq C.$$

To see that this is the case, observe that $p(\cdot)$ is continuous (since it is in LH_0) and so there exist $y_1, y_2 \in \overline{B}$ for which $p(y_1) = p_-(B)$ and $p(y_2) = p_+(B)$. And since

$$d(x_0, y_0) \leq A_0(d(x_0, y_k) + r) \leq A_0 d(x_0, y_k) + \frac{1}{2} d(x_0, y_0)$$

for $k = 1, 2$, we have that $d(x_0, y_k) \geq (2A_0)^{-1} d(x_0, y_0)$, so the LH_∞ condition implies

$$p_+(B) - p_-(B) \leq |p(y_1) - p_\infty| + |p(y_2) - p_\infty| \leq \frac{C}{\log(e + (2A_0)^{-1} d(x_0, y_0))}.$$

Using this together with the lower mass bound,

$$\mu(B_3) \leq C d^{\log_2 C_\mu} \mu(B_0) = C(2A_0 d(x_0, y_0))^{\log_2 C_\mu} \mu(B_0) \leq C(e + A_0 d(x_0, y_0))^{log_2 C_\mu},$$

we get that

$$\begin{aligned} \mu(B_3)^{p_+(B)-p_-(B)} &\leq \left[C(e + A_0 d(x_0, y_0))^{log_2 C\mu}\right]^{C/\log(e+(2A_0)^{-1}d(x_0,y_0))} \\ &\leq C e^{C \log_2 C_\mu \log(e + A_0 d(x_0,y_0))/\log(e+(2A_0)^{-1}d(x_0,y_0))} \leq C e^{C \log_2 C_\mu}. \end{aligned}$$

This last inequality is from the bound on (2.4). Since the above bound is independent of B,

$$[\mu(B) \|w\chi_{B_3}\|_{p(\cdot)}]^{p_+(B)-p_-(B)} \leq C \|w\chi_B\|_{p(\cdot)}^{p_+(B)-p_-(B)}.$$

If we apply Lemma 2.11 on the left and the bound just derived on the right, we obtain

$$\|w\chi_B\|_{p(\cdot)}^{p_-(B)-p_+(B)} \leq C(1 + \|w\chi_{B_0}\|^{-1})^{p_+ - p_-}. \qquad \square$$

We can now prove the following lemma, which will allow us to apply Lemma 3.1 to weights in variable exponent spaces.

Lemma 3.4. *Given an exponent $p(\cdot) \in LH$ and a weight $w \in A_{p(\cdot)}$, we have that $W(\cdot) = w(\cdot)^{p(\cdot)} \in A_\infty$.*

Proof. Fix a ball B and a measurable set $E \subseteq B$. We will show that

$$\frac{\mu(E)}{\mu(B)} \leq C \left(\frac{W(E)}{W(B)}\right)^{1/p_+}, \tag{3.3}$$

which by Lemma 3.1 is sufficient to show $W(\cdot) \in A_\infty$. We will prove this in three cases. Consider first the case that $\|w\chi_B\|_{p(\cdot)} \leq 1$. By Lemma 3.2,

$$\frac{\mu(E)}{\mu(B)} \leq C \frac{\|w\chi_E\|_{p(\cdot)}}{\|w\chi_B\|_{p(\cdot)}} \leq C \frac{\|w\chi_E\|_{p(\cdot)}}{\|w\chi_B\|_{p(\cdot)}^{p_-(B)/p_+(B)} \|w\chi_B\|_{p(\cdot)}^{1-p_-(B)/p_+(B)}}.$$

If we appeal to Lemma 2.5 for the inequalities $\|w\chi_E\|_{p(\cdot)} \leq W(E)^{1/p_+(B)}$ and $\|w\chi_B\|_{p(\cdot)}^{p_-(B)} \geq W(E)$, then apply Lemma 3.3 on the remaining term, we get that

$$\frac{\mu(E)}{\mu(B)} \leq C \left(\frac{W(E)}{W(B)}\right)^{1/p_+} \|w\chi_B\|^{p_-(B)/p_+(B)-1} \leq C \left(\frac{W(E)}{W(B)}\right)^{1/p_+}.$$

Now considering the case $\|w\chi_E\|_{p(\cdot)} \le 1 \le \|w\chi_B\|_{p(\cdot)}$, by the same lemmas as before,

$$\frac{\mu(E)}{\mu(B)} \le C\frac{\|w\chi_E\|_{p(\cdot)}}{\|w\chi_B\|_{p(\cdot)}} \le C\frac{\|w\chi_E\|_{p(\cdot)}}{\|w\chi_B\|_{p(\cdot)}^{p_-(B)/p_+(B)}\|w\chi_B\|_{p(\cdot)}^{1-p_-(B)/p_+(B)}}$$
$$\le C\frac{W(E)^{1/p_+}}{\|w\chi_B\|^{p_-(B)/p_+(B)}\|w\chi_B\|^{1-p_-(B)/p_+(B)}},$$

which, given $\|w\chi_B\|_{p(\cdot)} \ge 1$ and $p_+ \ge p_+(B)$, yields

$$\frac{\mu(E)}{\mu(B)} \le C\left(\frac{W(E)}{W(B)}\right)^{1/p_+}.$$

The third case is $\|w\chi_E\|_{p(\cdot)} \ge 1$. Let $\lambda = \|w\chi_B\|_{p(\cdot)} \ge \|w\chi_E\|$. Since $p(\cdot) \in \mathrm{LH}_\infty$, by Lemma 2.10 with $d\mu$ replaced by $W(x)\,d\mu$, for all $t > 1$ there exists a constant C_t for which

$$\int_B \frac{W(x)}{\lambda^{p_\infty}}\,d\mu \le C_t\int_B \frac{W(x)}{\lambda^{p(x)}}\,d\mu + \int_B \frac{W(x)}{(e+d(x_0,x))^{tp_\infty}}\,d\mu. \tag{3.4}$$

The first integral on the right hand side is less than 1 by Lemma 2.4. We claim that the same is true of the second term for sufficiently large t independent of B. This is obvious if $W(X) < \infty$, since

$$\int_X \frac{W(x)}{(e+d(x_0,x))^{tp_\infty}}\,d\mu \le Ce^{-tp_\infty}W(X),$$

which may be made arbitrarily small. If on the other hand $W(X) = \infty$, let $B_k = B(x_0, 2^k)$. Then by Lemma 2.5,

$$\begin{aligned}\int_X \frac{W(x)}{(e+d(x_0,x))^{tp_\infty}}\,d\mu &\le e^{-tp_\infty}W(B_0) + C\sum_{k=1}^\infty \int_{B_k\setminus B_{k-1}} \frac{W(x)}{(e+d(x_0,x))^{tp_\infty}}\,d\mu \\ &\le e^{-tp_\infty}W(B_0) + C\sum_{k=1}^\infty 2^{-ktp_\infty}W(B_k) \\ &\le e^{-tp_\infty}W(B_0) + C\sum_{k=1}^\infty 2^{-ktp_\infty}\max\left\{\|w\chi_{B_k}\|_{p(\cdot)}^{p_+}, \|w\chi_{B_k}\|_{p(\cdot)}^{p_-}\right\} \\ &\le e^{-tp_\infty}W(B_0) + C\sum_{k=1}^\infty 2^{-ktp_\infty}\|w\chi_{B_k}\|_{p(\cdot)}^{p_+}.\end{aligned}$$

The last inequality comes from the fact that $\|w\chi_{B_k}\|_{p(\cdot)} > 1$ for all k sufficiently large, by continuity of the measure $dW = W(x)\,d\mu$ and the fact that $X = \bigcup_{k=1}^\infty B_k$. By Lemma 3.2,

$$\|w\chi_{B_k}\|_{p(\cdot)} \le C\frac{\mu(B_k)}{\mu(B_0)}\|w\chi_{B_0}\|_{p(\cdot)} \le C2^{k\log_2 C_\mu}.$$

Combining these two estimates yields

$$\int_X \frac{W(x)}{(e+d(x_0,x))^{tp_\infty}}\,d\mu \le e^{-tp_\infty}W(B_0) + C\sum_{k=1}^\infty 2^{kp_+\log_2 C_\mu - ktp_\infty}. \tag{3.5}$$

For $t > p_\infty/\log_2 C_\mu^{p_+}$ the sum converges, and choosing t sufficiently large (independent of B) makes the right hand side less than 1. Thus the right hand side of (3.4) is bounded, and so we may rearrange to obtain

$$W(B)^{1/p_\infty} \le C\|w\chi_B\|_{p(\cdot)}. \tag{3.6}$$

Now repeating the argument switching B with E and $p(\cdot)$ with p_∞, we get

$$1 \le \int_E \frac{W(x)}{\lambda^{p(x)}}\,d\mu \le C_t \int_E \frac{W(x)}{\lambda^{-p_\infty}}\,d\mu + \int_E \frac{W(x)}{(e+d(x_0,x))^{tp_\infty}}\,d\mu.$$

As before, we can make the rightmost term less than 1/2, so that

$$\lambda^{p_\infty} = \|w\chi_E\|_{p(\cdot)}^{p_\infty} \le CW(E). \tag{3.7}$$

Then by Lemma 3.2,

$$\frac{\mu(E)}{\mu(B)} \le C\frac{\|w\chi_E\|_{p(\cdot)}}{\|w\chi_B s\|_{p(\cdot)}} \le C\left(\frac{W(E)}{W(B)}\right)^{1/p_\infty} \le C\left(\frac{W(E)}{W(B)}\right)^{1/p_+}. \qquad \square$$

From the latter stages of the proof of Lemma 3.4, we may pull the following corollary.

Corollary 3.5. *Given an exponent $p(\cdot) \in LH$, if $w \in A_{p(\cdot)}$ is a weight satisfying $\|w\chi_B\|_{p(\cdot)} \ge 1$ on a ball B, then $\|w\chi_B\|_{p(\cdot)} \approx W(B)^{1/p_\infty}$.*

We conclude this section with a lemma that will allow for the reduction from our main result to the unweighted case.

Lemma 3.6. *If $p(\cdot) \in LH$ and $p_- > 1$, then $1 \in A_{p(\cdot)}$.*

Proof. Fix a ball B. If $\mu(B) \le 1$, then by Lemma 2.5,

$$\|\chi_B\|_{p(\cdot)}^{p_+(B)} \le \int_B 1^{p(x)}\,d\mu = \mu(B),$$

which implies

$$\|\chi_B\|_{p(\cdot)} \le \mu(B)^{1/p_+(B)},$$

and by the same argument applied to $p'(\cdot)$,

$$\|\chi_B\|_{p'(\cdot)} \le \mu(B)^{1/(p')_+(B)} = \mu(B)^{1-1/p_-(B)}.$$

Thus (applying Lemma 2.11)

$$\|\chi_B\|_{p(\cdot)}\|\chi_B\|_{p'(\cdot)} \le \left[\mu(B)^{p_-(B)-p_+(B)}\right]^{1/p_+(B)p_-(B)} \mu(B) \le K\mu(B),$$

which is the desired inequality. Suppose now $\mu(B) > 1$. By an argument that is essentially the same as the proof of Corollary 3.5 with $w = 1$, we get that

$$\|\chi_B\|_{p(\cdot)}\|\chi_B\|_{p'(\cdot)} \le K\mu(B)^{1/p_\infty + 1/p'_\infty} = K\mu(B). \qquad \square$$

4. Dyadic cubes

Important to the proofs of many results of variable exponent spaces in $\mathbb{R}^n$ are the dyadic cubes of the form

$$Q = [m_1 2^{-k}, (m_1+1)2^{-k}) \times \cdots \times [m_n 2^{-k}, (m_n+1)2^{-k}), \quad m_1, \ldots, m_n \in \mathbb{Z}.$$

Due to the usefulness of dyadic objects in many areas of harmonic analysis, a great deal of effort has gone into developing similar systems in metric and quasi-metric spaces, for example [6, 24, 28]. We will use the form of Hytönen and Kairema's construction [24] presented in [3].

Theorem 4.1. *There exist constants $C_d > 0, d_0 > 1$, and $0 < \epsilon < 1$ depending on X, a family $\mathcal{D} = \bigcup_{k\in\mathbb{Z}} \mathcal{D}_k$, called the* dyadic grid *on X of subsets of X, called* dyadic cubes, *and a collection $\{x_c(Q)\}_{Q\in\mathcal{D}}$ of points such that:*

(1) *For every $k \in \mathbb{Z}$ the cubes in $\mathcal{D}_k$ are pairwise disjoint and $X = \bigcup_{Q\in\mathcal{D}_k} Q$. We will refer to the cubes in $\mathcal{D}_k$ as cubes in the* kth generation.
(2) *If $Q_1, Q_2 \in \mathcal{D}$, then either $Q_1 \cap Q_2 = \varnothing$, or $Q_1 \subseteq Q_2$, or $Q_2 \subseteq Q_1$.*
(3) *For any $Q_1 \in \mathcal{D}_k$, there exists at least one $Q_2 \in \mathcal{D}_{k-1}$, which is called* a child *of Q_1, such that $Q_2 \subseteq Q_1$, and there exists exactly one $Q_3 \in \mathcal{D}_{k+1}$, which is called* a parent *of Q_1, such that $Q_1 \subseteq Q_3$.*
(4) *If Q_2 is a child of Q_1, then $\mu(Q_2) \geq \epsilon\mu(Q_1)$.*
(5) *For every k and $Q \in \mathcal{D}_k$, $B(x_c(Q), d_0^k) \subseteq Q \subseteq B(x_c(Q), C_d d_0^k)$.*

In general, we may freely switch back and forth between the settings of cubes and balls. Consider, for example, the following equivalent formulation of the $A_{p(\cdot)}$ condition.

Lemma 4.2. (The $A_{p(\cdot)}$ condition for cubes) *Given a dyadic grid $\mathcal{D}$ and $p(\cdot) \in LH$, if $w \in A_{p(\cdot)}$, then there exists a constant K such that for any $Q \in \mathcal{D}$,*

$$\|w\chi_Q\|_{p(\cdot)}\|w^{-1}\chi_Q\|_{p'(\cdot)} \leq K\mu(Q).$$

Proof. Fix $Q \in \mathcal{D}_k$. Then by Theorem 4.1, the $A_{p(\cdot)}$ condition, and the lower mass bound,

$$\begin{aligned}\|w\chi_Q\|_{p(\cdot)}\|w^{-1}\chi_Q\|_{p'(\cdot)} &\leq \|w\chi_{B(x_c(Q),C_d d_0^k)}\|_{p(\cdot)}\|w^{-1}\chi_{B(x_c(Q),C_d r d_0^k)}\|_{p'(\cdot)}\\ &\leq K\mu(B(x_c(Q), Cd_0^k)) \leq C\mu(B(x_c(Q), d_0^k)) \leq C\mu(Q).\end{aligned}$$

The constant C is independent of k. □

In general, the argument in the proof of Lemma 4.2, in which we expand cubes to fill balls and then apply the lower mass bound to shrink back to cubes, may be used to show that any previously stated result is also true when balls are replaced by cubes. In particular, Lemmas 3.1 and 3.3 hold in this way. Another object which it is convenient to recast into a dyadic form is the maximal operator.

Definition 4.3. Given a weight σ and a dyadic grid $\mathcal{D}$, define the *weighted dyadic maximal operator* $M_\sigma^{\mathcal{D}}$ with respect to $\mathcal{D}$ by

$$M_\sigma^{\mathcal{D}} f(x) = \sup_{\substack{Q\ni x\\ Q\in\mathcal{D}}} \fint_Q |f(y)|\, d\sigma$$

for any locally integrable function f. When $\sigma = 1$, we will denote $M_\sigma^{\mathcal{D}}$ simply by $M^{\mathcal{D}}$.

The weighted dyadic maximal operator satisfies the same weak- and strong-type inequalities as the classical maximal operator. Given a fixed grid $\mathcal{D}$ and weight σ, for each $\lambda > 0$, we define the set

$$X_\lambda^{\mathcal{D}} = \{x \in X : M_\sigma^{\mathcal{D}} f(x) > \lambda\}.$$

Then the following lemma holds.

Lemma 4.4. *Given a dyadic grid $\mathcal{D}$ on X and a weight σ, the dyadic maximal operator $M_\sigma^{\mathcal{D}}$ is weak $(1,1)$: for $f \in L^1(\sigma)$ and all $\lambda > 0$,*

$$\sigma\left(X_\lambda^{\mathcal{D}}\right) \leq \frac{1}{\lambda}\int_X |f(x)|\, d\sigma.$$

Further, for $1 < p < \infty$, $M_\sigma^{\mathcal{D}}$ is strong (p,p): there exists a constant C depending on p and X such that for any $f \in L^p(\sigma)$,

$$\int_X M_\sigma^{\mathcal{D}} f(x)^p \, d\sigma \le C \int_X |f(x)|^p \, d\sigma.$$

Proof. For each integer n, define the truncated maximal operator

$$M_\sigma^n f(x) = \sup_{\substack{x \in Q \in \mathcal{D}_k \\ k \le n}} \fint_Q |f(y)| \, d\sigma.$$

Observe that for every $x \in X$, the sequence $\{M_\sigma^k f(x)\}$ increases to $M_\sigma^{\mathcal{D}} f(x)$. Certainly, it is increasing and bounded; if $M_\sigma^{\mathcal{D}} f(x) < \infty$, then for any $\epsilon > 0$ there exists a cube Q for which

$$M_\sigma^{\mathcal{D}} f(x) - \epsilon \le \fint_Q |f(y)| \, d\sigma;$$

but then for any n greater than the generation of Q,

$$\fint_Q |f(y)| \, d\sigma \le M_\sigma^n f(x),$$

and so the sequence converges. A similar argument shows that if $M_\sigma^{\mathcal{D}} f(x) = \infty$, then $M_\sigma^n f(x)$ can be made greater than any integer.

Therefore, by the monotone convergence theorem, it suffices to prove the weak-type inequality for the truncated maximal operator. To that end, fix $\lambda > 0$. If $M_\sigma^n f(x) > \lambda$, then there exists a cube Q_x containing x such that

$$\fint_{Q_x} |f(y)| \, d\sigma > \lambda,$$

and Q_x is of generation at most n. Without loss of generality, take Q_x to be the maximal of all such cubes, and let its generation be k. Since there are countably many dyadic cubes, the set $\{Q_x \colon x \in X\}$ may be enumerated as $\{Q_j\}$. If $Q_i \cap Q_j \ne \varnothing$ for some $i \ne j$, then we have some containment $Q_i \subseteq Q_j$ (without loss of generality), and thus $Q_i = Q_j$ by maximality, so the cubes are mutually disjoint. Then

$$\sigma\left(\{x \in X \colon M_\sigma^n f(x) > \lambda\}\right) = \sum_j \sigma(Q_j) \le \frac{1}{\lambda} \sum_j \int_{Q_j} |f(y)| \, d\sigma \le \int_X |f(y)| \, d\sigma.$$

This proves the weak-type inequality.

For the strong-type inequality,

$$\fint_Q |f(y)| \, d\sigma \le \frac{1}{\sigma(Q)} \|f\|_{L^\infty(\sigma)} \int_Q d\sigma = \|f\|_{L^\infty(\sigma)}.$$

Now fix $1 < p < \infty$ and $f \in L^1(\sigma) \cap L^\infty(\sigma)$. Without loss of generality, assume $f\sigma \ne 0$. Then $M_\sigma^{\mathcal{D}} f \in L^{1,\infty}(\sigma) \cap L^\infty(\sigma)$, and consequently by Tonelli's theorem,

$$\begin{aligned} \int_X M^{\mathcal{D}} f(x)^p \, d\sigma &= \int_0^\infty p\lambda^{p-1} \sigma\left(\{x \in X \colon M_\sigma^{\mathcal{D}} f(x) > \lambda\}\right) d\lambda \\ &\le C \int_0^{\|M_\sigma^{\mathcal{D}} f\|_{L^\infty(\sigma)}} \lambda^{p-2} \, d\lambda < \infty. \end{aligned}$$

Thus $0 < \|M_\sigma^{\mathcal{D}} f\|_{L^p(\sigma)} < \infty$. Hence, by the weak-type inequality, Tonelli's Theorem, and Hölder's inequality,

$$\begin{aligned}\int_X M_\sigma^{\mathcal{D}} f(x)^p \sigma(x)\,d\mu &= p\int_0^\infty \lambda^{p-1}\sigma\left(\{x\in X\colon M_\sigma^{\mathcal{D}} f(x)>\lambda\}\right)\,d\lambda\\ &\le p\int_0^\infty \lambda^{p-2}\int_X |f(x)|\,d\sigma\,d\lambda\\ &= p\int_X |f(x)|\int_{\{\lambda\colon M_\sigma^{\mathcal{D}} f(x)>\lambda\}} \lambda^{p-2}\,d\lambda\,d\sigma\\ &\le \frac{p}{p-1}\int_X |f(x)|[M_\sigma^{\mathcal{D}} f(x)]^{p-1}\,d\sigma\\ &\le C\|f\|_{L^p(\sigma)}\|M_\sigma^{\mathcal{D}} f\|_{L^p(\sigma)}^{p-1}.\end{aligned}$$

Rearranging, we obtain that

$$\int_X M_\sigma^{\mathcal{D}} f(x)^p\,d\sigma \le C\int_X |f(x)|^p\,d\sigma,$$

which is the desired strong-type inequality. For general functions $f \in L^p(X)$, the desired inequality follows from an approximation argument if we use Lemma 2.6 and the monotone convergence theorem. □

We now prove the Calderón–Zygmund decomposition for the maximal operator over spaces of homogeneous type. This result is known, but since we could not find the precise formulation we wanted, for completeness we include the proof here.

Lemma 4.5. (Calderón–Zygmund Decomposition) *If $\mu(X) = \infty$, given a weight $\sigma \in A_\infty$, let $\mathcal{D}$ be a dyadic grid on X. If $f \in L^1_{\mathrm{loc}}(\sigma)$ is such that $⨍_{Q^k} |f(x)|\sigma(x)\,d\mu \to 0$ for any nested sequence $\{Q_k \in \mathcal{D}_k\}_{k=0}^\infty$, where each Q_k is a child of Q_{k+1}, then for each $\lambda > 0$, there exists a (possibly empty) set $\{Q_j\}$, called the Calderón–Zygmund (CZ) cubes of f at height λ, of pairwise disjoint dyadic cubes and a constant $C_{CZ} = C_{CZ}(\mathcal{D}, X, \sigma) > 1$, independent of λ, such that*

$$X_\lambda^{\mathcal{D}} = \bigcup_j Q_j.$$

Moreover, for each j,

$$\lambda < ⨍_{Q_j} |f(x)|\,d\sigma \le C_{CZ}\lambda. \tag{4.1}$$

If $\{Q_j^k\}$ are the Calderón–Zygmund cubes at height a^k for $k \in \mathbb{Z}$ and $a > C_{CZ}$, define $E_j^k = Q_j^k \setminus X_{a^{k+1}}^{\mathcal{D}}$. These sets are pairwise disjoint for all j and k, and $\sigma(E_j^k) \ge \frac{a-C_{CZ}}{a}\sigma(Q_j^k)$.

If $\mu(X) < \infty$, then the Calderón–Zygmund cubes may be constructed for any function $f \in L^1_{\mathrm{loc}}(\sigma)$ and at any height $\lambda > ⨍_X |f(y)|\,d\sigma = \lambda_0$, with (4.1) still holding. In this case, the sets E_j^k are defined only for $k > \log_a \lambda_0$, and are pairwise disjoint with $\sigma(E_j^k) \ge \frac{a-C_{CZ}}{a}\sigma(Q_j^k)$.

Proof. Suppose first $\mu(X) = \infty$ and fix $\lambda > 0$. If $X_\lambda^{\mathcal{D}}$ is empty, then take $\{Q_j\}$ to be the empty set. Otherwise, fix $x \in X_\lambda^{\mathcal{D}}$. Then x is contained in exactly one cube

Q_k^x of each generation k and $M_\sigma^{\mathcal{D}} f(x) > \lambda$, so there exists at least one k for which

$$\fint_{Q_k^x} |f(y)|\, d\sigma > \lambda. \tag{4.2}$$

Since by assumption

$$\lim_{k\to\infty} \fint_{Q_k^x} |f(y)|\, d\sigma \to 0,$$

we may take k to be the largest integer for which (4.2) holds. Let $\{Q_x : x \in X_\lambda^{\mathcal{D}}\}$ be the set of all such maximal cubes. As in the proof of Lemma 4.4, this set must be countable and mutually disjoint. Clearly, $X_\lambda^{\mathcal{D}}$ is contained in the union of these cubes. Conversely, given any $z \in Q_x$ for some x, we have that

$$M_\sigma^{\mathcal{D}} f(z) \geq \fint_{Q_x} |f(y)|\, d\sigma > \lambda,$$

and so $z \in X_\lambda^{\mathcal{D}}$; consequently,

$$X_\lambda^{\mathcal{D}} = \bigcup_j Q_j.$$

We now wish to show the inequalities (4.1). The first holds by choice of Q_j. For the second, the maximality of each Q_j ensures that its parent, $\widehat{Q}_j$, satisfies

$$\fint_{\widehat{Q}_j} |f(y)|\, d\sigma \leq \lambda.$$

From this fact together with Lemma 4.1 and the lower mass bound,

$$\fint_{Q_j} |f(y)|\sigma(y)\, d\mu \leq \frac{\sigma(\widehat{Q}_j)}{\sigma(Q_j)}\lambda \leq \frac{\sigma(B(x_c(\widehat{Q}_j), C d_0^{k+1}))}{\sigma(B(x_c(Q_j), d_0^k))}\lambda \leq C d_0^{\log_2 C_\mu}\lambda,$$

which is the second inequality in (4.1).

Now fix $a > C_{CZ}$ and consider the Calderón–Zygmund cubes $\{Q_j^k\}$ at heights a^k for $k \in \mathbb{Z}$. For simplicity, we define $X_k = X_{a^k}^{\mathcal{D}}$. Observe that $X_{k+1} \subset X_k$. Consequently, given any Q_i^{k+1}, the set $\{Q_k^x\}$ (constructed above) for an arbitrary $x \in Q_i^{k+1}$ contains Q_i^{k+1}, and so there exists j such that $Q_i^{k+1} \subset Q_j^k$.

We claim that this implies that the sets E_j^k are pairwise disjoint for all j, k. To see this, consider two arbitrary sets $E_{j_1}^{k_1}$ and $E_{j_2}^{k_2}$ and suppose without loss of generality that $k_1 \leq k_2$. By the above argument, there exists j_3 such that $Q_{j_2}^{k_2} \subset Q_{j_3}^{k_1}$. If $j_3 = j_1$, then $k_1 \neq k_2$ and so disjointness arises from the containment $E_{j_2}^{k_2} \subset X_{k_2} \subset X_{k_1}$; otherwise, the disjointness of Q_j^k for fixed k implies that for $E_{j_1}^{k_1}$ and $E_{j_2}^{k_2}$.

Now fix Q_j^k; we have that

$$\sigma(Q_j^k) = \sigma(Q_j^k \cap X_{k+1}) + \sigma(E_j^k). \tag{4.3}$$

By the properties listed above,

$$\begin{aligned}\sigma(Q_j^k \cap X_{k+1}) &= \sum_{i: Q_i^{k+1} \subset Q_j^k} \sigma(Q_i^{k+1}) \leq \frac{1}{a^{k+1}} \sum_{i: Q_i^{k+1} \subset Q_j^k} \int_{Q_i^{k+1}} |f(y)|\, d\sigma \\ &\leq \frac{1}{a^{k+1}} \int_{Q_j^k} |f(y)|\, d\sigma \leq \frac{C_{CZ}}{a}\sigma(Q_j^k).\end{aligned}$$

After plugging this into (4.3) and rearranging, we obtain

$$\sigma(E_j^k) \geq \frac{a - C_{CZ}}{a}\sigma(Q_j^k),$$

which is the desired inequality.

For $\mu(X) < \infty$, the proof is the same, with one exception. Since X is bounded, for all cubes Q sufficiently large, $Q = X$. As such, choosing $\lambda > \fint_X |f(y)|\,d\sigma$ ensures that we may find maximal cubes as before. □

5. Necessity

In this section we prove the necessity of the $A_{p(\cdot)}$ condition in Theorem 1.13. Actually, we will prove necessity in Conjecture 1.14, but by the monotonicity of the norm, we get that the strong-type inequality implies the weak-type, so to prove necessity in both results it suffices to demonstrate that any weight satisfying the latter is in $A_{p(\cdot)}$.

To that end, let w be such a weight and fix a ball $B \subseteq X$. First, we will show that w is $p(\cdot)$-integrable on B. Supposing to the contrary, since $p_+ < \infty$ we have from Lemma 2.3 that $\|w\chi_B\|_{p(\cdot)} = \infty$. Fix $x \in B$ and choose any ball E with $x \in E \subseteq B$. If we choose $f = \chi_E$ then $Mf(x) \geq \frac{\mu(E)}{\mu(B)}\chi_B$. Then for each $t < \frac{\mu(E)}{\mu(B)}$ the weak-type inequality implies that

$$t\|w\chi_B\|_{p(\cdot)} \leq \|t\chi_{\{x\in X\,:\,Mf(x)>t\}}w\|_{p(\cdot)} \leq C\|w\chi_E\|_{p(\cdot)}.$$

Thus the right hand side must be infinite, and so by Lemma 2.3,

$$\int_E w(x)^{p(x)}\,d\mu = \infty.$$

Letting E shrink to x and applying the Lebesgue Differentiation Theorem (since μ is Borel regular; see [2, Theorem 1.4]), we find that $w(x)^{p(x)} = \infty$ and thus $w(x) = \infty$ for almost every x, contrary to the definition of a weight. It follows that w is locally $p(\cdot)$-integrable.

Now we show that $w \in A_{p(\cdot)}$. We first assume that $\|w^{-1}\chi_B\|_{p'(\cdot)} < \infty$; later, we will see that this is necessarily the case. By the homogeneity of both the weak-type inequality and the $A_{p(\cdot)}$ condition in w, we can assume that $\|w^{-1}\chi_B\|_{p'(\cdot)} = 1$.

We partition B into the sets

$$F_0 = \{x \in B\colon p'(x) < \infty\}, \quad F_\infty = \{x \in B\colon p'(x) = \infty\}.$$

By the definition of the norm, for any $\lambda \in (\frac{1}{2}, 1)$,

$$1 < \rho_{p'(\cdot)}\left(\frac{w^{-1}\chi_B}{\lambda}\right) = \int_{F_0}\left(\frac{w(x)^{-1}}{\lambda}\right)^{p'(x)} d\mu + \lambda^{-1}\|w^{-1}\chi_{F_\infty}\|_\infty.$$

One of the terms on the right must be greater than $\frac{1}{2}$. More specifically, one of the following must be true: either $\|w^{-1}\chi_{F_\infty}\|_\infty \geq \frac{1}{2}$, or there exists $\lambda_0 \in (\frac{1}{2}, 1)$ for which $\int_{F_0}\left(\frac{w(x)^{-1}}{\lambda}\right)^{p'(x)} d\mu \geq \frac{1}{2}$ for any $\lambda \in [\lambda_0, 1)$. Suppose for now it is the first.

Fix $s > \|w^{-1}\chi_{F_\infty}\|_\infty^{-1} = \operatorname{ess\,inf}_{x\in F_\infty} w(x)$. There exists a subset $E \subseteq F_\infty$ with $\mu(E) > 0$ such that $w(E) \subseteq (0, s]$. Choose the function $f = \chi_E$. Since $p(\cdot)$ is identically 1 on F_∞,

$$\|fw\|_{p(\cdot)} = \|w\chi_E\|_{p(\cdot)} = w(E).$$

Further, we see that for all $x \in B$,

$$Mf(x) \geq \frac{\mu(E)}{\mu(B)}.$$

Thus if we fix $t < \frac{\mu(E)}{\mu(B)}$, the weak-type inequality implies that

$$t\|w\chi_B\|_{p(\cdot)} \leq t\|w\chi_{\{x:\, Mf(x)>t\}}\|_{p(\cdot)} \leq C\|fw\|_{p(\cdot)} = Cw(E).$$

If we take the supremum over all such t and rearrange, we get that

$$\frac{1}{\mu(B)}\|w\chi_B\|_{p(\cdot)} \leq C\frac{w(E)}{\mu(E)} \leq Cs.$$

Now taking the infimum over all such s, we get

$$\frac{1}{\mu(B)}\|w\chi_B\|_{p(\cdot)} \leq C\|w^{-1}\chi_{F_\infty}\|_\infty^{-1} \leq 2C.$$

Since $\|w^{-1}\chi_B\|_{p'(\cdot)} = 1$, this is the $A_{p(\cdot)}$ condition on B.

We now consider the case that

$$\int_{F_0} \left(\frac{w(x)^{-1}}{\lambda}\right)^{p'(x)} d\mu \geq \frac{1}{2}$$

for all $\lambda \in [\lambda_0, 1)$. If we define $F_R = \{x \in F_0 \colon p'(x) < R\}$ for $R > 1$, by the monotone convergence theorem for $L^{p(\cdot)}$ norms (Lemma 2.7) we may find R sufficiently large that

$$\int_{F_R} \left(\frac{w(x)^{-1}}{\lambda_0}\right)^{p'(x)} d\mu > \frac{1}{3}.$$

Further, since $\|w^{-1}\chi_B\|_{p'(\cdot)} = 1$, by Lemma 2.3,

$$\begin{aligned}\int_{F_R} \left(\frac{w(x)^{-1}}{\lambda_0}\right)^{p'(x)} d\mu &\leq \int_{F_R} \left(\frac{2}{\lambda_0}\right)^{p'(x)} \left(\frac{w(x)^{-1}}{2}\right)^{p'(x)} d\mu \\ &\leq \left(\frac{2}{\lambda_0}\right)^{R} \int_{F_R} \left(\frac{w(x)^{-1}}{2}\right)^{p'(x)} d\mu \leq \left(\frac{2}{\lambda_0}\right)^{R} < \infty.\end{aligned}$$

Now define the function

$$G(\lambda) = \int_{F_R} \left(\frac{w(x)^{-1}}{\lambda}\right)^{p'(x)} d\mu.$$

Then we know from the above computations that $\frac{1}{3} < G(\lambda_0) < \infty$ and by the dominated convergence theorem that G is continuous on $[\lambda_0, 1]$. If $G(1) \geq \frac{1}{3}$, then by Lemma 2.3, for any $\lambda \in [\lambda_0, 1)$,

$$\frac{1}{3\lambda} \leq \frac{1}{\lambda}\int_{F_R} w(x)^{-p'(x)}\, d\mu \leq G(\lambda) \leq \lambda^{-R} < \infty.$$

Now by taking λ sufficiently close to 1, we may make $\lambda^{-R} \leq 2$, so that

$$\frac{1}{3} \leq \int_{F_R} \left(\frac{w(x)^{-1}}{\lambda}\right)^{p'(x)} d\mu \leq 2. \tag{5.1}$$

On the other hand, if $G(1) < \frac{1}{3}$, then by continuity there is some $\lambda \in (\lambda_0, 1)$ such that $G(\lambda) = \frac{1}{3}$, and so by choosing this λ we get that (5.1) holds in this case as well.

Having fixed λ, we now choose our function to be

$$f(x) = \frac{w(x)^{-p'(x)}}{\lambda^{p'(x)-1}}\chi_{F_R}.$$

Then

$$\rho_{p(\cdot)}(fw) = \int_{F_R} \left(\frac{w(x)^{-1}}{\lambda}\right)^{p'(x)} d\mu \leq 2.$$

Hence, by the proof of Lemma 2.3, $\|fw\|_{p(\cdot)} \leq 2^{1/(p')_-}$. On the other hand, for all $x \in B$,

$$Mf(x) \geq \fint_B f(x)\,d\mu = \frac{\lambda}{\mu(B)}\int_{F_R}\left(\frac{w(x)^{-1}}{\lambda}\right)^{p'(x)} d\mu \geq \frac{\lambda}{3\mu(B)}.$$

Thus for $t < \frac{\lambda}{3\mu(B)}$, by the weak-type inequality,

$$C \geq C\|fw\|_{p(\cdot)} \geq t\|w\chi_{\{x\,:\,Mf(x)>t\}}\|_{p(\cdot)} \geq t\|w\chi_B\|_{p(\cdot)},$$

which after taking the supremum over all such t is the $A_{p(\cdot)}$ condition.

It remains to show that $w \in A_{p(\cdot)}$ if $\|w^{-1}\chi_B\|_{p'(\cdot)} = \infty$. To that end, fix $\epsilon > 0$ and define the weight $w_\epsilon(x) = w(x) + \epsilon$. Then $w_\epsilon^{-1} \leq \epsilon^{-1} < \infty$ and so $\|w_\epsilon^{-1}\chi_B\|_{p'(\cdot)} < \infty$. We observe that

$$\|w_\epsilon\chi_{\{x\in X\,:\,Mf(x)>t\}}\|_{p(\cdot)} \leq \|w\chi_{\{x\in X\,:\,Mf(x)>t\}}\|_{p(\cdot)} + \epsilon\|\chi_{\{x\in X\,:\,Mf(x)>t\}}\|_{p(\cdot)}.$$

Since $p(\cdot) \in \mathrm{LH}$, M satisfies the weak type inequality on $L^{p(\cdot)}(X,\mu)$. This is a result of the sufficiency argument (Section 6) if $p_- > 1$, and in general it is one case in the main result of [16]. Consequently,

$$\leq C\|fw\|_{p(\cdot)} + C\|\epsilon f\|_{p(\cdot)} \leq 2C\|fw_\epsilon\|_{p(\cdot)}.$$

This shows that w_ϵ satisfies the weak-type inequality, and does so with a constant depending only on the weak-type inequality constants of w and 1, both of which are independent of ϵ. From the argument with $\|w^{-1}\chi_B\|_{p'(\cdot)} < \infty$, it follows that $w_\epsilon \in A_{p(\cdot)}$. In fact, careful inspection of the previous argument will show that

$$\|w\chi_B\|_{p(\cdot)}\|w_\epsilon^{-1}\chi_B\|_{p'(\cdot)} \leq \|w_\epsilon\chi_B\|_{p(\cdot)}\|w_\epsilon^{-1}\chi_B\|_{p'(\cdot)} \leq K\mu(B)$$

with K depending only on $p(\cdot)$ and the weak-type inequality constant (in the F_∞ case the dependency is only on the latter, while the F_0 case involves $(p')_-$). Since as we said before this is independent of ϵ, we have that K is independent of ϵ. Thus, since w_ϵ^{-1} increases to w^{-1} pointwise, by Lemma 2.7, we get that $w \in A_{p(\cdot)}$. While this completes the proof of necessity, it is of note that $w \in A_{p(\cdot)}$ in turn implies that the assumption $\|w^{-1}\chi_B\|_{p'(\cdot)} < \infty$ must have been true originally.

6. Sufficiency

In this section we prove sufficiency in Theorem 1.13. We first assume that $\mu(X) = \infty$; the finite measure case is much simpler, as we will later show. Consider the following lemma, which is proved in [24, 25].

Lemma 6.1. *There exists a finite family $\{\mathcal{D}_i\}_{i=1}^N$ of dyadic grids such that*

$$Mf(x) \leq C\sum_{i=1}^N M^{\mathcal{D}_i}f(x)$$

for any function f and almost every $x \in X$.

As a result of Lemma 6.1, to prove the boundedness of M it suffices to prove the boundedness of $M^{\mathcal{D}}$ for an arbitrary dyadic grid $\mathcal{D}$. To that end, fix an exponent $p(\cdot)$ with $1 < p_- \le p_+ < \infty$, a weight $w \in A_{p(\cdot)}$, and a function f; without loss of generality we may assume that f is nonnegative and that $\|fw\|_{p(\cdot)} = 1$. It is useful to define the weights $W(\cdot) = w(\cdot)^{p(\cdot)}$ and $\sigma(\cdot) = w(\cdot)^{-p'(\cdot)}$, both of which are in A_∞ by Lemma 3.4 and hence doubling by Lemma 3.1.

We will want to form the Calderón–Zygmund cubes of f (with respect to μ). In order to do so, we must show that $\fint_{Q_k} |f(x)|\,d\mu \to 0$ as $k \to \infty$ for any nested sequence $\{Q_k\}_{k=1}^\infty$ with $Q_{k-1} \subseteq Q_k \in \mathcal{D}_k$. Fix such a sequence; considering $k = 1$, we have as a consequence of W being doubling that

$$W(Q_1) \le W(B(x_c(Q_1), C_d d_0)) \le C_W^{\log_2 C_d} W(B(x_c(Q_1), d_0)).$$

By a similar argument, for any k,

$$\frac{1}{W(Q_k)} \le \frac{C}{W(B(x_c(Q_k), C_d d_0^k))}.$$

Combining these two estimates and applying Lemma 3.1, we get

$$\frac{W(Q_1)}{W(Q_k)} \le C \frac{W(B(x_c(Q_1), d_0))}{W(B(x_c(Q_k), C_d d_0^k))} \le C \left(\frac{\mu(B(x_c(Q_1, d_0))}{\mu(B(x_c(Q_k), C_d d_0^k))} \right)^\delta.$$

If we rearrange and apply the lower mass bound,

$$W(Q_k) \ge C\mu(B(x_c(Q_k), C_d d_0^k))^\delta \ge C\mu(B(x_c(Q_1), C d_0^k))^\delta.$$

As $k \to \infty$, by continuity of μ and the fact that $X = \bigcup_{k=1}^\infty B(x_c(Q_1), C d_0^k)$, the right side approaches $C\mu(X)^\delta = \infty$, and thus $W(Q_k) \to \infty$. By Lemma 2.8, the $A_{p(\cdot)}$ condition, and Lemma 2.5 respectively, for all k sufficiently large,

$$\fint_{Q_k} f(x)\,d\mu \le C\|fw\|_{p(\cdot)} \mu(Q_k)^{-1} \|w^{-1}\chi_{Q_k}\|_{p'(\cdot)} \le C\|w\chi_{Q_k}\|_{p(\cdot)}^{-1} \le CW(Q_k)^{-1/p_+}.$$

This gives us the desired limit.

Decompose f as $f = f_1 + f_2$ where $f_1 = f\chi_{\{f\sigma^{-1} > 1\}}$ and $f_2 = f\chi_{\{f\sigma^{-1} \le 1\}}$. By sublinearity, $M^{\mathcal{D}} f \le M^{\mathcal{D}} f_1 + M^{\mathcal{D}} f_2$, and by Lemma 2.5, for $i = 1, 2$,

$$\int_X |f_i(x)|^{p(x)} w(x)^{p(x)}\,d\mu \le \|f_i w\|_{p(\cdot)} \le 1. \tag{6.1}$$

Hence, by Lemma 2.3, to prove the desired inequality it suffices to show that there exists a constant C depending on X, $p(\cdot)$, and w such that

$$\int_X M^{\mathcal{D}} f_i(x)^{p(x)} w(x)^{p(x)}\,d\mu \le C, \quad i = 1, 2. \tag{6.2}$$

We begin with the estimate of (6.2) for f_1. Choose $a > C_{CZ}$ and for each $k \in \mathbb{Z}$ let

$$X_k = \{x \in X \colon M^{\mathcal{D}} f_1(x) > a^{k+1}\}.$$

Since $f \in L^1_{\mathrm{loc}}$ and $\fint_{Q^k} f(x)\,d\mu \to 0$ as $k \to \infty$, $M^{\mathcal{D}} f_1$ is finite almost everywhere, and so

$$\{x \in X \colon Mf_1(x) > 0\} = \bigcup_k X_k \setminus X_{k+1}$$

up to a set of measure zero. Let $\{Q_j^k\}$ be the CZ cubes of f_1 at height a^k with respect to μ. Then by Lemma 4.5, for all k,

$$X_k = \bigcup_j Q_j^k. \tag{6.3}$$

Define the sets $E_j^k = Q_j^k \setminus X_k$, as in Lemma 4.5. Then from (6.3) we have

$$X_k \setminus X_{k+1} = \bigcup_j E_j^k.$$

We now estimate:

$$\begin{aligned}
\int_X M^{\mathcal{D}} f_1(x)^{p(x)} w(x)^{p(x)}\, d\mu &= \sum_k \int_{X_k \setminus X_{k+1}} M^{\mathcal{D}} f_1(x)^{p(x)} w(x)^{p(x)}\, d\mu \\
&\le a^{2p_+} \sum_k \int_{X_k \setminus X_{k+1}} a^{kp(x)} w(x)^{p(x)}\, d\mu \\
&\le C \sum_{k,j} \int_{E_j^k} \left(\fint_{Q_j^k} f_1(y)\, d\mu \right)^{p(x)} w(x)^{p(x)}\, d\mu \\
&= C \sum_{k,j} \int_{E_j^k} \left(\int_{Q_j^k} f_1(y)\sigma(y)^{-1}\sigma(y)\, d\mu \right)^{p(x)} \mu(Q_j^k)^{-p(x)} w(x)^{p(x)}\, d\mu. \qquad (6.4)
\end{aligned}$$

Since either $f_1\sigma^{-1} \ge 1$ or $f_1\sigma^{-1} = 0$,

$$\begin{aligned}
\int_{Q_j^k} f_1(y)\sigma(y)^{-1}\sigma(y)\, d\mu &\le \int_{Q_j^k} (f_1(y)\sigma(y)^{-1})^{p(y)/p_-(Q_j^k)} \sigma(y)\, d\mu \\
&\le \int_{Q_j^k} (f_1(y)\sigma(y)^{-1})^{p(y)} \sigma(y)\, d\mu \le \int_{Q_j^k} f_1(y)^{p(y)}\, d\mu \le 1.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{k,j} \int_{E_j^k} \left(\int_{Q_j^k} f_1(y)\sigma(y)^{-1}\sigma(y)\, d\mu \right)^{p(x)} \mu(Q_j^k)^{-p(x)} w(x)^{p(x)}\, d\mu \\
&\le \sum_{k,j} \left(\int_{Q_j^k} (f_1(y)\sigma(y)^{-1})^{p(y)/p_-(Q_j^k)} \sigma(y)\, d\mu \right)^{p_-(Q_j^k)} \int_{E_j^k} \mu(Q_j^k)^{-p(x)} w(x)^{p(x)}\, d\mu.
\end{aligned}$$

If we multiply and divide by $\sigma(Q_j^k)$ and apply Hölder's inequality with exponent $p_-(Q_j^k)/p_-$, we get

$$\begin{aligned}
&\le C \sum_{k,j} \left(\fint_{Q_j^k} (f_1(y)\sigma(y)^{-1})^{p(y)/p_-} \sigma(y)\, d\mu \right)^{p_-} \\
&\qquad \cdot \int_{E_j^k} \sigma(Q_j^k)^{p_-(Q_j^k)} \mu(Q_j^k)^{-p(x)} w(x)^{p(x)}\, d\mu. \qquad (6.5)
\end{aligned}$$

Assume for the moment that

$$\int_{Q_j^k} \sigma(Q_j^k)^{p_-(Q_j^k)} \mu(Q_j^k)^{-p(x)} w(x)^{p(x)}\, d\mu \le C\sigma(Q_j^k). \tag{6.6}$$

Since $\mu(Q_j^k) \le C\mu(E_j^k)$ by Lemma 4.5 and $\sigma \in A_\infty$ by Lemma 3.4 applied to $w^{-1} \in A_{p'(\cdot)}$, we have from Lemma 3.1 (applied to cubes instead of balls) that $\sigma(Q_j^k) \le C\sigma(E_j^k)$. Thus (6.5) is bounded by

$$\begin{aligned} C\sum_{k,j}\left(\fint_{Q_j^k}(f_1(y)\sigma(y)^{-1})^{p(y)/p_-}\,d\sigma\right)^{p_-}\sigma(E_j^k) \\ \le C\sum_{k,j}\int_{E_j^k} M_\sigma^{\mathcal{D}}((f_1\sigma^{-1})^{p(\cdot)/p_-})(x)^{p_-}\sigma(x)\,d\mu \\ \le C\int_X M_\sigma^{\mathcal{D}}((f_1\sigma^{-1})^{p(\cdot)/p_-})(x)^{p_-}\sigma(x)\,d\mu. \end{aligned}$$

By Lemma 4.4 and (6.2),

$$\begin{aligned} &\le C\int_X f_1(x)^{p(x)}\sigma(x)^{-p(x)}\sigma(x)\,d\mu \\ &= C\int_X f_1(x)^{p(x)}w(x)^{p(x)}\,d\mu \le C. \end{aligned}$$

We now justify (6.6). Observe that the left-hand side is dominated by

$$\begin{aligned} &\left(\frac{\sigma(Q_j^k)}{\|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)}}\right)^{p_-(Q_j^k)} \\ &\cdot\int_{Q_j^k}\|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)}^{p_-(Q_j^k)-p(x)}\|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)}^{p(x)}\mu(Q_j^k)^{-p(x)}w(x)^{p(x)}\,d\mu. \end{aligned} \tag{6.7}$$

We will bound (6.7) by showing that, under our hypotheses, it reduces to the $A_{p(\cdot)}$ condition. First, we show that

$$\|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)}^{p_-(Q_j^k)-p(x)} \le C. \tag{6.8}$$

If $\|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)} > 1$, then $C = 1$ works, so assume otherwise. Then

$$\begin{aligned} p(x) - p_-(Q_j^k) &= \frac{p'(x)}{p'(x)-1} - \frac{(p')_+(Q_j^k)}{(p')_+(Q_j^k)-1} \\ &= \frac{(p')_+(Q_j^k)-p'(x)}{[p'(x)-1][(p')_+(Q_j^k)-1]} \le \frac{(p')_+(Q_j^k)-(p')_-(Q_j^k)}{[(p')_- - 1]^2}, \end{aligned}$$

and so by Lemma 3.3, we obtain (6.8). We would also like to prove the bound

$$\left(\frac{\sigma(Q_j^k)}{\|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)}}\right)^{p_-(Q_j^k)} \le C\sigma(Q_j^k). \tag{6.9}$$

If $\|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)} > 1$, then by Lemma 2.5,

$$\left(\frac{\sigma(Q_j^k)}{\|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)}}\right)^{p_-(Q_j^k)} \le \left(\sigma(Q_j^k)^{1-1/(p')_+(Q_j^k)}\right)^{p_-(Q_j^k)} = \sigma(Q_j^k).$$

If on the other hand $\|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)} \leq 1$, then by Lemma 2.5 (applied twice) and Lemma 3.3 (applied to cubes),

$$\begin{aligned}
\left(\frac{\sigma(Q_j^k)}{\|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)}}\right)^{p_-(Q_j^k)} &\leq \left(\|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)}^{(p')_-(Q_j^k)-1}\right)^{p_-(Q_j^k)} \\
&\leq \left(\|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)}^{(p')_-(Q_j^k)-1+(p')_+(Q_j^k)-(p')_+(Q_j^k)}\right)^{p_-(Q_j^k)} \\
&\leq C\left(\|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)}^{(p')_+(Q_j^k)-1}\right)^{p_-(Q_j^k)} \\
&\leq C\left(\sigma(Q_j^k)^{\frac{(p')_+(Q_j^k)-1}{(p')_+(Q_j^k)}}\right)^{p_-(Q_j^k)} \\
&\leq C\left(\sigma(Q_j^k)^{\frac{p_-(Q_j^k)'-1}{p_-(Q_j^k)'}}\right)^{p_-(Q_j^k)} = C\sigma(Q_j^k).
\end{aligned}$$

Applying both (6.8) and (6.9) to (6.7), we have that in order to demonstrate (6.6) it suffices to show

$$\int_{Q_j^k} \|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)}^{p(x)} \mu(Q_j^k)^{-p(x)} w(x)^{p(x)}\, d\mu \leq C. \tag{6.10}$$

By Lemma 2.3, this is equivalent to bounding

$$\|(C\mu(Q_j^k))^{-1}\|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)} w\chi_{Q_j^k}\|_{p(\cdot)} = \frac{1}{C\mu(Q_j^k)}\|w\chi_{Q_j^k}\|_{p(\cdot)}\|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)}.$$

But by Lemma 4.2 this is, as claimed, the $A_{p(\cdot)}$ condition. Since $w \in A_{p(\cdot)}$, (6.6) holds for any k and j. This completes the proof of (6.2) for f_1.

We now proceed to show the corresponding bound for f_2. Recall that 1, σ, and W are all in A_∞; from now on, we will use properties of A_∞ without reference.

We would like to fix a particular LH_∞ base point x_0. Let $\{Q_j^k\}$ now represent the CZ cubes of f_2 with respect to μ. Choose a nested tower of cubes $\{Q_{k,0}\}$. Since A_∞ weights are doubling, we have that $\mu(Q_{k,0})$, $\sigma(Q_{k,0})$, and $W(Q_{k,0})$ all go to infinity, and as a result we may find a cube $Q_{k_0,0} = Q_0 \in \mathcal{D}_{k_0}$ such that $\mu(Q_0)$, $\sigma(Q_0)$, and $W(Q_0) \geq 1$. By Lemma 1.9, we may fix $x_0 = x_c(Q_0)$. Let $N_0 = 2A_0C_d$, where C_d is as in Theorem 4.1, and define the sets

$$\begin{aligned}
\mathscr{F} &= \{(k,j)\colon Q_j^k \subseteq Q_0\}, \\
\mathscr{G} &= \{(k,j)\colon Q_j^k \not\subseteq Q_0 \text{ and } d(x_0, x_c(Q_j^k)) < N_0 d_0^k\}, \\
\mathscr{H} &= \{(k,j)\colon Q_j^k \not\subseteq Q_0 \text{ and } d(x_0, x_c(Q_j^k)) \geq N_0 d_0^k\}.
\end{aligned}$$

Observe that $\mathscr{F} \cup \mathscr{G} \cup \mathscr{H} = \mathbb{Z} \times \mathbb{N}$, so that every CZ cube Q_j^k has indices in one of the three sets. By repeating the argument used to obtain (6.4) with f_2 in place of

f_1, we may split the corresponding sum into three parts:

$$\int_X M^{\mathcal{D}} f_2(x)^{p(x)} w(x)^{p(x)}\,d\mu \le C\sum_{k,j}\int_{E_j^k}\left(\fint_{Q_j^k} f_2(y)\sigma(y)\sigma(y)^{-1}\,d\mu\right)^{p(x)} w(x)^{p(x)}\,d\mu$$
$$= C\left(\sum_{(k,j)\in\mathscr{F}} + \sum_{(k,j)\in\mathscr{G}} + \sum_{(k,j)\in\mathscr{H}}\right) = C(I_1+I_2+I_3).$$

We will bound each of these three sums in turn, beginning with I_1. Using that $f_2\sigma^{-1}\le 1$ to eliminate f_2 and then applying (6.6), we get

$$\begin{aligned}
I_1 &\le \sum_{(k,j)\in\mathscr{F}}\int_{E_j^k}\left(\fint_{Q_j^k}\sigma(y)\,d\mu\right)^{p(x)} w(x)^{p(x)}d\mu\\
&\le \sum_{(k,j)\in\mathscr{F}}\int_{E_j^k}\sigma(Q_j^k)^{p(x)-p_-(Q_j^k)}\sigma(Q_j^k)^{p_-(Q_j^k)}\mu(Q_j^k)^{-p(x)}w(x)^{p(x)}\,d\mu\\
&\le \sum_{(k,j)\in\mathscr{F}}(1+\sigma(Q_j^k))^{p_+(Q_j^k)-p_-(Q_j^k)}\int_{E_j^k}\sigma(Q_j^k)^{p_-(Q_j^k)}\mu(Q_j^k)^{-p(x)}w(x)^{p(x)}\,d\mu\\
&\le C(1+\sigma(Q_0))^{p_+-p_-}\sum_{(k,j)\in\mathscr{F}}\sigma(Q_j^k)\\
&\le C(1+\sigma(Q_0))^{p_+-p_-}\sum_{(k,j)\in\mathscr{F}}\sigma(E_j^k)\\
&\le C(1+\sigma(Q_0)))^{p_+-p_-}\sigma(Q_0),
\end{aligned}$$

which is a constant independent of Q_j^k and f.

Now to estimate I_2, pick $(k,j)\in\mathscr{G}$. Note that if $x_c(Q_j^k)\in Q_0$, then since $Q_j^k\not\subseteq Q_0$ we must have that

$$Q_0\subseteq Q_j^k\subseteq B(x_c(Q_j^k),A_0(C_d+1)N_0d_0^k).$$

On the other hand, if $x_c(Q_j^k)\notin Q_0\supseteq B(x_0,d_0^{k_0})$, then by the definition of $\mathscr{G}$,

$$d_0^{k_0}\le d(x_0,x_c(Q_j^k))\le N_0d_0^k.$$

As a result, since $x_0\in B(x_c(Q_j^k),N_0d_0^k)$ and $x\in B(x_0,C_dd_0^{k_0})$, for any $x\in Q_0$,

$$d(x,x_c(Q_j^k))\le A_0(d(x,x_0)+d(x_0,x_c(Q_j^k)))\le A_0(C_dd_0^{k_0}+N_0d_0^k)\le A_0(C_d+1)N_0d_0^k.$$

It follows that $Q_0\subseteq B(x_c(Q_j^k),A_0(C_d+1)N_0d_0^k)=B_j^k$ for any $(k,j)\in\mathscr{G}$. Consequently, we have that $W(B_j^k),\sigma(B_j^k)\ge 1$. Note also that by doubling and Lemma 4.1, $\mu(Q_j^k)\approx\mu(B_j^k)$. By Lemma 2.3 we also have that $\|w^{-1}\chi_{Q_0}\|_{p'(\cdot)}\ge 1$, and so by Corollary 3.5 (applied to $w^{-1}\in A_{p'(\cdot)}$),

$$\begin{aligned}
\mu(Q_j^k)^{-1}\le C\mu(B_j^k)^{-1}&\le C\mu(Q_0)^{-1}\left(\frac{\sigma(Q_0)}{\sigma(B_j^k)}\right)^{1/p'_\infty}\\
&\le C\|w^{-1}\chi_{B_j^k}\|_{p'(\cdot)}^{-1}\le C\|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)}^{-1}.
\end{aligned}$$

It follows from this inequality and Lemma 2.8 that

$$\fint_{Q_j^k} f_2(y)\,d\mu \le C\|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)}^{-1}\|f_2 w\|_{p(\cdot)}\|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)} \le C.$$

Given this, we may apply Lemma 2.10 with the exponents $p(\cdot)$ and p_∞ to estimate:

$$\begin{aligned} I_2 &\le C \sum_{(k,j)\in\mathscr{G}} \int_{E_j^k} \left(C^{-1} \fint_{Q_j^k} f_2(y)\,d\mu \right)^{p(x)} w(x)^{p(x)}\,d\mu \\ &\le C_t \sum_{(k,j)\in\mathscr{G}} \int_{E_j^k} \left(\fint_{Q_j^k} f_2(y)\,d\mu \right)^{p_\infty} w(x)^{p(x)}\,d\mu \\ &\qquad + \sum_{(k,j)\in\mathscr{G}} \int_{E_j^k} \frac{w(x)^{p(x)}}{(e+d(x_0,x))^{tp_-}}\,d\mu. \end{aligned} \tag{6.11}$$

Arguing as we did in the proof of Lemma 3.4 to obtain inequality (3.5), we may choose t sufficiently large (depending only on X, Q_0, $p(\cdot)$, and w) so that

$$\sum_{(k,j)\in\mathscr{G}} \int_{E_j^k} \frac{w(x)^{p(x)}}{(e+d(x_0,x))^{tp_-}}\,d\mu \le \int_X \frac{w(x)^{p(x)}}{(e+d(x_0,x))^{tp_-}}\,d\mu \le 1. \tag{6.12}$$

We now need only bound the first term of (6.11). But we have that

$$\begin{aligned} &\sum_{(k,j)\in\mathscr{G}} \int_{E_j^k} \left(\fint_{Q_j^k} f_2(y)\,d\mu \right)^{p_\infty} w(x)^{p(x)}\,d\mu \\ &= \sum_{(k,j)\in\mathscr{G}} \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} f_2(y)\sigma(y)^{-1}\sigma(y)\,d\mu \right)^{p_\infty} \left(\frac{\sigma(Q_j^k)}{\mu(Q_j^k)} \right)^{p_\infty} W(E_j^k). \end{aligned}$$

Now invoking (3.6) (applied to σ and then W, with cubes) as well as the $A_{p(\cdot)}$ condition,

$$\begin{aligned} \sigma(Q_j^k)^{p_\infty-1} = \sigma(Q_j^k)^{p_\infty/p'_\infty} &\le C\|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)}^{p_\infty} \\ &\le C\left(\frac{\mu(Q_j^k)}{\|w\chi_{Q_j^k}\|_{p(\cdot)}} \right)^{p_\infty} \le C\frac{\mu(Q_j^k)^{p_\infty}}{W(Q_j^k)}. \end{aligned} \tag{6.13}$$

If we apply this estimate, Lemmas 4.4 (since by assumption $p_- > 1$ and we must have $p_\infty \geq p_-$) and 2.10, and that $\sigma(Q_j^k) \leq C\sigma(E_j^k)$, then we get

$$\sum_{(k,j)\in\mathscr{G}} \left(\frac{1}{\sigma(Q_j^k)}\int_{Q_j^k} f_2(y)\sigma(y)^{-1}\sigma(y)\,d\mu\right)^{p_\infty}\left(\frac{\sigma(Q_j^k)}{\mu(Q_j^k)}\right)^{p_\infty} W(E_j^k). \tag{6.14}$$

$$\leq C\sum_{(k,j)\in\mathscr{G}} \left(\frac{1}{\sigma(Q_j^k)}\int_{Q_j^k} f_2(y)\sigma(y)^{-1}\sigma(y)\,d\mu\right)^{p_\infty} \sigma(Q_j^k)W(Q_j^k)^{-1}W(E_j^k)$$

$$\leq C\sum_{(k,j)\in\mathscr{G}} \left(\frac{1}{\sigma(Q_j^k)}\int_{Q_j^k} f_2(y)\sigma(y)^{-1}\sigma(y)\,d\mu\right)^{p_\infty} \sigma(E_j^k)$$

$$\leq C\sum_{(k,j)\in\mathscr{G}} \int_{E_j^k} M_\sigma(f_2\sigma^{-1})(x)^{p_\infty}\sigma(x)\,d\mu$$

$$\leq C\int_X M_\sigma(f_2\sigma^{-1})(x)^{p_\infty}\sigma(x)\,d\mu \tag{6.15}$$

$$\leq C\int_X (f_2(x)\sigma^{-1}(x))^{p_\infty}\sigma(x)\,d\mu \tag{6.16}$$

$$\leq C_t\int_X (f_2(x)\sigma(x))^{p(x)}\sigma(x)\,d\mu + \int_X \frac{\sigma(x)}{(e+d(x_0,x))^{tp_-}}\,d\mu$$

$$\leq C_t\int_X f_2(x)^{p(x)}w(x)^{p(x)}\,d\mu + \int_X \frac{\sigma(x)}{(e+d(x_0,x))^{tp_-}}\,d\mu. \tag{6.17}$$

The second term is bounded by a constant independent of Q_j^k and f, by an argument identical to that used to prove (6.12) with σ in place of W. By (6.1), the first term is also bounded by a constant, and thus I_2 is as well.

We now estimate I_3. Central to this part of the proof will be that $d(x_0,x)$ is essentially constant on Q_j^k; that is,

$$\sup_{x\in Q_j^k} d(x_0,x) \leq R \inf_{x\in Q_j^k} d(x_0,x), \tag{6.18}$$

for some constant $R \geq 1$ independent of k and j. In fact, we will show that (6.18) is true with Q_j^k replaced by the ball $A_j^k = N_0^{-1}B_j^k \supseteq Q_j^k$. To that end, fix $(k,j) \in \mathscr{H}$ and choose $x \in A_j^k$. We have that

$$\begin{aligned} d(x_0,x) &\leq A_0[d(x_0,x_c(Q_j^k)) + d(x_c(Q_j^k),x)] \\ &\leq A_0[d(x_0,x_c(Q_j^k)) + C_d d_0^k] \leq \left(A_0 + \frac{1}{2}\right) d(x_0,x_c(Q_j^k)). \end{aligned}$$

Conversely,

$$\begin{aligned} d(x_0,x_c(Q_j^k)) &\leq A_0[d(x,x_c(Q_j^k)) + d(x_0,x)] \\ &= \frac{1}{2}N_0 d_0^k + A_0 d(x_0,x) \leq \frac{1}{2}d(x_0,x_c(Q_j^k)) + A_0 d(x_0,x), \end{aligned}$$

and so by rearranging terms,

$$d(x_0,x_c(Q_j^k)) \leq 2A_0 d(x_0,x).$$

It follows that $d(x_0,x_c(Q_j^k)) \approx d(x_0,x)$. This is equivalent to (6.18).

To now estimate I_3, we need to divide $\mathscr{H}$ into two subsets,

$$\mathscr{H}_1 = \{(k,j) \in \mathscr{H} \colon \sigma(Q_j^k) \leq 1\}, \quad \mathscr{H}_2 = \{(k,j) \in \mathscr{H} \colon \sigma(Q_j^k) > 1\}.$$

We sum first over $\mathscr{H}_1$. Let $x_+ \in \overline{A_j^k}$ be the point which (by continuity of $p(\cdot) \in \mathrm{LH}_0$) satisfies $p_+(A_j^k) = p(x_+)$. Then by the LH_∞ condition and (6.18), for all $x \in Q_j^k$,

$$\begin{aligned} |p_+(Q_j^k) - p(x)| &\leq |p(x_+) - p_\infty| + |p(x) - p_\infty| \\ &\leq \frac{C_\infty}{\log(e + d(x_0, x_+))} + \frac{C_\infty}{\log(e + d(x_0, x))} \\ &\leq C_\infty \left[\frac{1}{\log(e + (RA_0)^{-1} d(x_0, x))} + \frac{1}{\log(e + d(x_0, x))} \right] \\ &\leq \frac{C_\infty (RA_0 + 1)}{\log(e + d(x_0, x))}. \end{aligned}$$

This provides the necessary condition to apply Lemma 2.9, from which (bounding the second term with (6.12) as before) we get

$$\sum_{(k,j)\in\mathscr{H}_1} \int_{E_j^k} \left(\fint_{Q_j^k} f_2(y)\, d\mu \right)^{p(x)} w(x)^{p(x)}\, d\mu \leq C_t \sum_{(k,j)\in\mathscr{H}_1} \int_{E_j^k} \left(\fint_{Q_j^k} f_2(y)\, d\mu \right)^{p_+(Q_j^k)} + 1.$$

By appealing to Lemma 2.11 for the inequality

$$\mu(Q_j^k)^{p(x) - p_+(Q_j^k)} \leq C,$$

we may bound the sum on the right by

$$C \sum_{(k,j)\in\mathscr{H}_1} \int_{E_j^k} \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} f_2(y) \sigma(y)^{-1} \sigma(y)\, d\mu \right)^{p_+(Q_j^k)} \sigma(Q_j^k)^{p_+(Q_j^k)} \mu(Q_j^k)^{-p(x)} w(x)^{p(x)}\, d\mu,$$

and since $f_2 \sigma^{-1} \leq 1$, by Lemma 2.10 we may continue to estimate

$$\begin{aligned} &\leq C \sum_{(k,j)\in\mathscr{H}_1} \int_{E_j^k} \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} f_2(y) \sigma(y)^{-1} \sigma(y)\, d\mu \right)^{p_\infty} \sigma(Q_j^k)^{p_+(Q_j^k)} \mu(Q_j^k)^{-p(x)} w(x)^{p(x)}\, d\mu \\ &\quad + C \sum_{(k,j)\in\mathscr{H}_1} \sigma(Q_j^k)^{p_+(Q_j^k)} \mu(Q_j^k)^{-p(x)} \frac{w(x)^{p(x)}}{(e + d(x_0, x))^{tp_-}}\, d\mu = CJ_1 + CJ_2. \end{aligned}$$

To estimate J_2 we use that $\sigma(Q_j^k) \leq 1$, then apply (6.6)—together with the fact that $\sigma(Q_j^k) \leq C\sigma(E_j^k)$, as used in the f_1 argument—and subsequently (6.18), to get that

$$\begin{aligned} J_2 &\leq \sum_{(k,j)\in\mathscr{H}_1} \sup_{x\in E_j^k} (e + d(x_0, x))^{-tp_-} \int_{E_j^k} \sigma(Q_j^k)^{p_-(Q_j^k)} \mu(Q_j^k)^{-p(x)} w(x)^{p(x)}\, d\mu \\ &\leq C \sum_{(k,j)\in\mathscr{H}_1} \sup_{x\in E_j^k} (e + d(x_0, x))^{-tp_-} \sigma(E_j^k) \\ &\leq C \sum_{(k,j)\in\mathscr{H}_1} \int_{E_j^k} \frac{\sigma(x)}{(e + d(x_0, x))^{tp_-}}\, d\mu \\ &\leq C \int_X \frac{\sigma(x)}{(e + d(x_0, x))^{tp_-}}\, d\mu, \end{aligned}$$

which is the same quantity as the second term in (6.17), which we argued was bounded by a constant at the end of the estimate for I_2.

Similarly, to estimate J_1 we may use that $\sigma(Q_j^k) \leq 1$ and (6.6) to get that

$$J_1 \leq C \sum_{(k,j)\in\mathscr{H}_1} \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} f_2(y)\sigma(y)^{-1}\sigma(y)\,d\mu \right)^{p_\infty} \sigma(Q_j^k).$$

Again using that $\sigma(Q_j^k) \leq C\sigma(E_j^k)$, we get that

$$\begin{aligned} &\leq C \sum_{(k,j)\in\mathscr{H}_1} \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} f_2(y)\sigma(y)^{-1}\sigma(y)\,d\mu \right)^{p_\infty} \sigma(E_j^k) \\ &\leq C \int_X M_\sigma(f_2\sigma^{-1})(x)^{p_\infty}\sigma(x)\,d\mu. \end{aligned}$$

But this is yet another quantity that appears near the end of the I_2 estimate, and thus it is bounded by a constant. This completes the estimate for $\mathscr{H}_1$.

Finally, we now estimate the sum over $\mathscr{H}_2$. By Lemma 2.8,

$$\int_{Q_j^k} f_2(y)\,d\mu \leq c\|f_2 w\|_{p(\cdot)}\|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)} \leq c\|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)}.$$

Thus we can apply Lemma 2.10 to get

$$\begin{aligned} &\sum_{(k,j)\in\mathscr{H}_2} \int_{E_j^k} \left(\fint_{Q_j^k} f_2(y)\,d\mu \right)^{p(x)} w(x)^{p(x)}\,d\mu \\ &\leq C \sum_{(k,j)\in\mathscr{H}_2} \int_{E_j^k} \left(c\|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)}^{-1} \int_{Q_j^k} f_2(y)\,d\mu \right)^{p(x)} \left(\frac{\|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)}}{\mu(Q_j^k)} \right)^{p(x)} w(x)^{p(x)}\,d\mu \\ &\leq C \sum_{(k,j)\in\mathscr{H}_2} \int_{E_j^k} \left(\|w\chi_{Q_j^k}\|_{p'(\cdot)}^{-1} \int_{Q_j^k} f_2(y)\,d\mu \right)^{p_\infty} \left(\frac{\|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)}}{\mu(Q_j^k)} \right)^{p(x)} w(x)^{p(x)}\,d\mu \\ &\quad + \sum_{(k,j)\in\mathscr{H}_2} \int_{E_j^k} \left(\frac{\|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)}}{\mu(Q_j^k)} \right)^{p(x)} \frac{w(x)^{p(x)}}{(e+d(x_0,x))^{tp_-}}\,d\mu \\ &= K_1 + K_2. \end{aligned}$$

To estimate K_2, note that $1 \leq \sigma(Q_j^k) \leq C\sigma(E_j^k)$, so $\sigma(E_j^k) > \epsilon$ for some fixed constant $\epsilon > 0$. Therefore, by (6.10) and (6.18) we have that

$$\begin{aligned} K_2 &\leq \epsilon^{-1} \sum_{(k,j)\in\mathscr{H}_2} \sup_{x\in E_j^k} (e+d(x_0,x))^{-tp_-}\epsilon \int_{Q_j^k} \left(\frac{\|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)}}{\mu(Q_j^k)} \right)^{p(x)} w(x)^{p(x)}\,d\mu \\ &\leq C \sum_{(k,j)\in\mathscr{H}_2} \sup_{x\in E_j^k} (e+d(x_0,x))^{-tp_-}\sigma(E_j^k) \\ &\leq C \int_X \frac{\sigma(x)}{(e+d(x_0,x))^{tp_-}}\,d\mu, \end{aligned}$$

which as we argued in J_2 and I_2 is bounded by a constant.

To estimate K_1, we use (3.6) to get

$$\|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)}^{-p_\infty}\sigma(Q^k)j)^{p_\infty} \leq C\sigma(Q_j^k)^{p_\infty - p_\infty/p'_\infty} = C\sigma(Q_j^k).$$

Therefore, applying (6.10) and that $\sigma(Q_j^k) \le C\sigma(E_j^k)$, we have

$$\begin{aligned} K_1 &= \sum_{(k,j)\in\mathscr{H}_2} \int_{E_j^k} \left(\frac{1}{\sigma(Q_j^k)}\int_{Q_j^k} f_2(y)\,d\mu\right)^{p_\infty} \|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)}^{p(x)-p_\infty} \frac{\sigma(Q_j^k)^{p_\infty}}{\mu(Q_j^k)^{p(x)}} w(x)^{p(x)}\,d\mu \\ &\le C \sum_{(k,j)\in\mathscr{H}_2} \left(\frac{1}{\sigma(Q_j^k)}\int_{Q_j^k} f_2(y)\,d\mu\right)^{p_\infty} \sigma(Q_j^k) \int_{Q_j^k} \|w^{-1}\chi_{Q_j^k}\|_{p'(\cdot)}^{p(x)} \mu(Q_j^k)^{-p(x)} w(x)^{p(x)}\,d\mu \\ &\le C \sum_{(k,j)\in\mathscr{H}_2} \left(\frac{1}{\sigma(Q_j^k)}\int_{Q_j^k} f_2(y)\,d\mu\right)^{p_\infty} \sigma(E_j^k) \\ &\le C \int_X M_\sigma(f_2\sigma^{-1})(x)^{p_\infty}\sigma(x)\,d\mu. \end{aligned}$$

This last term is the same quantity that appeared in (6.15), which as we argued in the estimates for J_2 and I_2 is bounded by a constant. This completes the estimate for I_3, and thus gives us the desired estimate for f_2, completing our proof of the sufficiency of the $A_{p(\cdot)}$ condition in Theorem 1.13 for the strong-type inequality.

The finite case. If $\mu(X) < \infty$, we may apply the same proof as in the infinite case, with some modifications. For each $i = 1, 2$, in accordance with Lemma 4.5, we may only construct the CZ cubes at heights greater than $\lambda_0 = \fint_X f_i\,d\mu$. Note that with the assumption that $\|fw\|_{p(\cdot)} = 1$ as before, we have from Lemma 2.8 that

$$\lambda_0 \le 4\mu(X)^{-1}\|f_i w\|_{p(\cdot)}\|w^{-1}\|_{p'(\cdot)} \le 4\mu(X)^{-1}\|w^{-1}\|_{p'(\cdot)}.$$

By Lemma 2.3, this is bounded by a constant, since from the $A_{p(\cdot)}$ condition with $B = X$,

$$\|w^{-1}\|_{p'(\cdot)} \le C\mu(X)\|w\|_{p(\cdot)}^{-1}.$$

Fix $a = 2C_{CZ}$ and let Q_j^k denote, as before, the CZ cubes of f_i at height a^k, where $k \ge k_0 = \lfloor \log_a \lambda_0 + 1\rfloor$. These cubes cover only $X_{k_0} = \{x \in X\colon M^{\mathcal{D}} f_i(x) > \lambda_0\}$. If, however, we define

$$X_0 = \{M^{\mathcal{D}} f_i(x) \le \lambda_0\} = X \setminus X_{k_0}$$

then

$$X = \left(\bigcup_{k=k_0}^{\infty} X_k \setminus X_{k+1}\right) \bigcup X_0.$$

Thus the analogous argument to (6.4) proceeds as

$$\begin{aligned} &\int_X M^{\mathcal{D}} f_i(x)^{p(x)} w(x)^{p(x)}\,d\mu \\ &= \int_{X_0} M^{\mathcal{D}} f_i(x)^{p(x)} w(x)^{p(x)}\,d\mu + \sum_{k=k_0}^{\infty} \int_{X_k\setminus X_{k+1}} M^{\mathcal{D}} f_i(x)^{p(x)} w(x)^{p(x)}\,d\mu \\ &\le \lambda_0 W(X) + C \sum_{k\ge k_0, j} \int_{E_j^k} \left(\int_{Q_j^k} f_i\sigma^{-1}\sigma\,d\mu\right)^{p(x)} \mu(Q_j^k)^{-p(x)} w(x)^{p(x)}\,d\mu. \end{aligned}$$

Since λ_0 is bounded by a constant, the first term depends only on X, $\mathcal{D}$, and $p(\cdot)$. For f_1, the second term may be controlled by an argument identical to that of the infinite case.

The f_2 case, on the other hand, simplifies greatly: essentially, we just choose $Q_0 = X$, and so $I_2 = I_3 = 0$. More explicitly, since $f_2\sigma^{-1} \le 1$, if $\sigma(X) \ge 1$, then by

(6.6) and the fact that $\sigma(Q_j^k) \le C\sigma(E_j^k)$, the second term in the above expression is bounded by

$$\begin{aligned}
&\sum_{k\ge k_0,j} \int_{E_j^k} \sigma(Q_j^k)^{p(x)} \mu(Q_j^k)^{-p(x)} w(x)^{p(x)}\, d\mu \\
&\le \sum_{k\ge k_0,j} \int_{E_j^k} \sigma(X)^{p(x)} \left(\frac{\sigma(Q_j^k)}{\sigma(X)}\right)^{p(x)} \mu(Q_j^k)^{-p(x)} w(x)^{p(x)}\, d\mu \\
&\le \sigma(X)^{p_+} \sum_{k\ge k_0,j} \int_{E_j^k} \left(\frac{\sigma(Q_j^k)}{\sigma(X)}\right)^{p_-(Q_j^k)} \mu(Q_j^k)^{-p(x)} w(x)^{p(x)}\, d\mu \\
&\le \sigma(X)^{p_+-p_-} \sum_{k\ge k_0,j} C\sigma(E_j^k) \le C\sigma(X)^{p_+-p_-+1}.
\end{aligned}$$

If $\sigma(X) < 1$, simply exchange p_+ with p_-. This proves sufficiency for $\mu(X) < \infty$.

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