On the Hardy number of comb domains

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Abstract. Let $H^p(\mathbb{D})$ be the Hardy space of all holomorphic functions on the unit disk $\mathbb{D}$ with exponent $p > 0$. If $D \neq \mathbb{C}$ is a simply connected domain and f is the Riemann mapping from $\mathbb{D}$ onto D, then the Hardy number of D, introduced by Hansen, is the supremum of all p for which $f \in H^p(\mathbb{D})$. Comb domains are a well-studied class of simply connected domains that, in general, have the form of the entire plane minus an infinite number of vertical rays. In this paper we study the Hardy number of a class of comb domains with the aid of the quasi-hyperbolic distance and we establish a necessary and sufficient condition for the Hardy number of these domains to be equal to infinity. Applying this condition, we derive several results that show how the mutual distances and the distribution of the rays affect the finiteness of the Hardy number. By a result of Burkholder our condition is also necessary and sufficient for all moments of the exit time of Brownian motion from comb domains to be infinite.

Kampa-alueiden Hardyn luku

Tiivistelmä. Olkoon $H^p(\mathbb{D})$ yksikkökiekolla $\mathbb{D}$ määriteltyjen holomorfisten funktioiden Hardyn avaruus, jonka eksponentti on $p > 0$. Jos $D \neq \mathbb{C}$ on yhdesti yhtenäinen alue, ja f on Riemannin kuvaus kiekolta $\mathbb{D}$ alueelle D, niin Hansenin määrittelemä alueen D Hardyn luku on niiden eksponenttien p pienin yläraja, joilla $f \in H^p(\mathbb{D})$. Kampa-alueet käsittävät yhdesti yhtenäisten alueiden paljon tutkitun alaluokan. Niiden yleinen muoto on taso, josta on poistettu ääretön määrä pystysuoria säteitä. Tässä työssä tutkimme eräiden kampa-alueiden Hardyn lukua kvasihyperbolisen etäisyyden avulla ja löydämme riittävän ja välttämättömän ehdon sille, että näiden alueiden Hardyn luku on ääretön. Tämän ehdon avulla johdamme useita tuloksia, jotka osoittavat, kuinka säteiden keskinäiset etäisyydet ja jakauma vaikuttavat Hardyn luvun äärellisyyteen. Burkholderin klassisen tuloksen perusteella ehtomme on myös riittävä ja välttämätön sille, että Brownin liikkeen kampa-alueesta poistumisajan kaikki momentit ovat äärettömiä.

1. Introduction

A classical problem in geometric function theory is to find geometric conditions for a holomorphic function on the unit disk to belong in Hardy spaces (see e.g. [1, 11, 12, 14, 17, 20, 19]). In this paper we study this problem in the case of conformal mappings from the unit disk onto a comb domain. The Hardy space with exponent $p > 0$ [8, p. 1–2] is denoted by $H^p(\mathbb{D})$ and is defined to be the set of all holomorphic functions, f, on the unit disk $\mathbb{D}$ that satisfy the condition

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < +\infty.$$

The fact that a function f belongs to some $H^p(\mathbb{D})$ imposes a restriction on its growth and this restriction is stronger as p increases. That is, if $p > q$ then $H^p(\mathbb{D}) \subset H^q(\mathbb{D})$.

In [11] Hansen studied the problem of determining the numbers p for which $f \in H^p(\mathbb{D})$ by studying $f(\mathbb{D})$. For this purpose he introduced a number which he called the Hardy number of a region. Since we study comb domains, we only state the definition in the case of simply connected domains. Let $D \neq \mathbb{C}$ be a simply connected domain and f be a Riemann mapping from $\mathbb{D}$ onto D. The Hardy number of D, or equivalently of f, is defined by

$$\mathrm{h}(D) = \sup\{p > 0 \colon f \in H^p(\mathbb{D})\}.$$

We note that this definition is independent of the choice of the Riemann mapping onto D. It is known that every conformal mapping on $\mathbb{D}$ belongs to $H^p(\mathbb{D})$ for all $p \in (0, 1/2)$ [8, p. 50]. This implies that $\mathrm{h}(D)$ lies in $[1/2, +\infty]$.

There is no general method for computing the Hardy number but there are some ways to estimate it for certain types of domains. In [11] Hansen gave a lower bound for the Hardy number of an arbitrary region and improved this bound for simply connected domains. Moreover, he determined the exact value of the Hardy number of starlike [11] and spiral-like regions [12]. In [21] Poggi-Corradini studied the Hardy number of Kœnigs mappings. He also proved [20] for a certain class of functions, which give a geometric model for the self-mappings of $\mathbb{D}$, that the Hardy number is equal to infinity if and only if the image region does not contain a twisted sector. Furthermore, in [9] and [17] Essén, and Kim and Sugawa, respectively, gave a description of the Hardy number of a plane domain in terms of harmonic measure. In [15] the current author gave a formula for the Hardy number of a simply connected domain in terms of hyperbolic distance. Finally, Burkholder [7] studied the Hardy number of a domain in relation with the exit time of Brownian motion (see also [5]). More precisely, if D is a simply connected domain, then we define the number $\widetilde{h}(D)$ to be the supremum of all $p > 0$ for which the p-th moment of the exit time of Brownian motion is finite. Then Burkholder proved in [7] that

$$(1.1) \qquad\qquad\qquad \widetilde{h}(D) = \mathrm{h}(D)/2.$$

Comb domains furnish an interesting class of simply connected domains and thus they have been studied from various points of view. For example, they have been studied in relation with the angular derivative (see [13], [16] and references therein), the harmonic measure [3] and the semigroups of holomorphic functions [4]. Moreover, in [6] Boudabra and Markowsky studied the moments of the exit time of planar Brownian motion from comb domains.

Let $\{x_n\}_{n \in \mathbb{Z}}$ be a strictly increasing sequence of real numbers such that $x_0 = 0$ and

$$\inf_{n \in \mathbb{Z}}(x_n - x_{n-1}) > 0.$$

Also, let $\{c_n\}_{n \in \mathbb{Z}}$ be a sequence of positive numbers such that for some constants $m, M > 0$,

$$m \leq c_n \leq M$$

for every $n \in \mathbb{Z}$. We consider comb domains of the form (see Figure 1)

$$D = \mathbb{C} \backslash \bigcup_{n \in \mathbb{Z}} \{x_n + iy \colon |y| \geq c_n\}.$$
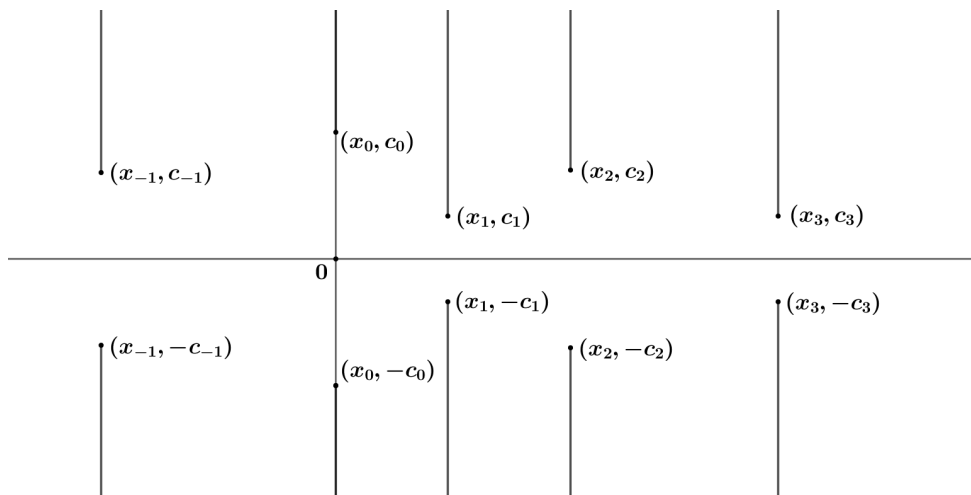
Figure 1. Comb domain.

Since we want to find conditions for the Hardy number of such comb domains to be equal to infinity, we can simplify the problem in the following way. First, we observe that if

$$D_m = \mathbb{C} \backslash \bigcup_{n \in \mathbb{Z}} \{x_n + iy \colon |y| \geq m\} \quad \text{and} \quad D_M = \mathbb{C} \backslash \bigcup_{n \in \mathbb{Z}} \{x_n + iy \colon |y| \geq M\},$$

then $D_m \subseteq D \subseteq D_M$ and hence $\mathrm{h}(D_M) \leq \mathrm{h}(D) \leq \mathrm{h}(D_m)$ (see [11]). Moreover, since the Hardy number is invariant under affine mappings (see [11]), we have $\mathrm{h}(D_m) = \mathrm{h}(D_M) = \mathrm{h}(D)$. Therefore, it suffices to study comb domains of the form

$$D_c = \mathbb{C} \backslash \bigcup_{n \in \mathbb{Z}} \{x_n + iy \colon |y| \geq c\},$$

where $c > 0$. However, we can do more simplifications. We observe that if

$$D_c^+ = D \cap \{z \colon \mathrm{Re}\, z > -x_1\} \quad \text{and} \quad D_c^- = D \cap \{z \colon \mathrm{Re}\, z < -x_{-1}\},$$

then $\mathrm{h}(D_c) = +\infty$ if and only if $\mathrm{h}(D_c^+) = +\infty$ and $\mathrm{h}(D_c^-) = +\infty$. This follows from Proposition 8 in [6] and (1.1). Furthermore, $\mathrm{h}(D_c^-) = \mathrm{h}(-D_c^-)$. Therefore, it suffices to study the Hardy number of comb domains of the form D_c^+. Finally, since the Hardy number is invariant under affine mappings, without loss of generality, we suppose that $c = 1$ and the infimum of the differences $x_n - x_{n-1}$ is greater than 1. So, henceforth we consider comb domains C of the form

$$(1.2) \qquad C = \{z \colon \mathrm{Re}\, z > -x_1\} \backslash \bigcup_{n \in \mathbb{N} \cup \{0\}} \{x_n + iy \colon |y| \geq 1\},$$

where $x_0 = 0$ and $\{x_n\}_{n \in \mathbb{N}}$ is a strictly increasing sequence of positive numbers such that

$$\lim_{n \to +\infty} x_n = +\infty \quad \text{and} \quad \inf_{n \in \mathbb{N}} (x_n - x_{n-1}) > 1.$$

First, we establish a necessary and sufficient condition for $\mathrm{h}(C)$ to be equal to infinity by studying the Euclidean distances between the rays. For every $n \in \mathbb{N}$, we denote these distances by

$$\alpha_n = x_n - x_{n-1}.$$

Theorem 1.1. *Let C be a comb domain of the form* (1.2). *Then* $h(C) = +\infty$ *if and only if*

$$\lim_{n \to +\infty} \frac{\sum_{i=1}^{n} \log \alpha_i}{\log x_n} = +\infty \quad \textit{or, equivalently,} \quad \lim_{n \to +\infty} \frac{\sum_{i=1}^{n} \log \alpha_i}{\log \sum_{i=1}^{n} \alpha_i} = +\infty.$$

An immediate consequence is that if the sequence α_n is bounded then $h(C) = +\infty$. So, we actually study the case of α_n being unbounded. By applying Theorem 1.1 we can examine how the mutual distances and the distribution of the rays affect the finiteness of the Hardy number. First, we consider the case of α_n growing at a subexponential rate and prove that $h(C)$ is always equal to infinity.

Theorem 1.2. *If*

$$\lim_{n \to +\infty} \frac{\log \alpha_n}{n} = 0,$$

then $h(C) = +\infty$.

This result is stronger than the corollary of the main theorem of Boudabra and Markowsky in [6], where they approach the problem by studying the moments of the exit time of the Brownian motion. In fact, their main theorem implies that if α_n grows at most polynomially in n then the Hardy number is infinite. However, it does not cover all subexponential sequences. For the proof of Theorem 1.2 see Section 4.

Next, we explain why the assumption in Theorem 1.2 cannot be relaxed. Theorem 1.2 covers all the cases when α_n grows at a subexponential rate, even those in which the sequence α_n oscillates very rapidly. For example, one can take $\alpha_n = 2$ when n is odd and $\alpha_n = n^p$ when n is even and $p > 0$.

If the sequence α_n is of exponential type, i.e. $\alpha_n = e^{cn}$ for every $n \in \mathbb{N}$, and hence there are no sharp oscillations, then Theorem 1.1 implies that the Hardy number is equal to infinity. However, if we allow wild oscillations and suppose that $\alpha_n \leq e^n$, then the Hardy number might be finite. Actually, we construct such an example in Theorem 1.4. Therefore, in order to obtain a general result in case $\alpha_n \leq e^n$, we need to suppose that there are no wild oscillations. By imposing that

$$\lim_{n \to +\infty} \alpha_n = +\infty$$

and thus preventing sharp oscillations of α_n, we prove that $h(C) = +\infty$. In fact, a more general result is true.

Theorem 1.3. *Let $\{b_n\}_{n \in \mathbb{N}}$ be an increasing sequence of positive numbers such that $\inf_{n>1}(b_n - b_{n-1}) > 0$. Let $\alpha_n \leq e^{b_n}$ for every $n \in \mathbb{N}$. If*

$$\lim_{n \to +\infty} \frac{\log \alpha_n}{b_n - b_{n-1}} = +\infty,$$

then $h(C) = +\infty$.

An interesting case, as we already remarked, is when $b_n = n$.

Corollary 1.1. *Let $\alpha_n \leq e^n$ for every $n \in \mathbb{N}$. If*

$$\lim_{n \to +\infty} \alpha_n = +\infty,$$

then $h(C) = +\infty$.

Next, we prove that the assumption in Theorem 1.3 is sharp. In other words, if there are wild oscillations of α_n, then the Hardy number might be finite.

Theorem 1.4. *Let $\{b_n\}$ be as Theorem 1.3. There is a comb domain C such that $\alpha_n \leq e^{b_n}$ for every $n \in \mathbb{N}$,*

$$\liminf_{n \to +\infty} \frac{\log \alpha_n}{b_n - b_{n-1}} < +\infty$$

and $\mathrm{h}(C) < +\infty$.

Theorem 1.3 covers a variety of cases such as α_n being comparable to e^{n^p} for some $p > 0$, α_n being comparable to $e^{e^{n^k}}$ for some $k < 1$ and α_n being comparable to $e^{e^{n/\log n}}$. However, it does not apply if α_n is comparable to e^{e^n}. In this case, despite the fact that there are no wild oscillations of α_n, Theorem 1.1 implies that $\mathrm{h}(C)$ is finite.

Theorem 1.5. *If α_n is comparable to e^{e^n} for every $n \in \mathbb{N}$, then* $\mathrm{h}(C) < +\infty$.

Therefore, the Hardy number of C might be finite when the sequence α_n oscillates very quickly or if it goes to infinity rapidly enough like α_n being comparable to e^{e^n}.

Remark 1.1. Note that by (1.1) all the results above concerning the Hardy number provide us with information about the finiteness of the moments of the exit time of Brownian motion from comb domains. In fact, $\widetilde{h}(D)$ is equal to infinity if and only if $\mathrm{h}(D)$ is equal to infinity.

Remark 1.2. By Corollary 1.1 the Riemann mapping from $\mathbb{D}$ onto the comb domain C with $x_n = e^n$ belongs to every $H^p(\mathbb{D})$ space. However, it does not belong to $BMOA$ (see [22]). So, it is an example which ensures that

$$BMOA \subsetneq \bigcap_{p>0} H^p(\mathbb{D}).$$

In Section 2, we introduce some preliminaries such as notions and results in hyperbolic geometry and their connection with the Hardy number. In Section 3, we prove Theorem 1.1 and applying this, in Section 4, we prove all the other theorems stated above.

2. Preliminary results

2.1. Hyperbolic distance. The hyperbolic distance between two points z, w in the unit disk $\mathbb{D}$ (see [2, p. 11-28]) is defined by

$$d_{\mathbb{D}}(z, w) = \log \frac{1 + \left|\frac{z-w}{1-z\bar{w}}\right|}{1 - \left|\frac{z-w}{1-z\bar{w}}\right|}.$$

It can also be defined on any simply connected domain $D \neq \mathbb{C}$ in the following way: If f is a Riemann mapping of $\mathbb{D}$ onto D and $z, w \in D$, then $d_D(z, w) = d_{\mathbb{D}}(f^{-1}(z), f^{-1}(w))$. Also, for a set $E \subset D$, we define $d_D(z, E) = \inf\{d_D(z, w) : w \in E\}$.

2.2. Quasi-hyperbolic distance. Let $D \neq \mathbb{C}$ be a simply connected domain. The hyperbolic distance between $z_1, z_2 \in D$ can be estimated by the quasi-hyperbolic distance which is defined by

$$\delta_D(z_1, z_2) = \inf_{\gamma: z_1 \to z_2} \int_\gamma \frac{|dz|}{d(z, \partial D)},$$

where the infimum ranges over all the paths γ connecting z_1 to z_2 in D and $d\,(z, \partial D)$ denotes the Euclidean distance of z from ∂D. It is known [2, p. 33–36] that

$$(2.1) \qquad\qquad \frac{1}{2}\delta_D \leq d_D \leq 2\delta_D.$$

2.3. Hardy number and hyperbolic distance. In [15] the current author proves that the Hardy number of a simply connected domain can be found with the aid of hyperbolic distance in the following way.

Theorem 2.1. *Let D be a simply connected domain containing the origin. If $F_r = D \cap \{|z| = r\}$ for $r > 0$, then*

$$\mathrm{h}\,(D) = \liminf_{r\to+\infty} \frac{d_D\,(0, F_r)}{\log r}.$$

2.4. The Stolz–Cesaro theorem. Next, we state a generalized form of the Stolz–Cesaro theorem which we apply in Section 4. For the proof see [18] and [10, p. 263–266].

Theorem 2.2. *Let $\{b_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $\sum_{n=1}^{+\infty} b_n = +\infty$. For any real sequence $\{a_n\}_{n\in\mathbb{N}}$, it is true that*

$$\limsup_{n\to+\infty} \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \leq \limsup_{n\to+\infty} \frac{a_n}{b_n}$$

and

$$\liminf_{n\to+\infty} \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \geq \liminf_{n\to+\infty} \frac{a_n}{b_n}.$$

3. A necessary and sufficient condition

In this section we give a necessary and sufficient condition for the Hardy number to be equal to infinity. First, we prove two auxiliary lemmas which give an upper and a lower estimate for $\mathrm{h}(C)$.

Lemma 3.1. *Let C be a comb domain of the form described in Section 1. If $K = 4\log((1 + \sqrt{5})/2)$, then*

$$\mathrm{h}(C) \leq \liminf_{n\to+\infty} \left(\frac{4\sum\limits_{i=1}^{n} \log \alpha_i}{\log x_n} + \frac{nK}{\log x_n} + 4 \right).$$

Proof. Let $r > 0$. There exists a number $n \in \mathbb{N}$ such that $x_{n-1} < r \leq x_n$. Due to the symmetry of C with respect to the real axis and the uniqueness of the hyperbolic geodesic in simply connected domains, the hyperbolic geodesic between 0 and r in C is the line segment $[0, r]$. Therefore, we have

$$(3.1) \qquad\qquad d_C(0, r) = d_C(0, x_{n-1}) + d_C(x_{n-1}, r)$$

(see [2, p. 14]). Applying (2.1) and letting m_{i-1} denote the midpoint of the interval $[x_{i-1}, x_i]$ for every $i \in \mathbb{N}$, we infer that

$$d_C(0, x_{n-1}) \leq 2\delta_C(0, x_{n-1}) = 2 \int_0^{x_{n-1}} \frac{dx}{d(x, \partial C)} = 2 \sum_{i=1}^{n-1} \int_{x_{i-1}}^{x_i} \frac{dx}{d(x, \partial C)}$$

$$= 4 \sum_{i=1}^{n-1} \int_{x_{i-1}}^{m_{i-1}} \frac{dx}{d(x, \partial C)} = 4 \sum_{i=1}^{n-1} \int_{x_{i-1}}^{m_{i-1}} \frac{dx}{\sqrt{1 + (x - x_{i-1})^2}}$$

$$= 4 \sum_{i=1}^{n-1} \operatorname{arcsinh}(m_{i-1} - x_{i-1}) = 4 \sum_{i=1}^{n-1} \operatorname{arcsinh}\left(\frac{x_i - x_{i-1}}{2}\right)$$

$$(3.2) \qquad = 4 \sum_{i=1}^{n-1} \log\left(\frac{\alpha_i}{2} + \sqrt{\left(\frac{\alpha_i}{2}\right)^2 + 1}\right).$$

Recall that the domain C has the property that $\inf\limits_{n \in \mathbb{N}} \alpha_n > 1$, which implies that, for every $i \in \mathbb{N}$,

$$\frac{\alpha_i}{2} > \frac{1}{2}.$$

Therefore,

$$\sqrt{\left(\frac{\alpha_i}{2}\right)^2 + 1} = \sqrt{\left(\frac{\alpha_i}{2}\right)^2 + 4\left(\frac{1}{2}\right)^2} \leq \frac{\alpha_i}{2}\sqrt{5}$$

and hence

$$(3.3) \qquad \log\left(\frac{\alpha_i}{2} + \sqrt{\left(\frac{\alpha_i}{2}\right)^2 + 1}\right) \leq \log\left(\frac{\alpha_i}{2}\left(1 + \sqrt{5}\right)\right) = \log \alpha_i + \frac{K}{4},$$

where $K = 4\log((1 + \sqrt{5})/2)$. Combining (3.2) with (3.3), we deduce that

$$(3.4) \qquad d_C(0, x_{n-1}) \leq 4 \sum_{i=1}^{n-1} \log \alpha_i + (n-1)K.$$

Now, in order to find an upper estimate for $d_C(x_{n-1}, r)$, we consider the following cases.

Case 1: If $r \in \left(x_{n-1}, \frac{x_{n-1} + x_n}{2}\right]$, then

$$d_C(x_{n-1}, r) \leq 2\delta_C(x_{n-1}, r) = 2 \int_{x_{n-1}}^r \frac{dx}{\sqrt{1 + (x - x_{n-1})^2}}$$

$$= 2\operatorname{arcsinh}(r - x_{n-1}) \leq 2\operatorname{arcsinh} r.$$

Case 2: If $r \in \left(\frac{x_{n-1} + x_n}{2}, x_n\right]$, then

$$d_C(x_{n-1}, r) \leq d_C(x_{n-1}, x_n) \leq 2\delta_C(x_{n-1}, x_n) = 2 \int_{x_{n-1}}^{x_n} \frac{dx}{d(x, \partial C)}$$

$$= 4 \int_{x_{n-1}}^{\frac{x_{n-1} + x_n}{2}} \frac{dx}{\sqrt{1 + (x - x_{n-1})^2}} = 4\operatorname{arcsinh}\left(\frac{x_n - x_{n-1}}{2}\right)$$

$$\leq 4\operatorname{arcsinh} r.$$

Therefore, it follows that in both cases,

$$(3.5) \qquad d_C(x_{n-1}, r) \leq 4\operatorname{arcsinh} r.$$

Recall that $r > x_{n-1}$. So, by (3.1), (3.4) and (3.5) we derive that

$$\frac{d_C(0, r)}{\log r} \leq \frac{4 \sum\limits_{i=1}^{n-1} \log \alpha_i}{\log x_{n-1}} + \frac{(n-1)K}{\log x_{n-1}} + \frac{4 \operatorname{arcsinh} r}{\log r}.$$

This in conjunction with Theorem 2.1 gives

$$\mathrm{h}(C) \leq \liminf_{r \to +\infty} \frac{d_C(0, r)}{\log r} \leq \liminf_{n \to +\infty} \left(\frac{4 \sum\limits_{i=1}^{n} \log \alpha_i}{\log x_n} + \frac{nK}{\log x_n} + 4 \right)$$

and the proof is complete. $\square$

Lemma 3.2. *Let C be a comb domain of the form described in Section 1. Then*

$$\mathrm{h}(C) \geq \liminf_{n \to +\infty} \frac{\sum\limits_{i=1}^{n} \log \alpha_i}{\log x_n} - 1.$$

Proof. If $F_r = C \cap \{|z| = r\}$, by Theorem 2.1 we have

$$\mathrm{h}(C) = \liminf_{r \to +\infty} \frac{d_C(0, F_r)}{\log r} = \lim_{n \to +\infty} \frac{d_C(0, F_{r_n})}{\log r_n},$$

where $\{r_n\}$ is an increasing sequence of positive numbers. The hyperbolic distance $d_C(0, F_{r_n})$ is attained on some component of F_{r_n} lying in the vertical strip $\{z \colon x_{i_n} < \operatorname{Re} z < x_{i_n+1}\}$, where $\{x_{i_n}\}$ is a subsequence of $\{x_n\}$. If we denote this component by $F_{r_n}^{x_{i_n}}$ then

(3.6) $$\mathrm{h}(C) = \lim_{n \to +\infty} \frac{d_C(0, F_{r_n}^{x_{i_n}})}{\log r_n}.$$

Since C is symmetric with respect to the real axis, without loss of generality, we suppose that $F_{r_n}^{x_{i_n}}$ lies on the upper half-plane or intersects the positive real axis (see Figure 2). If $\mathrm{h}(C) = +\infty$, the result is trivial. Hence, we suppose that $\mathrm{h}(C) < +\infty$ and take the following cases.
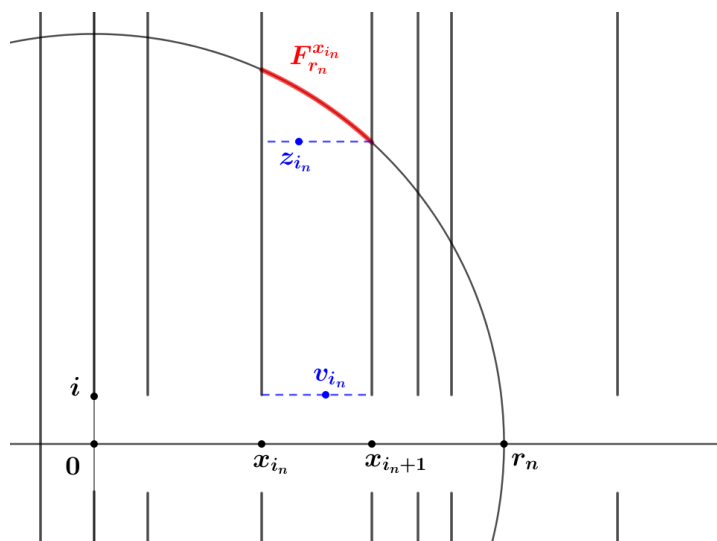

Figure 2. The component $F_{r_n}^{x_{i_n}}$.

Case 1: For infinitely many n, $F_{r_n}^{x_{i_n}}$ is the component of F_{r_n} containing r_n. By passing to a subsequence we assume that this is the case for all n. The hyperbolic geodesic between 0 and $F_{r_n}^{x_{i_n}}$ passes from some point u_{i_n} of the line segment $\{x_{i_n} + iy : -1 < y < 1\}$. If m_{j-1} denotes the midpoint of the interval $[x_{j-1}, x_j]$, then we have

$$d_C\left(0, F_{r_n}^{x_{i_n}}\right) \geq d_C\left(0, u_{i_n}\right) \geq \frac{1}{2}\delta_C\left(0, u_{i_n}\right) \geq \frac{1}{2}\int_0^{x_{i_n}} \frac{dx}{d\left(x, \partial C\right)}$$

$$= \frac{1}{2}\sum_{j=1}^{i_n}\int_{x_{j-1}}^{x_j} \frac{dx}{d\left(x, \partial C\right)} = \sum_{j=1}^{i_n}\int_{x_{j-1}}^{m_{j-1}} \frac{dx}{\sqrt{1 + (x - x_{j-1})^2}}$$

$$= \sum_{j=1}^{i_n}\operatorname{arcsinh}\left(\frac{x_j - x_{j-1}}{2}\right) = \sum_{j=1}^{i_n}\log\left(\frac{\alpha_j}{2} + \sqrt{\left(\frac{\alpha_j}{2}\right)^2 + 1}\right)$$

$$(3.7) \qquad \geq \sum_{j=1}^{i_n}\log\alpha_j.$$

Since $x_{i_n} < r_n < x_{i_n+1}$, by (3.7) and (3.6) it follows that

$$\mathrm{h}(C) \geq \liminf_{n \to +\infty} \frac{\sum_{j=1}^{i_n}\log\alpha_j}{\log x_{i_n+1}} = \liminf_{n \to +\infty}\left(\frac{\sum_{j=1}^{i_n+1}\log\alpha_j}{\log x_{i_n+1}} - \frac{\log\alpha_{i_n+1}}{\log x_{i_n+1}}\right)$$

$$(3.8) \qquad \geq \liminf_{n \to +\infty} \frac{\sum_{j=1}^{i_n+1}\log\alpha_j}{\log x_{i_n+1}} - 1 \geq \liminf_{n \to +\infty} \frac{\sum_{j=1}^{n}\log\alpha_j}{\log x_n} - 1.$$

Case 2: For infinitely many n, $F_{r_n}^{x_{i_n}}$ is not the component of F_{r_n} containing r_n. By passing to a subsequence we suppose that this is the case for all n. First, suppose that for $k = 2/(3\,\mathrm{h}(C))$,

$$(3.9) \qquad x_{i_n+1} \leq k\frac{r_n}{\log r_n}$$

for infinitely many n. The hyperbolic geodesic between 0 and $F_{r_n}^{x_{i_n}}$ passes from some point v_{i_n} of the line segment $(x_{i_n} + i, x_{i_n+1} + i)$ and some point z_{i_n} of the line segment $\left(x_{i_n} + i\sqrt{r_n^2 - x_{i_n+1}^2}, x_{i_n+1} + i\sqrt{r_n^2 - x_{i_n+1}^2}\right)$. Thus, we have

$$d_C(0, F_{r_n}^{x_{i_n}}) \geq d_C(v_{i_n}, z_{i_n}) \geq \frac{1}{2}\delta_C(v_{i_n}, z_{i_n}) \geq \frac{1}{2}\int_1^{\sqrt{r_n^2 - x_{i_n+1}^2}} \frac{dx}{d(x, \partial C)}$$

$$= \frac{\sqrt{r_n^2 - x_{i_n+1}^2} - 1}{x_{i_n+1} - x_{i_n}} \geq \frac{\sqrt{r_n^2 - x_{i_n+1}^2} - 1}{x_{i_n+1}}$$

$$\geq \frac{1}{k}\frac{\log r_n}{r_n}\left(\sqrt{r_n^2 - k^2\left(\frac{r_n}{\log r_n}\right)^2} - 1\right),$$

where we applied (3.9). This in combination with (3.6) implies that

$$\mathrm{h}(C) \geq \frac{1}{k}\liminf_{n \to +\infty} \frac{\sqrt{r_n^2 - k^2\left(\frac{r_n}{\log r_n}\right)^2} - 1}{r_n} = \frac{1}{k} = \frac{3}{2}\,\mathrm{h}(C),$$

which is a contradiction. Therefore,

$$(3.10) \qquad\qquad x_{i_n+1} > k\frac{r_n}{\log r_n}$$

for all but finitely many n. So, working as in Case 1, we have

$$d_C(0, F_{r_n}^{x_{i_n}}) \geq \sum_{j=1}^{i_n} \log\alpha_j.$$

By this and (3.6), it follows that

$$h(C) \geq \liminf_{n\to+\infty} \frac{\displaystyle\sum_{j=1}^{i_n} \log\alpha_j}{\log r_n} = \liminf_{n\to+\infty} \left(\frac{\displaystyle\sum_{j=1}^{i_n} \log\alpha_j}{\log x_{i_n+1}} \frac{\log x_{i_n+1}}{\log r_n} \right)$$

$$\geq \liminf_{n\to+\infty} \left(\frac{\displaystyle\sum_{j=1}^{i_n} \log\alpha_j}{\log x_{i_n+1}} \frac{\log k + \log r_n - \log(\log r_n)}{\log r_n} \right)$$

$$= \liminf_{n\to+\infty} \frac{\displaystyle\sum_{j=1}^{i_n} \log\alpha_j}{\log x_{i_n+1}} \geq \liminf_{n\to+\infty} \frac{\displaystyle\sum_{j=1}^{n} \log\alpha_j}{\log x_n} - 1,$$

where we applied (3.10) and (3.8). Consequently, in any case we obtain the desired result. $\qquad\square$

Next, we prove our main theorem.

Proof of Theorem 1.1. Suppose that $h(C) = +\infty$. If

$$\liminf_{n\to+\infty} \frac{n}{\log x_n} < +\infty,$$

then by Lemma 3.1 we deduce that

$$\lim_{n\to+\infty} \frac{\displaystyle\sum_{i=1}^{n} \log\alpha_i}{\log x_n} = +\infty.$$

Now, suppose that

$$\liminf_{n\to+\infty} \frac{n}{\log x_n} = +\infty.$$

Recall that $\inf_{n\in\mathbb{N}} \alpha_n = l > 1$. So, we have

$$\frac{\displaystyle\sum_{i=1}^{n} \log\alpha_i}{\log x_n} > \frac{n}{\log x_n} \log l$$

and thus

$$\lim_{n\to+\infty} \frac{\displaystyle\sum_{i=1}^{n} \log\alpha_i}{\log x_n} = +\infty$$

in both cases. The other direction is direct by Lemma 3.2. $\qquad\square$

4. Consequent results

In this section we prove several results derived by Theorem 1.1. They are all stated in Section 1. First, we show that if the sequence α_n grows at a subexponential rate, then the Hardy number is equal to infinity.

Proof of Theorem 1.2. Let $A_n = \max\limits_{1 \leq j \leq n} \alpha_j$. First, we prove that our assumption implies that

$$\lim_{n \to +\infty} \frac{\log A_n}{n} = 0.$$

Suppose, on the contrary, that it is false. Then there are a constant $\delta > 0$ and a subsequence $\{A_{k_n}\}_{n \in \mathbb{N}}$ of $\{A_n\}_{n \in \mathbb{N}}$ such that, for every $n \in \mathbb{N}$,

$$\frac{\log A_{k_n}}{k_n} \geq \delta.$$

For every $n \in \mathbb{N}$ there is an $m_n \in \mathbb{N}$ such that $A_{k_n} = \alpha_{m_n}$ and $1 \leq m_n \leq k_n$.

Case 1: If there is a constant $K > 0$ such that $m_n \leq K$ for every $n \in \mathbb{N}$, then

$$0 \leq \frac{\log A_{k_n}}{k_n} = \frac{\log \alpha_{m_n}}{k_n} \leq \frac{\max\limits_{1 \leq i \leq K} \log \alpha_i}{k_n}.$$

So, taking limits as $n \to +\infty$, we derive that

$$\lim_{n \to +\infty} \frac{\log A_{k_n}}{k_n} = 0,$$

which is a contradiction.

Case 2: If $m_n \to +\infty$, then there is a subsequence $\{m_{l_n}\}_{n \in \mathbb{N}}$ such that $m_{l_n} \to +\infty$ and m_{l_n} is strictly increasing with respect to n. Thus,

$$\delta \leq \frac{\log A_{k_{l_n}}}{k_{l_n}} = \frac{\log \alpha_{m_{l_n}}}{k_{l_n}} = \frac{\log \alpha_{m_{l_n}}}{m_{l_n}} \frac{m_{l_n}}{k_{l_n}} \leq \frac{\log \alpha_{m_{l_n}}}{m_{l_n}}.$$

Taking limits as $n \to +\infty$, we infer that $\delta \leq 0$, which is a contradiction. Therefore,

$$(4.1) \qquad \lim_{n \to +\infty} \frac{\log A_n}{n} = 0.$$

Recall that $\inf \alpha_n = l > 1$. Since $\alpha_n \leq A_n$ for every $n \in \mathbb{N}$ and $\{A_n\}_{n \in \mathbb{N}}$ is an increasing sequence, we have

$$\frac{\sum\limits_{i=1}^{n} \log \alpha_i}{\log \sum\limits_{i=1}^{n} \alpha_i} \geq \frac{n \log l}{\log \sum\limits_{i=1}^{n} A_i} \geq \frac{n \log l}{\log n + \log A_n} = \frac{\log l}{\frac{\log n}{n} + \frac{\log A_n}{n}}.$$

Taking limits as $n \to +\infty$, by (4.1) we deduce that

$$\lim_{n \to +\infty} \frac{\sum\limits_{i=1}^{n} \log \alpha_i}{\log \sum\limits_{i=1}^{n} \alpha_i} = +\infty.$$

Thus, Theorem 1.1 implies that $h(C) = +\infty$. $\qquad \square$

The following corollary of Theorem 1.2 is the corollary of Theorem 4 in [6, p. 3]. Let $\{x_n\}_{n\in\mathbb{Z}}$ be an increasing sequence of distinct real numbers without accumulation point in $\mathbb{R}$ and $\{c_n\}_{n\in\mathbb{Z}}$ be an associated sequence of positive numbers, and let

$$D = \mathbb{C}\backslash \bigcup_{n\in\mathbb{Z}} \{x_n + iy \colon |y| \geq c_n\}.$$

Corollary 4.1. *Let $\alpha_n = x_n - x_{n-1}$. Suppose that $\inf_{n\in\mathbb{Z}} \alpha_n > 0$ and $\{c_n\}_{n\in\mathbb{Z}}$ is bounded. If*

$$\sum_{j=1}^{+\infty} (\max_{|n|\leq j} \alpha_n^2)\theta^j < +\infty$$

for every $\theta \in (0,1)$, then $\widetilde{h}(D) = +\infty$.

Proof. By assumption, there is a constant $c > 0$ such that $c_n \leq c$ for every $n \in \mathbb{Z}$. Thus,

$$D \subseteq \mathbb{C}\backslash \bigcup_{n\in\mathbb{Z}} \{x_n + iy \colon |y| \geq c\} := D_c$$

and $h(D) \geq h(D_c)$. By this and (1.1), it suffices to prove that $h(D_c) = +\infty$ or, equivalently, $h(D_c^-) = h(D_c^+) = +\infty$ (see Section 1). Without loss of generality, we suppose that $c = 1$ and $\inf \alpha_n > 1$. We have

$$\sum_{j=1}^{+\infty} \alpha_j^2\theta^j \leq \sum_{j=1}^{+\infty} (\max_{|n|\leq j} \alpha_n^2)\theta^j < +\infty$$

for every $\theta \in (0,1)$. This implies that, for every $\theta \in (0,1)$,

$$\lim_{n\to+\infty} \alpha_n^2\theta^n = 0$$

and hence for every $\theta \in (0,1)$ there is an $n_0(\theta) \in \mathbb{N}$ such that for $n \geq n_0$,

$$\alpha_n\theta^{n/2} < 1$$

or, equivalently,

$$\frac{\log \alpha_n}{n} < \frac{1}{2}\log\frac{1}{\theta}.$$

Set $\varepsilon = (1/2)\log(1/\theta)$. So, for every $\varepsilon > 0$ there is an $n_0(\varepsilon) \in \mathbb{N}$ such that for $n \geq n_0$,

$$\frac{\log \alpha_n}{n} < \varepsilon.$$

By Theorem 1.2, we deduce that $h(D_c^+) = +\infty$. Working with α_n for $n < 0$ in the same way as above, we infer that $h(D_c^-) = +\infty$ and thus, it follows that $h(D_c) = +\infty$. $\square$

Next, we prove Theorem 1.3 which implies that if the sequence $\log \alpha_n$ grows at a subexponential rate and there are no wild oscillations of α_n, then the Hardy number is equal to infinity.

Proof of Theorem 1.3. Since $\{b_n\}_{n\in\mathbb{N}}$ is an increasing sequence, we have

$$(4.2) \qquad \log\sum_{i=1}^{n} \alpha_i \leq \log\sum_{i=1}^{n} e^{b_i} \leq \log(ne^{b_n}) = \log n + b_n.$$

By assumption, $\inf_{n>1}(b_n - b_{n-1}) = r > 0$. This implies that

$$(4.3) \qquad b_n = \sum_{i=2}^{n}(b_i - b_{i-1}) + b_1 \geq (n-1)r + b_1.$$

So, for every $n \in \mathbb{N}$, we have

$$0 \leq \frac{\log n}{b_n} \leq \frac{\log n}{(n-1)r + b_1}$$

and thus

$$\lim_{n \to +\infty} \frac{\log n}{b_n} = 0.$$

By this and (4.2), we obtain the following estimates

$$\liminf_{n \to +\infty} \frac{\sum_{i=1}^{n} \log \alpha_i}{\log \sum_{i=1}^{n} \alpha_i} \geq \liminf_{n \to +\infty} \left(\frac{\sum_{i=1}^{n} \log \alpha_i}{b_n} \frac{b_n}{\log n + b_n} \right) = \liminf_{n \to +\infty} \frac{\sum_{i=1}^{n} \log \alpha_i}{b_1 + \sum_{i=2}^{n}(b_i - b_{i-1})}$$

$$\geq \liminf_{n \to +\infty} \frac{\log \alpha_n}{b_n - b_{n-1}} = +\infty.$$

In the last inequality we applied Theorem 2.2. Therefore, Theorem 1.1 implies that $h(C) = +\infty$. $\qquad\square$

Next, we prove that the condition of Theorem 1.3 is sharp.

Proof of Theorem 1.4. Fix a $c > 1$ and let $\{b_{k_m}\}_{m \in \mathbb{N}}$ be a subsequence of $\{b_m\}_{m \in \mathbb{N}}$ such that

$$(4.4) \qquad b_{k_m} \geq \sum_{i=1}^{m-1} b_{k_i}$$

for every $m \geq 2$. Moreover, we observe that (4.3) implies that

$$(4.5) \qquad \frac{k_m}{b_{k_m}} \leq \frac{1}{r} + \frac{r - b_1}{r b_{k_m}}.$$

We consider a comb domain with

$$\alpha_n = \begin{cases} c, & n \notin \{k_m : m \in \mathbb{N}\} \\ e^{b_{k_m}}, & n = k_m \text{ for some } m \in \mathbb{N} \end{cases}.$$

Applying (4.4) and (4.5), we have the following estimates

$$I_{k_m} := \frac{\sum_{i=1}^{k_m} \log \alpha_i}{\log \sum_{i=1}^{k_m} \alpha_i} = \frac{(k_m - m)\log c + \sum_{i=1}^{m} b_{k_i}}{\log \left((k_m - m)c + \sum_{i=1}^{m} e^{b_{k_i}} \right)}$$

$$\leq \frac{k_m \log c + 2b_{k_m}}{b_{k_m}} \leq \left(\frac{1}{r} + \frac{r - b_1}{r b_{k_m}} \right) \log c + 2.$$

This implies that

$$\liminf_{n \to +\infty} \frac{\sum_{i=1}^{n} \log \alpha_i}{\log \sum_{i=1}^{n} \alpha_i} \leq \liminf_{m \to +\infty} I_{k_m} \leq 2 + \frac{1}{r} \log c$$

and hence by Theorem 1.1 we derive that $\mathrm{h}(C) < +\infty$. Finally, it follows that

$$\liminf_{n\to+\infty} \frac{\log \alpha_n}{b_n - b_{n-1}} \leq \liminf_{m\to+\infty} \frac{\log \alpha_{k_m+1}}{b_{k_m+1} - b_{k_m}} \leq \frac{\log c}{r} < +\infty$$

and the proof is complete. $\qquad\square$

Finally, we prove that if α_n is comparable to e^{e^n}, then the Hardy number is finite.

Proof of Theorem 1.5. By assumption there are constants $c_1, c_2 > 0$ such that, for every $n \in \mathbb{N}$,

$$c_1 e^{e^n} \leq \alpha_n \leq c_2 e^{e^n}.$$

So, it follows that

$$\liminf_{n\to+\infty} \frac{\sum_{i=1}^n \log \alpha_i}{\log \sum_{i=1}^n \alpha_i} \leq \liminf_{n\to+\infty} \frac{\sum_{i=1}^n \log \alpha_i}{\log \alpha_n} \leq \liminf_{n\to+\infty} \frac{n\log c_2 + \sum_{i=1}^n e^i}{\log c_1 + e^n}$$

$$= \frac{e}{e-1} \liminf_{n\to+\infty} \frac{e^n}{\log c_1 + e^n} = \frac{e}{e-1} < +\infty.$$

By Theorem 1.1 we deduce that $\mathrm{h}(C) < +\infty$. $\qquad\square$

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