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On big pieces approximations of parabolic hypersurfaces

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Abstract. Let Σ be a closed subset of $\mathbb{R}^{n+1}$ which is parabolic Ahlfors–David regular and assume that Σ satisfies a 2-sided corkscrew condition. Assume, in addition, that Σ is either time-forwards Ahlfors–David regular, time-backwards Ahlfors–David regular, or parabolic uniform rectifiable. We then first prove that Σ satisfies a *weak synchronized two cube condition*. Based on this we are able to revisit the argument of Nyström and Strömqvist (2009) and prove that Σ contains *uniform big pieces of $Lip(1,1/2)$ graphs*. When Σ is parabolic uniformly rectifiable the construction can be refined and in this case we prove that Σ contains *uniform big pieces of regular parabolic $Lip(1,1/2)$ graphs*. Similar results hold if $\Omega\subset\mathbb{R}^{n+1}$ is a connected component of $\mathbb{R}^{n+1}\setminus\Sigma$ and in this context we also give a parabolic counterpart of the main result of Azzam et al. (2017) by proving that if Ω is a one-sided parabolic chord arc domain, and if Σ is parabolic uniformly rectifiable, then Ω is in fact a parabolic chord arc domain. Our results give a flexible parabolic version of the classical (elliptic) result of David and Jerison (1990) concerning the existence of uniform big pieces of Lipschitz graphs for sets satisfying a two disc condition.

Parabolisten hyperpintojen likiarvioiminen suurten osien mielessä

Tiivistelmä. Olkoon Σ avaruuden $\mathbb{R}^{n+1}$ suljettu osajoukko, joka toteuttaa sekä parabolisen Ahlforsin–Davidin säännöllisyysehdon että kaksipuolisen korkkiruuviehdon. Oletetaan lisäksi, että Σ on joko ennakoivasti tai takautuvasti Ahlforsin–Davidin säännöllinen taikka parabolisesti tasaisesti suoristuva. Todistamme, että tällöin Σ toteuttaa *heikon tahdistetun kahden kuution ehdon*. Tämän avulla voimme palata Nyströmin ja Strömqvistin (2009) päättelyyn ja osoittaa, että Σ sisältää *$Lip(1,1/2)$-kuvaajien tasaisen suuria osia*. Tapauksessa, jossa Σ on parabolisesti tasaisesti suoristuva, tarkastelua voidaan hienontaa, ja tällöin osoitamme, että Σ sisältää jopa *säännöllisten parabolisten $Lip(1,1/2)$-kuvaajien tasaisen suuria osia*. Vastaavat tulokset ovat voimassa, jos $\Omega\subset\mathbb{R}^{n+1}$ on joukon $\mathbb{R}^{n+1}\setminus\Sigma$ yhtenäinen komponentti, ja tässä tilanteessa saamme myös parabolisen vastineen Azzamin ym. (2017) päätulokselle osoittamalla, että jos Ω on yksipuolinen parabolinen jännekaarialue ja Σ on parabolisesti tasaisesti suoristuva, niin Ω on itse asiassa parabolinen jännekaarialue. Tuloksemme tarjoavat joustavan parabolisen vastineen Davidin ja Jerisonin (1990) klassisille (elliptisille) tuloksille, jotka koskevat Lipschitzin kuvaajien tasaisen suurten osien olemassaoloa kahden kiekon ehdon toteuttavissa joukoissa.

1. Introduction

An important result due to David and Jerison [DJ] states that if $\Sigma\subset\mathbb{R}^{n+1}$ is a closed set which is Ahlfors–David regular with respect to the surface measure

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$\sigma = \mathcal{H}^n\lfloor\Sigma$, (i.e., the restriction of n-dimensional Hausdorff measure to Σ), and if Σ satisfies what they call a two disc condition, then Σ contains uniform big pieces of Lipschitz graphs, see [DJ]. This result and its ramifications have had deep impact on the theory of elliptic boundary value problems and on the analysis of and on uniformly rectifiable sets. Indeed, if Ω is one component of $\mathbb{R}^n \setminus \Sigma$, and if, in addition, Ω is an NTA-domain in the sense of [JK], then the result of David and Jerison implies that the harmonic measure on $\partial\Omega$ belongs to the Muckenhoupt class A_∞ defined with respect to σ; equivalently, that the Dirichlet problem for Laplace's equation is solvable in such domains, with L^p boundary data. Furthermore, the results of [DJ], combined with the monumental works of David and Semmes [DS, DS1], have led to additional characterizations of uniform rectifiability: see, e.g. [HMM, GMT].

In this paper we are interested in parabolic counterparts of the result of David and Jerison. In general the theory of parabolic boundary value problems, and the analysis of and on parabolic uniformly rectifiable sets, is less developed compared to the elliptic counterparts and there are essentially only two strains of main results in the field: the results due to Hofmann, Lewis, Murray, Silver, see [H1, HL, HL1, LM, LS] and the results due to Hofmann, Lewis, Nyström, see [HLN1, HLN2].

To indicate the scope of the present paper, we give rough statements of three theorems to be proved; more precise statements, as well as definitions of our terminology, will be given in the sequel.

Theorem 1. *If Σ is parabolic ADR and satisfies a "weak time synchronized two cube condition", then Σ contains big pieces of $Lip(1, 1/2)$ graphs.*

This "weak time-synchronized two cube condition" is automatically satisfied in the presence of two sided corkscrews and parabolic uniform rectifiability, see Theorem 3.2. In fact, when Σ is parabolic uniformly rectifiable, we can transfer regularity from the set Σ to the approximating graph, which gives the additional subtle t-regularity required for boundedness of parabolic singular integrals and for parabolic potential theory.

Theorem 2. *If Σ is parabolic uniformly rectifiable and satisfies the two-sided corkscrew condition, then Σ contains big pieces of regular $Lip(1, 1/2)$ graphs. If in addition, Σ is time-symmetric ADR, and $\Sigma = \partial\Omega$ is the boundary of an open set $\Omega \subset \mathbb{R}^{n+1}$ satisfying an interior corkscrew condition, then Σ satisfies a uniform interior big pieces of regular $Lip(1, 1/2)$ graphs condition.*

Corollary 1. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set satisfying an interior corkscrew condition. If $\Sigma = \partial\Omega$ is parabolic uniformly rectifiable, time-symmetric ADR, and satisfies the two-sided corkscrew condition, then caloric measure ω is absolutely continuous with respect to "surface measure" σ on Σ, the parabolic "Poisson kernel" $d\omega/d\sigma$ verifies a uniform scale invariant weak reverse Hölder estimate, and the L^p (initial)-Dirichlet problem for the heat equation is solvable in Ω, for some $p < \infty$.*

Theorem 3. *If Ω is a one-sided parabolic chord arc domain, whose boundary is parabolic uniformly rectifiable and time-symmetric ADR, then Ω is a (two-sided) chord arc domain. Moreover, the caloric measure of Ω satisfies a (local) A_∞ condition.*

A few comments are in order concerning Corollary 1, and Theorem 3. By the main result of [GH] (and the maximum principle), in the setting of Theorem 3, and of the second part of Theorem 2, we immediately deduce that caloric measure satisfies a local, scale-invariant weak-A_∞ condition with respect to the natural parabolic analogue of surface measure on Σ. In the setting of Theorem 3, caloric measure is doubling (by a fairly routine extension of the results of [FGS] essentially following [HLN2]), and so in that case the weak-A_∞ condition immediately improves to (strong) A_∞. Furthermore (again see [GH]), the (weak) A_∞ condition is equivalent to L^p solvability of the Dirichlet problem, for some $p < \infty$. Prior to this result, L^p solvability for finite p had not been established even for parabolic chord arc domains with parabolic uniformly rectifiable boundaries.

In [H1, HL, LM, LS], the authors established the correct notion of (time-dependent) regular parabolic Lipschitz graphs from the point of view of parabolic singular integrals and parabolic measure. To expand a bit on this, recall that $\psi\colon \mathbb{R}^{n-1}\times\mathbb{R}\to\mathbb{R}$ is called $Lip(1,1/2)$ (or "parabolic Lipschitz", and we shall sometimes simply write $\psi \in PLip$) with constant b, if

$$|\psi(x,t)-\psi(y,s)| \le b(|x-y|+|t-s|^{1/2}) \tag{1.1}$$

whenever $(x,t)\in\mathbb{R}^n$, $(y,s)\in\mathbb{R}^n$. An open set $\Omega\subset\mathbb{R}^{n+1}$ is said to be an (unbounded) $Lip(1,1/2)$ (or $PLip$) graph domain, with constant b, if

$$\Omega=\Omega_\psi=\{(x,x_n,t)\in\mathbb{R}^{n-1}\times\mathbb{R}\times\mathbb{R}\colon x_n>\psi(x,t)\} \tag{1.2}$$

for some $Lip(1,1/2)$ function ψ having $Lip(1,1/2)$ constant bounded by b. A function $\psi=\psi(x,t)\colon\mathbb{R}^{n-1}\times\mathbb{R}\to\mathbb{R}$ is called a *regular parabolic* $Lip(1,1/2)$ function (and we shall write $\psi\in RPLip$) with parameters b_1 and b_2, if ψ satisfies

$$\begin{aligned} &\text{(i)} \quad |\psi(x,t)-\psi(y,t)|\le b_1|x-y|, \quad x,y\in\mathbb{R}^{n-1},\ t\in\mathbb{R},\\ &\text{(ii)} \quad D^t_{1/2}\psi\in BMO(\mathbb{R}^n), \quad \|D^t_{1/2}\psi\|_*\le b_2<\infty. \end{aligned} \tag{1.3}$$

It is well known, and essentially due to Strichartz [Stz] (but see also [HL, H2]), that if $\psi\in RPLip$ with parameters b_1 and b_2, then ψ is $Lip(1,1/2)$ with constant $b=b(b_1,b_2)$. Here $D^t_{1/2}\psi(x,t)$ denotes the 1/2 derivative in t of $\psi(x,\cdot), x\in\mathbb{R}^{n-1}$ fixed, and $BMO(\mathbb{R}^n)$ is the usual parabolic BMO space consisting of all $f\in L^1_{loc}(\mathbb{R}^n)$ (modulo constants) such that

$$\|f\|_* := \sup_R \fint_R |f(x,t)-f_R|\,\mathrm{d}x\,\mathrm{d}t<\infty,$$

where R denotes a parabolic cube in $\mathbb{R}^n$, having dimensions $r\times\cdots\times r\times r^2$ for some $r>0$, and $f_R:=\fint_R f$.

This half derivative in time can be defined by way of the Fourier transform (at least for compactly supported ψ), or by the formula

$$D^t_{1/2}\psi(x,t)\equiv\hat{c}\int_{\mathbb{R}}\frac{\psi(x,s)-\psi(x,t)}{|s-t|^{3/2}}\,\mathrm{d}s \tag{1.4}$$

for properly chosen $\hat{c}$.

As noted above, every $RPLip$ function is, in particular, $Lip(1,1/2)$, i.e. the $RPLip$ condition is stronger than $Lip(1,1/2)$. In fact, it is strictly stronger: there are examples of functions ψ which are $Lip(1,1/2)$ but not $RPLip$, see [LS], [KW].

We call $\Omega\subset\mathbb{R}^{n+1}$ an (unbounded) regular parabolic $Lip(1,1/2)$ graph domain (or simply an $RPLip$ graph domain), with constants (b_1,b_2), if (1.2) holds for some regular parabolic $Lip(1,1/2)$ function ψ having constants (b_1,b_2). An important insight in [KW, H1, HL, LM, LS], is that from the perspective of parabolic singular integrals and parabolic measure, the $Lip(1,1/2)$ condition alone does not suffice; instead the problems have to be framed in the context of *regular* parabolic $Lip(1,1/2)$ graph domains and this induces additional complexity in the parabolic setting compared to the elliptic situation.

In [HLN1, HLN2] the third and fifth author, together with Lewis, introduced a notion of parabolic uniformly rectifiable sets and proved the existence of big pieces of regular parabolic Lipschitz graphs under the additional assumption that Σ is Reifenberg flat in the parabolic sense. These results were the first of their kind in the context of parabolic problems and the studies [HLN1, HLN2] were motivated by the study of parabolic or caloric measures in rough domains. Still, up to very recently no systematic and correct studies of parabolic uniformly rectifiable sets have appeared in the literature. In the series including [BHHLN1, BHHLN2], and the present paper, we attempt to rectify this by conducting a thorough and detailed study of these objects. In particular, in [BHHLN1] we prove, among other things, that parabolic uniformly rectifiable sets satisfy a corona decomposition with respect to *regular* $Lip(1, 1/2)$ graphs. In [BHHLN2], we obtain a converse to this result from [BHHLN1], as we prove that corona decomposition with respect to *regular* $Lip(1, 1/2)$ graphs implies parabolic uniformly rectifiability. This converse is a consequence of more general results established in [BHHLN2]. In combination, [BHHLN1] and [BHHLN2] prove that, just as in the elliptic setting of [DS] and [DS1], we can characterize parabolic uniform rectifiability in terms of the existence of a corona decomposition with respect to an appropriate family of graphs (*regular* $Lip(1, 1/2)$ graphs). In addition we obtain that all sufficiently "nice" parabolic singular integral operators are L^2 bounded on a parabolic uniformly rectifiable set.

It is true that in [RN1, RN2, RN3], the author took on the ambitious challenge to develop the theory of parabolic uniformly rectifiable sets. Unfortunately though, in [RN1, RN2] the author either gives no proofs of his claims or supplies proofs which have gaps, a few of which, pertaining to [RN1], we pinpoint in [BHHLN1]. For now, let us point out three such errors or gaps in [RN2], as these are directly relevant to the results in the present paper. First, [RN2, Lemma 6.2] is essentially our Theorem 2 stated above, and is stated in [RN2] without proof, except for the claim that it is essentially proved in [HLN1]. In fact, had that been the case, the authors of [HLN1] would have stated their results that way. To be sure, our proof here follows that of [HLN1] to some extent, but an additional non-trivial idea, borrowed from [DS], is also used, in order to remove the extra flatness assumption (mentioned above) imposed in [HLN1]. Second, [RN2, Theorem 3.1] is essentially our Corollary 1 above, and relies on [RN2, Lemma 5.3], which is a parabolic version of a deep (elliptic) result of [BL]. However, the argument in [RN2] relies on an application of Safonov's time-backwards (i.e., non time-lagged) Harnack inequality (see [SY]) to solutions which do *not* vanish on Σ (and thus to which Safonov's result is inapplicable in any case), in a domain which need not verify the Harnack Chain condition, a setting in which Safonov's result has not been proved. Consequently, the proof of the parabolic version of the result of [BL] (which may be found in [GH]) is rather more delicate than in the elliptic case, as one is forced to account for the time lag in the parabolic Harnack inequality. Finally, in [RN2, Theorem 6.1], there is a claim (without proof) that a 2-sided corkscrew condition yields big pieces of $Lip(1, 1/2)$ graphs (and even interior big pieces), via the method of [DJ], without any mention of time-synchronization (even in a weak sense). It is not clear to the present authors how such a result might be proved, given the distinguished nature of the time direction in parabolic problems. Perhaps it is true, but a proof should be given. We have not checked in detail the validity of the argument in [RN3], as the result claimed there is proved using a method entirely different to ours in [BHHLN2].

In [HLN1, HLN2] the assumption that Σ is Reifenberg flat in the parabolic sense was motivated by the particular applications considered but this assumption may often seem too restrictive in other contexts. Therefore in [NS] the fifth author, together with Strömqvist, set out on the path to find and develop the parabolic analogue of the result of David and Jerison [DJ] mentioned above. In [NS] it is proved that if $\Sigma \subset \mathbb{R}^{n+1}$ is a closed set which is Ahlfors–David regular in the parabolic sense, see Definition 1 below, and if Σ satisfies what the authors called a synchronized two cube condition, then Σ contains uniform big pieces of $Lip(1,1/2)$ graphs by adapting the original arguments of [DJ].

To elaborate on the synchronized two cube condition, if $\Sigma \subset \mathbb{R}^{n+1}$ is parabolic Ahlfors–David regular in the sense of Definition 1, then Σ is said to satisfy a synchronized two cube condition with constant $\gamma_1 \in (0,1)$ if there exist, for all $(X,t) \in \Sigma$, $T_0 < t < T_1$ and $0 < r < \operatorname{diam} \Sigma$, two parabolic cubes $Q_\rho(X_1,t_1)$, $Q_\rho(X_2,t_2)$, both contained in $Q_r(X,t)$, such that $Q_\rho(X_1,t_1) \cap (\mathbb{R}^n \times (T_0,T_1))$ and $Q_\rho(X_2,t_2) \cap (\mathbb{R}^n \times (T_0,T_1))$ belong to different connected components of $\mathbb{R}^{n+1} \setminus \Sigma$, and

$$\gamma_1 r \leq \rho < r,\ t_1 = t = t_2. \tag{1.5}$$

Note that the condition as stated in (1.5) is quite rigid as the two cubes $Q_\rho(X_1,t_1)$, $Q_\rho(X_2,t_2)$ have to satisfy $t_1 = t = t_2$, where t is the time component of the original point (X,t) fixed on the boundary. A more flexible condition would be to relax (1.5) and to assume that Σ instead satisfies a weak synchronized two cube condition with constant γ_1, in the sense that there exist, for all $(X,t) \in \Sigma$, $T_0 < t < T_1$ and $0 < r < \operatorname{diam} \Sigma$, two parabolic cubes $Q_\rho(X_1,t_1)$, $Q_\rho(X_2,t_2)$, as above and both contained in $Q_r(X,t)$, but with (1.5) replaced by

$$\gamma_1 r \leq \rho < r,\ t_1 = t_2. \tag{1.6}$$

(1.6) is weaker compared to (1.5) as the cubes $Q_\rho(X_1,t_1)$, $Q_\rho(X_2,t_2)$ still have to have the same time coordinate but this coordinate makes no explicit reference to time coordinate of the original point (X,t) fixed on the boundary.

The discussion of the weak synchronized two cube condition leads us to the main contributions of this paper. First, assuming that $\Sigma \subset \mathbb{R}^{n+1}$ is a closed set which is parabolic Ahlfors–David regular, and satisfies the general (i.e., not necessarily synchronized) 2-cube (i.e., corkscrew) condition, we prove that certain natural additional geometrical assumptions imply a self-improvement of the corkscrew property, namely that if in addition Σ is either time-forwards Ahlfors–David regular, time-backwards Ahlfors–David regular, or parabolic uniform rectifiable, then in fact Σ satisfies the weak time-synchronized two cube condition discussed above. Second, we show that the results of [NS] continue to hold with the strong time synchronized two cube condition replaced by the weak version; more precisely, using the weak synchronized two cube condition, and revisiting the argument in [NS], we are able to establish uniform big pieces of $Lip(1,1/2)$ graphs. Third, assuming that $\Sigma \subset \mathbb{R}^{n+1}$ is parabolic uniform rectifiable and satisfies the weak synchronized two cube condition we are able to establish not only uniform big pieces of $Lip(1,1/2)$ graphs but also uniform big pieces of *regular* $Lip(1,1/2)$ graphs. This is what we need from the perspective of parabolic singular integrals and parabolic measure. Note that the latter conclusion was also established (in partial form) in [NS], where the final part of the argument was left out and the authors referred to the corresponding arguments in [HLN1]. Strictly speaking, the argument referred to in [HLN1] applies only if the norm of the Carleson measure underlying the notion of parabolic uniform rectifiability is sufficiently small, depending on the dimension n and the constant defining the Ahlfors–David regularity, and thus, the proof in [NS] applies in the presence of such a size restriction. In this paper we remove this size restriction, and spell out the details of the argument using a parabolic version of a summation approach introduced in [DS]. Note that if, as in [HLN1], Σ has the separation property and is δ-Reifenberg flat, then Σ satisfies a synchronized two cube condition. This implication can not be reversed. Hence, in particular and as already noted in [NS], our result generalizes Theorem 1 in [HLN1] beyond the hypothesis of Σ being Reifenberg flat.

In addition, we give a parabolic counterpart of the main result in [AHMNT] by proving that if $\Omega \subset \mathbb{R}^{n+1}$ is a domain defined as a connected component of $\mathbb{R}^{n+1} \setminus \Sigma$, if Ω is a one-sided parabolic chord arc domain (see Definition 12), and if Σ is parabolic uniformly rectifiable, then Ω is in fact a parabolic chord arc domain (see Definition 13). To prove this we use [BHHLN2, Theorem 4.16] and [BHHLN2, Theorem 4.15(iii)], and hence also [BHHLN1], to first conclude that if Σ is parabolic uniformly rectifiable, then Σ satisfies the parabolic bilateral weak geometric lemma, from which we then deduce the existence of exterior corkscrew points (and hence the chord-arc condition) more or less as in the elliptic case treated in [AHMNT], using the Harnack chain condition.

The rest of the paper is organized as follows. In Section 2 we introduce the geometric notions and terminology used in the paper. In Section 3 we state the results proved in the paper: Theorems 3.1–3.3, and Theorems 3.4–3.5 with their respective corollaries. In particular, Theorems 3.1 and 3.2 give geometric criteria for the existence of weak time-synchronized corkscrew points, and Theorem 3.3 provides the geometric foundation for Theorem 3.7. Theorem 3.4 (a precise version of "Theorem 1" stated above), and Theorem 3.5 and Corollary 3.2 (together a precise version of "Theorem 2" stated above) are the main results of the present work. In Section 3, we also briefly discuss, for the record, applications of our geometric results to the study of parabolic/caloric measure along the lines of [NS] and [GH]. In particular, we give Theorem 3.6, which is the precise version of "Corollary 1" stated above, and we present Theorem 3.7, a precise version of "Theorem 3" stated above. Section 4 is devoted to the proofs of Theorems 3.1–3.3 and Theorem 3.4 is proved in Section 5. The proof of Theorem 3.5 is given in Section 6. In Section 7, we present two counter-examples to show that our weak time-synchronization hypotheses are strict improvements over those in [NS].

2. Preliminaries and geometrical notions

Points in Euclidean space-time $\mathbb{R}^{n+1}$ are denoted by $\mathbf{X} := (X, t) = (x_1, \dots, x_n, t)$, where $X = (x_1, \dots, x_n) \in \mathbb{R}^n$ and t represents the time-coordinate. We will always assume that $n \geq 1$. We let $\bar{E}, \partial E$, be the closure and boundary of the set $E \subset \mathbb{R}^{n+1}$. $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$ and we let $|X| = \langle X, X \rangle^{1/2}$ be the Euclidean norm of X. We let $\|(X,t)\| := |X| + |t|^{1/2}$ denote the parabolic length of a space-time vector $\mathbf{X} = (X, t)$. Given $(X, t), (Y, s) \in \mathbb{R}^{n+1}$, we set

$$d_p(X, t, Y, s) := \|(X, t) - (Y, s)\| = |X - Y| + |t - s|^{1/2},$$

and we define $d_p(X, t, E)$ to equal the parabolic distance, defined with respect to $d_p(\cdot, \cdot)$, from $(X, t) \in \mathbb{R}^{n+1}$ to E. We let

$$Q_r(X, t) := \{(Y, s) \in \mathbb{R}^{n+1} \colon |y_i - x_i| < r, |t - s| < r^2\},$$

whenever $(X,t) \in \mathbb{R}^{n+1}$, $r > 0$, and we call $Q_r(X,t)$ a parabolic cube of "length" r. We may sometimes leave the center implicit, and write simply Q_r to denote such a cube. We also introduce the time-forward and time-backwards halves of $Q_r(X,t)$ as follows:

$$Q_r^+(X,t) := Q_r(X,t) \cap \{(Y,s) \in \mathbb{R}^{n+1} \colon s \geq t\},$$
$$Q_r^-(X,t) := Q_r(X,t) \cap \{(Y,s) \in \mathbb{R}^{n+1} \colon s \leq t\}.$$

We let $\mathrm{d}x$ denote Lebesgue n-measure on $\mathbb{R}^n$ and given a number $\eta \geq 0$, we let $\mathcal{H}^\eta$ denote standard η-dimensional Hausdorff measure. We also define *parabolic* Hausdorff measure of *homogeneous* dimension η, denoted $\mathcal{H}^\eta_{\mathrm{p}}$, in the same way that one defines standard Hausdorff measure, but using coverings by *parabolic* cubes, i.e., for $\delta > 0$, and for $A \subset \mathbb{R}^{n+1}$, we set

$$\mathcal{H}^\eta_{\mathrm{p},\delta}(A) := \inf \sum_k \mathrm{diam}_p(A_k)^\eta\,,$$

where the infimum runs over all countable coverings of A, denoted $(A_k)_k$, with $\mathrm{diam}(A_k) \leq \delta$ for all k, and then define

$$\mathcal{H}^\eta_{\mathrm{p}}(A) := \lim_{\delta \to 0^+} \mathcal{H}^\eta_{\mathrm{p},\delta}(A)\,.$$

As is the case for classical Hausdorff measure, $\mathcal{H}^\eta_{\mathrm{p}}$ is a Borel regular measure. We refer the reader to [EG2, Chapter 2] for a discussion of the basic properties of standard Hausdorff measure. The arguments in [EG2] adapt readily to treat $\mathcal{H}^\eta_{\mathrm{p}}$. In particular, one obtains a measure equivalent to $\mathcal{H}^\eta_{\mathrm{p}}$ if one defines $H^\eta_{\mathrm{p},\delta}$ in terms of coverings by arbitrary sets of parabolic diameter at most δ, rather than cubes. As in the classical setting, we define the parabolic homogeneous dimension of a set $A \subset \mathbb{R}^{n+1}$ by

$$\mathcal{H}_{\mathrm{p,dim}}(A) := \inf \{0 \leq \eta < \infty \mid \mathcal{H}^\eta(A) = 0\}\,.$$

We observe that $\mathcal{H}_{\mathrm{p,dim}}(\mathbb{R}^d) = d+1$; in particular $\mathcal{H}_{\mathrm{p,dim}}(\mathbb{R}^{n+1}) = n+2$.

Given a closed set $\Sigma \subset \mathbb{R}^{n+1}$ of homogeneous dimension $\mathcal{H}_{\mathrm{p,dim}}(\Sigma) = n+1$, we then define "surface measure" on Σ by

$$\sigma = \sigma_\Sigma := \mathcal{H}^{n+1}_{\mathrm{p}}\lfloor_\Sigma. \tag{2.1}$$

We observe that this measure is apparently different to the one typically used in previous work on parabolic equations with time-varying boundaries; see, e.g., [KW, LM, HL, HL1, HLN1, HLN2]. In those works, the following version of "surface measure" was used: given a closed set $\Sigma \subset \mathbb{R}^{n+1}$, for a Borel subset $E \subset \Sigma$, we set

$$\sigma^{\mathbf{s}}(E) := \iint_E \mathrm{d}\sigma_t\,\mathrm{d}t, \tag{2.2}$$

where $\mathrm{d}\sigma_t$ denotes the restriction of $\mathcal{H}^{n-1}$ to the time slice $E \cap (\mathbb{R}^n \times \{t\})$. It turns out that in the cases of greatest interest to us, the "slice" measure $\sigma^{\mathbf{s}}$, and the measure σ defined in (2.1), are equivalent (similar observations have been made previously in [He] and [MP]), although they need not be equivalent in general.

Remark 2.1. Some further remarks are in order.

(i) If $\sigma^{\mathbf{s}}$ (or for that matter *any* measure $\mathfrak{m}$ defined on Σ) satisfies the parabolic Ahlfors–David regularity (p-ADR) condition (see Definition 1 below), then so does σ, and in that case the two measures are of course equivalent. This follows easily from the definition of $\mathcal{H}^{n+1}_{\mathrm{p}}$ measure, and it is really just the same phenomenon that occurs in the classical (elliptic) case; see [DS].

(ii) Consequently, if Σ is a $Lip(1,1/2)$ graph, then $\sigma \approx \sigma^{\mathbf{s}}$. In particular, on a hyperplane $\mathcal{P} \subset \mathbb{R}^{n+1}$ parallel to the t-axis, which we may identify with Euclidean space $\mathbb{R}^n$, we have that $\mathcal{H}^n\lfloor_{\mathcal{P}} \approx \mathcal{H}^{n+1}_{\mathrm{p}}\lfloor_{\mathcal{P}}$, since the former is just n-dimensional Lebesgue measure on $\mathcal{P}$, which is parabolic ADR on $\Sigma = \mathcal{P}$.

(iii) If $\mathcal{P}$ is a hyperplane parallel to the t-axis, and if π is the orthogonal projection operator onto $\mathcal{P}$, then $\mathcal{H}^{n+1}_{\mathrm{p}}$ measure does not increase under the action of π. In particular, by virtue of item (ii), we have for any Borel set A that $\mathcal{H}^n(\pi(A)) = \mathcal{H}^{n+1}_{\mathrm{p}}(\pi(A)) \leq \mathcal{H}^{n+1}_{\mathrm{p}}(A)$.

(iv) If Σ is parabolic uniformly rectifiable (p-UR; see Definition 5 below), where we can initially define p-UR with respect either to σ, or to $\sigma^{\mathbf{s}}$, then the two measures are equivalent.

(v) On the other hand, the measures are not equivalent in general, even in the p-ADR setting. In fact, $\sigma^{\mathbf{s}} \lesssim \sigma$, but the other direction does not need to hold.

Item (iii) follows exactly as in the classical case (see [EG2, pp. 75–76]), as one may readily verify using that the orthogonal projection operator is Lipschitz with norm 1 with respect to the parabolic metric, i.e., $\|\pi(X,t) - \pi(Y,s)\| \leq \|(X,t) - (Y,s)\|$. Items (iv) and (v) are non-trivial. We shall provide details of the proofs of the latter two facts in our forthcoming paper [BHHLN1]. See also [He] and [MP].

As above, $\Sigma \subset \mathbb{R}^{n+1}$ will denote a closed set. For $(X,t) \in \Sigma$ and $r > 0$, we shall denote a "surface cube" on Σ by

$$\Delta(X,t,r) := \Sigma \cap Q_r(X,t),$$

and its time-forward and time-backward halves by

$$\Delta^+(X,t,r) := \Delta(X,t,r)) \cap \{(Y,s) \in \mathbb{R}^{n+1} \colon s \geq t\},$$
$$\Delta^-(X,t,r) := \Delta(X,t,r)) \cap \{(Y,s) \in \mathbb{R}^{n+1} \colon s \leq t\}.$$

The extremal time coordinates of Σ will be denoted by $T_0 = \inf\{t \colon \exists (X,t) \in \Sigma\}$ and $T_1 = \sup\{t \colon \exists (X,t) \in \Sigma\}$. When we consider an open set $\Omega \subset \mathbb{R}^{n+1}$, we shall define T_0 and T_1 relative to $\Sigma = \partial\Omega$.

Given a set $A \subset \mathbb{R}^{n+1}$, we denote its topological interior by $\mathtt{int}(A)$.

2.1. Parabolic Ahlfors–David regular sets.

Definition 1. (Parabolic Ahlfors–David regularity) Let $\Sigma \subset \mathbb{R}^{n+1}$ be a closed set. We say that a measure $\mathfrak{m}$ defined on Σ is *parabolic Ahlfors–David regular*, parabolic ADR for short (or simply p-ADR, or just ADR) with constant $M \geq 1$, if

$$M^{-1}\, r^{n+1} \leq \mathfrak{m}(\Delta(X,t,r)) \leq M\, r^{n+1}, \tag{2.3}$$

whenever $0 < r < \operatorname{diam}\Sigma$, $(X,t) \in \Sigma$, $T_0 < t < T_1$ and where $\operatorname{diam}\Sigma$ is the (parabolic) diameter of Σ (which may be infinite). As noted above (see Remark 2.1 (i)), if (2.3) holds for any measure $\mathfrak{m}$ on Σ, then it holds for σ as in (2.1), i.e. for a possibly different but still universal choice of M,

$$M^{-1}\, r^{n+1} \leq \sigma(\Delta(X,t,r)) \leq M\, r^{n+1}, \tag{2.4}$$

and in this case we simply say that Σ is parabolic ADR (p-ADR, or just ADR).

Definition 2. (Time-forward/time-backward/time-symmetric ADR) Let $\Sigma \subset \mathbb{R}^{n+1}$ be a closed set which is parabolic ADR as in Definition 1 above. We say that Σ is *parabolic time-forward ADR*, or TFADR for short, if $T_1 = \infty$ and there exists a uniform constant $M' \geq 1$, such that for each $(X,t) \in \Sigma$ with $T_0 < t$ we have

$$\sigma(\Delta^+(X,t,r)) \geq (M')^{-1} r^{n+1}.$$

Similarly, we say that Σ is *parabolic time-backward ADR*, or parabolic TBADR for short, if $T_0 = -\infty$ and there exists a uniform constant $M' \geq 1$, such that for each $(X,t) \in \Sigma$ with $t < T_1$ we have

$$\sigma(\Delta^-(X,t,r)) \geq (M')^{-1} r^{n+1}.$$

If Σ is both time-forwards ADR and time-backwards ADR, we say that Σ is *time-symmetric ADR* (TSADR for short).

Definition 3. (Dyadic cubes on an ADR set) If Σ is ADR, then $(\Sigma, d_p, d\sigma)$ is a space of homogeneous type Σ and as such admits a parabolic dyadic decomposition (see [Ch] for the construction, as well as [HK] for an alternative approach; the original construction, in the elliptic ADR setting, appears in [D1], [D2]). That is, there exists a constant $\alpha > 0$ such that for each $k \in \mathbb{Z}$ there is a collection of Borel sets, $\mathbb{D}_k$, which we will call (dyadic) cubes, such that

$$\mathbb{D}_k := \{\mathcal{Q}_j^k \subset \Sigma \colon j \in \mathfrak{I}_k\},$$

where $\mathfrak{I}_k$ denotes some index set depending on k (if Σ is unbounded, then we may simply take $\mathfrak{I}_k$ to be the set of positive integers, for each k), satisfying:

(i) $\Sigma = \bigcup_j \mathcal{Q}_j^k$ for each $k \in \mathbb{Z}$.
(ii) If $m \geq k$ then either $\mathcal{Q}_i^m \subset \mathcal{Q}_j^k$ or $\mathcal{Q}_i^m \cap \mathcal{Q}_j^k = \emptyset$.
(iii) For each (j,k) and each $m < k$, there is a unique i such that $\mathcal{Q}_j^k \subset \mathcal{Q}_i^m$.
(iv) $\operatorname{diam}\big(\mathcal{Q}_j^k\big) \lesssim 2^{-k}$.
(v) $\mathcal{Q}_j^k \supset \Sigma \cap Q_{\alpha 2^{-k}}(Z_j^k, t_j^k)$ for some $(Z_j^k, t_j^k) \in \Sigma$ (the "center" of $\mathcal{Q}_j^k$).

The dyadic cubes also enjoy a "thin boundary property", but we shall not make use of that fact in the present work.

Remark 2.2. To avoid possible confusion, let us note that we shall deal with four sorts of parabolic cubes in the sequel, each with distinct notation: the cubes $Q_r = Q_r(X,t) \subset \mathbb{R}^{n+1}$, and the surface cubes $\Delta = \Delta(X,t,r) := Q_r(X,t) \cap \Sigma$, defined above; the *dyadic* "cubes" on Σ, as in Definition 3, which we denote by the calligraphic $\mathcal{Q}$, and finally, n-dimensional parabolic cubes, defined on the hyperplane $\mathcal{P} := \mathbb{R}^{n-1} \times \{0\} \times \mathbb{R} \cong \mathbb{R}^n$, which we define analogously to Q_r in one less spatial dimension, and which we denote by $I_r = I_r(x,t)$ for $(X,t) = (x,0,t) \in \mathcal{P}$ (equivalently $I_r(x,t) := Q_r(x,0,t) \cap \mathcal{P}$).

Mildly abusing notation, we write $\ell(Q_r) := r$, $\ell(I_r) := r$, $\ell(\Delta(X,t,r)) := r$, and $\ell(\mathcal{Q}) := 2^{-k}$ when $\mathcal{Q} \in \mathbb{D}_k$.

We shall also use the letter I, and sometimes J, to denote a *dyadic* parabolic cube in $\mathcal{P} \cong \mathbb{R}^n$; in particular, such a cube has dimensions $2^m \times \cdots \times 2^m \times 2^{2m}$ for some integer m, and in this case we write $\ell(I) = 2^m$. We apologize for the fact that this notation for side length differs from that for the cubes I_r, by a factor of 2.

2.2. Parabolic uniform rectifiability.

Definition 4. Assume that $\Sigma \subset \mathbb{R}^{n+1}$ is parabolic ADR in the sense of Definition 1. Let

$$\beta(Z,\tau,r) := \inf_P \left(r^{-n-1} \iint_{\Delta(Z,\tau,r)} \left(\frac{d(Y,s,P)}{r} \right)^2 \mathrm{d}\sigma(Y,s) \right)^{1/2},$$

whenever $(Z,\tau)\in\Sigma$, $r>0$, and where the infimum is taken with respect to all n dimensional planes P containing a line parallel to the t axis. Let

$$\mathrm{d}\nu(Z,\tau,r) := \beta^2(Z,\tau,r)\ \mathrm{d}\sigma(Z,\tau)\, r^{-1}\,\mathrm{d}r. \tag{2.5}$$

We say that ν is a Carleson measure on $\Delta(Y,s,R)\times(0,R)$ if there exists $\tilde M<\infty$ such that

$$\nu(\Delta(X,t,\rho)\times(0,\rho))\le \tilde M\,\rho^{n+1}, \tag{2.6}$$

whenever $(X,t)\in\Sigma$ and $Q_\rho(X,t)\subset Q_R(Y,s)$. The least such $\tilde M$ in (2.6) is called the Carleson norm of ν on $\Delta(Y,s,R)\times(0,R)$.

Definition 5. (Parabolic uniform rectifiability) Assume that $\Sigma\subset\mathbb{R}^{n+1}$ is parabolic ADR in the sense of Definition 1 with constant M. Let ν be defined as in (2.5). Then Σ is *parabolic uniformly rectifiable*, parabolic UR (or simply p-UR) for short, with UR constants $(M,\tilde M)$ if

$$\|\nu\| := \sup_{(X,t)\in\Sigma,\ \rho>0} \rho^{-n-1}\nu(\Delta(X,t,\rho)\times(0,\rho))\le\tilde M. \tag{2.7}$$

2.3. Corkscrews and the weak time-synchronized two cube condition. In the following definitions, Definitions 6-10, we consistently assume that $\Sigma\subset\mathbb{R}^{n+1}$ is a closed set.

Definition 6. (Corkscrew, 2-cube condition) Let $\gamma_0\in(0,1)$ be given. We say that Σ satisfies a *corkscrew condition* (more precisely, *2-sided corkscrew condition*, or *2-cube condition*) with constant γ_0, if there exists, for all $(X,t)\in\Sigma$, $T_0<t<T_1$ and $0<r<\operatorname{diam}\Sigma$, two parabolic cubes $Q_\rho(X_1,t_1)$, $Q_\rho(X_2,t_2)$, both contained in $Q_r(X,t)$, such that $Q_\rho(X_1,t_1)\cap(\mathbb{R}^n\times(T_0,T_1))$ and $Q_\rho(X_2,t_2)\cap(\mathbb{R}^n\times(T_0,T_1))$ belong to different connected components of $\mathbb{R}^{n+1}\setminus\Sigma$, and with

$$\gamma_0 r\le\rho<r.$$

Definition 7. (Weak time-synchronized 2-cube condition) Let $\gamma_1\in(0,1)$ be given. We say that Σ satisfies a weak time-synchronized two cube condition with constant γ_1, if there exist, for all $(X,t)\in\Sigma$, $T_0<t<T_1$ and $0<r<\operatorname{diam}\Sigma$, two parabolic cubes $Q_\rho(X_1,t_1)$, $Q_\rho(X_2,t_2)$, both contained in $Q_r(X,t)$, such that $Q_\rho(X_1,t_1)\cap(\mathbb{R}^n\times(T_0,T_1))$ and $Q_\rho(X_2,t_2)\cap(\mathbb{R}^n\times(T_0,T_1))$ belong to different connected components of $\mathbb{R}^{n+1}\setminus\Sigma$, and with

$$\gamma_1 r\le\rho<r,\quad t_1=t_2\,.$$

Remark. The (strong) synchronized 2-cube condition considered in [NS] entailed the further requirement that the cubes $Q_\rho(X_1,t_1)$ and $Q_\rho(X_2,t_2)$ be synchronized also with $Q_r(X,t)$, i.e., $t_1=t=t_2$.

Definition 8. (Interior corkscrew condition) Let $\gamma_0\in(0,1)$ be given. Let $\Omega\subset\mathbb{R}^{n+1}$ be an open set with boundary $\partial\Omega=\Sigma$. We say that Ω satisfies an *interior corkscrew condition* with constant γ_0, if there exists, for all $(X,t)\in\Sigma$, $T_0<t<T_1$ and $0<r<\operatorname{diam}\Sigma$, a parabolic cube $Q_\rho(X_1,t_1)$, contained in $Q_r(X,t)$, such that $Q_\rho(X_1,t_1)\cap(\mathbb{R}^n\times(T_0,T_1))\subset\Omega$ and with

$$\gamma_0 r\le\rho<r.$$

Definition 9. (Corkscrew condition w.r.t. an open set Ω) Let $\gamma_0\in(0,1)$ be given. Let $\Omega\subset\mathbb{R}^{n+1}$ be an open set with boundary $\partial\Omega=\Sigma$. We say that Ω (or sometimes, in keeping with previous terminology, $\partial\Omega$) satisfies a *corkscrew condition* (more precisely *2-sided corkscrew condition*) with constant γ_0, if there exists, for all

$(X,t) \in \Sigma$, $T_0 < t < T_1$ and $0 < r < \operatorname{diam}\Sigma$, two parabolic cubes $Q_\rho(X_1,t_1)$, $Q_\rho(X_2,t_2)$, both contained in $Q_r(X,t)$, such that $Q_\rho(X_1,t_1) \cap (\mathbb{R}^n \times (T_0,T_1)) \subset \Omega$ and $Q_\rho(X_2,t_2) \cap (\mathbb{R}^n \times (T_0,T_1)) \subset \mathbb{R}^{n+1} \setminus \overline{\Omega}$, and with

$$\gamma_0 r \le \rho < r.$$

Definition 10. (Weak time-synchronized 2-cube condition w.r.t. an open set) Let $\gamma_1 \in (0,1)$ be given. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with boundary $\partial\Omega = \Sigma$. We say that Ω (or sometimes, in keeping with previous terminology, $\partial\Omega$) satisfies a *weak time-synchronized two cube condition* with constant γ_1, if there exist, for all $(X,t) \in \partial\Omega$, $T_0 < t < T_1$ and $0 < r < \operatorname{diam}\Sigma$, two parabolic cubes $Q_\rho(X_1,t_1)$, $Q_\rho(X_2,t_2)$, both contained in $Q_r(X,t)$, such that $Q_\rho(X_1,t_1) \cap (\mathbb{R}^n \times (T_0,T_1)) \subset \Omega$, $Q_\rho(X_2,t_2) \cap (\mathbb{R}^n \times (T_0,T_1)) \subset \mathbb{R}^{n+1} \setminus \overline{\Omega}$, and with

$$\gamma_1 r \le \rho < r, \quad t_1 = t_2.$$

Remark. We observe that in Definition 9 (resp. 10), we are assuming that $\Sigma = \partial\Omega$ satisfies Definition 6 (resp. 7), but with the additional requirement that one of the stipulated components of $\mathbb{R}^{n+1} \setminus \partial\Omega$ lies in Ω, at every scale and at every boundary point.

2.4. Harnack chains and parabolic chord arc domains. In the following definitions, Definitions 11–13, we consistently assume that $\Sigma \subset \mathbb{R}^{n+1}$ is a closed set and that $\Omega \subset \mathbb{R}^{n+1}$ is a connected open set (a domain) with boundary $\partial\Omega = \Sigma$. In addition we will simply assume $\operatorname{diam}\Sigma = \infty$, $T_0 = -\infty$ and $T_1 = \infty$, to avoid tedious notation. If T_0 or T_1 is finite, the interested reader can formulate the localized versions of the definitions.

Definition 11. (Harnack chain condition) We say that Ω is Harnack chain connected (or that it satisfies the *Harnack chain condition*) with constants $\kappa > 100$ and $C_* \ge 1$ if the following holds. For every $(U_1,s_1),(U_2,s_2) \in \Omega$, with

$$(s_2 - s_1)^{1/2} \ge \kappa^{-1} d_p((U_1,s_1),(U_2,s_2))$$

there exists a chain of parabolic cubes $\{Q_i\}_{i=1}^{\ell}$, $Q_i = Q_{r_i}(X_i,t_i)$, $i = 1,2,\dots,\ell$ with $Q_i \in \Omega$, such that

(i) $(U_1,s_1) \in Q_1$ and $(U_2,s_2) \in Q_\ell$,
(ii) $Q_{i+1} \cap Q_i \neq \emptyset$, $i = 1,2,\dots,\ell-1$,
(iii) $(C_*)^{-1}\operatorname{diam}(Q_i) \le d(Q_i,\partial\Omega) \le C_*\operatorname{diam}(Q_i)$,
(iv) $t_{i+1} - t_i \ge (C_*)^{-1} r_i^2$ and
(v) the length of the chain, ℓ, satisfies $\ell \le C_* \log_2\left(2 + \frac{d((U_1,s_1),(U_2,s_2))}{\min_{i=1,2} d((U_i,s_i),\partial\Omega)}\right)$.

Definition 12. (One-sided parabolic chord-arc domain) We say that Ω is a one-sided parabolic chord arc domain with constants (M,γ_0,κ,C^*) if

(a) $\partial\Omega$ is parabolic Ahlfors–David regular with constant M,
(b) Ω satisfies the interior corkscrew condition with constant γ_0,
(c) Ω satisfies the Harnack chain condition with constants (κ, C_*).

Definition 13. (Parabolic chord-arc domain) We say that Ω is a parabolic chord arc domain with constants (M,γ_0,κ,C^*) if

(a) $\partial\Omega$ is parabolic Ahlfors–David regular with constant M,
(b) $\partial\Omega$ satisfies the (two-sided) corkscrew condition with constant γ_0,
(c) Ω satisfies the Harnack chain condition with constants (κ, C_*).

Note that the only difference between Definition 12 and Definition 13 relates to the corkscrew conditions stated in (b): in Definition 12 only interior corkscrews in the sense of Definition 8 are assumed while in Definition 13 the (full) corkscrew condition in the sense of Definition 9 is assumed.

2.5. Uniform big pieces. Assume that $\Sigma \subset \mathbb{R}^{n+1}$ is parabolic ADR in the sense of Definition 1. Let in the following π denote the orthogonal projection onto the plane $\{(x, x_n, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} : x_n = 0\}$. At instances we identify $\mathbb{R}^n$ with $\mathbb{R}^{n-1} \times \{0\} \times \mathbb{R}$, and put

$$I_r(z, \tau) = \{(y, s) \in \mathbb{R}^n \colon |y_i - z_i| < r,\ i = 1, \dots, n-1,\ |s - \tau| < r^2\}$$

for $(z, \tau) \in \mathbb{R}^n$, $r > 0$.

Definition 14. (Uniform big pieces of $Lip(1, 1/2)$ graphs). We say that Σ contains uniform big pieces of $Lip(1, 1/2)$ graphs with constants (ϵ, b) if the following condition holds: Given $(X, t) \in \Sigma$, $T_0 < t < T_1$ and $0 < R < \operatorname{diam} \Sigma$, there exists, after a possible rotation in the space variable, a $Lip(1, 1/2)$ function ψ with constant b, and $\epsilon > 0$, such that

$$\mathcal{H}^n(\pi(\Sigma_\psi \cap \Delta(X, t, R))) \geq \epsilon R^{n+1}, \tag{2.8}$$

where

$$\Sigma_\psi := \{(x, x_n, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} \colon x_n = \psi(x, t)\}. \tag{2.9}$$

Remark 2.3. Note that (2.8) implies, as Hausdorff measure does not increase under projections, that

$$\sigma(\Sigma_\psi \cap \Delta(X, t, R)) \geq \epsilon R^{n+1}. \tag{2.10}$$

Definition 15. (Uniform big pieces of $RPLip$ graphs) We say that Σ contains uniform big pieces of *regular parabolic* $Lip(1, 1/2)$ ($RPLip$ for short) graphs with constants (ϵ, b_1, b_2) if (2.8) and (2.9) hold whenever $(X, t) \in \Sigma$, $T_0 < t < T_1$ and $0 < R < \operatorname{diam} \Sigma$, but with a regular parabolic $Lip(1, 1/2)$ function ψ, satisfying (1.3) with constants b_1, b_2, and for $\epsilon > 0$.

Definition 16. (Interior big pieces of $Lip(1, 1/2)$ graphs) Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with boundary $\partial\Omega = \Sigma$. We then say that $\partial\Omega$ satisfies a uniform interior big pieces of $Lip(1, 1/2)$ graphs condition with constants

$$\epsilon > 0,\ b \geq 0,\ C \geq 1,\ c > 0,\ A > 0,$$

if the following holds: given $(\hat{X}, \hat{t}) = (\hat{x}, \hat{x}_n, \hat{t}) \in \Omega$, we can find a $Lip(1, 1/2)$ function ψ with constant b, and a domain $\tilde{\Omega}$, such that with $d := d(\hat{X}, \hat{t}, \Sigma)$, we have

(i) $Q_{\epsilon d}(\hat{X}, \hat{t}) \subset \tilde{\Omega} \subset \Omega \cap Q_{Cd}(\hat{X}, \hat{t})$.

(ii) After a possible rotation in the space variables we have

$$\tilde{\Omega} = \{(y, y_n, s) \colon (y, s) \in I_{cd}(x, t),\ \psi(y, s) < y_n < \hat{x}_n + Ad\},$$

where $(X, t) = (x, x_n, t)$ is some point in $\Sigma \cap Q_{Cd}(\hat{X}, \hat{t})$ with

$$\Delta_{d/2}(X, t) \subseteq \Sigma \cap Q_{Cd}(\hat{X}, \hat{t}).$$

(iii) $\mathcal{H}^n\left(\pi(\Sigma \cap \partial\tilde{\Omega}) \cap Q_{cd}(x, t)\right) \geq \epsilon d^{n+1}$.

Definition 17. (Interior big pieces of $RPLip$ graphs) Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with boundary $\partial\Omega = \Sigma$. We then say that $\partial\Omega$ satisfies a uniform interior big pieces of regular parabolic $Lip(1, 1/2)$ graphs condition with constants $(\epsilon, b_1, b_2, C, c, A)$ if the following hold. Given $(\hat{X}, \hat{t}) \in \Omega$, we can find a domain $\tilde{\Omega}$ such that (i)–(iii) of Definition 16 hold for some regular parabolic $Lip(1, 1/2)$ function ψ with constants (b_1, b_2) and for some constant $b = b(b_1, b_2)$.

3. Statement of main results

We first prove the following two theorems concerning additional weak geometrical assumptions beyond parabolic ADR which imply that Σ satisfies the weak time-synchronized two cube condition. We consider these theorems elementary but important.

Theorem 3.1. *Let Σ be a closed subset of $\mathbb{R}^{n+1}$ which is parabolic ADR with constant M and assume that Σ satisfies a corkscrew condition in the sense of Definition 6 with constant γ_0. Assume, in addition, that Σ is either time-forwards ADR with constant M' or time-backwards ADR with constant M'. Then Σ satisfies the weak time-synchronized two cube condition in the sense of Definition 7 with $\gamma_1 = \gamma_1(n, M, \gamma_0, M')$. Furthermore, given $(X, t) \in \Sigma$, $T_0 < t < T_1$ and $0 < r < \operatorname{diam} \Sigma$, and if Σ is time-forwards ADR or time-backwards ADR, then the two synchronized cubes in the weak time-synchronized two cube condition can be constructed to be contained in $Q_r^+(X, t)$ and $Q_r^-(X, t)$, respectively.*

Theorem 3.2. *Let Σ be a closed subset of $\mathbb{R}^{n+1}$ which is parabolic ADR with constant M and assume that Σ satisfies a corkscrew condition in the sense of Definition 6 with constant γ_0. Assume, in addition, that Σ is parabolic UR in the sense of Definition 4 with UR constants $(M, \tilde{M})$. Then Σ satisfies the weak time-synchronized two cube condition in the sense of Definition 7 with $\gamma_1 = \gamma_1(n, M, \gamma_0, \tilde{M})$.*

We are able to prove the following parabolic counterpart of the result in [AHMNT].

Theorem 3.3. *Let Ω be a one-sided parabolic chord arc domain with constants $(M, \gamma_0, \kappa, C^*)$. If, in addition, Σ is parabolic uniformly rectifiable with constants $(M, \tilde{M})$, then Ω is a parabolic chord arc domain with constants $(M, \hat{\gamma}_0, \kappa, C^*)$, where $\hat{\gamma}_0 = \hat{\gamma}_0(n, M, \tilde{M}, \gamma_0, \kappa, C^*)$.*

Concerning uniform big pieces of $Lip(1, 1/2)$ graphs we prove the following.

Theorem 3.4. *Let Σ be a closed subset of $\mathbb{R}^{n+1}$ which is parabolic ADR with constant M. Assume that Σ satisfies the weak time-synchronized two cube condition in the sense of Definition 7 with $\gamma_1 \in (0, 1)$. Then Σ contains uniform big pieces of $Lip(1, 1/2)$ graphs with constants (ϵ, b) depending only n, M and γ_1.*

Corollary 3.1. *Let Σ be a closed subset of $\mathbb{R}^{n+1}$ which is parabolic ADR with constant M, and let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with boundary $\partial\Omega = \Sigma$. Assume that $\partial\Omega$ satisfies a corkscrew condition in the sense of Definition 9 with constant γ_0 and that $\partial\Omega$ is time-symmetric ADR in the sense of Definition 2 with constant M'. Then $\partial\Omega$ satisfies a uniform interior big pieces of $Lip(1, 1/2)$ graphs condition with constants (ϵ, b, C, c, A) depending only on n, M, γ_0 and M'.*

Concerning uniform big pieces of regular parabolic $Lip(1, 1/2)$ graphs we prove the following.

Theorem 3.5. *Let Σ be a closed subset of $\mathbb{R}^{n+1}$ which is parabolic UR with constants $(M, \tilde{M})$, and which satisfies the corkscrew condition in the sense of Definition 6 with constant γ_0. Then Σ contains uniform big pieces of $RPLip$ graphs with constants (ϵ, b_1, b_2) depending only on $n, M, \tilde{M}$ and γ_0.*

Corollary 3.2. *Let Σ be a closed subset of $\mathbb{R}^{n+1}$ which is parabolic UR with constants $(M, \tilde{M})$. let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with boundary $\partial\Omega = \Sigma$. Assume that Ω satisfies a corkscrew condition in the sense of Definition 9 with constant γ_0 and that $\partial\Omega$ is time-symmetric ADR in the sense of Definition 2 with constant M'. Then $\partial\Omega$ satisfies a uniform interior big pieces of RPLip graphs condition with constants $(\epsilon, b_1, b_2, C, c, A)$ depending only on for some $n, M, \tilde{M}, \gamma_1$ and M'.*

Naturally Theorems 3.1–3.3 and Theorems 3.4–3.5, along with their corollaries, have applications to the study of parabolic/caloric measure. Given an open set $\Omega \subset \mathbb{R}^{n+1}$, and a point $(X, t) \in \Omega$, we let $\omega(X, t, \cdot)$ denote caloric measure for Ω with pole at (X, t). Then in particular, combining Corollary 3.2 with the results of [GH], we have the following. For simplicity, we state the result in the case that $\operatorname{diam}\Sigma = \infty$, $T_0 = -\infty$ and $T_1 = \infty$. In the case that T_0 or T_1 is finite, one may modify the formulation appropriately; see [GH].

Theorem 3.6. *Let $\Omega \subset \mathbb{R}^{n+1}$ and $\Sigma = \partial\Omega$ be as in Corollary 3.2. Then caloric measure is absolutely continuous with respect to σ, and satisfies a local weak reverse Hölder condition. More precisely, there are constants $C \geq 1$, $\lambda > 0$ depending on the constants in Corollary 3.2, such that, given $(X_0, t_0) \in \Sigma$ and $r > 0$, we have for every $(X, t) \in \Omega \setminus Q_{4r}(X_0, t_0)$ that $\omega(X, t, \cdot) \ll \sigma$ on $\Delta_r(X_0, t_0) = \Sigma \cap Q_r(X_0, t_0)$, with $d\omega(X, t, \cdot)/d\sigma =: h$ satisfying*

$$\begin{aligned} \left(\rho^{-n-1} \iint_{\Delta_\rho(Y,s)} h^{1+\lambda} d\sigma\right)^{1/(1+\lambda)} &\leq C\rho^{-n-1} \iint_{\Delta_{2\rho}(Y,s)} h \, d\sigma \\ &= C\rho^{-n-1} \omega(X, t, \cdot)\left(\Delta_{2\rho}(Y, s)\right), \end{aligned} \tag{3.1}$$

whenever $(Y, s) \in \Sigma$ and $Q_{2\rho}(Y, s) \subset Q_r(X_0, t_0)$, where $\Delta_\rho(Y, s) = Q_\rho(Y, s) \cap \Sigma$, and $\Delta_{2\rho}(Y, s) = Q_{2\rho}(Y, s) \cap \Sigma$. Equivalently, we obtain solvability of the Dirichlet problem[1] with L^p (lateral) boundary data, for some $p < \infty$.

We remark that the results in [GH] are stated and proved with underlying measure σ given by our measure σ^{s} defined as in (2.2), however, all the arguments in [GH] carry over with this measure replaced by our σ measure defined as in (2.1).

Next, we state another application, in the context of parabolic chord arc domains. To set the stage, let Σ be a closed subset of $\mathbb{R}^{n+1}$ which is parabolic ADR with constant M, let $\Omega \subset \mathbb{R}^{n+1}$ be a connected component of $\mathbb{R}^{n+1} \setminus \Sigma$ and assume that $\operatorname{diam}\Sigma = \infty$, $T_0 = -\infty$ and $T_1 = \infty$. Using the Wiener criterion in [EG1][2] we can conclude that any point $(X, t) \in \partial\Omega$ is regular for the bounded continuous Dirichlet problem for the heat equation, as well as the adjoint heat equation, in Ω. Using this and exhausting Ω by bounded sets, and applying Perron–Wiener–Brelot type arguments, one can conclude that the bounded continuous Dirichlet problems for the heat equation, as well as the adjoint heat equation, in Ω always have unique solutions.

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[1] See [GH] for the precise formulation of the L^p Dirichlet problem (and initial-Dirichlet problem in the case that T_0 is finite).

[2] In the initial discussion of the Dirichlet problem in Section 4 in [NS] the correct assumption is of course that Σ should be parabolic time-backward parabolic ADR, not only parabolic ADR. Indeed, if Σ is parabolic time-backward parabolic ADR and $\Omega \subset \mathbb{R}^{n+1}$ is a connected component of $\mathbb{R}^{n+1} \setminus \Sigma$, then the uniform capacity estimate stated in [NS] can be verified and using the Wiener criterion in [EG1] one can conclude that if $\operatorname{diam}\Sigma = \infty$, $T_0 = -\infty$ and $T_1 = \infty$, then any point $(X, t) \in \partial\Omega$ is regular for the bounded continuous Dirichlet problem for the heat equation in Ω.

Recall that $\omega(\hat{X},\hat{t},\cdot)$ is the caloric measure, at $(\hat{X},\hat{t})\in\Omega$, associated to the heat equation in Ω. For (X,t), $r>0$, and $A\geq 100$, we define

$$\Gamma_A^+(X,t,r)=\{(Y,s)\colon |Y-X|^2\leq A(s-t),\ s-t\geq 5r^2\}. \tag{3.2}$$

Definition 18. Let $(X,t)\in\partial\Omega$, $r>0$, and consider $(\hat{X},\hat{t})\in\Omega\cap\Gamma_A^+(X,t,4r)$. We say that $\omega(\cdot)=\omega(\hat{X},\hat{t},\cdot)$ satisfies a *reverse Hölder* condition (equivalently the A_∞ condition) on $\partial\Omega\cap Q_r(X,t)$, with constants L and $\lambda>0$ if the following is true: ω is a doubling measure, i.e.,

$$\omega(Q_{2\rho}(\tilde{X},\tilde{t}))\lesssim\omega(Q_\rho(\tilde{X},\tilde{t})),$$

and $\mathrm{d}\omega/\mathrm{d}\sigma=h$ exists on $\Delta(X,t,r)$ with

$$\iint_{\Delta(\tilde{X},\tilde{t},\rho)} h^{1+\lambda}\,\mathrm{d}\sigma\leq L\sigma(Q_\rho(\tilde{X},\tilde{t}))^{-\lambda}(\omega(Q_\rho(\tilde{X},\tilde{t})))^{1+\lambda} \tag{3.3}$$

whenever $(\tilde{X},\tilde{t})\in\partial\Omega$, $Q_{2\rho}(\tilde{X},\tilde{t})\subset Q_r(X,t)$.

The following theorem is an immediate consequence of the combination of Theorem 3.3 (which gives the (2-sided) corkscrew condition), Theorem 3.2 (which gives the weak time-synchronized two cube condition), Corollary 3.2 (which gives the uniform interior big pieces of $RPLip$ graphs condition), the doubling property of parabolic measure (which can be proved as in [HLN2]), and a familiar argument based on the maximum principle. We note that the following is a parabolic analogue of the main result of [HM], although our approach is based on the much more efficient method of [AHMNT], using big pieces technology.

Theorem 3.7. *Suppose that Ω is a one-sided parabolic chord arc domain with constants (M,γ_0,κ,C^*), and with boundary $\partial\Omega=:\Sigma$. Assume also that* $\operatorname{diam}\Sigma=\infty$, *$T_0=-\infty$ and $T_1=\infty$,and that Σ is time-symmetric ADR and parabolic uniformly rectifiable with constants $(M,\tilde{M})$. Let $(X,t)\in\partial\Omega$, $r>0$, $A\geq 100$, and consider $(\hat{X},\hat{t})\in\Omega\cap\Gamma_A^+(X,t,4r)$. Then $\omega(\hat{X},\hat{t},\cdot)$ is a doubling measure in the sense that there exists a constant $c=c(n,M,\tilde{M},\gamma_0,\kappa,C^*,A)$ such that*

$$\omega(\hat{X},\hat{t},\Delta(\tilde{X},\tilde{t},2\rho))\leq c\omega(\hat{X},\hat{t},\Delta(\tilde{X},\tilde{t},\rho)), \tag{3.4}$$

for all $(\tilde{X},\tilde{t})\in\partial\Omega$, $Q_\rho(\tilde{X},\tilde{t})\subset Q_{2r}(X,t)$. Furthermore, $\omega(\hat{X},\hat{t},\cdot)$ satisfies the reverse Hölder condition (i.e., the A_∞ condition) on $\Delta(X,t,r)$ in the sense of Definition 18 with constants L and $\lambda>0$ depending only on $(n,M,\tilde{M},\gamma_0,\kappa,C^*,A)$.

In the following sections we give the proofs of Theorems 3.1–3.3, and Theorems 3.4–3.5 with their corollaries, in the case that $\operatorname{diam}\Sigma=\infty$, $T_0=-\infty$ and $T_1=\infty$. If T_0 or T_1 is finite, the proofs are completely analogous and in this case the difference is that all sets occurring have to be intersected with $\mathbb{R}^n\times(T_0,T_1)$ and the notation will be more cumbersome.

We conclude this section by recalling the following elementary lemma from [GH] which we shall use in the sequel.

Lemma 3.1. [GH] *Let Σ be a closed subset of $\mathbb{R}^{n+1}$ which is parabolic ADR with constant M. Assume that Σ is time-forward ADR or time-backwards ADR with constant M'. Then there exist constants $a_1\in(0,1/2)$, $a_2\in(0,1)$, both depending only on n,M and M' such that the following is true. Let $(X,t)\in\Sigma$. If Σ is time-forward ADR, then*

$$\sigma(\Delta^+(X,t,r)\cap\{(Y,s)\colon s<t+(a_1r)^2\})\geq a_2r^{n+1}$$

and if Σ *is time-backwards ADR, then*

$$\sigma(\Delta^-(X,t,r)\cap\{(Y,s)\colon s<t-(a_1r)^2\})\geq a_2r^{n+1}.$$

Proof. See [GH]. □

4. Proof of the geometric theorems

In this section we prove Theorem 3.1, Theorem 3.2, and Theorem 3.3; when $\operatorname{diam}\Sigma=\infty$, $T_0=-\infty$ and $T_1=\infty$.

4.1. Proof of Theorem 3.1. Let Σ be a closed subset of $\mathbb{R}^{n+1}$ which is parabolic ADR with constant M and assume that Σ satisfies a corkscrew condition in the sense of Definition 6 with constant γ_0. Assume, in addition, that Σ is time-backwards ADR with constant M'. That case in turn implies the time-forward case, by the change of variable $t\to -t$. Our goal is to prove that Σ satisfies the weak time-synchronized two cube condition in the sense of Definition 7 with $\gamma_1=\gamma_1(n,M,\gamma_0,M')$.

Let $(X,t)\in\Sigma$, $r>0$. We first apply the corkscrew condition at (X,t) and on the scale r/C_1 where C_1 a large constant to be chosen, to produce two cubes

$$Q_1:=Q_{\gamma_0r/C_1}(Y_1,s_1),\quad Q_2:=Q_{\gamma_0r/C_1}(Y_2,s_2),$$

both contained in $Q_{r/C_1}(X,t)$, but belonging to different connected components of $\mathbb{R}^{n+1}\setminus\Sigma$. If $s_1=s_2$, then we are done and hence we can without loss of generality assume that $s_1<s_2$. Let $\delta:=s_2-s_1$.

Assume that $\delta\leq(\gamma_0r/2C_1)^2$. In this case it follows readily that we can find two cubes Q_1' and Q_2', both of size $\gamma_0r/(2C_1)$, $Q_1'\subset Q_1$, $Q_2'\subset Q_2$, such that the centers of Q_1' and Q_2' have the same time coordinate and we are done.

Assume that $\delta>(\gamma_0r/2C_1)^2$. Using that Q_1 and Q_2 are contained in different connected components of $\mathbb{R}^{n+1}\setminus\Sigma$ we see that the line connecting (Y_1,s_1) and (Y_2,s_2) intersects Σ at some point $(Z_1,\tau_1)\in\Sigma$. Set $\delta'=\tau_1-s_1$. Our strategy is now to use Lemma 3.1 to produce a chain of cubes, starting with a cube centred at (Z_1,τ_1), such that the terminal cube in the chain has time coordinate very close to s_1. To start the construction of the chain we let

$$\Delta_1:=\Delta(Z_1,\tau_1,\gamma_0r/C_2)$$

where $C_2>C_1$ is yet an other large constant to be chosen. Applying Lemma 3.1 to Δ_1 we can pick

$$(Z_2,\tau_2)\in\Delta_1\cap\{(X,t)\in\Sigma\colon t<\tau_1-a^2\gamma_0^2r^2/(C_2^2)\},$$

where a is the constant denoted by a_1 in the statement of Lemma 3.1. Also, let

$$\Delta_2:=\Delta(Z_2,\tau_2,\gamma_0r/C_2).$$

We can now repeat this argument with (Z_1,τ_1) replaced by (Z_2,τ_2) to iteratively produce a sequence of points $(Z_i,\tau_i)\in\Sigma$ and we let N be the first integer such that $|\tau_N-s_1|<(\gamma_0r/C_2)^2$. At (Z_N,τ_N) we apply the corkscrew condition at scale $\gamma_0r/(2C_2)$ to produce a corkscrew cube Q_0 centered at (Y_0,s_0), contained in a component of $\mathbb{R}^{n+1}$ which is different the component containing Q_1, of parabolic size $\gamma_0^2r/(2C_2)$, and such that $Q_0\subset Q_{\gamma_0r/(2C_2)}(Z_N,\tau_N)$. Then, as in the case $\delta\leq(\gamma_0r/2C_1)^2$ it follows readily that Q_1 contains a cube of size $\gamma_0^2r/(2C_2)$ with the same time coordinate as Q_0.

To complete the proof it only remains to show how to choose C_1 and C_2 appropriately to ensure that Q_0 lies inside $Q_r(X,t)$. However, it is easy to see that

$$N\frac{a^2\gamma_0^2 r^2}{C_2^2} \le s_1 - s_2 \implies N \le \frac{C_2^2(s_1-s_2)}{a^2\gamma_0^2 r^2} \le \frac{4C_2^2}{C_1^2\gamma_0^2 a^2}.$$

This implies that

$$\|Z_1 - Z_N\| \le \frac{4C_2^2}{C_1^2 a^2}\frac{\gamma_0 r}{C_2} = \frac{4C_2\gamma_0 r}{C_1^2 a^2} \quad \text{and} \quad \|Z_1 - X\| \le \frac{r}{C_1}.$$

Hence, if we choose $C_1 > 100$, and $C_2 = \max\{C_1+1, C_1^2 a^2/(40\gamma_0)\}$ (here we can make γ_0 smaller than a if necessary), then $\|Z_N - X\| \le r/50$ and consequently

$$\|(X - Y_0, t - s_0)\| \le r/25.$$

This proves that $Q_0 \subset Q_r(X,t)$.

To see that the corkscrew cube constructed can be constructed as to be contained in $Q_r^-(X,t)$ we first apply Lemma 3.1 and then repeat the same argument above, but with (X,t,r) replaced by (X',t',r') where $(X',t') \in \Delta^-(X,t,r/100)$ and where $r' = r'(a_1,r)$ is chosen so that $\Delta(X',t',r') \subseteq \Delta^-(X,t,r/100)$. This completes the proof of Theorem 3.1.

4.2. Proof of Theorem 3.2. We introduce for $(Z,\tau) \in \Sigma, r > 0$,

$$\beta_\infty(Z,\tau,r) := \inf_P \sup_{(Y,s)\in\Delta(Z,\tau,r)} \frac{d(Y,s,P)}{r}, \tag{4.1}$$

where the infimum is taken over all n-planes P containing a line parallel to the t axis. Given $(Z,\tau), r$ as above, in display (2.2) in [HLN1] it is proved that

$$\beta_\infty(Z,\tau,r)^{n+3} \le 16^{n+3}\beta^2(Z,\tau,2r). \tag{4.2}$$

We also consider the dyadic versions

$$\beta_\infty(\mathcal{Q}) := \inf_P \operatorname{diam}(\mathcal{Q})^{-1} \sup_{\{(Y,s)\in k\mathcal{Q}\}} \operatorname{dist}(Y,s,P), \tag{4.3}$$

and

$$\beta(\mathcal{Q}) = \beta_2(\mathcal{Q}) := \inf_P \left(\operatorname{diam}(\mathcal{Q})^{-d-2}\int_{2k\mathcal{Q}} \operatorname{dist}^2(Y,s,P)\,d\sigma(Y)\right)^{1/2}, \tag{4.4}$$

where $\mathcal{Q}$ is a dyadic cube as in Definition 3, k is a sufficiently large number to be chosen, and for $k \ge 1$ we define the "dilate" $k\mathcal{Q} := \{(Y,s) \in \Sigma\colon \operatorname{dist}(Y,s,\mathcal{Q}) \le k\operatorname{diam}(\mathcal{Q})\}$. We then have the dyadic version of (4.2), by the same argument:

$$\beta_\infty(\mathcal{Q})^{n+3} \le C\beta^2(\mathcal{Q}), \tag{4.5}$$

where $C = C(n, ADR)$.

By definition, since Σ is p-UR, we have that $\beta^2(X,t,r)\,d\sigma(X,t)\,dr/r$ is a Carleson measure on $\Sigma \times (0,\infty)$, which readily implies in turn (in fact is equivalent to) the fact that $\beta(\mathcal{Q})$ satisfies the dyadic Carleson measure condition

$$\sup_{\mathcal{Q}} \frac{1}{\sigma(\mathcal{Q})}\sum_{\mathcal{Q}'\subset\mathcal{Q}} \beta^2(\mathcal{Q}')\,\sigma(\mathcal{Q}') =: \|\beta\|_{\mathcal{C}} < \infty. \tag{4.6}$$

Using (4.5), one may readily verify (basically via Tchebychev's inequality) that (4.6) implies a Carleson packing condition for "non-flat" cubes, as follows: given $\varepsilon > 0$, there is a constant $C_\varepsilon < \infty$ such that

$$\sup_{\mathcal{Q}} \frac{1}{\sigma(\mathcal{Q})} \sum_{\mathcal{Q}' \subset \mathcal{Q}} \alpha_\varepsilon(\mathcal{Q}') \le C_\varepsilon, \tag{4.7}$$

where

$$\alpha_\varepsilon(\mathcal{Q}') := \begin{cases} \sigma(\mathcal{Q}'), & \text{if } \beta_\infty(\mathcal{Q}') \ge \varepsilon, \\ 0, & \text{if } \beta_\infty(\mathcal{Q}') < \varepsilon. \end{cases}$$

Consider a cube $Q_r(X,t)$ centered on Σ. By the standard properties of the dyadic system, there is a dyadic cube

$$\mathcal{Q}_0 \subset \Delta(X,t,r/10) = Q_{r/10}(X,t) \cap \Sigma,$$

with $\ell(\mathcal{Q}_0) \approx r$. Fix ε suitably small to be chosen, and note that as a consequence of the packing condition (4.7), there is a dyadic subcube $\mathcal{Q}_1 \subset \mathcal{Q}_0$ with

$$c_\varepsilon r \le r_1 := \operatorname{diam}(\mathcal{Q}_1) < r/100,$$

such that $\beta_\infty(\mathcal{Q}_1) < \varepsilon$. Fixing $\mathbf{X}^1 = (X^1, t^1) = (x^1, x^1_n, t^1) \in \mathcal{Q}_1$, we see that by the definition of dyadic β_∞, see (4.3), there is a hyperplane P_1 parallel to the t-axis such that

$$\operatorname{dist}(Y,s,P_1) < \epsilon r_1, \quad \forall\,(Y,s) \in \Delta_1 := Q_{10r_1}(\mathbf{X}^1) \cap \Sigma. \tag{4.8}$$

provided that k is chosen large enough, depending on the constants in the construction of the dyadic system in Definition 3. By translation we may suppose that $\mathbf{X}^1 = (0,0)$, and by a spatial rotation we may suppose that $P_1 = \mathbb{R}^{n-1} \times \{0\} \times \mathbb{R}$. Set $Q_1 := Q_{10r_1}(\mathbf{X}^1) = Q_{10r_1}(0,0)$, define

$$Q_1^{up} := Q_1 \cap \{y_n \ge \varepsilon r_1\}, \quad Q_1^{down} := Q_1 \cap \{y_n \le -\varepsilon r_1\}$$

and observe that $Q_1^{up} \cap \Sigma = \emptyset = Q_1^{down} \cap \Sigma$, by (4.8). By the (2-sided) corkscrew condition (Definition 6), we see that Q_1^{up} and Q_1^{down} lie in distinct connected components of $\mathbb{R}^{n+1} \setminus \Sigma$, call them Ω^+ and Ω^- respectively, provided that we fix ε small enough depending on the constant γ_0 in Definition 6. In particular, we choose $\varepsilon < 1$, and then define

$$Q^\pm := Q_{r_1}(0, \pm 3r_1, 0) \subset \Omega^\pm \cap Q_1.$$

Since $Q_1 \subset Q_r(X,t)$, and $r_1 \approx r$, the conclusion of Theorem 3.2 follows.

4.3. Proof of Theorem 3.3. The proof of Theorem 3.3 has similarities with the proof of Theorem 3.2. Let $k \ge 2$. We introduce the *bilateral* dyadic β_∞ numbers

$$b\beta_\infty(\mathcal{Q}) := \operatorname{diam}(\mathcal{Q})^{-1} \inf_P \left\{ \sup_{\mathbf{Y} \in k\mathcal{Q}} \operatorname{dist}(\mathbf{Y}, P) + \sup_{\mathbf{Z} \in P \cap B(\mathbf{X}_{\mathcal{Q}}, k \operatorname{diam}(\mathcal{Q}))} \operatorname{dist}(\mathbf{Z}, \Sigma) \right\},$$

where $\mathbf{X}_{\mathcal{Q}}$ is the "center" of the dyadic cube $\mathcal{Q} \subset \Sigma$, as in Definition 3 (v). We say that Σ satisfies the bilateral weak geometric lemma with parameter ϵ, if there exists $M_\epsilon > 0$ such that for every dyadic cube $\mathcal{R} \in \mathbb{D}(\Sigma)$,

$$\sum_{\substack{\mathcal{Q} \subseteq \mathcal{R} \\ b\beta_\infty(\mathcal{Q}) > \epsilon}} \sigma(\mathcal{Q}) \le M_\epsilon \sigma(\mathcal{R}).$$

Since Σ is parabolic UR we can apply [BHHLN2, Theorem 4.16] and [BHHLN2, Theorem 4.15(iii)] to conclude that Σ satisfies the parabolic bilateral weak geometric lemma, for every fixed $\varepsilon > 0$, where $k \ge 2$ is at our disposal, and will eventually be chosen large enough. We now follow one of the two arguments in [AHMNT].

Let $Q_r(X,t)$ be centered on Σ, and let $\varepsilon > 0$ be a sufficiently small number to be chosen. Following the proof of Theorem 3.2 in the preceding subsection, we may again construct a dyadic cube $\mathcal{Q}_1$, of diameter $r_1 \approx r$, for which now $b\beta_\infty(\mathcal{Q}_1) < \varepsilon$, along with slightly modified versions of the cubes $Q^\pm$ as above, still disjoint from Σ, and contained in the same cube Q_1 as before, but now off-set in time, so that

$$Q^\pm := Q_{r_1}(0, \pm 3r_1, \pm r_1^2).$$

In addition, by the interior corkscrew condition, choosing ε small enough we may assume without loss of generality that $Q^+ \subset \Omega$. If Q^- lies in a different connected component of $\mathbb{R}^{n+1} \setminus \Sigma$ than does Q^+, we are done. Otherwise if both $Q^\pm \subset \Omega$, then by the Harnack Chain condition we may connect the points $\mathbf{Y}^\pm := (0, \pm 3r_1, \pm r_1^2)$ by a chain of cubes $\{Q^m\}_m$ of uniformly bounded cardinality, with

$$Q^m \subset \Omega \cap Q_{Cr_1}(0,0), \quad \text{and} \quad \ell(Q^m) \approx \operatorname{dist}(Q^m, \Sigma) \geq cr_1,$$

for every m, and with c, C each depending on the constants in the Harnack chain condition. For $k \gg C$, and $\varepsilon \ll c$, we contradict the fact that $b\beta_\infty(\mathcal{Q}_1) < \varepsilon$. The proof of Theorem 3.3 is complete.

5. The proof of Theorem 3.4 and Corollary 3.1

We here prove Theorem 3.4 and Corollary 3.1. We will give the proofs only in the case when $\operatorname{diam}\Sigma = \infty$, $T_0 = -\infty$ and $T_1 = \infty$. Throughout the section we assume that Σ is a closed subset of $\mathbb{R}^{n+1}$, which is parabolic ADR with constant M, and we assume that Σ satisfies the weak time-synchronized two cube condition in the sense of Definition 7 with $\gamma_1 \in (0,1)$.

It is true that the proof of Theorem 3.4 has substantial overlap with the corresponding result in [NS] and the difference is that in our proof we have to be even more careful as we only assume that Σ satisfies the weak time-synchronized two cube condition while in [NS] it is assumed that Σ satisfies the synchronized two cube condition. For the convenience of the reader we in the following give what we believe is a sufficiently detailed presentation of the proof of Theorem 3.4 and we try to highlight the key differences in the proof compared to [NS].

We have divided our presentation into three subsections, Subsections 5.1–5.3. In Subsection 5.1 we reduce the proof of Theorem 3.4 to Proposition 5.1. In Subsection 5.2 we prove Corollary 3.1 and in Subsection 5.3 we prove Proposition 5.1.

5.1. Reducing Theorem 3.4 to Proposition 5.1. The argument in this subsection follows closely its counterpart in [DJ], but of course adapted to the parabolic setting. We start by redefining

$$M \text{ to equal } \max\{M, \sqrt{n}\gamma_1, 4n\}. \tag{5.1}$$

Based on (5.1) we can without loss of generality assume that Σ is parabolic ADR with constant M and that there exist, for all $(X,t) \in \Sigma$ and $R > 0$, two parabolic cubes $Q_\rho(X_1,t_1)$, $Q_\rho(X_2,t_2)$, both contained in $Q_R(X,t)$ but belonging to different connected components of $\mathbb{R}^{n+1} \setminus \Sigma$, and with

$$\rho = M^{-1}R, \quad t_1 = t' = t_2. \tag{5.2}$$

Consider the points (X_1,t'), (X_2,t'), and consider the line in $\mathbb{R}^n \times \{t'\}$ connecting (X_1,t') and (X_2,t'). As (X_1,t'), (X_2,t') belong to different connected components of $\mathbb{R}^{n+1} \setminus \Sigma$, this line meets Σ at a point which we denote by (X',t'). Let $\delta_i := ||X_i - X'||$, $i = 1,2$, and note that $M^{-1}R \leq \delta_i \leq R$. We will construct the big piece of $Lip(1,1/2)$ graph to be contained in the set of points on Σ which are reached by lines emanating from points in $Q_\rho(X_1,t_1)$ and which are parallel to the line connecting X_1 and X_2. It is clear that we can translate and re-scale our setting about the point (X',t') and in particular we can in the following and without loss of generality assume that

$$R = 2M \quad \text{and} \quad (X',t') = (0,0).$$

Through this (X_1,t'), (X_2,t') are mapped to $(Y_1,0)$, $(Y_2,0)$, the corkscrew cubes $Q_\rho(X_1,t_1)$, $Q_\rho(X_2,t_2)$ are mapped to $Q_2(Y_1,0)$, $Q_2(Y_2,0)$, and Σ is mapped to a new closed set having the same quantitative properties as Σ: for simplicity we will, with an abuse of notation, also use the notation Σ for this set.

Consider the time-independent hyperplane $\mathcal{P}$ which passes through $(0,0)$ and is orthogonal to $(Y_1,0)$. Then by construction we can after a possible rotation in the spatial coordinates, represent points in $\mathbb{R}^{n+1}$ as $\mathbf{X}=(x,x_n,t)\in\mathbb{R}^{n-1}\times\mathbb{R}\times\mathbb{R}$, and in this coordinate system $\mathbf{Y}_1 := (Y_1,0)$ and $\mathbf{Y}_2 := (Y_2,0)$ are represented by

$$\mathbf{Y}_1 = (0,\overline{M}_1,0), \quad \mathbf{Y}_2 = (0,\overline{M}_2,0), \quad \text{respectively,}$$

where $2 \le \overline{M}_i \le 2M$. We may then identify the hyperplane $\mathcal{P}$ with $\mathbb{R}^{n-1}\times\{0\}\times\mathbb{R}$. Let π denote the orthogonal projection onto this plane and let $\pi^\perp$ denote the orthogonal projection onto the normal to the plane. Let $\mathcal{U}$ be the component of $\mathbb{R}^{n+1}\setminus\Sigma$ containing $\mathbf{Y}_1$.

Given $(z,\tau)\in\mathbb{R}^n$ we let

$$I_r(z,\tau) = \{(y,s)\in\mathbb{R}^n \colon |y_i - z_i| < r,\ i=1,\dots,n-1,\ |s-\tau| < r^2\}.$$

Define $I^0 := \overline{I_1(0,0)}$, and set $\overline{M} := \overline{M}_1$,

$$I^{\overline{M}} := \{\mathbf{X} \colon (x,t)\in I^0,\ x_n = \overline{M}\}. \tag{5.3}$$

By construction, $I^{\overline{M}}$ is a closed n-dimensional parabolic cube contained in the same component as $Q_2(\mathbf{Y}_1)$ (namely $\mathcal{U}$), and $d(I^{\overline{M}},\Sigma)\ge 1$. We also note that

$$D := \pi\left(Q_{1/2}(\mathbf{Y}_2)\right) = \frac{1}{2}I^0,$$

and $\sigma(D) = \mathcal{H}^n(D) = 2^{-n-1}$. In particular, choosing

$$\gamma = 2^{-n-2}, \tag{5.4}$$

we have

$$\sigma(D) \ge 2\gamma. \tag{5.5}$$

Note that any line in the x_n direction connecting $D\times\{x_n=-M\}$ with $I^{\overline{M}}$ has to intersect $Q_{1/2}(\mathbf{Y}_2)$ and $Q_2(\mathbf{Y}_1)$, thus it also has to intersect Σ.

Given $h>0$ we introduce

$$\Gamma = \Gamma_h := \{\mathbf{X}\in\mathbb{R}^{n+1} \colon x_n \ge h\|(x,t)\|\},$$

i.e. Γ is a parabolic cone with aperture h, and we let

$$S := \{\mathbf{X}\in\Sigma : -M \le x_n \le \overline{M},\ \text{and if } \mathbf{Y}\in\mathbf{X}+\Gamma,\ y_n = \overline{M}, \text{ then } \mathbf{Y}\in I^{\overline{M}}\}. \tag{5.6}$$

Note that $S\subset\Sigma$, and that $\pi(S)\subset I^0$. Also, if in this construction we choose $h\ge 6M$, it follows that if $\mathbf{X}=(x,x_n,t)\in\Sigma$, $-M\le x_n\le\overline{M}$, and if $(x,t)\in D$, then $\mathbf{Y}\in I^{\overline{M}}$ whenever $\mathbf{Y}\in\mathbf{X}+\Gamma$ is such that $y_n=\overline{M}$. Indeed, for such a point $\mathbf{Y}$, we have

$$3M \ge \overline{M}+M \ge y_n - x_n \ge h\|(y,s)-(x,t)\| \ge h(\|(y,s)\| - \|(x,t)\|).$$

Hence

$$3M \geq h(\|(y,s)\| - 1/2) \implies 1 \geq \|(y,s)\|,$$

as $h \geq 6M$ and the last conclusion in the display implies that $(y,s) \in I^0$. In particular, $D \subset \pi(S)$ and thus by (5.5),

$$\mathcal{H}^n(\pi(S) \cap I^0) = \mathcal{H}^n(\pi(S)) \geq 2\gamma. \tag{5.7}$$

To prove Theorem 3.4 it suffices to prove the following proposition.

Proposition 5.1. *Let γ be as in (5.4), (5.5). Then there exists $h > 0$, depending only on n and M, such that if we let $\Gamma = \Gamma_h$, and if we define*

$$W := \{(x,t) \in I^0 \colon \exists\ \mathbf{X} = (x, x_n, t) \in S,\ (\mathbf{X} + \Gamma) \cap S = \{\mathbf{X}\}\}$$

(so that in particular, $W \subset \pi(S)$), then $\mathcal{H}^n(\pi(S) \setminus W) \leq \gamma$.

We defer the proof of Proposition 5.1 until Subsection 5.3 below.

Remark 5.1. Let us record a remark summarizing the preceding observations. Set

$$W' := \big\{\mathbf{X} = (x, x_n, t) \in S \colon (\mathbf{X} + \Gamma) \cap S = \{\mathbf{X}\},\ \text{and}\ \pi(\mathbf{X}) = (x,t) \in I^0\big\}$$

(thus, $\pi(W') = W$), and define

$$\Omega' := \mathtt{int}\left(\bigcup_{\mathbf{X} \in W'} (\mathbf{X} + \Gamma)\right).$$

Then

$$\Omega' \cap \{y_n < \overline{M} + 2\} \subset \mathcal{U}$$

(recall that $\mathcal{U}$ is the component of $\mathbb{R}^{n+1} \setminus \Sigma$ containing $\mathbf{Y}_1$), and $\partial\Omega'$ is given by a $Lip(1, 1/2)$ graph $\{(y, \psi(y,s), s)\}$, where ψ has $Lip(1,1/2)$ norm equal to h. Note that $W' \subset \Sigma \cap \partial\Omega'$, and thus

$$\pi(\Sigma \cap \partial\Omega') \cap I^0 \supset \pi(W') = W\,.$$

Also, by Proposition 5.1, we have $\mathcal{H}^n(\pi(S) \setminus W) \leq \gamma$, and therefore by (5.7),

$$\mathcal{H}^n(\pi(\Sigma \cap \partial\Omega') \cap I^0) \geq \mathcal{H}^n(W) \geq \gamma$$

Furthermore, if for some $N \geq 2$, we have that $Q_N(\mathbf{Y}_1) \subset \mathcal{U}$, then

$$\Omega' \cap \{y_n < \overline{M} + N\} \subset \mathcal{U}\,. \tag{5.8}$$

Thus, taking Proposition 5.1 for granted, we conclude that there is a Lip (1,1/2) graph G with constant h such that $\mathcal{H}^n(\pi(\Sigma \cap G) \cap I^0) \geq \gamma$. Thus, conditioned on Proposition 5.1 the proof of Theorem 3.4 is complete.

Proposition 5.1 is essentially Lemma 2.1 in [NS], and we again emphasize that the difference now is that in the present proof of this key result we assume only that Σ satisfies the *weak* time-synchronized two cube condition, while in [NS] it is assumed that Σ satisfies the (strong) synchronized two cube condition. This weaker assumption will force us to revisit certain subtleties of the proofs in [NS] and [DJ].

5.2. Proof of Corollary 3.1. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with $\partial\Omega = \Sigma$, and assume that Ω satisfies a corkscrew condition in the sense of Definition 9 with constant γ_0 and that Σ is time-symmetric ADR in the sense of Definition 2 with constant M'. Consider $(\hat{X}, \hat{t}) \in \Omega$ and let $(X', t') \in \partial\Omega$ be a point such that

$$d_p(\hat{X}, \hat{t}, X', t') = d_p(\hat{X}, \hat{t}, \partial\Omega) =: d.$$

Our hypotheses are invariant under the change of variable $t \mapsto -t$, so without loss of generality we may assume that $t' \geq \hat{t}$.

Let N be a sufficiently large number to be chosen. If $t' - \hat{t} \leq (N^{-2}h^{-1}d)^2$, then we translate in time so that $(X', t') = (X', 0)$. Otherwise, if $t' - \hat{t} > (N^{-2}h^{-1}d)^2$, then using TBADR, and iterating Lemma 3.1, we may find $(X'', t'') \in \Sigma$ such that

$$d_p(\hat{X}, \hat{t}, X'', t'') \approx d$$

(depending implicitly on the constants in Lemma 3.1), with $|t'' - \hat{t}| \leq (N^{-2}h^{-1}d)^2$. In this case, we translate in time so that $(X'', t'') = (X'', 0)$. We set $\widetilde{X} := X'$ in the first case, and $\widetilde{X} := X''$ in the second. In either case, upon application of the corkscrew condition at $(\widetilde{X}, 0)$, we can produce a corkscrew cube

$$Q_0 = Q_0(\mathbf{X}^0), \quad \text{for some } \mathbf{X}^0 = (X^0, t^0) \in \mathbb{R}^{n+1} \setminus \Sigma,$$

of (parabolic) diameter $N^{-2}h^{-1}\gamma_0 d/100$, whose distance to $(\widetilde{X}, 0)$ is no more than $N^{-2}h^{-1}d$, and which is contained in a connected component of $\mathbb{R}^{n+1} \setminus \Sigma$ that does *not* contain $(\hat{X}, \hat{t})$. We define the point

$$\mathbf{X}^1 = (X^1, t^1) := (\hat{X}, t^0)$$

and we construct the subcube

$$Q_1(X^1, t^1) =: Q_1 \subset Q_d(\hat{X}, \hat{t}),$$

of (parabolic) diameter $N^{-2}h^{-1}\gamma_0 d/100$ (i.e., equal to that of Q^0). Note that by construction, for N large we have $(X^1, t^1) \in \Omega$, and in fact

$$d_p(X^1, t^1, \hat{X}, \hat{t}) = |t^0 - \hat{t}|^{1/2} \lesssim N^{-2}h^{-1}d \ll d.$$

Since $\mathbf{X}^1$ and $\mathbf{X}^0$ lie in different connected components of $\mathbb{R}^{n+1} \setminus \Sigma$, the line connecting them meets Σ, say at the point $\mathbf{X}^2 = (X^2, t^0)$ (here we are using that $t^1 = t^0$), and by a translation in the space variables, we may suppose that $X^2 = 0$. Let $\mathcal{P}$ denote the hyperplane through $(X^2, t^0) = (0, t^0)$ orthogonal to the line joining $\mathbf{X}^1$ to $\mathbf{X}^0$, and note that since $t^1 = t^0$, the plane $\mathcal{P}$ is parallel to the t-axis. Letting π denote projection onto $\mathcal{P}$, we have by construction that $\pi(\mathbf{X}^1) = \pi(\mathbf{X}^0)$. We perform a rotation in the spatial variables, so that $\mathcal{P} = \mathbb{R}^{n-1} \times \{0\} \times \mathbb{R}$, and so that in this new coordinate system, for N large,

$$(\hat{X}, \hat{t}\,) = (0, \hat{x}_n, \hat{t}) = (0, \kappa d, ad^2), \quad \text{with } \frac{1}{2} \leq \kappa \leq \kappa_0, \text{ and } |a| \leq (N^2 h)^{-2},$$

where κ_0 is uniformly controlled from above, and

$$\mathbf{X}^2 \cong \pi(\mathbf{X}^2) = \pi(\mathbf{X}^1) = \pi(\mathbf{X}^0) = (0, t_0) = (0, \tilde{a}d^2), \quad \text{with } |\tilde{a}| \lesssim (N^2 h)^{-2},$$

After making a possible slight adjustment in diameter, by a purely dimensional factor $c(n)$, we may assume that Q_0 and Q_1 have been rotated so that their faces are parallel to the coordinate hyperplanes in the new coordinate system.

Clearly, there is a constant $C \geq 1$ such that $Q_d(\hat{X}, \hat{t})$, Q_0 and Q_1 are all contained in $Q_{Cd}(0, t_0)$. Furthermore, we can view Q_0 and Q_1 as weak time-synchronized corkscrew cubes relative to $Q_{Cd}(0, t_0)$, so using the boundary point $(0, t_0)$ in place of the origin, we can run the argument above (as in the proof of Theorem 3.4), with corkscrew cubes Q_0 and Q_1 at point $(0, t_0)$ and scale $2d$, to obtain the interior domain (see Remark 5.1):

$$\tilde{\Omega} = \{(Y, s) \colon (y, s) = \pi(Y, s) \in I_*,\ \psi(y, s) < y_n < \kappa_1 d\} \subseteq \Omega, \tag{5.9}$$

where ψ is a $Lip(1,1/2)$ function with norm h, and where

$$I_* := \pi(Q_*), \quad Q_* := Q_{(Nh)^{-1}d}(0,0,t^0), \quad \kappa_1 := \kappa + c(n)/200 \tag{5.10}$$

(so that $\kappa_1 d = \hat{x}_n + c(n)d/200$), and

$$\mathcal{H}^n(\pi(\Sigma \cap \partial\tilde{\Omega}) \cap I_*) \geq \epsilon d^{n+1}, \tag{5.11}$$

for some $\epsilon = \epsilon(n, \gamma_0, M')$. We should note that, when running the argument as in the proof of Theorem 3.4, we perform a parabolic rescaling, and then we "undo" the parabolic rescaling to obtain the set $\tilde{\Omega}$ above. We observe that by construction,

$$Q_{N^{-2}h^{-1}d}(\hat{X}, \hat{t}) \subset \tilde{\Omega}, \tag{5.12}$$

provided that we choose N large enough.

This concludes the proof of Corollary 3.1.

Remark 5.2. For future reference, let us record some additional observations. To begin, letting G denote the graph of ψ, we may find a point $\mathbf{X}^* = (x^*, x_n^*, t^*) \in \Sigma \cap G$ such that $\pi(\mathbf{X}^*) \in I_*$: just choose $\mathbf{X}^*$ in the un-rescaled version of the set W' in Remark 5.1. Note that such an $\mathbf{X}^*$ lies below the bottom face of $Q_{c(n)d}(\hat{X}, \hat{t})$ (by construction of G, since $\mathbf{X}^* \in \Sigma$), hence we see that $x_n^* \leq (\kappa - c(n))d$. Since $\operatorname{diam}(I_*) \lesssim (Nh)^{-1}d$, and since ψ has $Lip(1,1/2)$ norm h, we find that

$$\sup_{(y,s)\in 100I_*} \psi(y,s) \leq (\kappa - c(n) + CN^{-1})d \leq (\kappa - c(n)/2)d, \tag{5.13}$$

for N large enough, and therefore with $c_1 = c(n)/2$, we have

$$\hat{x}_n - \sup_{(y,s)\in 100I_*} \psi(y,s) \geq c_1 d \approx N \operatorname{diam}(I_*). \tag{5.14}$$

We note also that Q_* is centered on Σ, at $\mathbf{X}^2 = (0,0,t_0))$.

5.3. Proof of Proposition 5.1. We roughly follow the argument in [DJ], as adapted to the parabolic setting in [NS], with some modest technical refinements to deal with the fact that the time-synchronization in our 2-cube condition holds only weakly. As above, we identify $\mathbb{R}^n$ with the hyperplane $\mathcal{P} = \{x_n = 0\}$. For any

$$\mathbf{X} \in \{\mathbf{X} = (x, x_n, t) \colon (x,t) \in I^0,\ x_n \in [-M, \bar{M}]\},$$

we let $L(\mathbf{X})$ denote the open line segment in the x_n direction which connects $\mathbf{X}$ to $(x, \bar{M}, t)$. If $\mathbf{X} \in \Sigma$, then the length of $L(\mathbf{X})$ is at least $d(I^{\bar{M}}, \Sigma) \geq 1$. Define $\tilde{\mathcal{G}}$ to be the closure of the set of all such points $\mathbf{X} \in \Sigma$ which satisfy $L(\mathbf{X}) \cap \Sigma = \emptyset$, and, recalling that the set S is defined in (5.6), we let

$$\mathcal{G} := \tilde{\mathcal{G}} \cap S \subset \Sigma.$$

Given $A \subset \mathbb{R}^n$, set

$$\nu(A) := \sigma\left(\pi^{-1}(A) \cap Q_{2M}(0,0)\right),$$

where again π denotes the orthogonal projection onto $\mathcal{P} = \mathbb{R}^{n-1} \times \{0\} \times \mathbb{R}$, and note that ν defines a Borel measure with total mass

$$\|\nu\| \leq \sigma\left(Q_{2M}(0,0)\right) \leq C$$

(since σ is ADR). For $(x,t) \in I^0$, define

$$\mathcal{M}(x,t) = \sup\left\{\frac{1}{\mathcal{H}^n(I)}\,\sigma(\pi^{-1}(I) \cap Q_{2M}(0,0)) \colon I \text{ contains } (x,t)\right\},$$

where the supremum runs over all parabolic cubes $I \subset \mathcal{P}$ with $(x,t) \in I$, so that $\mathcal{M}(x,t) = \mathcal{M}\nu(x,t)$, the parabolic Hardy–Littlewood maximal function of ν. We let

N^* be a suitably large constant to be chosen momentarily. Then, by the standard weak-type bounds, we have

$$\mathcal{B} := \{(x,t) \in \mathbb{R}^n \colon \mathcal{M}(x,t) \geq N^*\} \text{ satisfies } \mathcal{H}^n(\mathcal{B}) \leq C/N^*,$$

for some constant $C = C(n,M) \geq 1$. Then for $N^* = N^*(n,M,\gamma)$ large enough, we have

$$\mathcal{H}^n(\mathcal{B}) \leq \gamma/2. \tag{5.15}$$

We fix N^* with respect to (5.15). In particular, since γ is a purely dimensional constant previously fixed (see (5.4), (5.5)) it follows that N^* is from now on a fixed constant depending only on n, M.

Having fixed γ and N^*, there will appear, in the construction to be outlined, four important constants: Λ_0, Λ_1, Λ_2, and Λ_3, with $1 \leq \Lambda_i < \infty$ for $i \in \{0,1,2,3\}$. In general all constants appearing will depend at most on n, M, Λ_0, Λ_1, Λ_2, and Λ_3. We will choose the degrees of freedom Λ_0, Λ_1, Λ_2, and Λ_3 to depend only on n, M, γ and N^*, and hence to depend only on n, M. Furthermore, Λ_i for $i \in \{0,1,2,3\}$, will be chosen to be of the form 2^{N_i} for some integer $N_i \geq 1$.

We let, for Λ_1 fixed and as above, we choose a dyadic number $A = 2^{k_0}$ large enough that

$$A > 2M\Lambda_1. \tag{5.16}$$

With A fixed, we define, for $j \in \{0,1,\ldots\}$,

$$\begin{aligned}\Sigma_j = \{(x,t) \in I^0 \colon & \text{there exist } \mathbf{X} \in \mathcal{G} \text{ and } \mathbf{Y} \in S \text{ such that} \\ & \mathbf{X} = (x,x_n,t),\ \mathbf{Y} \in \mathbf{X} + \Gamma \text{ and } A^{-j} \leq y_n - x_n < A^{-j+1}\}.\end{aligned}$$

If $\mathbf{X} = (x,x_n,t) \in S$, there exists a maximal $\hat{x}_n$ such that $(x,\hat{x}_n,t) \in S$. This follows since $x_n \leq \bar{M}$ if $(x,x_n,t) \in S$, $I^{\bar{M}} \subset \mathcal{U}$ and S is closed. Thus $(x,\hat{x}_n,t) \in \mathcal{G}$, which shows that $\pi(\mathcal{G}) = \pi(S)$. When $(x,t) \in \pi(S) \setminus W$ we have $[(\mathbf{X}+\Gamma) \setminus \{\mathbf{X}\}] \cap S \neq \emptyset$ whenever $\mathbf{X} = (x,x_n,t) \in S$. In particular, this is true for $\hat{\mathbf{X}} = (x,\hat{x}_n,t) \in \mathcal{G}$, the maximal point constructed above, so there exists $\mathbf{Y} \in S \setminus \{\hat{\mathbf{X}}\}$ such that $\mathbf{Y} \in \hat{\mathbf{X}} + \Gamma$. By our restriction on A we have

$$\pi(S) \setminus W \subset \textstyle\bigcup_j \Sigma_j.$$

Furthermore, as by construction $\mathcal{H}^n(\mathcal{B}) \leq \gamma/2$, the proof of Proposition 5.1 is reduced to proving that

$$\mathcal{H}^n\left(\bigcup_j \Sigma_j \cap (\mathbb{R}^n \setminus \mathcal{B})\right) \leq \gamma/2. \tag{5.17}$$

To continue the proof we will need the following lemma.

Lemma 5.1. *Let $\varepsilon > 0$ be given. Let Λ_1 be as above and define A as in* (5.16)*. Then there exist Λ_2, and Λ_3 as above, and an integer $N_0 = N_0(\epsilon,\Lambda_2) \geq 1$, such that if we let $\Lambda_0 = \Lambda_2^{\tilde{N}_0}$, for some $\tilde{N}_0 \geq N_0$, and if we restrict h to satisfy $h \geq 2A\Lambda_0\Lambda_1\Lambda_3$, then the following is true. Let $j \geq 0$ and $I \subset I^0$ be a dyadic cube of length $\ell(I) = A^{-j}$. Then the number of dyadic cubes J of length $\ell(J) = \Lambda_0^{-1}A^{-j}$ that are contained in I and satisfy $J \cap (\Sigma_j \cap (\mathbb{R}^n \setminus \mathcal{B})) \neq \emptyset$, is less than $\varepsilon\Lambda_0^{n+1}$.*

Remark on the proof. This is Lemma 2.2 in [NS] and its proof does not rely directly on any two cube condition. On the other hand, [NS, Lemma 2.2] is deduced as a consequence of Lemma 2.5 in [NS], whose proof does make nominal use of the synchronized two cube condition. However, a careful examination of the argument reveals that the weak synchronized two cube condition is sufficient. We omit further details. □

Let $\epsilon > 0$ be a degree of freedom to be fixed in (5.29) below. To proceed with the proof of (5.17), given $j \geq 0$ we dyadically subdivide I^0 into (non-overlapping) dyadic cubes $\{I_{j,l}\}_l$ of length $\ell(I_{j,l}) = A^{-j}$. Note that there are $A^{(n+1)j}$ such cubes, since I^0 is a unit cube. We then subdivide each cube $I_{j,l}$ further and for Λ_0 as in Lemma 5.1, we let $\{J_{j,l,k}\}_{k=1}^{k_l}$ denote the so constructed set of dyadic cube of length $\ell(J_{j,l,k}) = \Lambda_0^{-1}A^{-j}$, satisfying $J_{j,l,k} \subseteq I_{j,l}$ and $J_{j,l,k} \cap \left(\Sigma_j \cap (\mathbb{R}^n \setminus \mathcal{B})\right) \neq \emptyset$. By Lemma 5.1 we have, for each $I_{j,l}$, that the cardinality k_l of the collection $\{J_{j,l,k}\}_k$ is at most $\varepsilon\Lambda_0^{n+1}$. We then have

$$\begin{aligned}\mathcal{H}^n\left(\textstyle\bigcup_j \Sigma_j \cap (\mathbb{R}^n \setminus \mathcal{B})\right) &\leq \sum_j \sum_l \mathcal{H}^n\left(I_{j,l} \cap \left(\Sigma_j \cap (\mathbb{R}^n \setminus \mathcal{B})\right)\right)\\ &\leq \sum_j \sum_l \sum_{k=1}^{k_l} \mathcal{H}^n\left(J_{j,l,k}\right). \qquad (5.18)\end{aligned}$$

Hence, to prove (5.17), it suffices to show that

$$\sum_j \sum_l \sum_{k=1}^{k_l} \mathcal{H}^n\left(J_{j,l,k}\right) \leq \gamma/2. \qquad (5.19)$$

To prove (5.19) we will associate, to each $J_{j,l,k}$, a surface $S(J_{j,l,k})$, and we intend to estimate the measure of $|J_{j,l,k}|$ in terms of the measures of the sets $\{S(J_{j,l,k})\}$. The surfaces will not be uniquely defined but as we will see we will make the construction so that $S(J_{j,l,k}) \cap S(J_{j',l',k'}) = \emptyset$ whenever $(j,l,k) \neq (j',l',k')$, thus enabling efficient summation.

To proceed with the construction of the surface $S(J_{j,l,k})$, consider $J := J_{j,l,k}$ and choose any $\mathbf{X} \in \mathcal{G}$ and $\mathbf{Y} \in S$ such that $\pi(\mathbf{X}) \in J$, $\mathbf{Y} \in \mathbf{X} + \Gamma$ and $A^{-j} \leq y_n - x_n < A^{-j+1}$. Applying the weak time-synchronized corkscrew condition, at $\mathbf{Y}$ and at scale $\Lambda_1^{-1}A^{-j}$, we see that there exists a cube $Q \subset \mathbb{R}^{n+1}$ of length

$$\ell(Q) = R_j := M^{-1}\Lambda_1^{-1}A^{-j},$$

with center $\mathbf{U}$ and contained in $Q_{\Lambda_1^{-1}A^{-j}}(\mathbf{Y})$, and such that Q belongs to a component of $\mathbb{R}^{n+1} \setminus \Sigma$ different from $\mathcal{U}$. We recall that $\mathcal{U}$ is the component that contains $I^{\bar{M}}$. However, in contrast to [NS] the t-coordinates of $\mathbf{U}$ and $\mathbf{Y}$ *do not necessarily coincide*. This turns out to be harmless. Given J we let

$$\hat{J} = \hat{J}(J) := I_{\Lambda_1^{-1}R_j}(\pi(\mathbf{U})). \qquad (5.20)$$

We also introduce

$$\mathcal{S} := (\Sigma \cap Q_{2M}(0,0)) \cup (I^0 \times \{x_n = -A\})$$

and we recall that $A \geq 2M$. Given $J = J_{j,l,k}$ we define $S(J)$ to be the set of all $\mathbf{V} \in \mathcal{S}$ such that $\pi(\mathbf{V}) \in \hat{J} = \hat{J}(J) = I_{\Lambda_1^{-1}R_j}(\pi(\mathbf{U}))$, with $v_n < u_n - R_j$, where $\mathbf{U} = (u, u_n, \tau)$, and such that the open line segment joining $\mathbf{V}$ to $\pi(\mathbf{V}) + (0, u_n - R_j, 0)$ does not meet Σ. By construction, since $\pi(\mathcal{S}) \supset \hat{J}$, we have

$$\pi(S(J)) = \hat{J}, \quad \text{and} \quad \hat{J} \subset 2J, \qquad (5.21)$$

where the latter holds since we have chosen h very large.

To proceed, let K_j be the number of cubes $\{I_{j,l}\}$ that contain at least one of the $J_{j,l,k}$. Then, given ϵ, $\tilde{N}_0 \geq N_0$ and h as stated in Lemma 5.2 below, and using Lemma 5.1,

$$\sum_l \sum_{k=1}^{k_l} \mathcal{H}^n\left(J_{j,l,k}\right) \leq K_j \varepsilon \Lambda_0^{n+1}(2\Lambda_0^{-1}A^{-j})^{n+1} = K_j \varepsilon 2^{n+1} A^{-j(n+1)}. \tag{5.22}$$

Fix j, l and assume that the collection $\{J_{j,l,k}\}_k$ is non-empty. Since $J_{j,l,k} \subset I_{j,l}$, it follows that for all $k \in \{1, \ldots, k_l\}$, one has $\pi(S(J_{j,l,k})) \subset 2I_{j,l}$, and therefore by (5.20) and (5.21),

$$\mathcal{H}^n\left(\pi\left(\textstyle\bigcup_{k=1}^{k_l} S(J_{j,l,k})\right) \cap 2I_{j,l}\right) \geq (2\Lambda_1^{-1}R_j)^{n+1}. \tag{5.23}$$

Hence, summing the inequality in (5.23) over l

$$\begin{aligned}\sum_l \mathcal{H}^n\left(\pi\left(\textstyle\bigcup_{k=1}^{k_l} S(J_{j,l,k})\right) \cap 2I_{j,l}\right) &\geq K_j(2\Lambda_1^{-1}R_j)^{n+1} \\ &= K_j(M^{-1}\Lambda_1^{-2})^{n+1}2^{n+1}A^{-j(n+1)}.\end{aligned} \tag{5.24}$$

Combining (5.22) and (5.24), and using that for each given j, the fattened cubes $\{2I_{j,l}\}_l$ have bounded overlaps, we see that

$$\sum_l \sum_{k=1}^{k_l} \mathcal{H}^n(J_{j,l,k}) \leq C\varepsilon(M\Lambda_1^2)^{n+1}\, \mathcal{H}^n\left(\pi\left(\textstyle\bigcup_l \bigcup_{k=1}^{k_l} S(J_{j,l,k})\right)\right) \tag{5.25}$$

for all $j \geq 0$, where the constant $C = C(n)$. Hence, summing in j we have

$$\sum_j \sum_l \sum_{k=1}^{k_l} \mathcal{H}^n(J_{j,l,k}) \leq C\varepsilon(M\Lambda_1^2)^{n+1} \sum_j \mathcal{H}^n\left(\pi\left(\textstyle\bigcup_l \bigcup_{k=1}^{k_l} S(J_{j,l,k})\right)\right). \tag{5.26}$$

To complete the proof we will need the following lemma, Lemma 5.2.

Lemma 5.2. *Let $\varepsilon > 0$ be given. Let Λ_2, Λ_3, N_0, be as in the statement of Lemma 5.1. Then there exists an integer $\tilde{N}_0 \geq N_0$, depending only on n, M, Λ_2, Λ_3, such that if we let $\Lambda_0 = \Lambda_2^{\tilde{N}_0}$, $\Lambda_1 = 2^{\tilde{N}_0}$, define A as in* (5.16), *and if we restrict h to satisfy $h \geq 2A\Lambda_0\Lambda_1\Lambda_3$, then*

$$S(J_{j,l,k}) \cap S(J_{j',l',k'}) = \emptyset,\ \forall\, l, k, l', k',\ \text{whenever } j \neq j'.$$

Proof. This is Lemma 2.3 in [NS], whose proof relies in turn on Lemma 2.4 in [NS]. Neither of the proofs of these two Lemmata relies on a two cube condition, hence Lemma 5.2 generalizes immediately to our setting. □

We can now use Lemma 5.2 to complete the proof of (5.19) and hence the proof of Proposition 5.1. Recall that by definition,

$$S(J_{j,l,k}) \subset \mathcal{S} = (\Sigma \cap Q_{2M}(0,0)) \cup (I^0 \times \{x_n = -A\}).$$

Hence, using that $\mathcal{H}^n$ and $\mathcal{H}_{\mathrm{p}}^{n+1}$ are the same on a hyperplane parallel to the t-axis, and that (parabolic) Hausdorff measure does not increase under a projection (see Remark 2.1 (ii) and (iii)), and then Lemma 5.2, we deduce that

$$\begin{aligned}\sum_j \mathcal{H}^n\left(\pi\left(\textstyle\bigcup_l \bigcup_{k=1}^{k_l} S(J_{j,l,k})\right)\right) &\leq \sum_j \mathcal{H}_{\mathrm{p}}^{n+1}\left(\textstyle\bigcup_l \bigcup_{k=1}^{k_l} S(J_{j,l,k})\right) \\ &\leq \mathcal{H}_{\mathrm{p}}^{n+1}\big(\Sigma \cap Q_{2M}(0,0)\big) + \mathcal{H}^n\left(I^0 \times \{x_n = -A\}\right) \leq C,\end{aligned} \tag{5.27}$$

since Σ is ADR. Together (5.26) and (5.27) imply the bound

$$\sum_j \sum_l \sum_{k=1}^{k_l} \mathcal{H}^n\left(J_{j,l,k}\right) \leq C\varepsilon(M\Lambda_1^2)^{n+1}, \tag{5.28}$$

where $C = C(n)$, $1 \leq C < \infty$. Let now ε be defined through the relation

$$C\varepsilon(M\Lambda_1^2)^{n+1} = \gamma/2. \tag{5.29}$$

Then $\varepsilon = \varepsilon(n, M, \Lambda_1, \gamma) = \varepsilon(n, M, \gamma) = \varepsilon(n, M)$ and we see that Lemma 5.1 holds with $h = 2A\Lambda_0\Lambda_1\Lambda_3$ and, by construction, $h = h(n, M)$. In particular, the proof of Proposition 5.1 is now complete.

6. The proof of Theorem 3.5 and Corollary 3.2

In this section we give the proof of Theorem 3.5 when $\operatorname{diam}\Sigma = \infty$, $T_0 = -\infty$ and $T_1 = \infty$, followed by a sketch of the refinements to this argument needed to prove Corollary 3.2. The proof will be a combination of ideas in [HLN1] and [DS].

In the following C will denote a positive constant satisfying $1 \leq C < \infty$. We write $c_1 \lesssim c_2$ if c_1/c_2 is bounded from above by a positive constant depending at most on n, M and γ_1 if not otherwise stated. We write $c_1 \sim c_2$ if $c_1 \lesssim c_2$ and $c_2 \lesssim c_1$.

Proof of Theorem 3.5. Let Σ be a closed subset of $\mathbb{R}^{n+1}$ which is parabolic ADR with constant M. Assume that Σ is parabolic UR with constants $(M, \|\nu\|)$, and that and that Σ satisfies a 2-sided corkscrew condition as in Definition 6. Then by Theorem 3.2, Σ satisfies the weak synchronized two cube condition in the sense of Definition 7 with $\gamma_1 \in (0, 1)$. If necessary, we shrink γ_1 slightly so that any rotation $\varrho(Q)$ of a corkscrew cube Q does not intersect Σ, and in fact retains the Whitney property that $\operatorname{diam}(\varrho(Q)) \approx \operatorname{diam}(Q) \approx \operatorname{dist}(\varrho(Q), \Sigma)$, with uniform implicit constants.

To start the proof of Theorem 3.5, let $(X, t) \in \Sigma$ and $R > 0$. By Theorem 3.4 there exists, after possibly a rotation in the spatial variables, a coordinate system and $Lip(1, 1/2)$ function ψ^* with constant $b^* = b^*(n, M)$ such that if we let π denote the orthogonal projection onto the plane $\{(y, y_n, s) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} \colon y_n = 0\}$, then

$$\sigma(F) \geq \mathcal{H}^n(\pi(F)) \geq \epsilon R^{n+1} \quad \text{where } F := \Sigma_{\psi^*} \cap \Delta(X, t, R) \tag{6.1}$$

and

$$\Sigma_{\psi^*} := \{(y, y_n, s) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} \colon y_n = \psi^*(y, s)\}.$$

To prove Theorem 3.5 we need to invoke the Carleson measure condition used in the very definition of parabolic uniform rectifiability. Let

$$f(Z, \tau) = \int_0^{100R} \gamma(Z, \tau, r) r^{-1}\, \mathrm{d}r, \quad (Z, \tau) \in \Sigma.$$

Then, using (2.6) we see that

$$\iint_{\Delta(X,t,100R)} f(Z, \tau)\, \mathrm{d}\sigma(Z, \tau) \leq \|\nu\|(100R)^{n+1}.$$

Using this and weak estimates we see that if $A = 1000\epsilon^{-1}$, then

$$\begin{aligned} \sigma\left(\{(Z, \tau) \in \Delta(X, t, 100R) \colon f(Z, \tau) \geq A^{n+1}\|\nu\|\}\right) &\leq (100R/A)^{n+1} \\ &\leq (\epsilon R/10)^{n+1}. \end{aligned} \tag{6.2}$$

Using this inequality, (6.1) and the fact that Hausdorff measure does not increase under a projection, we deduce the existence of a closed set $F_1 = F_1(A)$ with $F_1 \subset F$, such that

$$f(Z,\tau) \leq A^{n+1}\|\nu\|, \quad (Z,\tau) \in F_1, \tag{6.3}$$

and

$$\mathcal{H}^n(\pi(F_1)) \geq \frac{\epsilon}{2}R^{n+1}. \tag{6.4}$$

We construct the approximating graph by extending ψ^* off $\pi(F_1)$. To do this we again identify $\mathbb{R}^{n-1} \times \{0\} \times \mathbb{R}$ with $\mathbb{R}^n$, and put

$$I_r(z,\tau) = \{(y,s) \in \mathbb{R}^n\colon |y_i - z_i| < r,\ i = 1,\dots,n-1,\ |s-\tau| < r^2\},$$

whenever $(z,\tau) \in \mathbb{R}^n$, $r > 0$. Let $\{\bar{I}_i = \overline{I_{r_i}(\hat{x}_i,\hat{t}_i)}\}$ be a Whitney decomposition of $\mathbb{R}^n \setminus \pi(F_1)$ into (n-dimensional parabolic) cubes, such that $I_i \cap I_j = \emptyset, i \neq j$, and

$$10^{-10n} d(I_i, \pi(F_1)) \leq r_i \leq 10^{-8n} d(I_i, \pi(F_1)). \tag{6.5}$$

Let $\{v_i\}$ be a partition of unity adapted to $\{I_i\}$, i.e.,

(a) $\sum v_i \equiv 1$ on $\mathbb{R}^n \setminus \pi(F_1)$,
(b) $v_i \equiv 1$ on I_i and $v_i \equiv 0$ in $\mathbb{R}^n \setminus \overline{I_{2r_i}(\hat{x}_i,\hat{t}_i)}$ for all i,
(c) v_i is infinitely differentiable on $\mathbb{R}^n$ with

$$r_i^{-l}\left|\frac{\partial^l}{\partial x^l}v_i\right| + r_i^{-2l}\left|\frac{\partial^l}{\partial t^l}v_i\right| \leq c(l,n) \ \text{ for } \ l = 1,2,\dots. \tag{6.6}$$

In (c), $\frac{\partial^l}{\partial x^l}$ denotes an arbitrary partial derivative with respect to the space variable x and of order l. Next, for each i we fix $(x_i', t_i') \in \pi(F_1)$ with

$$\rho_i := d((x_i',t_i'), I_i) = d(\pi(F_1), I_i) \approx r_i \approx \operatorname{diam}(I_i), \tag{6.7}$$

where the last two equivalences are standard properties of Whitney cubes. We set $\Lambda = \{i\colon \bar{I}_i \cap \overline{I_{2R}(x,t)} \neq \emptyset\}$, where $(y,s) \cong (y,0,s)$ is the projection of (Y,s) onto $\mathbb{R}^n \cong \mathbb{R}^{n-1} \times \{0\} \times \mathbb{R}$. We now let

$$\psi(y,s) = \begin{cases} \psi^*(y,s), (y,s) \in \pi(F_1), \\ \sum_{i\in\Lambda} \big(\psi^*(x_i',t_i') + \mu b^* \rho_i\big) v_i(y,s), \quad (y,s) \in \mathbb{R}^n \setminus \pi(F_1), \end{cases} \tag{6.8}$$

where μ is a non-negative constant which may be taken equal to 0, in the case of Theorem 3.5, and which will be chosen sufficiently large in the case of Corollary 3.2. Then, $\psi \equiv 0$ on $\mathbb{R}^n \setminus Q_{4R}(X,t)$, and

$$\mathcal{H}^n(\pi(F_1)) \geq \frac{\epsilon}{2}R^{n+1}, \quad F_1 \subset \Sigma_\psi \cap \Delta(X,t,R), \tag{6.9}$$

where

$$\Sigma_\psi := \{(y,y_n,s) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}\colon y_n = \psi(y,s)\}.$$

We intend to prove that the function ψ is a regular parabolic $Lip(1,1/2)$ function with constants $b_1 = b_1(n,M,\tilde{M})$, $b_2 = b_2(n,M,\tilde{M})$.

Since ψ^* is a $Lip(1,1/2)$ function with constant $b^* = b^*(n,M)$, one can use (6.5)–(6.8) and a standard Whitney extension argument (see [St, Ch. VI]) to conclude that (1.1) holds with b_1 replaced by Cb^*. To verify this, the more delicate case occurs when (y,s) is in the closure of two cubes say I_i, I_j with $i \in \Lambda, j \notin \Lambda$. However this case follows easily from the fact that $|\psi^*| \le cb^* R$ and $|\partial v_k/\partial y_l|(y,s) \le c/R$ for $1 \le l \le n-1$, $k \in \{i,j\}$. Hence it remains only to prove that

$$\|D^t_{1/2}\psi\|_* \le b_2 \ \text{ for some } b_2 = b_2(n, M, \tilde{M}). \tag{6.10}$$

Let β_ψ, ν_ψ, be as in the statement Definition 4 but with Σ replaces by Σ_ψ as the underlying closed set. To prove (6.10) the key step is to prove that

$$\|\nu_\psi\| \lesssim (1 + \|\nu\|). \tag{6.11}$$

Once (6.11) is established, one can repeat the proof in [HLN1, pp. 368–373] to conclude that (6.10) holds with $b_2 \approx 1 + \|\nu\|$, thus completing the proof of Theorem 3.5.

It therefore remains to give the proof of (6.11). To start, we make an elementary observation of a geometric nature. Indeed, we first note that

$$d(Y, s, \Sigma) \lesssim (1 + b^*)\, d\big(y, s, \pi(F_1)\big), \quad \forall\, (Y, s) \in \Sigma_\psi \cap \overline{Q_{100R}(X, t)}. \tag{6.12}$$

Indeed, this inequality is trival when $(Y, s) \in F_1$, so assume $(Y, s) = (y, s, \psi^*(y, s))$ with $(y, s) \in \bar{I}_i$ for some i. Then $d(y, s, \pi(F_1)) \approx \rho_i \approx d\big((y, s), (x_i', t_i')\big)$, by (6.7). Consequently, since ψ^* is Lip (1,1/2) with constant b^*,

$$d(Y, s, \Sigma) \le d\big((y, s, \psi^*(y, s)), (x_i', t_i', \psi^*(x_i', t_i')\big) \lesssim (1 + b^*)\rho_i. \tag{6.13}$$

This proves (6.12).

In the following $K \gg 1$ is a degree of freedom. Given $(Z, \tau, r) \in \Sigma \times (0, \infty)$ we let $P_{(Z,\tau,r)}$ be a time-independent plane which realizes $\beta(Z, \tau, Kr)$.

Consider

$$(Z, \tau) \in F_1 \ \text{ and } \ r > 0 \ \text{ such that } \ \overline{Q_r(Z, \tau)} \subset Q_{80R}(X, t). \tag{6.14}$$

Given $i \in \Lambda$, let $(X_i', t_i') \in F_1$ be such that $\pi(X_i', t_i') = (x_i', t_i')$ where $(x_i', t_i') \in \pi(F_1)$ realizes the distance from I_i to $\pi(F_1)$. Let $\mathcal{Q}_i$ be a dyadic cube on Σ (see Definition 3) containing (X_i', t_i') with $\ell(\mathcal{Q}_i) \approx \rho_i$. Furthermore, let

$$\Gamma_i = \{(y, \psi(y, s), s) \colon (y, s) \in \overline{I_i}\}.$$

Then

$$\sigma(\Gamma_i) \approx \rho_i^{n+1}, \tag{6.15}$$

(here we are using σ to denote the surface measure both on Σ and on Σ_ψ), and

$$\rho_i = d(x_i', t_i', I_i) = d\big(I_i, \pi(F_1)\big) \sim \ell(I_i) \sim \ell(\mathcal{Q}_i) \sim d(X_i', t_i', \Gamma_i) \gtrsim d(\mathcal{Q}_i, \Gamma_i), \tag{6.16}$$

where in the next-to-last step we have used that Σ_ψ is a *Lip*$(1, 1/2)$ graph. Using this notation we see that

$$\beta_\psi^2(Z, \tau, r) \lesssim r^{-(n+1)} \iint_{\Sigma_\psi \cap Q_r(Z,\tau)} \left(\frac{d(Y, s, P_{(Z,\tau,r)})}{r}\right)^2 \mathrm{d}\sigma(Y, s). \tag{6.17}$$

Introducing

$$T(Z, \tau, r) := r^{-(n+1)} \iint_{F_1 \cap Q_r(Z,\tau)} \left(\frac{d(Y, s, P_{(Z,\tau,r)})}{r}\right)^2 \mathrm{d}\sigma(Y, s),$$

$$T_i(Z, \tau, r) := r^{-(n+1)} \iint_{\Gamma_i \cap Q_r(Z,\tau)} \left(\frac{d(Y, s, P_{(Z,\tau,r)})}{r}\right)^2 \mathrm{d}\sigma(Y, s), \tag{6.18}$$

we can continue the estimate in (6.17) and conclude that

$$\beta_\psi^2(Z,\tau,r) \lesssim T(Z,\tau,r) + \sum_{i\in I(Z,\tau,r)} T_i(Z,\tau,r),$$

where $I(Z,\tau,r) := \{i\colon Q_r(Z,\tau)\cap\Gamma_i \neq \emptyset\}$. By construction

$$T(Z,\tau,r) \lesssim \beta^2(Z,\tau,Kr). \tag{6.19}$$

To handle the sum over $i \in I(Z,\tau,r)$ we will combine arguments from [DS] and [HLN1].

Let $i \in I(Z,\tau,r)$. Then

$$\rho_i \lesssim r \quad \text{and} \quad d(Z,\tau,\mathcal{Q}_i) \lesssim r. \tag{6.20}$$

Choose a $(Z_i,\tau_i) \in \overline{\mathcal{Q}}_i$ which minimizes the distance from $\overline{\mathcal{Q}}_i$ to $P_{(Z,\tau,r)}$, i.e.

$$h_i := \inf_{(Y,s)\in\mathcal{Q}_i} d(Y,s,P_{(Z,\tau,r)}) = d(Z_i,\tau_i,P_{(Z,\tau,r)}). \tag{6.21}$$

For $(Z_i,\tau_i) \in \overline{\mathcal{Q}}_i$ fixed as above, choose $\mathbf{Z}_{(Z,\tau,r)} \in P_{(Z,\tau,r)}$ so that

$$h_i = d(Z_i,\tau_i,P_{(Z,\tau,r)}) = d(Z_i,\tau_i,\mathbf{Z}_{(Z,\tau,r)}). \tag{6.22}$$

Using this notation and the triangle inequality, we write

$$T_i(Z,\tau,r) \lesssim \tilde{T}_i(Z,\tau,r) + \hat{T}_i(Z,\tau,r), \tag{6.23}$$

where

$$\tilde{T}_i(Z,\tau,r) := r^{-(n+1)} \iint_{\Gamma_i\cap Q_r(Z,\tau)} \left(\frac{d(Y,s,Z_i,\tau_i)}{r}\right)^2 \mathrm{d}\sigma(Y,s),$$
$$\hat{T}_i(Z,\tau,r) := r^{-(n+1)} \iint_{\Gamma_i\cap Q_r(Z,\tau)} \left(\frac{d(Z_i,\tau_i,\mathbf{Z}_{(Z,\tau,r)})}{r}\right)^2 \mathrm{d}\sigma(Y,s).$$

We then have

$$\tilde{T}_i(Z,\tau,r) \lesssim (\rho_i/r)^{n+3}, \quad \hat{T}_i(Z,\tau,r) \lesssim (\rho_i/r)^{n+1}(h_i/r)^2, \tag{6.24}$$

where we have used (6.12) in the first estimate. Combining (6.19) and (6.24) we can conclude that if (Z,τ,r) is as in (6.14), then

$$\beta_\psi^2(Z,\tau,r) \lesssim \beta^2(Z,\tau,Kr) + \sum_{i\in I(Z,\tau,r)} \left(\frac{\rho_i}{r}\right)^{n+3} + \sum_{i\in I(Z,\tau,r)} \left(\frac{\rho_i}{r}\right)^{n+1} \left(\frac{h_i}{r}\right)^2. \tag{6.25}$$

We first treat the last term in (6.25), following the argument in [DS, pp. 86–87]. Given $i \in I(Z,\tau,r)$ we set $J(i) := \{j\colon \mathcal{Q}_j \subset \mathcal{Q}_i\}$, and define

$$\mathcal{N}_i(Y,s) := \sum_{j\in J(i)} 1_{\mathcal{Q}_j}(Y,s)$$

for $(Y,s) \in \Sigma$. Then, as in [DS] we have

$$\fint\!\!\!\fint_{\mathcal{Q}_i} \mathcal{N}_i \,\mathrm{d}\sigma \lesssim 1, \tag{6.26}$$

and

$$\sum_i \mathcal{N}_i(Y,s)^{-2} 1_{\mathcal{Q}_i}(Y,s) \lesssim 1\,. \tag{6.27}$$

We sketch the proof of the latter estimate, as follows. If $\mathcal{N}_i(Y,s) = \infty$, then trivially $\mathcal{N}_i(Y,s)^{-2} = 0$. Otherwise, if $\mathcal{N}_i(Y,s) < \infty$, then there are only finitely many terms in the sum defining $\mathcal{N}_i(Y,s)$. Note also that for each k, there is at most one $\mathcal{Q}_j \in \mathbb{D}_k$ such that $(Y,s) \in \mathcal{Q}_j$. Thus, $\mathcal{N}_i(Y,s)$ equals the number of dyadic generations k such that there is a cube $\mathcal{Q}_j \in \mathbb{D}_k$, with $(Y,s) \in \mathcal{Q}_j \subset \mathcal{Q}_i$. For the smallest $\mathcal{Q}_i$ containing (Y,s), we have $\mathcal{N}_i(Y,s) = 1$, for the next smallest $\mathcal{N}_i(Y,s) = 2$, etc., so that the sum in (6.27) is controlled by $\sum_{k=1}^{\infty} k^{-2}$.

Following [DS], we write

$$\rho_i^{n+1} h_i^2 = \rho_i^{n+1} h_i^2 \left(\fint\!\!\fint_{\mathcal{Q}_i} d\sigma \right)^3 = \rho_i^{n+1} h_i^2 \left(\fint\!\!\fint_{\mathcal{Q}_i} \mathcal{N}_i^{-2/3} \mathcal{N}_i^{2/3}\, \mathrm{d}\sigma \right)^3 .$$

By Hölder's inequality, (6.21), and (6.26), we then deduce that

$$\begin{aligned} \rho_i^{n+1} h_i^2 &\lesssim \iint_{\mathcal{Q}_i} (d(Y,s,P_{(Z,\tau,r)}))^2\, \mathcal{N}_i(Y,s)^{-2} 1_{\mathcal{Q}_i}(Y,s)\ \mathrm{d}\sigma(Y,s) \left(\fint\!\!\fint_{\mathcal{Q}_i} \mathcal{N}_i\ \mathrm{d}\sigma \right)^2 \\ &\lesssim \iint_{\mathcal{Q}_i} (d(Y,s,P_{(Z,\tau,r)}))^2\, \mathcal{N}_i(Y,s)^{-2} 1_{\mathcal{Q}_i}(Y,s)\ \mathrm{d}\sigma(Y,s). \end{aligned}$$

Hence, summing over i, using (6.20) and (6.27), we obtain

$$\begin{aligned} \sum_{i \in I(Z,\tau,r)} (\rho_i/r)^{n+1} (h_i/r)^2 &\lesssim r^{-n-3} \iint_{\Sigma \cap Q_{Kr}(Z,\tau)} (d(Y,s,P_{(Z,\tau,r)}))^2\ \mathrm{d}\sigma(Y,s) \\ &\approx \beta^2(Z,\tau,Kr), \end{aligned} \tag{6.28}$$

provided that K is chosen large enough, depending on the implicit constants in (6.20). In particular,

$$\beta_\psi^2(Z,\tau,r) \lesssim \beta^2(Z,\tau,Kr) + \sum_{i \in I(Z,\tau,r)} (\rho_i/r)^{n+3} \tag{6.29}$$

for all $(Z,\tau) \in F_1$ and $r > 0$ such that $\overline{Q_r(Z,\tau)} \subset Q_{80R}(X,t)$.

For given $(\hat{Z},\hat{\tau}) \in \Sigma_\psi$ and $\hat{r} > 0$ such that with $Q_{\hat{r}}(\hat{Z},\hat{\tau}) \subset Q_{20R}(X,t)$, we integrate (6.29) over $F_1 \cap Q_{\hat{r}}(\hat{Z},\hat{\tau})$. If $F_1 \cap Q_{\hat{r}}(\hat{Z},\hat{\tau}) = \emptyset$ the following inequality is trivially true. Using (6.29)

$$\begin{aligned} \nu_\psi(F_1 \cap Q_{\hat{r}}(\hat{Z},\hat{\tau}) \times (0,\hat{r})) &= \int_0^{\hat{r}} \iint_{F_1 \cap Q_{\hat{r}}(\hat{Z},\hat{\tau})} \beta_\psi^2(Z,\tau,r)\ \mathrm{d}\sigma(Z,\tau) r^{-1}\ \mathrm{d}r \\ &\lesssim \nu(F_1 \cap Q_{\hat{r}}(\hat{Z},\hat{\tau}) \times (0,K\hat{r})) \\ &\quad + \int_0^{\hat{r}} \iint_{F_1 \cap Q_{\hat{r}}(\hat{Z},\hat{\tau})} \sum_{i \in I(Z,\tau,r)} (\rho_i/r)^{n+3}\ \mathrm{d}\sigma(Z,\tau)\, r^{-1}\ \mathrm{d}r \\ &=: I + II. \end{aligned} \tag{6.30}$$

By our assumptions, $I \lesssim \|\nu\|\, \hat{r}^{n+1}$. Note that $r_i'(Z,\tau) := d(Z,\tau,\Gamma_i) + \rho_i \lesssim r$, by (6.16) and (6.20). Thus, summing and interchanging the order of integration, we see that

$$II \leq \iint_{F_1 \cap Q_{\hat r}(\hat Z,\hat\tau)} \sum_{i\in I(\hat Z,\hat\tau,C\hat r)} \left(\int_{cr_i'(Z,\tau)}^{\hat r} (\rho_i/r)^{n+3} r^{-1}\, \mathrm{d}r \right) \mathrm{d}\sigma(Z,\tau)$$

$$\lesssim \sum_{i\in I(\hat Z,\hat\tau,C\hat r)} \iint_{F_1\cap Q_{\hat r}(\hat Z,\hat\tau)} \left(\frac{\rho_i}{r_i'(Z,\tau)} \right)^{n+3} \mathrm{d}\sigma(Z,\tau)$$

$$\lesssim \sum_{i\in I(\hat Z,\hat\tau,C\hat r)} \rho_i^{n+1} \lesssim \hat r^{n+1}. \tag{6.31}$$

Hence, combining (6.30) and (6.31), we have proved that

$$\nu_\psi(F_1 \cap Q_{\hat r}(\hat Z,\hat\tau) \times (0,\hat r)) \lesssim (1+\|\nu\|)\hat r^{n+1} \tag{6.32}$$

for all $(\hat Z,\hat\tau) \in \Sigma_\psi$ and $\hat r > 0$ such that $Q_{\hat r}(\hat Z,\hat\tau) \subset Q_{20R}(X,t)$.

Similarly, by repeating the argument between displays (2.30) and (2.32) in [HLN1] we first deduce that

$$\nu_\psi((\Sigma_\psi \setminus F_1) \cap Q_{\hat r}(\hat Z,\hat\tau) \times (0,\hat r)) \lesssim (1+\|\nu\|)\hat r^{n+1}, \tag{6.33}$$

and then, using also (6.32), we can conclude that

$$\nu_\psi(\Sigma_\psi \cap Q_{\hat r}(\hat Z,\hat\tau) \times (0,\hat r)) \lesssim (1+\|\nu\|)\hat r^{n+1}, \tag{6.34}$$

whenever $(\hat Z,\hat\tau) \in \Sigma_\psi$ and $\hat r > 0$ are such that with $Q_{\hat r}(\hat Z,\hat\tau) \subset Q_{20R}(X,t)$. The other cases can be handled by the observations in display (2.33) in [HLN1]. We omit further details and claim that the proof of (6.11), and hence the proof of Theorem 3.5, is complete. □

Proof of Corollary 3.2. Let $(\hat X,\hat t) \in \Omega$. We repeat the proof of Corollary 3.1 to construct a $Lip(1,1/2)$ function, which we now call ψ^*, along with the local graph subdomain $\tilde\Omega =: \tilde\Omega_{\psi^*} \subset \Omega$, defined as in (5.9) but with ψ^* in place of ψ, above the planar cube $I_* = \pi(Q_*)$ (see (5.10)). With ψ^* in hand, and using (5.11), we repeat the proof of Theorem 3.5, with $\Delta(X,t,R)$ replaced by $\Delta_* := Q_* \cap \Sigma$, and thus $R \approx d/(Nh)$ (we recall that Q_* is centered on Σ; see Remark 5.2)). Specifically, we construct ψ as in (6.8), now with $\mu = N^{1/2}$, where N is the suitably large constant in the proof of Corollary 3.1, and of course with $b^* = h$. By the proof of Theorem 3.5, ψ is a regular $Lip(1,1,2)$ (i.e., $RPLip$) graph, as desired. We now define $\tilde\Omega = \tilde\Omega_\psi$ again as in (5.9), this time with respect to ψ. To obtain the conclusion of Corollary 3.2, it remains only to verify that $\tilde\Omega_\psi \subset \Omega$, and that the corkscrew condition (5.12) holds for $\tilde\Omega = \tilde\Omega_\psi$. The former is easy: by construction (see (6.8)), $\psi^* \leq \psi$, pointwise in I_*, provided that N (hence also $\mu = N^{1/2}$) is chosen large enough. Thus, $\tilde\Omega_\psi \subset \tilde\Omega_{\psi^*}$, and we already know that in turn, $\tilde\Omega_{\psi^*} \subset \Omega$.

Let us now verify that (5.12) holds for $\tilde\Omega = \tilde\Omega_\psi$. To this end, observe first that in the proof of Theorem 3.5, by construction ψ has compact support in a ball of parabolic radius CR, and that the planar Whitney cubes I_i have "length" $r_i \approx \rho_i \lesssim R$. In the present setting, this means that $\rho_i \lesssim d/(Nh)$ for all i. Since we have chosen $\mu = N^{1/2}$, and since $b^* = h$, this means that by construction (see (6.8)), applying (5.13) to ψ^*, we have

$$\sup_{(y,s)\in 100I_*} \psi(y,s) \leq \sup_{(y,s)\in 100I_*} \psi^*(y,s) + CN^{1/2}hN^{-1}h^{-1}d$$
$$\leq \left(\kappa - c(n) + CN^{-1} + CN^{-1/2}\right) d \leq (\kappa - c(n)/2))d,$$

for N large enough, and therefore with $c_1 = c(n)/2$, we have

$$\hat{x}_n - \sup_{(y,s)\in 100I_*} \psi(y,s) \geq c_1 d\,. \tag{6.35}$$

Combining the latter estimate with the definition of $\tilde{\Omega} = \tilde{\Omega}_\psi$ (see (5.9)), we find that (5.12) holds for $\tilde{\Omega}_\psi$, provided that N is chosen large enough. $\square$

7. Two counterexamples

In [NS], the authors prove that a parabolic ADR set satisfying a synchronized two cube condition contains big pieces of *Lip*$(1,1/2)$ graphs. It is quite easy to see that any set satisfying a synchronized two cube condition is time-symmetric ADR. It turns out that this implication is not reversible. In particular, in light of Theorem 3.1, the weak time-synchronized two cube condition is *strictly* weaker than its strong counterpart. In this section, we construct two examples of time-symmetric ADR sets satisfying a (two-sided) corkscrew condition, which do *not* satisfy a synchronized two cube condition. Moreover, our examples are also parabolic UR. Importantly, these examples show that Theorems 3.4 and 3.5, and Corollaries 3.1 and 3.2, are strict improvements of the corresponding results in [NS][3].

The first example is rather simple: let Ω be the open region between the two graphs $\Gamma^\pm = \{(\psi^\pm(t), t\} \subset \mathbb{R}^2$, where

$$\psi^\pm(t) := \pm|t|^{1/2} \pm 1, \quad t \in \mathbb{R}.$$

Clearly, Ω is connected. Moreover, it is easy to check that the boundary $\Sigma = \Gamma^+ \cup \Gamma^-$ is time symmetric ADR (indeed, each of $\Gamma^\pm$ is a *Lip*$(1,1/2)$ graph), and satisfies the two sided corkscrew condition in the sense of Definition 9 (i.e., with one point interior to Ω and one exterior). On the other hand, Σ fails to have *synchronized* corkscrew points (one interior to Ω and one exterior) in the sense of Definition 10, at $t = 0$ (i.e., at the boundary points $(\pm 1, 0)$), since the interior corkscrew points get pushed to the side at large scales. Moreover, one may readily verify that each of the graphs $\Gamma^\pm$ is *regular Lip*$(1,1/2)$, and thus Σ is p-UR, by checking the regularity criterion of [Stz, Theorem 3.3], namely that each of $\psi^\pm$ satisfies the Carleson measure condition

$$\sup_{a\in\mathbb{R},\, h>0} \frac{1}{h} \int_{a-h}^{a+h} \int_{a-h}^{a+h} \frac{|\psi(t)-\psi(s)|^2}{|t-s|^2}\, \mathrm{d}s\, \mathrm{d}t \leq C.$$

We observe that the construction above does not provide a counter-example to the time-synchronized 2-cube condition in the weaker sense of Definition 7, in which one merely insists upon the existence of time-synchronized cubes in separate connected components of $\mathbb{R}^{n+1} \setminus \Sigma$ (not necessarily interior to one designated component). Our next example and construction addresses this issue. The construction will be set in $\mathbb{R}^2$.

To start the construction in $\mathbb{R}^2$ we in this example will identify the horizontal axis as the time axis, and the vertical axis as the spatial axis. However, we will continue to denote points by (X, t) where X refer to the spatial coordinate and t will refer to the time coordinate. Starting at $(0,0)$, we draw two line segment with slopes ± 2, traveling distance $1/4$ on the time axis in the positive direction. The endpoints of these two line segments will be $S_1 = \{\pm 1/2, 1/4\}$. Set $S_0 = \{(0,0)\}$,

[3] As noted in the introduction, Theorem 3.5 and Corollary 3.2 also improve the corresponding results in [NS] in a further sense, namely that in the present work we have removed the size constraint on the p-UR constants that was implicit in [NS].

and $S_1 = \{(1/2, 1/4), (1/2, -1/4)\}$. Also, we label the line segments constructed $\mathcal{G}_1$. We will construct sets $\mathcal{G}_k$ and S_k. We will refer to the set $\mathcal{G}_k$ as the set of "line segments of generation k", and we will refer to the set S_k as the set "branch points of generation k". We construct $\mathcal{G}_k$ and S_k inductively as follows. We set $t_0 = 0$ and for $k \geq 1$ we set

$$t_k = \sum_{n=1}^{k} \frac{1}{4^n}.$$

Starting with a branch point b of generation 1, draw two line segments, each with initial vertex b, one having slope 4, and the other slope -4, and each travelling t-distance $1/4^2 = 1/16$. Do this for each $b \in S_1$. The resulting line segments define the set $\mathcal{G}_2$. Additionally, the resulting branch points which define S_2 are

$$S_2 = \{(3/4, t_2), (1/4, t_2), (-1/4, t_2), (-3/4, t_2)\}.$$

Now, we iterate this process (see the figure below). From each branch point $b \in S_2$, we draw two line segments, one with slope 2^3, and one with slope -2^3, and each with t-length $1/4^3$. After doing this for all $b \in S_2$, the resulting lines define $\mathcal{G}_3$, and the resulting branch points define S_3. Proceeding inductively it is not hard to see that

$$S_n = \left\{ \left(\pm \frac{2k+1}{2^n}, t_n \right) \right\}_{k=0}^{2^{n-1}-1},$$

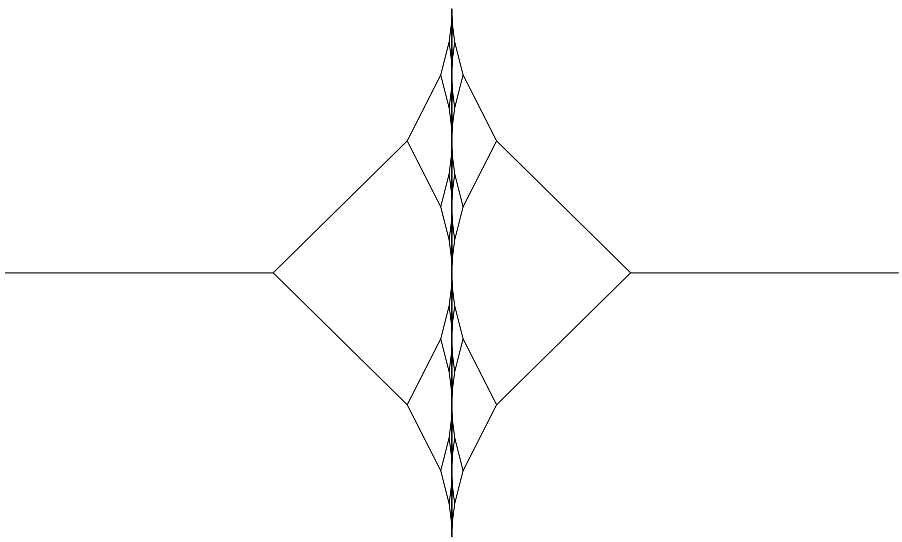
for $n \geq 2$. Note that, at generation k, the total distance travelled by the connected line segments of each previous generation is t_n and $t_n \to 1/3$ as $n \to \infty$.

We set

$$\Sigma_0 := \bigcup_{k=1}^{\infty} \bigcup_{l_\alpha \in \mathcal{G}_k} l_\alpha,$$

and we claim that

$$\overline{\Sigma_0} = \Sigma_0 \cup [[-1, 1] \times \{1/3\}].$$



To prove the claim, choose a point $a2^{-n}$ where $a \in \{\pm 1, \pm 2, \dots, \pm(2^n-1)\}$. It is easily seen that $a2^{-n}$ is the spatial coordinate of some branch point in $\bigcup_{k=1}^n S_k$. Let $a2^{-n}$ be the spatial coordinate of the branch point $(a2^{-n}, t_l) \in S_l$. Then, choosing a "child" branch point obtained by traveling down from $(a2^{-n}, t_l)$ on the line segment with initial point $(a2^{-n}, t_l)$ with slope -2^{l+1} to the lower branch point of the next generation, this "child" branch point has spatial coordinate

$$a2^{-n} - 2^{l+1}4^{-l-1} = a2^{-n} - 2^{-l-1}.$$

Next, suppose that we travel "up" on every subsequent branch point. Then the resulting spatial coordinates so obtained will converge to

$$a2^{-n} - 2^{-l-1} + \sum_{k=l+2}^{\infty} 2^{-l} = a2^{-n} - 2^{-l-1} + 2^{-l-1} = a2^{-n}.$$

Hence, $(a2^{-n}, 1/3)$ is a limit point of Σ_0. By the arbitrary nature of the spatial coordinate $a2^{-n}$ we see that

$$\bigcup_{n=1} \bigcup_{a=1}^{2^n-1} (\pm a2^{-n}, 1/3)$$

is in the closure of Σ_0. Hence, it is easy to see that the claim follows.

Now, add the ray $(-\infty, 0)$ to $\overline{\Sigma_0}$, and extend the resulting set by symmetry with respect to $t = 1/3$. We call the resulting set Σ. The preceding figure is a computer-generated image of the set Σ constructed (recall that the vertical axis represents X and that the horizontal axis represents t).

We will prove that Σ is parabolic UR and that Σ satisfies a corkscrew condition. First, let us focus on showing that it is parabolic ADR.

First, we will show the ADR bounds on surface cubes centered at $(1/3) \times [-1, 1]$. Choose $(t, X) \in (1/3) \times [-1, 1]$ and consider the surface cube $\Delta_R(t, x)$, $R > 0$.

Suppose first $R \geq 1$. We note that the measure of all of the line segments between $t = 0$ and $t = 1/3$ is

$$\sum_{n=1}^{\infty} 2^n 4^{-n} = 1.$$

Hence, the measure of all of the line segments between $t = 1/3$ and $t = 2/3$ is also 1. So, for $R \geq 1$,

$$\sigma(\Delta_R(X, t)) \leq 2 + 2(R-1)^2 \lesssim R^2,$$

where the factor $2(R-1)^2$ accounts for the possibility that the rays $(-\infty, 0)$ and $(1/3, \infty)$ intersect the surface cube. Now, suppose that $R \leq 1$, and that $R \approx 2^{-k}$ for some $k > 0$. We want to estimate the integers m such that the backward face of $Q_R(X, t)$ has t-coordinate $\approx t_m$, but

$$\sum_{n=1}^{m} 4^{-n} \approx \frac{1}{3} - 4^{-k} \implies \frac{1}{3} - \frac{4^{-m}}{3} \approx \frac{1}{3} - 4^{-k} \implies m \approx k.$$

If $\Delta_R(X, t)$ intersect segments of generation k, it will pick up approximately 2^{-k} of the total measure of the segments of that generation. Hence

$$\sigma(\Delta_R^-(X, t)) \approx 2^{-k} \sum_{n=k}^{\infty} 2^{-n} = 2^{-k}2^{1-k} \approx 2^{-2k} \approx R^2.$$

Because $\sigma(\Delta_R^-(X, t)) = \sigma(\Delta_R^+(X, t))$ by symmetry, this establishes the upper and lower ADR bounds for cubes centered on the vertical face $[-1, 1] \times \{1/3\}$.

Now, suppose that $(X,t) \in \Sigma \cap \{0 \le t \le 2/3\} \setminus \{t = 1/3\}$. Without loss of generality, we can assume that $0 < t < 1/3$. Let (X,t) lie on a line segment of generation k. First, we consider the case when $R \ge 2^{-k-1}$. Then, $\Delta_R(X,t)$ is contained a surface cube of size $\approx R$ (but greater than R) centered on the vertical face $[-1,1] \times \{1/3\}$. From this, we easily see that the upper ADR bound holds in this case. The lower ADR bound holds trivially in both the forward and backward directions. So, assume that $R < 2^{-k-1}$, so that the surface cube $\Delta_R(X,t)$ does not intersect the vertical face. In fact, we can see that the surface cube only intersects a uniformly bounded number of lines of generation $k-1$, k and $k+1$. So, it is easy to see that

$$\sigma(\Delta_R(X,t)) \approx R^2,\ \sigma(\Delta_R^+(X,t)) \approx R^2,\ \sigma(\Delta_R^-(X,t)) \approx R^2.$$

Now, the last case to consider is when (X,t) lies in one of the rays $(0,\infty) \times \{0\}$, $(1/3,\infty) \times \{0\}$. This case is easy to see. We leave the details to the reader.

Now, we show that the set is parabolic UR. First, we show Carleson measure estimates hold on the points of the vertical face $[-1,1] \times \{1/3\}$. Choose a point $(X,t) \in [-1,1] \times \{1/3\}$, and $R > 0$. Choose l to be the largest integer such that $R \le 2^{-l}$. We split

$$\begin{aligned}\int_0^R \int_{\Delta_R(X,t)} \beta^2(Y,s,r) \frac{\mathrm{d}\sigma(Y,s)\,\mathrm{d}r}{r} &\le \sum_{k=l}^{\infty} \int_0^{2^{-k}} \int_{\Delta_R(X,t)\cap \mathcal{G}_k} \beta^2(Y,s,r) \frac{\mathrm{d}\sigma(Y,s)\,\mathrm{d}r}{r} \\ &\quad + \sum_{k=l}^{\infty} \int_{2^{-k}}^{2^{-l}} \int_{\Delta_R(X,t)\cap \mathcal{G}_k} \beta^2(Y,s,r) \frac{\mathrm{d}\sigma(Y,s)\,\mathrm{d}r}{r} \\ &=: I + II.\end{aligned}$$

Here, $\mathcal{G}_k$ refers to both the "original" lines of generation k, and their reflections about $t = 1/3$. First, let us deal with term I. We note that for $(s,Y) \in \mathcal{G}_k$, is it easy to see that

$$\beta^2(Y,s,r) \lesssim \frac{1}{r^4} \int_0^{r^2} |2^k t^2|^2 \,\mathrm{d}t = \frac{2^{2k}}{r^4} \int_0^{r^2} t^2 \,\mathrm{d}t \approx 2^{2k} r^2.$$

Hence, as $\Delta_R(X,t)$ only intersects $\mathcal{G}_k$ for $k \ge l$,

$$\begin{aligned} I &\lesssim \sum_{k=l}^{\infty} \int_0^{2^{-k}} \int_{\Delta_R(X,t)\cap \mathcal{G}_k} 2^{2k} r^2 \frac{\mathrm{d}\sigma(Y,s)\,\mathrm{d}r}{r} \\ &\lesssim \sum_{k=l}^{\infty} \int_0^{2^{-k}} 2^{-l} 2^{-k} 2^{2k} r^2 \frac{\mathrm{d}\sigma(Y,s)\,\mathrm{d}r}{r} \approx 2^{-2l} \approx R^2.\end{aligned}$$

Now, we need to estimate term I. For this, we simply note that the β numbers are all uniformly bounded by a constant which depends only on ADR. Hence

$$\begin{aligned} II &\le \sum_{k=1}^{\infty} \int_{2^{-k}}^{2^{-l}} \int_{\Delta_R(X,t)\cap \mathcal{G}_k} \beta^2(Y,s,r) \frac{\mathrm{d}\sigma(Y,s)\,\mathrm{d}r}{r} \\ &\lesssim \sum_{k=l}^{\infty} \int_{2^{-k}}^{2^{-l}} \int_{\Delta_R(X,t)\cap \mathcal{G}_k} \frac{\mathrm{d}\sigma(Y,s)\,\mathrm{d}r}{r} \\ &\le \sum_{k=l} 2^{-l} 2^{-k} k \lesssim 2^{-2l} \approx R^2.\end{aligned}$$

So, we have appropriate Carleson measure bounds for surface cubes centered on the vertical face. Now, we need to prove the same estimates for points in $\Sigma \cap \{0 \le t \le$

$2/3\} \setminus ([-1,1] \times \{1/3\})$. Again, just as in proving the ADR bounds, we can reduce this to proving the bound for points $(X,t) \in \Sigma$ with $0 < t < 1/3$. Choose such a point (X,t), and suppose that it lies on a line segment of generation k. If we choose a scale $R \geq 2^{-k+1}$, then, again, there is a surface cube Δ_{CR} centered on the vertical face, containing $\Delta_R(X,t)$. Therefore

$$\nu[\Delta_R(X,t) \times (0,R)] \leq \nu[\Delta_{CR} \times (0,CR)] \lesssim R^2.$$

On the other hand, if $r < 2^{-k}$, then the appropriate Carleson measure bound follows immediately from estimating a term like I above. Finally, all that is left is to prove the Carleson measure estimate for surface cubes which are not centered at point with t-value between 0 and $2/3$. This case is easy, and we leave the details to the reader.

Now, we need to prove that Σ satisfies a two-sided corkscrew condition. Choose a point (X,t) on the vertical face, and a scale R. If $R \geq 1$, then $\Delta_R(X,t)$ will contain an portion of the ray $(0,\infty) \times \{0\}$ of t-length $\approx R$. It is easy to produce corkscrews in this case by considering points on the portion of the ray contained in the surface cube. Now, suppose that $R \leq 1$. then $R \approx 2^{-k}$ for some $k \geq 1$. It is easy to see that $\Delta_R(X,t)$ will completely contain a line segment in $\mathcal{G}_l$ for some $l \approx k$. At the midpoint of this line segment, it is also easy to see that we can produce corkscrews at scale 2^{-l}, each of parabolic size $\approx 2^l$. Now, consider the case where (X,t) lies on a line segment in $\mathcal{G}_k$. For $R < 2^{-k+1}$, it is trivial to show that we can produce corkscrews of size $\approx R$ which lie within $Q_R(X,t)$. Now, suppose that $R \geq 2^{-k+1}$. Then there is a surface cube $\Delta_{R/2}$ centered on the vertical face contained in $\Delta_R(X,t)$. By the work above, we can produce corkscrews relative to $\Delta_{R/2}$ of parabolic size $\approx R/2 \approx R$, which are clearly corkscrews relative to $\Delta_R(X,t)$. Finally, we need to consider when (X,t) lies outside of $\{(X,t)\colon 0 \leq t \leq 2/3\}$. But this case is trivial.

Note that as Σ contains a vertical face, it is impossible for Σ to satisfy a synchronized two-cube condition. However, Σ is parabolic UR, so in fact it satisfies a weak synchronized two cube condition.

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