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Injectivity of harmonic mappings with a specified injective holomorphic part

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Abstract. Let $F = H + \overline{G}$ be a locally injective and sense-preserving harmonic mapping of the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$, where H and G are holomorphic in $\mathbb{D}$ and $G(0) = 0$. The aim of this paper is studying interplay between properties of $F_\varepsilon := H + \varepsilon\overline{G}$, $\varepsilon \in \mathbb{C}$, and its holomorphic part H. In particular, several results dealing with the injectivity of F_ε are obtained.

Harmonisten kuvausten injektiivisyydestä, kun injektiivinen holomorfinen osa on annettu

Tiivistelmä. Olkoon $F = H + \overline{G}$ paikallisesti injektiivinen ja suunnistuksen säilyttävä yksikkökiekon $\mathbb{D}$ harmoninen kuvaus kompleksitasoon $\mathbb{C}$, missä H ja G ovat holomorfisia kiekolla $\mathbb{D}$ ja $G(0) = 0$. Tämän tutkimuksen tavoitteena on tarkastella kuvauksen $F_\varepsilon := H + \varepsilon\overline{G}$, missä $\varepsilon \in \mathbb{C}$, ja sen holomorfisen osan H välistä yhteyttä. Erityisesti saadaan useita kuvauksen F_ε injektiivisyyttä koskevia tuloksia.

1. Introduction

Let Ω be a non-empty domain in the complex plane $\mathbb{C}$. A twice continuously differentiable mapping $F\colon \Omega \to \mathbb{C}$ is said to be a *harmonic mapping* if F satisfies the Laplace equation

$$\partial\bar{\partial}F = 0 \quad \text{in } \Omega.$$

Here $\partial := \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)$ and $\bar{\partial} := \frac{1}{2}(\partial_x + \mathrm{i}\partial_y)$ are formal derivatives. A harmonic mapping F in Ω is said to be *sense-preserving* if

$$\mathrm{J}[F](z) := |\partial F(z)|^2 - |\bar{\partial}F(z)|^2 > 0, \quad z \in \Omega.$$

If F is a sense-preserving harmonic mapping, then the *complex dilatation*

$$\mu_F(z) := \frac{\bar{\partial}F(z)}{\partial F(z)}, \quad z \in \Omega,$$

is well defined and

$$\|\mu_F\|_\infty := \operatorname*{ess\,sup}_{z\in\Omega} |\mu_F(z)| = \sup_{z\in\Omega} |\mu_F(z)| \leq 1.$$

A sense-preserving harmonic mapping F is said to be *quasiconformal* if F is injective and $\|\mu_F\|_\infty < 1$. In particular, if $\|\mu_F\|_\infty = 0$, then F is an injective holomorphic

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mapping. Such a mapping F is usually called *conformal* or *univalent*. A mapping F is said to be *Lipschitz* (resp. *co-Lipschitz*) in Ω if F satisfies the following condition

$$\mathrm{L}^+(F) := \sup\left\{\Big|\frac{F(z)-F(w)}{z-w}\Big| \colon z,w\in\Omega,\ z\neq w\right\} < +\infty,$$

$$\left(\text{resp. } \mathrm{L}^-(F) := \inf\left\{\Big|\frac{F(z)-F(w)}{z-w}\Big| \colon z,w\in\Omega,\ z\neq w\right\} > 0\right).$$

A mapping F is said to be *bi-Lipschitz* if $\mathrm{L}^+(F) < +\infty$ and $\mathrm{L}^-(F) > 0$. In what follows suppose that $\Omega = \mathbb{D} := \mathbb{D}(0,1)$ where $\mathbb{D}(a,r) := \{z\in\mathbb{C}\colon |z-a|<r\}$ for any $a\in\mathbb{C}$ and $r>0$. Then each sense-preserving harmonic mapping F is represented uniquely by

$$F(z) = H(z) + \overline{G(z)}, \quad z\in\mathbb{D},$$

where H and G are holomorphic mappings in $\mathbb{D}$, $G(0)=0$, and consequently

$$\mathrm{J}[F](z) = |H'(z)|^2 - |G'(z)|^2 \quad \text{and} \quad \mu_F(z) = \frac{\overline{G'(z)}}{H'(z)}, \quad z\in\mathbb{D}.$$

Let us consider the following deformation of a harmonic mapping F in $\mathbb{D}$,

$$\mathbb{D}\ni z \mapsto F_\varepsilon(z) := H(z) + \varepsilon\overline{G(z)}, \quad \varepsilon\in\mathbb{C}. \tag{1.1}$$

Our purpose is studying interplay between properties of the harmonic mapping F_ε and its holomorphic part H under certain assumptions on $\varepsilon\in\mathbb{C}$. In particular, we focus our attention to the injectivity of F_ε under certain conditions on F and H. Section 2 contains results involving injectivity, quasiconformality and Lipschitz type properties of F_ε, provided $\varepsilon\in\mathbb{C}$ satisfies an additional inequality. We borrow here from results published in [9] and [10]. Most essential results are Theorems 2.2 and 2.3. In Section 3 we prove Theorem 3.1 relevant to the classical results of Clunie and Sheil-Small dealing with close-to-convex harmonic mappings; cf. [1]. The theorem yields Corollary 3.4, which seems to be of special interest. It provides a simple method of producing sense-preserving injective harmonic mappings of $\mathbb{D}$ onto close-to-convex domains; see Examples 3.5 and 3.6. Section 4 is devoted to the problem of homeomorphic extensions of harmonic mappings to the closed unit disk $\mathrm{cl}(\mathbb{D})$, where cl denotes the closure operator in the complex plane. We prove Corollaries 4.2 and 4.4 which refer to Theorems 2.2 and 2.3, respectively. However, the most sophisticated result here is Theorem 4.6, which is relevant to [10, Corollary 3.2].

2. Injectivity of harmonic mappings in the unit disk

For any $M\geq 1$ a domain Ω in $\mathbb{C}$ is said to be *rectifiably M-arcwise connected* if for all $z,w\in\Omega$ there exists an arc γ joining the points z and w in Ω with the length $|\gamma|_1 \leq M|w-z|$; cf. [7]. Note that Ω is a convex domain if and only if Ω is a rectifiably 1-arcwise connected domain. Write $\mathbb{Z}_{p,q} := \{k\in\mathbb{Z}\colon p\leq k\leq q\}$ for $p,q\in\mathbb{Z}$.

Lemma 2.1. *Given $M\geq 1$ let Ω be a rectifiably M-arcwise connected domain in $\mathbb{C}$. If $F\colon\Omega\to\mathbb{C}$ is a bi-Lipschitz mapping, then $F(\Omega)$ is a rectifiably M'-arcwise connected domain with $M' := M\,\mathrm{L}^+(F)/\,\mathrm{L}^-(F)$.*

Proof. Fix M, Ω and F satisfying the assumptions. Given $z_1,z_2\in F(\Omega)$, we see that $w_1 := F^{-1}(z_1), w_2 := F^{-1}(z_2)\in\Omega$ and

$$|z_1-z_2| = |F(w_1)-F(w_2)| \geq \mathrm{L}^-(F)|w_1-w_2|. \tag{2.1}$$

Since Ω is rectifiably M-arcwise connected, there exists an arc $\gamma\colon [0;1] \to \mathbb{C}$ joining the points w_1 and w_2 in Ω with the length $|\gamma|_1 \le M|w_1 - w_2|$. Setting $\sigma := F \circ \gamma$ we see that for every $n \in \mathbb{N}$ and every increasing sequence $\mathbb{Z}_{1,n} \ni k \mapsto t_k \in [0;1]$, if $t_0 = 0$ and $t_n = 1$ then

$$\begin{aligned}\sum_{k=1}^{n} |\sigma(t_k) - \sigma(t_{k-1})| &= \sum_{k=1}^{n} |F(\gamma(t_k)) - F(\gamma(t_{k-1}))| \\ &\le \mathrm{L}^+(F) \sum_{k=1}^{n} |\gamma(t_k) - \gamma(t_{k-1})| \le \mathrm{L}^+(F)|\gamma|_1.\end{aligned}$$

Hence and by (2.1),

$$|\sigma|_1 \le \mathrm{L}^+(F)|\gamma|_1 \le M\,\mathrm{L}^+(F)|w_1 - w_2| \le M\frac{\mathrm{L}^+(F)}{\mathrm{L}^-(F)}|z_1 - z_2| = M'|z_1 - z_2|.$$

Therefore the image $F(\Omega)$ is rectifiably M'-arcwise connected, which is the desired conclusion. $\square$

It is clear that the function

$$[0;1) \ni s \mapsto \lambda(s) := s \cdot \sqrt{\frac{(1-s)^2 + 1}{(1-s)^2 + s^2}} \tag{2.2}$$

satisfies the inequalities $0 < s < \lambda(s) < 1$ for every $s \in (0;1)$.

Theorem 2.2. *Let $F = H + \overline{G}$ be a quasiconformal harmonic mapping in $\mathbb{D}$ such that $F(\mathbb{D})$ is a convex domain and write $l := \lambda(\|\mu_F\|_\infty)$. Then for every $\varepsilon \in \mathbb{C}$ satisfying $|\varepsilon|l < 1$, $F_\varepsilon(\mathbb{D})$ is a rectifiably M_ε-arcwise connected domain with*

$$M_\varepsilon := \frac{1+l}{1-l} \cdot \frac{1+|\varepsilon|l}{1-|\varepsilon|l}\,, \tag{2.3}$$

F_ε is a quasiconformal and co-Lipschitz mapping and $F_\varepsilon \circ H^{-1}$ is a bi-Lipschitz mapping.

Proof. Fix a qusiconformal harmonic mapping $F\colon \mathbb{D} \to \mathbb{C}$ and $\varepsilon \in \mathbb{C}$. Suppose that $F(\mathbb{D})$ is a convex domain. From [9, Remark 2.3 and Lemma 2.4] it follows that H is injective and

$$|G(z_2) - G(z_1)| \le l|H(z_2) - H(z_1)|, \quad z_1, z_2 \in \mathbb{D}, \tag{2.4}$$

which yields,

$$|\varepsilon G(z_2) - \varepsilon G(z_1)| = |\varepsilon||G(z_2) - G(z_1)| \le |\varepsilon|l|H(z_2) - H(z_1)|, \quad z_1, z_2 \in \mathbb{D}.$$

Then

$$\begin{aligned}|F_\varepsilon(z_2) - F_\varepsilon(z_1)| &\le |H(z_2) - H(z_1)| + |\varepsilon G(z_2) - \varepsilon G(z_1)| \\ &\le (1 + |\varepsilon|l)|H(z_2) - H(z_1)|, \quad z_1, z_2 \in \mathbb{D},\end{aligned} \tag{2.5}$$

as well as

$$\begin{aligned}|F_\varepsilon(z_2) - F_\varepsilon(z_1)| &\ge |H(z_2) - H(z_1)| - |\varepsilon G(z_2) - \varepsilon G(z_1)| \\ &\ge (1 - |\varepsilon|l)|H(z_2) - H(z_1)|, \quad z_1, z_2 \in \mathbb{D}.\end{aligned} \tag{2.6}$$

Suppose now that $\varepsilon \in \mathbb{C}$ satisfies $|\varepsilon|l < 1$. Then

$$\mathrm{L}^+(F_\varepsilon \circ H^{-1}) \le 1 + |\varepsilon|l \quad \text{and} \quad \mathrm{L}^-(F_\varepsilon \circ H^{-1}) \ge 1 - |\varepsilon|l, \tag{2.7}$$

and so $F_\varepsilon \circ H^{-1}$ is a bi-Lipschitz mapping. In particular, $\mathrm{L}^+(F \circ H^{-1}) \le 1 + l$ and $\mathrm{L}^-(F \circ H^{-1}) \ge 1 - l$, from which

$$\mathrm{L}^+(H \circ F^{-1}) \le (1-l)^{-1} \quad \text{and} \quad \mathrm{L}^-(H \circ F^{-1}) \ge (1+l)^{-1}. \tag{2.8}$$

Since $H(\mathbb{D}) = (H \circ F^{-1})(F(\mathbb{D}))$ and the image $F(\mathbb{D})$ is rectifiably 1-arcwise connected, we deduce from (2.8) and Lemma 2.1 that $H(\mathbb{D})$ is rectifiably $(1+l)/(1-l)$-arcwise connected. Applying Lemma 2.1 once more we see by (2.7) that $F_\varepsilon(\mathbb{D})$ is a rectifiably M_ε-arcwise connected domain with M_ε given by the expression (2.3). Since $F(\mathbb{D})$ is a convex domain, we conclude from [8, Corollary 3.1] (see also [3, Theorem 2.5]) that there exists a constant $c > 0$ such that $|H'(z)| = |\partial F(z)| \ge c$ for $z \in \mathbb{D}$. Therefore $\mathrm{D}^-(H) := \inf(\{|H'(z)| \colon z \in \mathbb{D}\}) \ge c$, and in view of [10, Lemma 2.5] H is a co-Lipschitz mapping. Since $F_\varepsilon \circ H^{-1}$ is a bi-Lipschitz mapping and $F_\varepsilon = (F_\varepsilon \circ H^{-1}) \circ H$ we see that F_ε is a quasiconformal and co-Lipschitz mapping, which completes the proof. □

Theorem 2.3. *Given $M \ge 1$ suppose that $F = H + \overline{G}$ is a sense-preserving harmonic mapping in $\mathbb{D}$, its holomorphic part H is injective in $\mathbb{D}$ and $H(\mathbb{D})$ is a rectifiably M-arcwise connected domain. Then*

$$|G(z_2) - G(z_1)| \le M\|\mu_F\|_\infty |H(z_2) - H(z_1)|, \quad z_1, z_2 \in \mathbb{D}, \tag{2.9}$$

as well as for every $\varepsilon \in \mathbb{C}$,

$$\begin{aligned}(1 - M|\varepsilon|\|\mu_F\|_\infty)|H(z_2) - H(z_1)| &\le |F_\varepsilon(z_2) - F_\varepsilon(z_1)| \\ &\le (1 + M|\varepsilon|\|\mu_F\|_\infty)|H(z_2) - H(z_1)|, \quad z_1, z_2 \in \mathbb{D}.\end{aligned} \tag{2.10}$$

Moreover, for every $\varepsilon \in \mathbb{C}$ the following implications hold:

(i) *If $|\varepsilon|\|\mu_F\|_\infty \le 1$ and G'/H' is not a constant function, then F_ε is a sense-preserving harmonic mapping in $\mathbb{D}$.*
(ii) *If $M|\varepsilon|\|\mu_F\|_\infty \le 1$ and G'/H' is not a constant function, then F_ε is an injective mapping.*
(iii) *If $M|\varepsilon|\|\mu_F\|_\infty < 1$, then F_ε is a quasiconformal mapping such that $F_\varepsilon \circ H^{-1}$ is a bi-Lipschitz mapping with*

$$\mathrm{L}^+(F_\varepsilon \circ H^{-1}) \le 1 + M|\varepsilon|\|\mu_F\|_\infty \quad \text{and} \quad \mathrm{L}^-(F_\varepsilon \circ H^{-1}) \ge 1 - M|\varepsilon|\|\mu_F\|_\infty; \tag{2.11}$$

in particular, $F_\varepsilon(\mathbb{D})$ is a rectifiably M_ε-arcwise connected domain with

$$M_\varepsilon := M\frac{1 + M|\varepsilon|\|\mu_F\|_\infty}{1 - M|\varepsilon|\|\mu_F\|_\infty}. \tag{2.12}$$

Proof. Fix M and F satisfying the assumptions. Since the holomorphic part H of F is injective in $\mathbb{D}$, $H'(z) \ne 0$ for $z \in \mathbb{D}$, and consequently

$$\mu_F(z) = \frac{\overline{G'(z)}}{H'(z)}, \quad z \in \mathbb{D}. \tag{2.13}$$

Note that the estimations (2.9) and (2.10) for $\varepsilon := 1$ follow from [10, Lemma 3.1] which is valid even in the case where F is not sense-preserving. Therefore we can adopt the proof of the estimations for F replaced by F_ε where $\varepsilon \in \mathbb{C}$ is arbitrarily fixed. Setting $\phi_\varepsilon := \overline{\varepsilon} G \circ H^{-1}$ we see that the quantity $\mathrm{D}^+(\phi_\varepsilon)$, defined by [10, (2.5)], satisfies the following equality

$$\mathrm{D}^+(\phi_\varepsilon) = \sup_{z\in\Omega}\big(|\partial\phi_\varepsilon(z)| + |\bar\partial\phi_\varepsilon(z)|\big) = \|\mu_{F_\varepsilon}\|_\infty = |\varepsilon|\|\mu_F\|_\infty,$$

where $\Omega := H(\mathbb{D})$. Since $F_\varepsilon = \mathrm{I}[\phi_\varepsilon] \circ H$, where

$$\Omega \ni z \mapsto \mathrm{I}[\phi_\varepsilon](z) := z + \overline{\phi_\varepsilon(z)},$$

we conclude from [10, Lemma 2.4] that $\mathrm{L}^+(\mathrm{I}[\phi_\varepsilon]) \leq 1 + M\,\mathrm{D}^+(\phi_\varepsilon) = 1 + M|\varepsilon|\|\mu_F\|_\infty$, which implies the second estimation in (2.10). If $M|\varepsilon|\|\mu_F\|_\infty \geq 1$, then the first estimation in (2.10) is obvious. Otherwise, we have $M|\varepsilon|\|\mu_F\|_\infty < 1$. Applying [10, Lemma 2.4] once more we see that $\mathrm{L}^-(\mathrm{I}[\phi_\varepsilon]) \geq 1 - M\,\mathrm{D}^+(\phi_\varepsilon) = 1 - M|\varepsilon|\|\mu_F\|_\infty$, and so the first estimation in (2.10) holds in both cases.

Suppose that $M|\varepsilon|\|\mu_F\|_\infty < 1$. Then from (2.10) we derive the inequalities (2.11). Hence $F_\varepsilon \circ H^{-1}$ is a bi-Lipschitz mapping. Since $F_\varepsilon = (F_\varepsilon \circ H^{-1}) \circ H$ and $\|\mu_{F_\varepsilon}\|_\infty = |\varepsilon|\|\mu_F\|_\infty < 1$, it follows that F_ε is a quasiconformal mapping. Moreover, by Lemma 2.1, $F_\varepsilon(\mathbb{D})$ is a rectifiably M_ε-arcwise connected domain with M_ε given by the expression (2.12), which yields the implication (iii).

It remains to prove the properties (i) and (ii). If $\|\mu_F\|_\infty = 0$, then by (2.13), G'/H' is a constant function, which contradicts the assumptions of the implications (i) and (ii). Therefore $\|\mu_F\|_\infty > 0$, i.e., F is not a conformal mapping. Suppose that $|\mu_F(\zeta)| = \|\mu_F\|_\infty$ for a certain $\zeta \in \mathbb{D}$. From (2.13) it follows that

$$\left|\frac{G'(\zeta)}{H'(\zeta)}\right| = |\mu_F(\zeta)| = \|\mu_F\|_\infty = \sup_{z\in\mathbb{D}}\left|\frac{G'(z)}{H'(z)}\right|. \tag{2.14}$$

Then by the maximum principle for holomorphic functions we see that G'/H' is a constant function, which also contradicts the assumptions of the implications (i) and (ii). Thus

$$|\mu_F(z)| < \|\mu_F\|_\infty \neq 0, \quad z \in \mathbb{D}. \tag{2.15}$$

Suppose that $|\varepsilon|\|\mu_F\|_\infty \leq 1$. If $\varepsilon = 0$, then $\mathrm{J}[F_\varepsilon](z) = |H'(z)| > 0$ for $z \in \mathbb{D}$. In the opposite case we conclude from (2.13) and (2.15) that for every $z \in \mathbb{D}$,

$$\begin{aligned}\mathrm{J}[F_\varepsilon](z) &= |H'(z)|^2 - |\overline{\varepsilon}G'(z)|^2 = |H'(z)|^2(1 - |\overline{\varepsilon}|^2|\mu_F(z)|^2)\\ &> |H'(z)|^2(1 - |\overline{\varepsilon}|^2\|\mu_F\|_\infty^2) \geq 0.\end{aligned}$$

Therefore for every $\varepsilon \in \mathbb{C}$ satisfying $|\varepsilon|\|\mu_F\|_\infty \leq 1$, F_ε is a a sense-preserving harmonic mapping, as stated in the conclusion of implication (i). If $M|\varepsilon|\|\mu_F\|_\infty \leq 1$, then applying [10, Corollary 3.2] to F_ε we see that F_ε is an injective mapping, as stated in the conclusion of implication (ii). ☐

Remark 2.4. Under the hypotheses of Theorem 2.3 the condition "G'/H' is not a constant function" in the implications (i) and (ii) is equivalent to the following one:

$$|G(\zeta) - G(0)| \neq \|\mu_F\|_\infty|H(\zeta) - H(0)| \quad \text{for a certain } \zeta \in \mathbb{D}\setminus\{0\}. \tag{2.16}$$

Indeed, suppose that G'/H' is a constant function. Then there exists $c \in \mathbb{C}$ such that

$$G'(z) = cH'(z), \quad z \in \mathbb{D}. \tag{2.17}$$

which implies $\|\mu_F\|_\infty = |c|$ and $G(z) - G(0) = c(H(z) - H(0))$ for $z \in \mathbb{D}$. Hence for every $z \in \mathbb{D}$, $|G(z) - G(0)| = \|\mu_F\|_\infty|H(z) - H(0)|$. Therefore, by the contradiction law, the condition (2.16) implies that G'/H' is not a constant function.

Now, suppose that the condition (2.16) does not hold, i.e., $|G(z) - G(0)| = \|\mu_F\|_\infty|H(z) - H(0)|$ for $z \in \mathbb{D}$. By the maximum principle applied to the holomorphic function

$$\mathbb{D}\setminus\{0\} \ni z \mapsto \frac{G(z) - G(0)}{H(z) - H(0)}$$

there exists $\theta \in \mathbb{R}$ such that $G(z) - G(0) = \mathrm{e}^{\mathrm{i}\theta}\|\mu_F\|_\infty(H(z) - H(0))$ for $z \in \mathbb{D} \setminus \{0\}$. Hence the condition (2.17) holds with $c := \mathrm{e}^{\mathrm{i}\theta}\|\mu_F\|_\infty$, and so G'/H' is a constant function. Therefore, by the contradiction law, the condition "G'/H' is not a constant function" implies the condition (2.16).

Remark 2.5. Suppose that $F = H + \overline{G}$ is a sense-preserving harmonic mapping in $\mathbb{D}$ such that its holomorphic part H is a convex holomorphic mapping, i.e., H is injective in $\mathbb{D}$ and $H(\mathbb{D})$ is a convex domain. Then $H(\mathbb{D})$ is a rectifiably 1-arcwise connected domain, and thereby Theorem 2.3 is applicable in this case with $M := 1$. Furthermore, in the item (iii), F_ε is a co-Lipschitz mapping. To show this we can apply Theorem 2.2 for $F := H$ and $\varepsilon := 0$. Then H is a co-Lipschitz mapping. Since $F_\varepsilon \circ H^{-1}$ is a bi-Lipschitz mapping and $F_\varepsilon = (F_\varepsilon \circ H^{-1}) \circ H$, it follows that F_ε is a co-Lipschitz mapping.

3. Close-to-convex harmonic mappings

Let us recall that a harmonic mapping $F\colon \mathbb{D} \to \mathbb{C}$ is said to be *close-to-convex* provided F is injective and $F(\mathbb{D})$ is a close-to-convex domain, i.e., its complement $\mathbb{C} \setminus F(\mathbb{D})$ is a union of non-crossing half-lines; cf. [1, p. 13, 5.1], [4].

Theorem 3.1. *Let $F = H + \overline{G}$ be a sense-preserving harmonic mapping in $\mathbb{D}$ such that $H + \varepsilon_0 G$ is convex for a certain $\varepsilon_0 \in \mathbb{C}$ satisfying $|\varepsilon_0|\|\mu_F\|_\infty \le 1$. Then for every $\varepsilon \in \mathbb{C}$ the following implications hold:*

(i) *If $|\varepsilon|\|\mu_F\|_\infty \le 1$ and G'/H' is not a constant function, then F_ε is a sense-preserving and close-to-convex harmonic mapping in $\mathbb{D}$.*

(ii) *If $|\varepsilon|\|\mu_F\|_\infty < 1$, then F_ε is a quasiconformal and close-to-convex harmonic mapping in $\mathbb{D}$.*

Moreover, F is a close-to-convex harmonic mapping in $\mathbb{D}$.

Proof. Given F satisfying the assumptions, suppose first that G'/H' is not a constant function. As in the proof of Theorem 2.3 we show that the condition (2.15) holds. Let $a, b \in \mathbb{C}$ be arbitrarily fixed such that

$$|a\nu_F(z)| < 1 \quad \text{and} \quad |b\nu_F(z)| < 1, \quad z \in \mathbb{D}, \tag{3.1}$$

where $\nu_F(z) := G'(z)/H'(z)$ for $z \in \mathbb{D}$. Assume that $|a| \le |b|$ and $a \ne b$. Then $b \ne 0$ and $\operatorname{Re}(1 - a/b) > 0$, and so $1 - a/b = r\mathrm{e}^{\mathrm{i}\theta}$ for some $r > 0$ and $\theta \in (-\pi/2; \pi/2)$. For each $z \in \mathbb{D}$, $\operatorname{Re}(1/(1 + b\nu_F(z))) > 1/2$, which gives

$$\begin{aligned}
\operatorname{Re}\left(\mathrm{e}^{-\mathrm{i}\theta}\frac{1 + a\nu_F(z)}{1 + b\nu_F(z)}\right) &= \operatorname{Re}\left(\mathrm{e}^{-\mathrm{i}\theta}\frac{1 - a/b}{1 + b\nu_F(z)} + \mathrm{e}^{-\mathrm{i}\theta}\frac{a}{b}\right)\\
&= \operatorname{Re}\left(\frac{r}{1 + b\nu_F(z)}\right) + \operatorname{Re}\left(\mathrm{e}^{-\mathrm{i}\theta}\frac{a}{b}\right)\\
&> \frac{r}{2} + \operatorname{Re}\left(\mathrm{e}^{-\mathrm{i}\theta} - r\right) = \cos(\theta) - \frac{r}{2}.
\end{aligned}$$

On the other hand

$$r^2 = |1 - a/b|^2 = 1 - 2\operatorname{Re}(a/b) + (|a|/|b|)^2 \le 2r\cos(\theta).$$

Therefore

$$\operatorname{Re}\left(\mathrm{e}^{-\mathrm{i}\theta}\frac{1 + a\nu_F(z)}{1 + b\nu_F(z)}\right) > 0, \quad z \in \mathbb{D}. \tag{3.2}$$

If $|a| \geq |b|$ and $a \neq b$, then the inequality (3.2) holds with a and b replaced respectively by b and a, where $\theta \in (-\pi/2; \pi/2)$ is chosen such that $1 - b/a = r\mathrm{e}^{\mathrm{i}\theta}$ for a certain $r > 0$. Therefore,

$$\operatorname{Re}\left(\mathrm{e}^{\mathrm{i}\theta}\frac{1+a\nu_F(z)}{1+b\nu_F(z)}\right) = \left|\frac{1+a\nu_F(z)}{1+b\nu_F(z)}\right|^2 \operatorname{Re}\left(\mathrm{e}^{-\mathrm{i}\theta}\frac{1+b\nu_F(z)}{1+a\nu_F(z)}\right) > 0.$$

If $a = b$, then the inequality (3.2) evidently holds with $\theta := 0$. Thus the inequality (3.2) holds for all $a, b \in \mathbb{C}$ satisfying the condition (3.1) and suitably chosen $\theta \in (-\pi/2; \pi/2)$ in dependence of a and b.

Let $\varepsilon, \eta \in \mathbb{C}$ be arbitrarily fixed such that $|\varepsilon|\|\mu_F\|_\infty \leq 1$ and $|\eta| \leq 1$. Setting $a := \eta\overline{\varepsilon}$ and $b := \varepsilon_0$ we see by (2.15) that the condition (3.1) holds, and thereby there exists $\alpha \in \mathbb{R}$ such that

$$\operatorname{Re}\left(\mathrm{e}^{\mathrm{i}\alpha}\frac{H'(z)+\eta\overline{\varepsilon}G'(z)}{H'(z)+\varepsilon_0 G'(z)}\right) > 0, \quad z \in \mathbb{D}. \tag{3.3}$$

Since $\mathrm{e}^{-\mathrm{i}\alpha}(H + \varepsilon_0 G)$ is a convex holomorphic mapping, we conclude from (3.3) and the classical Kaplan univalence criterion [5], [11, pp. 51–52] that $H + \eta\overline{\varepsilon}G$ is a close-to-convex holomorphic mapping for every $\eta \in \operatorname{cl}(\mathbb{D})$. Since $F_\varepsilon = H + \varepsilon\overline{G} = H + \overline{\overline{\varepsilon}G}$ and $|\overline{\varepsilon}G'(0)| < |H'(0)|$, we deduce from [1, Lemma 5.15] that F_ε is a sense-preserving and close-to-convex harmonic mapping. Moreover, if $|\varepsilon|\|\mu_F\|_\infty < 1$, then $\|\mu_{F_\varepsilon}\|_\infty = |\varepsilon|\|\mu_F\|_\infty < 1$, and so F_ε is a quasiconformal mapping.

It remains to consider the case where G'/H' is a constant function. Since the mapping F is sense-preserving, there exists $c \in \mathbb{D}$ such that $G'(z) = cH'(z)$ for $z \in \mathbb{D}$. Then

$$G(z) - G(0) = c(H(z) - H(0)), \quad z \in \mathbb{D},$$

which yields

$$(H + \varepsilon_0 G)(z) = (1 + c\varepsilon_0)H(z) + \varepsilon_0(G(0) - cH(0)), \quad z \in \mathbb{D}. \tag{3.4}$$

Since $H + \varepsilon_0 G$ is a convex holomorphic function, we have $0 \neq (H + \varepsilon_0 G)'(0) = (1 + c\varepsilon_0)H'(0)$. Hence and by (3.4) we see that H is a convex holomorphic function. Thus for each $\varepsilon \in \mathbb{C}$,

$$\begin{aligned} F_\varepsilon(z) &= H(z) + \varepsilon\overline{G(z)} = H(z) + \varepsilon\overline{c}\,\overline{H(z)} + \varepsilon\overline{(G(0) - cH(0))} \\ &= L \circ H(z) + \varepsilon\overline{(G(0) - cH(0))}, \quad z \in \mathbb{D}, \end{aligned}$$

where $\mathbb{C} \ni z \mapsto L(z) := z + \varepsilon\overline{c}\,\overline{z}$ is an affine mapping. Suppose that $|\varepsilon|\|\mu_F\|_\infty < 1$. Then $|\varepsilon\overline{c}| < 1$, and so $L(H(\mathbb{D}))$ is a convex domain and $L \circ H$ is a quasiconformal mapping. Hence F_ε is a quasiconformal mapping and $F_\varepsilon(\mathbb{D})$ is a convex domain, and thereby close-to-convex. Therefore, F_ε is a quasiconformal and close-to-convex harmonic mapping, which completes the proof of the properties (i) and (ii).

Moreover, if $\|\mu_F\|_\infty < 1$, then using the property (ii) we see that $F = F_1$ is a close-to-convex harmonic mapping. If $\|\mu_F\|_\infty = 1$, then the last claim follows from [1, Theorem 5.17]. □

Corollary 3.2. *Let $F = H + \overline{G}$ be a sense-preserving harmonic mapping in $\mathbb{D}$ such that H is convex. Then for every $\varepsilon \in \mathbb{C}$ the implications* (i) *and* (ii) *of* Theorem 3.1 *hold.*

Proof. Fix F satisfying the assumptions. Setting $\varepsilon_0 := 0$ we see that $|\varepsilon_0|\|\mu_F\|_\infty = 0 \leq 1$ and $H + \varepsilon_0 G$ is convex. Thus the corollary follows from Theorem 3.1. □

Remark 3.3. Let $F = H + \overline{G}$ be a sense-preserving harmonic mapping in $\mathbb{D}$ such that H is convex and G'/H' is a constant function. Then $\|\mu_F\|_\infty = |c|$, where $c := G'(0)/H'(0)$. Analysing the last part of the proof of Theorem 3.1 we see that for every $\varepsilon \in \mathbb{C}$ satisfying $|\varepsilon|\|\mu_F\|_\infty < 1$, F_ε is a quasiconformal harmonic mapping and $F_\varepsilon(\mathbb{D})$ is a convex domain.

Corollary 3.4. *Let V be a conformal mapping in $\mathbb{D}$ and U be a non-constant holomorphic function in $\mathbb{D}$ such that $V(\mathbb{D})$ is a convex domain in $\mathbb{C}$ and*

$$\sup_{z\in\mathbb{D}} \operatorname{Re} U(z) \leq \frac{1}{2}. \tag{3.5}$$

Then the function $F := H + \overline{G}$ defined by the formulas

$$\mathbb{D} \ni z \mapsto G(z) := \int_0^z U(\zeta)V'(\zeta)\mathrm{d}\zeta \quad \textit{and} \quad \mathbb{D} \ni z \mapsto H(z) := V(z) - G(z) \tag{3.6}$$

is sense-preserving in $\mathbb{D}$ and satisfies $0 < \|\mu_F\|_\infty \leq 1$. Furthermore, for every $\varepsilon \in \mathbb{C}$ the implications (i) *and* (ii) *of* Theorem 3.1 *hold.*

Proof. Given V and U satisfying the assumption of the corollary we deduce from the formulas (3.6) that $G'(z) = U(z)V'(z)$ and $H'(z) = (1 - U(z))V'(z)$ for $z \in \mathbb{D}$. Since

$$\frac{G'(z)}{H'(z)} = \frac{U(z)}{1 - U(z)}, \quad z \in \mathbb{D},$$

we see that G'/H' is not a constant function. By (3.5) we have

$$\left|\frac{G'(z)}{H'(z)}\right|^2 = \frac{|U(z)|^2}{|1 - U(z)|^2} = \frac{|U(z)|^2}{1 - 2\operatorname{Re} U(z) + |U(z)|^2} \leq 1, \quad z \in \mathbb{D}.$$

Then from the maximum principle for holomorphic functions we see that

$$0 \leq \left|\frac{G'(z)}{H'(z)}\right| < \|\mu_F\|_\infty = \sup_{z\in\mathbb{D}} \left|\frac{G'(z)}{H'(z)}\right| \leq 1, \quad z \in \mathbb{D},$$

and thus F is a sense-preserving harmonic mapping in $\mathbb{D}$. Since $H + G = V$, $H + G$ is a convex function. Theorem 3.1 now implies that for every $\varepsilon \in \mathbb{C}$, if $|\varepsilon| \leq 1/\|\mu_F\|_\infty$, then $F_\varepsilon := H + \varepsilon\overline{G}$ is a sense-preserving and close-to-convex harmonic mapping. Moreover, if $|\varepsilon| < 1/\|\mu_F\|_\infty$, then F_ε is a quasiconformal mapping. ☐

Notice that Corollary 3.4 can be applied for constructing injective harmonic mappings, because the conditions on holomorphic functions U and V are not much restrictive. The following two examples illustrate this approach.

Example 3.5. Let V be a convex holomorphic mapping in $\mathbb{D}$ and Φ be a non-constant holomorphic function in $V(\mathbb{D})$ such that $\Phi(V(0)) = 0$ and

$$\sup_{\zeta\in V(\mathbb{D})} \operatorname{Re} \Phi'(\zeta) \leq \frac{1}{2}.$$

Then the function $U := \Phi' \circ V$ satisfies the condition (3.5). By the formulas (3.6) we have

$$F_\varepsilon(z) = V(z) - \Phi(V(z)) + \varepsilon\overline{\Phi(V(z))}, \quad z \in \mathbb{D}. \tag{3.7}$$

Corollary 3.4 now implies that for every $\varepsilon \in \mathbb{C}$, if $|\varepsilon| \leq 1$, then F_ε is a sense-preserving and close-to-convex harmonic mapping. Moreover, if $|\varepsilon| < 1$, then F_ε is a quasiconformal mapping. In particular, this is true for the functions

$$\mathbb{D} \ni z \mapsto V(z) := \frac{z}{z+1} \quad \text{and} \quad \mathbb{C} \ni \zeta \mapsto \Phi(\zeta) := \frac{\zeta^2}{2}.$$

Then (3.7) takes the following form

$$F_\varepsilon(z) = \frac{z}{z+1} - \frac{z^2}{2(z+1)^2} + \frac{\varepsilon}{2}\frac{\bar{z}^2}{(\bar{z}+1)^2}, \quad z \in \mathbb{D}.$$

Example 3.6. Setting

$$\mathbb{D} \ni z \mapsto V(z) := \log\frac{1+z}{1-z} \quad \text{and} \quad \mathbb{D} \ni z \mapsto U(z) := \frac{z}{z+1}$$

we see that V is a convex holomorphic mapping in $\mathbb{D}$ and U is a non-constant holomorphic function satisfying the inequality (3.5). By the formulas (3.6) we have

$$F_\varepsilon(z) = \frac{1}{2}\log\frac{1+z}{1-z} + \frac{z}{1+z} + \varepsilon\Big[\frac{1}{2}\log\frac{1+\bar{z}}{1-\bar{z}} - \frac{\bar{z}}{1+\bar{z}}\Big], \quad z \in \mathbb{D}.$$

Corollary 3.4 now implies that for every $\varepsilon \in \mathbb{C}$, if $|\varepsilon| \le 1$, then F_ε is a sense-preserving and close-to-convex harmonic mapping. Moreover, if $|\varepsilon| < 1$, then F_ε is a quasiconformal mapping.

4. Injectivity of harmonic mappings in the closed unit disk

Given holomorphic functions H and G in $\mathbb{D}$ suppose that they have the continuous extensions H^* and G^* to $\mathrm{cl}(\mathbb{D})$, respectively. Then for every $\varepsilon \in \mathbb{C}$ the function F_ε has the continuous extension F^*_ε to $\mathrm{cl}(\mathbb{D})$ defined by the formula

$$\mathrm{cl}(\mathbb{D}) \ni z \mapsto F^*_\varepsilon(z) := H^*(z) + \varepsilon\overline{G^*}(z), \quad z \in \mathrm{cl}(\mathbb{D}). \tag{4.1}$$

In what follows we will study the injectivity of the function F^*_ε in $\mathrm{cl}(\mathbb{D})$. We start from the following simple observation.

Lemma 4.1. *Let $F = H + \overline{G}$ be a sense-preserving harmonic mapping in $\mathbb{D}$, such that H is injective in $\mathbb{D}$, $H(\mathbb{D})$ is a bounded Jordan domain and the function $G \circ H^{-1}$ is uniformly continuous in $H(\mathbb{D})$. Then both the functions H and G have the continuous extensions H^* and G^* to $\mathrm{cl}(\mathbb{D})$, respectively. In particular, for every $\varepsilon \in \mathbb{C}$ the function F_ε has the continuous extension F^*_ε to $\mathrm{cl}(\mathbb{D})$.*

Proof. Under the assumptions of the lemma $H(\mathbb{D})$ is a bounded Jordan domain. From the Taylor–Osgood–Carethéodory theorem it follows that the conformal mapping H of $\mathbb{D}$ onto $H(\mathbb{D})$ has the continuous extension H^* to $\mathrm{cl}(\mathbb{D})$, which is a homeomorphism from $\mathrm{cl}(\mathbb{D})$ onto $\mathrm{cl}(H(\mathbb{D}))$; cf. [11, Theorem 9.10]. In particular, H is uniformly continuous in $\mathbb{D}$. Then the function G is uniformly continuous in $\mathbb{D}$ as a composition of uniformly continuous functions $G \circ H^{-1}$ and H. Therefore, G has the continuous extension G^* to $\mathrm{cl}(\mathbb{D})$. Consequently, the function F_ε has the continuous extension F^*_ε for every $\varepsilon \in \mathbb{C}$. □

Using Lemma 4.1 we derive from Theorem 2.2 the following result.

Corollary 4.2. *Let $F = H + \overline{G}$ be a quasiconformal harmonic mapping in $\mathbb{D}$ such that $F(\mathbb{D})$ is a bounded convex domain and write $l := \lambda(\|\mu_F\|_\infty)$. Then both the functions H and G have the continuous extensions H^* and G^* to $\mathrm{cl}(\mathbb{D})$, respectively.*

Moreover, for every $\varepsilon \in \mathbb{C}$,

$$(4.2)\qquad \begin{aligned}(1-|\varepsilon|l)|H^*(z)-H^*(w)| &\le |F_\varepsilon^*(z)-F_\varepsilon^*(w)|\\ &\le (1+|\varepsilon|l)|H^*(z)-H^*(w)|, \quad z,w\in \mathrm{cl}(\mathbb{D}).\end{aligned}$$

In particular, F_ε^* *is injective in* $\mathrm{cl}(\mathbb{D})$ *provided* $|\varepsilon|l<1$.

Proof. Given a quasiconformal harmonic mapping F in $\mathbb{D}$ assume that $F(\mathbb{D})$ is a bounded convex domain. Then $F(\mathbb{D})$ is a bounded Jordan domain, and thereby F has the homeomorphic extension F^* to $\mathrm{cl}(\mathbb{D})$; cf. [6, Theorem 8.2 in Chap. I §8]. Since $F(\mathbb{D})$ is a convex domain, H is injective, which was shown in [1, Corollary 5.8]; cf. also [9, Remark 2.3 and Lemma 2.4]. Moreover, by [9, Theorem 3.8], $H\circ F^{-1}$ is a bi-Lipschitz mapping, and in consequence, it has the homeomorphic extension to $\mathrm{cl}(F(\mathbb{D}))$. Therefore $H(\mathbb{D})$ is a bounded Jordan domain. By [1, Corollary 5.8],

$$|G\circ H^{-1}(z)-G\circ H^{-1}(w)|\le |z-w|, \quad w,z\in H(\mathbb{D}),$$

and so the function $G\circ H^{-1}$ is uniformly continuous in $H(\mathbb{D})$. Using now Lemma 4.1 we see that both the functions H and G have the continuous extensions H^* and G^* to $\mathrm{cl}(\mathbb{D})$, respectively, as well as for every $\varepsilon\in\mathbb{C}$ the function F_ε has the continuous extension F_ε^*. Then the inequalities (4.2) follow from (2.5) and (2.6). If additionally $|\varepsilon|l<1$, then by (4.2), F_ε^* is injective in $\mathrm{cl}(\mathbb{D})$, which completes the proof. □

Let us recall that a boundary point b of a simply connected domain $\Omega\subset\mathbb{C}$ is said to be a *simple boundary point of* Ω provided for each sequence $\mathbb{N}\ni n\mapsto a_n\in\Omega$ convergent to b there exist a continuous function $\gamma\colon [0;1]\to\mathbb{C}$ and an increasing sequence $\mathbb{N}\ni n\mapsto t_n\in[0;1]$ satisfying the following properties: $\gamma([0;1))\subset\Omega$, $\gamma(1)=b$, $t_n\to 1$ as $n\to+\infty$ and $\gamma(t_n)=a_n$ for $n\in\mathbb{N}$; cf. [12, Definition 14.16].

Lemma 4.3. *Let* Ω *be a bounded simply connected domain in* $\mathbb{C}$ *which is a rectifiably* M*-arcwise connected domain for a certain* $M\ge 1$. *Then* Ω *is a bounded Jordan domain.*

Proof. Given Ω satisfying the assumptions fix a boundary point b of Ω and a sequence $\mathbb{N}\ni n\mapsto a_n\in\Omega$ such that $a_n\to b$ as $n\to+\infty$. Write $t_n:=1-1/n$ for $n\in\mathbb{N}$. Since Ω is a bounded rectifiably M-arcwise connected domain, for each $n\in\mathbb{N}$ there exists an arc $\gamma_n\colon [t_n;t_{n+1}]\to\Omega$ joining the points a_n and a_{n+1}, i.e., $\gamma_n(t_n)=a_n$ and $\gamma_n(t_{n+1})=a_{n+1}$, with the length $|\gamma_n|_1\le M|a_{n+1}-a_n|$. Then there exists the unique function $\gamma\colon[0;1]\to\mathbb{C}$ such that $\gamma(1)=b$ and for every $n\in\mathbb{N}$,

$$\gamma(t)=\gamma_n(t), \quad t\in[t_n;t_{n+1}].$$

Hence $\gamma(t)\to b=\gamma(1)$ as $n\to+\infty$, because $a_n\to b$ as $n\to+\infty$. Therefore γ is a continuous function. Moreover, $\gamma([0;1))\subset\Omega$ and $\gamma(t_n)=a_n$ for $n\in\mathbb{N}$. Thus each boundary point b of Ω is a simple boundary point of Ω. Since Ω a bounded simply connected domain, it follows from the Riemman mapping theorem that there exists a conformal mapping H of $\mathbb{D}$ onto Ω; cf. [12, Theorem 14.8]. Then by [12, Theorem 14.19], H extends to the homeomorphism H^* of $\mathrm{cl}(\mathbb{D})$ onto $\mathrm{cl}(\Omega)$. In particular, Ω is a bounded Jordan domain, which is the desired conclusion. □

Using Lemmas 4.1 and 4.3 we derive from Theorem 2.3 the following result.

Corollary 4.4. *Given* $M\ge 1$ *suppose that* $F=H+\overline{G}$ *is a sense-preserving harmonic mapping in* $\mathbb{D}$, *its holomorphic part* H *is injective in* $\mathbb{D}$ *and* $H(\mathbb{D})$ *is a bounded rectifiably* M*-arcwise connected domain. Then both the functions* H *and* G *have the continuous extensions* H^* *and* G^* *to* $\mathrm{cl}(\mathbb{D})$, *respectively. Moreover, for* *every* $\varepsilon \in \mathbb{C}$,

$$(4.3)\qquad \begin{aligned}(1 - M|\varepsilon|\|\mu_F\|_\infty)|H^*(z_2) - H^*(z_1)| &\le |F^*_\varepsilon(z_2) - F^*_\varepsilon(z_1)| \\ &\le (1 + M|\varepsilon|\|\mu_F\|_\infty)|H^*(z_2) - H^*(z_1)|, \quad z_1, z_2 \in \mathrm{cl}(\mathbb{D}).\end{aligned}$$

In particular, F^*_ε *is injective in* $\mathrm{cl}(\mathbb{D})$ *provided* $M|\varepsilon|\|\mu_F\|_\infty < 1$.

Proof. Fix M, H and G satisfying the assumptions. By Lemma 4.3, $H(\mathbb{D})$ is a bounded Jordan domain. From the conclusion (2.9) of Theorem 2.3 it follows that

$$|G \circ H^{-1}(z) - G \circ H^{-1}(w)| \le M\|\mu_F\|_\infty |z - w|, \quad w, z \in H(\mathbb{D}),$$

and so the function $G \circ H^{-1}$ is uniformly continuous in $H(\mathbb{D})$. Using now Lemma 4.1 we see that both the functions H and G have the continuous extensions H^* and G^* to $\mathrm{cl}(\mathbb{D})$, respectively, as well as for every $\varepsilon \in \mathbb{C}$ the function F_ε has the continuous extension F^*_ε. Then the inequalities (4.3) follow from the ones (2.10) in Theorem 2.3. If additionally $M|\varepsilon|\|\mu_F\|_\infty < 1$, then by (4.3), F^*_ε is injective in $\mathrm{cl}(\mathbb{D})$, which completes the proof. □

Let $F = H + \overline{G}$ be a sense-preserving harmonic mapping in $\mathbb{D}$, such that its holomorphic part H is convex, i.e., H is injective in $\mathbb{D}$ and $H(\mathbb{D})$ is a convex domain. Suppose that G'/H' is not a constant function. Then for every $\varepsilon \in \mathbb{C}$ satisfying $|\varepsilon|\|\mu_F\|_\infty \le 1$ we can apply [10, Corollary 3.2] to the mapping F_ε. As a result we see that

$$(4.4)\qquad z \ne w \Rightarrow |H(z) - H(w)| > |\varepsilon||G(z) - G(w)|, \quad z, w \in \mathbb{D},$$

and, in consequence, F_ε is an injective mapping. Our aim is to extend the property (4.4) to the closed unit disk $\mathrm{cl}(\mathbb{D})$.

Lemma 4.5. *Let* $F = H + \overline{G}$ *be a sense-preserving harmonic mapping in* $\mathbb{D}$, *such that* H *is injective in* $\mathbb{D}$, $H(\mathbb{D})$ *is a bounded convex domain and* G'/H' *is not a constant function. Then both the functions* H *and* G *have the continuous extensions* H^* *and* G^* *to* $\mathrm{cl}(\mathbb{D})$, *respectively, which satisfy*

$$(4.5)\qquad z \ne w \Rightarrow |H^*(z) - H^*(w)| > |\varepsilon||G^*(z) - G^*(w)|, \quad z \in \mathbb{D},\ w \in \mathrm{cl}(\mathbb{D}),$$

for every $\varepsilon \in \mathbb{C}$ *such that* $|\varepsilon|\|\mu_F\|_\infty \le 1$.

Proof. Fix a harmonic mapping F satisfying the assumptions. Since $H(\mathbb{D})$ is a bounded convex domain, $H(\mathbb{D})$ is a Jordan domain; cf. [2]. From the Taylor–Osgood–Carethéodory theorem it follows that the conformal mapping H of $\mathbb{D}$ onto $H(\mathbb{D})$ has the continuous extension H^* to $\mathrm{cl}(\mathbb{D})$, which is a homeomorphism from $\mathrm{cl}(\mathbb{D})$ onto $\mathrm{cl}(H(\mathbb{D}))$; cf. [11, Theorem 9.10]. In particular, H is uniformly continuous in $\mathbb{D}$. Using the property (4.4) with $\varepsilon := 1$ we have

$$(4.6)\qquad |G(w) - G(z)| \le |H(w) - H(z)|, \quad w, z \in \mathbb{D}.$$

Hence G is uniformly continuous in $\mathbb{D}$, and therefore both the functions H and G can be extended to the continuous functions $H^*\colon \mathrm{cl}(\mathbb{D}) \to \mathbb{C}$ and $G^*\colon \mathrm{cl}(\mathbb{D}) \to \mathbb{C}$, respectively. Moreover, by (4.6) we see that

$$|G^*(w) - G^*(z)| \le |H^*(w) - H^*(z)|, \quad w, z \in \mathrm{cl}(\mathbb{D}).$$

Since the mapping H^* is injective, the function

$$\mathbb{D} \ni z \mapsto \omega_w(z) := \frac{\varepsilon(G(z) - G^*(w))}{H(z) - H^*(w)}$$

is well defined for all $w \in \mathbb{T}$ and $\varepsilon \in \mathbb{C}$. The property (4.5) evidently holds if $\varepsilon = 0$. It remains to consider the case where $\varepsilon \in \mathbb{C} \setminus \{0\}$ satisfies $|\varepsilon| \|\mu_F\|_\infty \le 1$. By (4.4) and the continuity of the functions H^* and G^* we see that for a given $w \in \mathbb{T}$,

$$(4.7) \qquad |\omega_w(z)| \le 1, \quad z \in \mathbb{D}.$$

Suppose that there exists $\zeta \in \mathbb{D}$ such that $|\omega_w(\zeta)| = 1$. From the maximum principle for holomorphic functions we conclude that ω_w is a constant function. Then there exists $c \in \mathbb{T}$ such that

$$\varepsilon(G(z) - G^*(w)) = c(H(z) - H^*(w)), \quad z \in \mathbb{D},$$

and consequently $G'(z)/H'(z) = c/\varepsilon$ for $z \in \mathbb{D}$. Thus G'/H' is a constant function, which contradicts the assumption. Therefore, there does not exist $z \in \mathbb{D}$ such that $|\omega_w(z)| = 1$. Combining this with (4.7) we get $|\omega_w(z)| < 1$ for $z \in \mathbb{D}$, which yields the inequality in (4.5) in the case where $z \in \mathbb{D}$ and $w \in \mathbb{T}$. If both $z, w \in \mathbb{D}$, then the implication in (4.5) follows from the property (4.4), which completes the proof. $\square$

Theorem 4.6. *Let $F = H + \overline{G}$ be a sense-preserving harmonic mapping in $\mathbb{D}$, such that H is injective in $\mathbb{D}$, $H(\mathbb{D})$ is a bounded convex domain and G'/H' is not a constant function. Then both the functions H and G have the continuous extensions H^* and G^* to* $\mathrm{cl}(\mathbb{D})$*, respectively, which satisfy*

$$(4.8) \qquad z \ne w \Rightarrow |H^*(z) - H^*(w)| > |\varepsilon||G^*(z) - G^*(w)|, \quad z, w \in \mathrm{cl}(\mathbb{D}),$$

for every $\varepsilon \in \mathbb{C}$ such that $|\varepsilon| \|\mu_F\|_\infty \le 1$. In particular, for each such ε the mapping $F^*_\varepsilon := H^* + \varepsilon\overline{G^*}$ is injective in $\mathrm{cl}(\mathbb{D})$.

Proof. Fix a harmonic mapping F satisfying the assumptions. From Lemma 4.5 it follows that both the functions H and G have continuous extensions H^* and G^*, respectively, which satisfy the property (4.5). Hence

$$(4.9) \qquad |H^*(u) - H^*(v)| \ge |\varepsilon||G^*(u) - G^*(v)|, \quad u, v \in \mathbb{T}.$$

As it was observed in the proof of Lemma 4.5, H^* is a homeomorphism of $\mathrm{cl}(\mathbb{D})$ onto $\mathrm{cl}(H(\mathbb{D}))$. Thus if $\varepsilon = 0$, then the theorem is evidently true. Thus we can confine ourselves to the case where $0 < |\varepsilon| \|\mu_F\|_\infty \le 1$.

Given $z, w \in \mathbb{T}$ assume that $z \ne w$. Since $\mathrm{cl}(H(\mathbb{D}))$ is a convex set, there exists a line segment $L \subset \mathrm{cl}(H(\mathbb{D}))$ which connects the points $H^*(z)$ and $H^*(w)$. Suppose first that $L \cap H(\mathbb{D}) \ne \emptyset$. Then $H(\zeta) \in L$ for a certain $\zeta \in \mathbb{D}$. Hence $z \ne \zeta \ne w$, and we deduce from the property (4.5) that

$$(4.10) \qquad \begin{aligned} |H^*(z) - H^*(w)| &= |H^*(z) - H^*(\zeta)| + |H^*(\zeta) - H^*(w)| \\ &> |\varepsilon||G^*(z) - G^*(\zeta)| + |\varepsilon||G^*(\zeta) - G^*(w)| \\ &\ge |\varepsilon||G^*(z) - G^*(w)|. \end{aligned}$$

Therefore the inequality in (4.8) holds provided $L \cap H(\mathbb{D}) \ne \emptyset$. Otherwise $L \cap H(\mathbb{D}) = \emptyset$. This means that the line segment L is included in the boundary of $H(\mathbb{D})$, i.e. $L \subset H^*(\mathbb{T})$. Since $H^*(z) \in L$ and $H^*(w) \in L$ there exists an arc $A \subset \mathbb{T}$ such that $z, w \in A$ and $H^*(A) = L$. Suppose that

$$(4.11) \qquad |H^*(\zeta) - H^*(w)| = |\varepsilon||G^*(\zeta) - G^*(w)|, \quad \zeta \in A.$$

Fix $\zeta \in A \setminus \{w\}$. Since $H^*(\zeta) \in L$ we deduce from (4.11) and (4.9) that

$$\begin{aligned}|G^*(z) - G^*(w)| &\leq |G^*(z) - G^*(\zeta)| + |G^*(\zeta) - G^*(w)| \\ &\leq \frac{1}{|\varepsilon|}\big(|H^*(z) - H^*(\zeta)| + |H^*(\zeta) - H^*(w)|\big) \\ &= \frac{1}{|\varepsilon|}|H^*(z) - H^*(w)| = |G^*(z) - G^*(w)|\end{aligned}$$

Hence

$$|G^*(z) - G^*(\zeta)| + |G^*(\zeta) - G^*(w)| = |G^*(z) - G^*(w)|,$$

and so the point $G^*(\zeta)$ belongs to the line segment joining the points $G^*(w)$ and $G^*(z)$. Therefore

$$G^*(\zeta) - G^*(w) = t'(G^*(z) - G^*(w)) \tag{4.12}$$

for a certain $t' \in (0;1]$. Also the point $H^*(\zeta)$ belongs to the line segment joining the points $H^*(w)$ and $H^*(z)$, which gives

$$H^*(\zeta) - H^*(w) = t''(H^*(z) - H^*(w)) \tag{4.13}$$

for a certain $t'' \in (0;1]$. Setting $\lambda := (G^*(z) - G^*(w))/(H^*(z) - H^*(w))$ we conclude from (4.12) and (4.13) that

$$G^*(\zeta) - G^*(w) = t'(G^*(z) - G^*(w)) = t'\lambda(H^*(z) - H^*(w)) = \lambda\frac{t'}{t''}(H^*(\zeta) - H^*(w)).$$

Hence and by (4.11),

$$\frac{1}{|\varepsilon|} = \frac{|G^*(\zeta) - G^*(w)|}{|H^*(\zeta) - H^*(w)|} = |\lambda|\frac{t'}{t''} = \frac{t'}{t''}\frac{|G^*(z) - G^*(w)|}{|H^*(z) - H^*(w)|} = \frac{1}{|\varepsilon|}\frac{t'}{t''},$$

and consequently $t' = t''$. Therefore

$$G^*(\zeta) - G^*(w) = \lambda(H^*(\zeta) - H^*(w)), \quad \zeta \in A. \tag{4.14}$$

From (4.11) it follows that the function

$$\mathbb{D} \cup (A \setminus \{w\}) \ni \zeta \mapsto \Psi(\zeta) := \frac{G^*(\zeta) - G^*(w)}{H^*(\zeta) - H^*(w)} \tag{4.15}$$

is holomorphic and bounded in $\mathbb{D}$, continuous at every point of $\mathbb{D} \cup (A \setminus \{w\})$ and $\Psi(\zeta) = \lambda$ for $\zeta \in A \setminus \{w\}$. Then by [12, Theorem 11.22], $\Psi(\zeta) = \lambda$ for $\zeta \in \mathbb{D}$. Thus G'/H' is a constant function in $\mathbb{D}$, which contradicts the assumption. Accordingly the condition (4.11) is not true. Then there exists $\zeta \in A \setminus \{w\}$ satisfying the following inequality

$$|H^*(\zeta) - H^*(w)| > |\varepsilon||G^*(\zeta) - G^*(w)|.$$

Combining this with (4.9) we obtain (4.10). Therefore the inequality in (4.8) holds provided $L \cap H(\mathbb{D}) = \emptyset$.

Taking into account both the cases we see that the implication in (4.8) holds provided both $z, w \in \mathbb{T}$. By Lemma 4.5 we know that the implication in (4.8) holds for all $z, w \in \mathrm{cl}(\mathbb{D})$ provided $z \in \mathbb{D}$ or $w \in \mathbb{D}$. This shows the property (4.8). Furthermore, from (4.8) it follows that for all $z, w \in \mathrm{cl}(\mathbb{D})$, if $z \neq w$, then

$$|F^*_\varepsilon(z) - F^*_\varepsilon(w)| \geq |H^*_\varepsilon(z) - H^*_\varepsilon(w)| - |\varepsilon||G^*_\varepsilon(z) - G^*_\varepsilon(w)| > 0.$$

This means that the mapping F^*_ε is injective in $\mathrm{cl}(\mathbb{D})$, which completes the proof. $\square$

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