

# Multiplicity and concentration of solutions to a fractional $p$ -Laplace problem with exponential growth

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**Abstract.** In this paper, we study the Schrödinger equation involving  $\frac{N}{s}$ -fractional Laplace as follows

$$\varepsilon^N (-\Delta)_{N/s}^s u + V(x)|u|^{\frac{N}{s}-2}u = f(u) \quad \text{in } \mathbb{R}^N,$$

where  $\varepsilon$  is a positive parameter,  $N = ps$ ,  $s \in (0, 1)$ . The nonlinear function  $f$  has the exponential growth and potential function  $V$  is a continuous function satisfying some suitable conditions. Our problem lacks of compactness. By using the Ljusternik–Schnirelmann theory, we obtain the existence, multiplicity and concentration of nontrivial nonnegative solutions for small values of the parameter.

## Ekspontiaalisesti kasvavan murtoasteisen $p$ -Laplacen ongelman ratkaisuiden monilukuisuus ja kasautuminen

**Tiivistelmä.** Tässä työssä tutkimme seuraavaa  $\frac{N}{s}$ -murtoasteisen Laplacen operaattorin sisältävää Schrödingerin yhtälöä

$$\varepsilon^N (-\Delta)_{N/s}^s u + V(x)|u|^{\frac{N}{s}-2}u = f(u) \quad \text{avaruudessa } \mathbb{R}^N,$$

missä  $\varepsilon$  on positiivinen parametri,  $N = ps$  ja  $s \in (0, 1)$ . Epälineaarinen funktio  $f$  kasvaa eksponentiaalisesti, ja potentiaali  $V$  on sopivat ehdot toteuttava jatkuva funktio. Tältä ongelmalta puuttuu kompaktisuusominaisuuksia. Ljusternikin–Schnirelmannin teorian avulla osoitamme pienillä parametriarvoilla epätriviaalien ei-negatiivisten ratkaisujen olemassaolon, monilukuisuuden ja kasautumisen.

## 1. Introduction and main results

In this paper, we first study the existence and concentration of nontrivial nonnegative solution for the following Schrödinger equation involving fractional  $\frac{N}{s}$ -Laplace:

$$(1.1) \quad \varepsilon^N (-\Delta)_{N/s}^s u(x) + V(x)|u|^{\frac{N}{s}-2}u = f(u) \quad \text{in } \mathbb{R}^N, \quad (P_\varepsilon)$$

where  $\varepsilon$  is small positive parameter,  $0 < s < 1$ ,  $2 \leq p < +\infty$ ,  $N = ps$ , the potential  $V$  is bounded below by  $V_0 > 0$ , the nonlinearity  $f$  has exponential critical growth, and  $(-\Delta)_p^s$  is the fractional  $p$ -Laplace operator which may be defined along a function  $\varphi \in C_0^\infty(\mathbb{R}^N)$  (up to a normalization constant) as

$$(-\Delta)_p^s \varphi(x) = 2 \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|\varphi(x) - \varphi(y)|^{p-2} (\varphi(x) - \varphi(y))}{|x - y|^{N+ps}} dy$$

for  $x \in \mathbb{R}^N$ , where  $B_\varepsilon(x)$  is a ball with center  $x$  and radius  $\varepsilon$ .

In order to study the problem (1.1), we need some assumptions on  $V$  and  $f$  as follows:

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(V)  $V: \mathbb{R}^N \rightarrow \mathbb{R}$  is a continuous function satisfying

$$V_\infty = \liminf_{|x| \rightarrow \infty} V(x) > V_0 = \inf_{x \in \mathbb{R}^N} V(x) > 0,$$

where  $V_\infty < \infty$  or  $V_\infty = \infty$ . This kind of hypothesis was introduced by Rabinowitz in [34].

( $f_1$ ) The nonlinearity  $f \in C^1(\mathbb{R})$  such that  $f(t) = 0$  for all  $t \in (-\infty, 0]$ ,  $f(t) > 0$  for all  $t > 0$  and there exist constants  $\alpha_0 \in (0, \alpha_*)$ ,  $b_1, b_2 > 0$  such that for any  $t \in \mathbb{R}$ ,

$$|f'(t)| \leq b_1|t|^{p-2} + b_2|t|^{p-2}\Phi_{N,s}(\alpha_0|t|^{N/(N-s)}),$$

where  $\Phi_{N,s}(y) = e^y - \sum_{i=0}^{j_p-2} \frac{y^i}{i!}$ ,  $j_p = \min\{j \in \mathbb{N}: j \geq p\}$  and  $\alpha_*$  is given in the Lemma 1.

( $f_2$ ) There exist  $\mu > \frac{N}{s}$  such that

$$f(t)t - \mu F(t) \geq 0$$

for all  $t \in \mathbb{R}$ , where  $F(t) = \int_0^t f(\tau) d\tau$ .

( $f_3$ )

$$\lim_{t \rightarrow 0^+} \frac{f'(t)}{t^{\frac{N}{s}-2}} = 0.$$

( $f_4$ ) There exists  $\gamma_1 > 0$  large enough such that  $F(t) \geq \gamma_1|t|^\mu$  for all  $t \geq 0$ .

( $f_5$ )  $\frac{f(t)}{t^{p-1}}$  is a strictly increasing function of  $t \geq 0$ .

**Remark 1.** (i) From the condition ( $f_3$ ), we have

$$\lim_{t \rightarrow 0^+} \frac{f(t)}{t^{\frac{N}{s}-1}} = \lim_{t \rightarrow 0^+} \frac{F(t)}{t^{\frac{N}{s}}} = 0.$$

(ii) The condition ( $f_5$ ) implies that the function  $\frac{1}{p}f(t)t - F(t)$  is an increasing function of  $t \geq 0$ . Indeed, we have

$$\left( \frac{1}{p}f(t)t - F(t) \right)'_t = \frac{f'(t)t - (p-1)f(t)}{p} > 0$$

for all  $t > 0$  due to the condition ( $f_5$ ).

(iii) The condition ( $f_5$ ) leads to that  $f(t) > 0$  for all  $t > 0$ . Indeed, we have  $\frac{f(t)}{t^{p-1}} > \lim_{t \rightarrow 0^+} \frac{f(t)}{t^{p-1}} = 0$  for all  $t > 0$ . Then  $f(t) > 0$  for all  $t > 0$ .

(iv) We have  $f(t)t$  is an increasing function on  $(0, +\infty)$ . From the condition ( $f_5$ ), we have

$$\frac{f(t_1)t_1}{t_1^p} < \frac{f(t_2)t_2}{t_2^p}$$

for all  $0 < t_1 < t_2$ . Then

$$f(t_1)t_1 < \left( \frac{t_2}{t_1} \right)^p f(t_1)t_1 < f(t_2)t_2.$$

We get the claim.

In 2019, Miyagaki and Pucci [28] have been studied the nonlocal Kirchhoff problem with critical Trudinger–Moser nonlinearity as follows:

$$-M \left( \iint_{\mathbb{R}^{2N}} |u(x) - u(y)|^2 H(x-y) dx dy + \int_{\mathbb{R}^N} V(x)|u|^2 dx \right) (\mathbb{L}_K u + V(x)u) = P(x)f(u)$$

in  $\mathbb{R}$ , where  $H$  satisfies the two following conditions:

- ( $h_1$ )  $mH \in L^1(\mathbb{R}^N)$ , where  $m(x) = \min\{|x|^2, 1\}$ ;
- ( $h_2$ ) there exists  $k_0 > 0$  such that  $H(x) \geq k_0|x|^{-2}$  for any  $x \in \mathbb{R}^N \setminus \{0\}$ .

The Kirchhoff function  $M: [0, +\infty) \rightarrow [0, +\infty)$  is continuous function and satisfies the condition as follows:

- ( $M_1$ ) For any  $\tau > 0$ , there exists  $\kappa = \kappa(\tau)$  such that  $M(t) \geq \kappa$  for all  $t \geq \tau$ .
- ( $M_2$ ) There exists  $\theta \geq 1$  such that  $tM(t) \leq \theta\mathcal{M}(t)$  for all  $t \in [0, +\infty)$ , where  $\mathcal{M}(t) = \int_0^t M(\tau) d\tau$ .
- ( $M_3$ ) The function  $\theta\mathcal{M}(t) - tM(t)$  is nondecreasing on  $[0, +\infty)$ .

The nonlinear function  $f$  satisfies the subcritical exponential growth or critical exponential growth,  $V$  and  $P$  satisfy some following conditions:

- (i) The potentials  $V$  and  $P$  are continuous and strictly positive in  $\mathbb{R}$ ;
- (ii) If  $\{A_n\}$  is a sequence of Borel sets of  $\mathbb{R}$ , with  $|A_n| \leq R$  for all  $n \in \mathbb{N}$  and some  $R > 0$ , then

$$\lim_{r \rightarrow \infty} \int_{A_n \cap B_r^c(0)} P(x) dx = 0, \quad \text{uniformly with respect to } n \in \mathbb{N},$$

where  $B_r^c(0)$  is the complement of the closed interval  $B_r = [-r, r]$ .

- (iii) The potential  $P$  is in  $L^\infty(\mathbb{R})$  and there exists  $C_0 > 0$  such that  $V(x) \geq C_0$  for all  $x \in \mathbb{R}$ .

In the work of Miyagaki and Pucci, the potential is bounded from below by a positive constant. In order to study their problem, they need the nonlinear function with the form  $P(x)f(u)$ , where  $P$  and  $V$  satisfy the conditions (i) to (iii). With these conditions, they get the compact embedding from the solution space  $E$  into the Lebesgue space with weight  $L_p^q(\mathbb{R}^N)$ ,  $q \in (2, +\infty)$ .

This paper was motivated by some work that have appeared in very recently related the Schrödinger equation involving fractional  $p$ -Laplace with the form

$$(1.2) \quad \varepsilon^{ps}(-\Delta)_p^s u(x) + V(x)|u|^{p-2}u = f(u) \quad \text{in } \mathbb{R}^N, \quad N > ps,$$

[8, 10, 7, 14, 9] and the work of Miyagaki and Pucci [28]. When  $p = 2$ , the equation (1.2) becomes fractional Schrödinger equation of the type

$$(1.3) \quad \varepsilon^{2s}(-\Delta)^s u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N,$$

which has been widely investigated in the last decade [3, 4, 5, 22, 37, 23, 36, 26, 35] and references therein. The study of (1.3) is strongly motivated by the seeking of standing waves solutions for the heat fractional Schrödinger equation

$$(1.4) \quad i\varepsilon \frac{\partial \psi}{\partial t} = (-\Delta)^s \psi + (V(x) + E)\psi - f(\psi) \quad \text{for } (x, t) \in \mathbb{R}^N \times \mathbb{R},$$

which the solution has the form  $\psi(x, t) = u(x)e^{-\frac{iEt}{\varepsilon}}$ , where  $E$  is a constant. The equation (1.4) is a fundamental equation of the fractional Quantum Mechanics.

When  $s \rightarrow 1$ , the equation (1.2) becomes

$$(1.5) \quad -\varepsilon^N \Delta_N u + V(x)|u|^{N-2}u = f(u) \quad \text{in } \mathbb{R}^N,$$

which arise in a lot of applications when  $\varepsilon = 1$ , such as image processing, non-Newtonian fluids and pseudo-plastic fluids. We refer the reader to [11, 13] for more details. In [34] Rabinowitz used the variational methods to prove the existence of positive solution to (1.5) for  $\varepsilon$  sufficiently small by assuming condition (V) and  $p = 2$ .

Later Wang [40] showed that these solutions concentrate at global minimum points of  $V(x)$  as  $\varepsilon \rightarrow 0$ . Denote  $M = \{x \in \mathbb{R}^N : V(x) = V_0\}$  and

$$M_\delta = \{x \in \mathbb{R}^N : \text{dist}(x, M) \leq \delta\} \quad \text{for } \delta > 0.$$

Using Lusternik–Schnirelmann category, Alves and Figueiredo [2] show that problem (1.5) has at least  $\text{cat}_{M_\delta}(M)$  positive solutions for small  $\varepsilon$  which  $f \in C^1(\mathbb{R})$  and satisfying the following conditions:

- (c<sub>1</sub>)  $f(t) = 0$  for all  $t \in (-\infty, 0]$  and  $f$  has critical growth at both  $+\infty$  and  $-\infty$ , that is there exists  $\alpha_0 > 0$  such that

$$\lim_{|s| \rightarrow +\infty} \frac{f(s)}{e^{\alpha|s|^{N/(N-1)}}} = 0 \quad \text{for } \alpha > \alpha_0$$

and

$$\lim_{|s| \rightarrow +\infty} \frac{f(s)}{e^{\alpha|s|^{N/(N-1)}}} = \infty \quad \text{for all } \alpha < \alpha_0.$$

- (c<sub>2</sub>)  $\lim_{s \rightarrow 0} \frac{|f'(s)|}{|s|^{N-2}} = 0$  and there exists  $C > 0$  such that

$$|f'(s)| \leq C \exp(\alpha_N s^{N/(N-1)})$$

for all  $s \geq 0$ , where  $\alpha_N = N w_N^{1/(N-1)}$  and  $w_N$  is the  $(N-1)$ -dimensional measure of the  $(N-1)$ -sphere.

- (c<sub>3</sub>) There exist constants  $p > N$  and  $\mu > 0$  large enough such that  $f(s) \geq \mu s^{p-1}$  for all  $s \geq 0$ .

- (c<sub>4</sub>) There exists  $C_1 > 0$  and  $\sigma \geq N$  such that

$$f'(s)s - (N-1)f(s) \geq C_1 s^\sigma$$

for all  $s \geq 0$ .

- (c<sub>5</sub>) There exists  $\theta > N$  such that

$$0 < \theta F(s) = \theta \int_0^s f(t) dt \leq s f(s)$$

for all  $s > 0$ .

- (c<sub>6</sub>) The function  $\frac{f(s)}{s^{N-1}}$  is strictly increasing in  $(0, +\infty)$ .

For more results about existence of solution to the problem (1.5), we refer the reader to [2] and references therein.

When  $s = \frac{1}{2}$  and  $N = 1$ , Alves, Do Ó and Miyagaki [1] studied the concentration of solution to the problem (1.1) with the following assumptions:

- (V)'  $V$  is bounded function and locally Hölder continuous and there exists  $V_0 > 0$  such that

- (i)  $V(x) \geq V_0$  for all  $x \in \mathbb{R}^N$ ,  
(ii) There exists a bounded interval  $\Lambda \subset \mathbb{R}$  such that

$$V_0 \equiv \inf_{\Lambda} V(x) < \min_{\partial\Lambda} V(x).$$

- (f<sub>1</sub>)'  $f: \mathbb{R} \rightarrow \mathbb{R}^+$  is a  $C^1$  function with  $f(t) = 0$  if  $t \leq 0$ .

- (f<sub>2</sub>)'  $f(t) = o(t)$  near original.

- (f<sub>3</sub>)'  $\frac{f(t)}{t}$  is an increasing function in  $\mathbb{R}^+$ .

- (f<sub>4</sub>)' There exists a constants  $p > 2$  and a suitable constant  $C_p > 0$  such that

$$f(t) \geq C_p t^{p-1} \quad \text{for all } t > 0.$$

For more result on Trundinger–Moser inequality and its application, we refer the readers to [16, 17, 18, 15, 24, 31, 39, 42, 27, 43, 20, 19] for more details.

Before starting our results, we recall some useful notations. Suppose that  $N = ps$  in our paper. The fractional Sobolev space  $W^{s,p}$  is defined by

$$W^{s,p}(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\},$$

where  $[u]_{s,p}$  denotes by the seminorm Gagliardo, that is

$$[u]_{s,p} = \left( \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{2N}} dx dy \right)^{1/p}.$$

$W^{s,p}(\mathbb{R}^N)$  is a uniformly convex Banach space (similar to [32]) with norm

$$\|u\| = \left( \|u\|_{L^p(\mathbb{R}^N)}^p + [u]_{s,p}^p \right)^{1/p}.$$

Set  $\eta > 0$ , we denote another norm on  $W^{s,p}(\mathbb{R}^N)$  as follows

$$\|u\|_\eta = \left( \eta \|u\|_{L^p(\mathbb{R}^N)}^p + [u]_{s,p}^p \right)^{1/p}.$$

Then  $\|\cdot\|$  and  $\|\cdot\|_\eta$  are two norms equivalent on  $W^{s,p}(\mathbb{R}^N)$ . For each  $\varepsilon > 0$ , let  $W_\varepsilon$  denote by the completion of  $C_0^\infty(\mathbb{R}^N)$ , with respect to the norm

$$\|u\|_{W_\varepsilon} = \left( [u]_{s,p}^p + \|u\|_{p,V,\varepsilon}^p \right)^{1/p}, \quad \|u\|_{p,V,\varepsilon}^p = \int_{\mathbb{R}^N} V(\varepsilon x) |u(x)|^p dx.$$

Then  $W_\varepsilon$  is uniformly convex Banach space (similar to [32, Lemma 10]), and then  $W_\varepsilon$  is a reflexive space. By the condition (V) and [29, Theorem 6.9], we have the embedding from  $W_\varepsilon$  into  $L^\nu(\mathbb{R}^N)$  is continuous for any  $\nu \in [\frac{N}{s}, +\infty)$ , then there exists a best constant  $S_{\nu,\varepsilon} > 0$  for all  $\nu \in [\frac{N}{s}, +\infty)$  as follows:

$$S_{\nu,\varepsilon} = \inf_{u \neq 0, u \in W_\varepsilon} \frac{\|u\|_{W_\varepsilon}}{\|u\|_{L^\nu(\mathbb{R}^N)}}.$$

This implies

$$(1.6) \quad \|u\|_{L^\nu(\mathbb{R}^N)} \leq S_{\nu,\varepsilon}^{-1} \|u\|_{W_\varepsilon} \quad \text{for all } u \in W_\varepsilon.$$

By [29, Theorem 6.9], we have the embedding from  $W^{s,N/s}(\mathbb{R}^N)$  into  $L^\nu(\mathbb{R}^N)$  is continuous for any  $\nu \in [\frac{N}{s}, +\infty)$ , then there exists a best constant  $A_{\nu,\eta} > 0$  for all  $\nu \in [\frac{N}{s}, +\infty)$  as follows:

$$A_{\nu,\eta} = \inf_{u \neq 0, u \in W^{s,N/s}(\mathbb{R}^N)} \frac{\|u\|_\eta}{\|u\|_{L^\nu(\mathbb{R}^N)}}.$$

This implies

$$(1.7) \quad \|u\|_{L^\nu(\mathbb{R}^N)} \leq A_{\nu,\eta}^{-1} \|u\|_\eta \quad \text{for all } u \in W^{s,N/s}(\mathbb{R}^N).$$

We denote  $\text{cat}_B(A)$  by the category of  $A$  with respect to  $B$ , namely the least integer  $k$  such that  $A \subset A_1 \cup \dots \cup A_k$ , where  $A_i$  ( $i = 1, \dots, k$ ) is closed and contractible in  $B$ . We set  $\text{cat}_B(\emptyset) = 0$  and  $\text{cat}_B(A) = +\infty$  if there is no integer with above property. We refer the reader to [41] for more details on Ljusternik–Schnirelmann theory. Now, we state the main result in this paper.

**Theorem 2.** *Let (V) and  $(f_1)$ – $(f_5)$  hold. Then for any  $\delta > 0$ , there exists  $\varepsilon_\delta > 0$  such that problem  $(P_\varepsilon)$  has at least  $\text{cat}_{M_\delta}(M)$  nontrivial nonnegative weak solutions for any  $0 < \varepsilon < \varepsilon_\delta$ . Moreover, if  $u_\varepsilon$  denotes one of these solutions and  $\eta_\varepsilon$  is its global maximum, then*

$$\lim_{\varepsilon \rightarrow 0^+} V(\eta_\varepsilon) = V_0.$$

**Remark 3.** Compare Theorem 2 with Theorem 1.1 [2], we do not need the condition of the form  $(c_4)$ . Therefore, when  $s \rightarrow 1^-$ , we get an improvement the result of Alves and Figueiredo [2].

**Remark 4.** We use the Nehari manifold, variational method, concentration compactness principle and Ljusternik-Schnirelmann theory to prove the main result. There are some difficulties in proving our theorem. The first difficulty is that the nonlinearity  $f$  has exponential critical growth. The second is to  $N = ps$ , then the fractional Sobolev embedding is the lack. Compare our work with the work of Alves and Figueiredo [2], we meet several new difficulties in employing the methods to deal with our problem due to the nonlocal property of equation (1.1). We recommend the readers to access the Lemma 4, Lemma 8, Lemma 10 and Lemma 12 and more results in our paper for that comment. Compare our work with the work of Miyagaki and Pucci [28], we only have continuous embedding since our nonlinear function does not contain the function  $P$  as in (i) to (iii). Hence, the method of Miyagaki and Pucci [28] is not enough to solve our problem.

The paper is organized as follows. In Section 2, we study the autonomous problem associated. In Section 3, we study the auxiliary problem. We prove the Palais–Smale condition for the energy functional and provide some tools which are useful to establish a multiplicity result. This allows us to show that the auxiliary problem has multiple solutions. In Section 4, we prove the existence of ground state solution to auxiliary problem. Finally, in Section 5, we complete the paper with the proof of Theorem 2.

## 2. Autonomous problem

In this section, we study the autonomous problem associated to (1.1) as following

$$(2.1) \quad (-\Delta)_{N/s}^s u + \eta |u|^{\frac{N}{s}-2} u = f(u) \quad \text{in } \mathbb{R}^N, \quad (P_\eta)$$

where  $\eta > 0$  is a constant.

We denote  $J_\eta: W^{s, N/s}(\mathbb{R}^N) \rightarrow \mathbb{R}$  by the corresponding energy functional for problem (2.1)

$$J_\eta(u) = \frac{1}{p} \|u\|_\eta^p - \int_{\mathbb{R}^N} F(u) dx.$$

From the condition  $(f_3)$ , there exist  $\tau > 0$  and  $\delta > 0$  such that for all  $|t| \leq \delta$ , we have

$$(2.2) \quad |f'(t)| \leq \tau |t|^{\frac{N}{s}-2}.$$

Moreover, from the condition  $(f_1)$  and  $f'$  is a continuous function, for each  $q \geq \frac{N}{s}$ , we can find a constant  $C = C(q, \delta) > 0$  such that

$$(2.3) \quad |f'(t)| \leq C |t|^{q-2} \Phi_{N,s}(\alpha_0 |t|^{N/(N-s)})$$

for all  $|t| \geq \delta$ . Combine (2.2) and (2.3), we get

$$(2.4) \quad |f'(t)| \leq \tau |t|^{\frac{N}{s}-2} + C |t|^{q-2} \Phi_{N,s}(\alpha_0 |t|^{N/(N-s)})$$

for all  $t \geq 0$ . Then we obtain

$$(2.5) \quad |f(t)| \leq \int_0^t |f'(s)| ds \leq \tau |t|^{\frac{N}{s}-1} + C |t|^{q-1} \Phi_{N,s}(\alpha_0 |t|^{N/(N-s)})$$

and

$$(2.6) \quad |F(t)| \leq \int_0^t |f(s)| ds \leq \tau |t|^{\frac{N}{s}} + C |t|^q \Phi_{N,s}(\alpha_0 |t|^{N/(N-s)})$$

for all  $t \geq 0$ .

In order to prove the result in this section, we need the following result:

**Lemma 1.** [43] *Let  $s \in (0, 1)$  and  $sp = N$ . Then for every  $0 \leq \alpha < \alpha_* \leq \alpha_{s,N}^*$ , the following inequality holds:*

$$\sup_{u \in W^{s,p}(\mathbb{R}^N), \|u\|_{W^{s,p}(\mathbb{R}^N)} \leq 1} \int_{\mathbb{R}^N} \Phi_{N,s}(\alpha |u|^{N/(N-s)}) dx < +\infty,$$

where  $\Phi_{N,s}(t) = e^t - \sum_{i=0}^{j_p-2} \frac{t^i}{i!}$ ,  $j_p = \min\{j \in \mathbb{N} : j \geq p\}$ . Moreover, for  $\alpha > \alpha_{s,N}^*$ ,

$$\sup_{u \in W^{s,p}(\mathbb{R}^N), \|u\|_{W^{s,p}(\mathbb{R}^N)} \leq 1} \int_{\mathbb{R}^N} \Phi_{N,s}(\alpha |u|^{N/(N-s)}) dx = +\infty.$$

**Remark 5.** From Lemma 1, if we use the norm  $\|\cdot\|_\eta$  on  $W^{s,N/s}(\mathbb{R}^N)$ , then we have

$$(\max\{1, \eta\})^{-1/p} \|u\|_\eta \leq \|u\|_{W^{s,p}(\mathbb{R}^N)} \leq (\min\{1, \eta\})^{-1/p} \|u\|_\eta,$$

then we get

$$\sup_{u \in W^{s,p}(\mathbb{R}^N), \|u\|_\eta \leq (\min\{1, \eta\})^{s/N}} \int_{\mathbb{R}^N} \Phi_{N,s}(\alpha |u|^{N/(N-s)}) dx < +\infty$$

for all  $0 \leq \alpha < \alpha_* \leq \alpha_{s,N}^*$ .

Using Lemma 1 and note that  $C_0^\infty(\mathbb{R}^N)$  is a density subspace of  $W^{s,p}(\mathbb{R}^N)$ , we see that  $J_\eta$  is well defined on  $W^{s,N/s}(\mathbb{R}^N)$ . Furthermore, we have

$$\begin{aligned} \langle J'_\eta(u), \varphi \rangle &= \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{\frac{N}{s}-2} (u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x - y|^{2N}} dx dy \\ &\quad + \eta \int_{\mathbb{R}^N} |u|^{\frac{N}{s}-2} u \varphi dx - \int_{\mathbb{R}^N} f(u) \varphi dx. \end{aligned}$$

**Lemma 2.** *Suppose that  $(f_1)$  and  $(f_3)$  hold. Then there exist constants positive  $t_0, \rho_0$  such that  $J_\eta(u) \geq \rho_0$  for all  $u \in W^{s,N/s}(\mathbb{R}^N)$ , with  $\|u\|_{W^{s,N/s}(\mathbb{R}^N)} = t_0$ .*

*Proof.* From (2.6), for some  $q > \frac{N}{s}$ , we have

$$|F(t)| \leq \tau |t|^{N/s} + C |t|^q \Phi_{N,s}(\alpha_0 |t|^{N/(N-s)})$$

for all  $t \in \mathbb{R}$ . Then we get

$$(2.7) \quad \begin{aligned} J_\eta(u) &= \frac{s}{N} \|u\|_\eta^{N/s} - \int_{\mathbb{R}^N} F(u) dx \\ &\geq \frac{s}{N} \|u\|_\eta^{N/s} - \tau \int_{\mathbb{R}^N} |u|^{N/s} dx - C \int_{\mathbb{R}^N} |u|^q \Phi_{N,s}(\alpha_0 |u|^{N/(N-s)}) dx. \end{aligned}$$

Using Hölder inequality, we have

$$(2.8) \quad \int_{\mathbb{R}^N} |u|^q \Phi_{N,s}(\alpha_0 |u|^{N/(N-s)}) dx \leq \left( \int_{\mathbb{R}^N} (\Phi_{N,s}(\alpha_0 |u|^{N/(N-s)}))^t dx \right)^{1/t} \|u\|_{L^{qt'}(\mathbb{R}^N)}^q,$$

where  $t > 1, t' > 1$  such that  $\frac{1}{t} + \frac{1}{t'} = 1$ . By Lemma 2.3 [25], for any  $\mathbf{b} > t$ , there exist a constant  $C(\mathbf{b}) > 0$  such that

$$(2.9) \quad (\Phi_{N,s}(\alpha_0 |u|^{N/(N-s)}))^t \leq C(\mathbf{b}) \Phi_{N,s}(\mathbf{b} \alpha_0 |u|^{N/(N-s)})$$

on  $\mathbb{R}^N$ . Denote by  $\mathfrak{d} = \min\{1, \eta\}$ , we get

$$(2.10) \quad \begin{aligned} & \int_{\mathbb{R}^N} (\Phi_{N,s}(\alpha_0 |u|^{N/(N-s)}))^t \leq C(\mathbf{b}) \int_{\mathbb{R}^N} \Phi_{N,s}(\mathbf{b} \alpha_0 |u|^{N/(N-s)}) dx \\ & = C(\mathbf{b}) \int_{\mathbb{R}^N} \Phi_{N,s}(\mathbf{b} \alpha_0 \mathfrak{d}^{-s/(N-s)} \|u\|_{\eta}^{N/(N-s)} |\mathfrak{d}^{s/N} u| \|u\|_{\eta}^{N/(N-s)}) dx. \end{aligned}$$

When  $\|u\|_{\eta}$  is small enough and  $\mathbf{b}$  near  $t$ , we have

$$(2.11) \quad \mathbf{b} \alpha_0 \mathfrak{d}^{-s/(N-s)} \|u\|_{\eta}^{N/(N-s)} \leq \beta_* < \alpha_*$$

by Remark 5, (2.10) and (2.11), there exists a constant  $D > 0$  such that

$$\left( \int_{\mathbb{R}^N} (\Phi_{N,s}(\alpha_0 |u|^{N/(N-s)}))^t dx \right)^{1/t} \leq D.$$

Since the embedding from  $W_{s,N/s}(\mathbb{R}^N) \rightarrow L^{qt'}(\mathbb{R}^N)$  is continuous, we get

$$(2.12) \quad \int_{\mathbb{R}^N} |u|^q \Phi_{N,s}(\alpha_0 |u|^{N/(N-s)}) dx \leq D A_{qt',\eta}^{-q} \|u\|_{\eta}^q < +\infty.$$

From (1.7), we have

$$(2.13) \quad \|u\|_{L^{N/s}(\mathbb{R}^N)} \leq A_{N/s,\eta}^{-1} \|u\|_{\eta} \quad \text{for all } u \in W^{s,N/s}(\mathbb{R}^N).$$

Hence, combine (2.7), (2.12) and (2.13), we obtain

$$(2.14) \quad \begin{aligned} J_{\eta}(u) & \geq \frac{s}{N} \|u\|_{\eta}^{N/s} - \tau A_{N/s,\eta}^{-N/s} \|u\|_{\eta}^{N/s} - C D A_{qt',\eta}^{-q} \|u\|_{\eta}^q \\ & = \|u\|_{\eta}^{N/s} \left[ \left( \frac{s}{N} - \tau A_{N/s,\eta}^{-N/s} \right) - C D A_{qt',\eta}^{-q} \|u\|_{\eta}^{q-\frac{N}{s}} \right]. \end{aligned}$$

We see  $\frac{s}{N} - \tau A_{N/s,\eta}^{-N/s} > 0$  for  $\tau$  small enough. Let

$$h(t) = \frac{s}{N} - \tau A_{N/s,\eta}^{-N/s} - C D A_{qt',\eta}^{-q} t^{q-\frac{N}{s}}, \quad t \geq 0.$$

We now prove there exists  $t_0 > 0$  small satisfying  $h(t_0) \geq \frac{1}{2}(\frac{s}{N} - \tau A_{N/s,\eta}^{-N/s})$ . We see that  $h$  is continuous function on  $[0, +\infty)$  and  $\lim_{t \rightarrow 0^+} h(t) = \frac{s}{N} - \tau A_{N/s,\eta}^{-N/s}$ , then there exists  $t_0$  such that  $h(t) \geq \frac{s}{N} - \tau A_{N/s,\eta}^{-N/s} - \varepsilon_1$  for all  $0 \leq t \leq t_0$ ,  $t_0$  is small enough such that  $\|u\|_{\eta} = t_0$  satisfies (2.11). If we choose  $\varepsilon_1 = \frac{1}{2}(\frac{s}{N} - \tau A_{N/s,\eta}^{-N/s})$ , we have

$$h(t) \geq \frac{1}{2} \left( \frac{s}{N} - \tau A_{N/s,\eta}^{-N/s} \right)$$

for all  $0 \leq t \leq t_0$ . Especially,

$$(2.15) \quad h(t_0) \geq \frac{1}{2} \left( \frac{s}{N} - \tau A_{N/s,\eta}^{-N/s} \right).$$



From (2.14) and (2.15), for  $\|u\|_\eta = t_0$ , we have

$$J_\eta(u) \geq \frac{t_0^{N/s}}{2} \cdot \left( \frac{s}{N} - \tau A_{N/s,\eta}^{-N/s} \right) = \rho_0. \quad \square$$

**Lemma 3.** *Suppose that (f<sub>4</sub>) hold. Then there exists a function  $v \in C_0^\infty(\mathbb{R}^N)$  with  $\|v\|_\eta > t_0$ , such that  $J_\eta(v) < 0$ , where  $t_0 > 0$  is the number given in Lemma 2.*

*Proof.* For all  $u \in C_0^\infty(\mathbb{R}^N)$  with  $\|u\|_\eta = 1$ , from (f<sub>4</sub>) and all  $t > 0$ , we obtain

$$J_\eta(tu) = \frac{st^{N/s}}{N} \|u\|_\eta^{N/s} - \int_{\mathbb{R}^N} F(tu) \, dx \leq \frac{st^{N/s}}{N} \|u\|_\eta^{N/s} - \gamma_1 t^\mu \int_{\mathbb{R}^N} |u(x)|^\mu \, dx.$$

By (1.7), for all  $\nu \in [\frac{N}{s}, \mu)$ , we have

$$0 < \frac{1}{A_{\nu,\eta} + \varepsilon} = \frac{\|u\|_\eta}{A_{\nu,\eta} + \varepsilon} \leq \|u\|_{L^\nu(\mathbb{R}^N)} \leq A_{\nu,\eta}^{-1} \|u\|_\eta = A_{\nu,\eta}^{-1} < +\infty,$$

where  $\varepsilon > 0$ . Since  $\mu > \frac{N}{s}$ ,  $t^{N/s}$  has growth smaller than  $t^\mu$  as  $t \rightarrow +\infty$ , then we have  $J_\eta(tu) \rightarrow -\infty$  as  $t \rightarrow +\infty$ . Taking  $v = \rho_1 u$ ,  $\rho_1 > t_0 > 0$  large enough, we have  $J_\eta(v) < 0$ ,  $\|v\|_\eta > t_0$ .  $\square$

From Lemma 2, Lemma 3 and a version of Mountain Pass Theorem without the Palais–Smale condition, we get a sequence  $\{u_n\} \subset W^{s,N/s}(\mathbb{R}^N)$  such that

$$J_\eta(u_n) \rightarrow c_\eta \quad \text{and} \quad J'_\eta(u_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

where the level  $c_\eta$  is characterized by

$$c_\eta = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\eta(\gamma(t))$$

and  $\Gamma = \{\gamma \in C([0, 1], W^{s,N/s}(\mathbb{R}^N)) : \gamma(0) = 0, J_\eta(\gamma(1)) < 0\}$ .

**Lemma 4.** *Let  $\{u_n\}$  be  $(PS)_{c_\eta}$  sequence for  $J_\eta$ . Then*

- (i) *there exists a constant  $C_{\gamma_1}$  such that  $\rho_0 \leq c_\eta \leq C_{\gamma_1}$ ,*
- (ii)  *$u_n \rightarrow u$  weakly in  $W^{s,N/s}(\mathbb{R}^N)$  and  $J'_\eta(u) = 0$ .*

*Proof.* We choose a function  $w \in W^{s,N/s}(\mathbb{R}^N) \setminus \{0\}$  such that  $\|w\|_{L^\mu(\mathbb{R}^N)} = 1$  and  $\|w\|_\eta = A_{\mu,\eta}$ . It means

$$\frac{\|w\|_\eta}{\|w\|_{L^\mu(\mathbb{R}^N)}} = A_{\mu,\eta} = \inf_{u \in W^{s,N/s}(\mathbb{R}^N), u \neq 0} \frac{\|u\|_\eta}{\|u\|_{L^\mu(\mathbb{R}^N)}}.$$

Then, from  $A_{N/s,\eta} \leq \frac{\|w\|_\eta}{\|w\|_{L^{N/s}(\mathbb{R}^N)}}$ , we get  $\|w\|_{L^{N/s}(\mathbb{R}^N)} \leq A_{N/s,\eta}^{-1}$ . We see that

$$\begin{aligned} c_\eta &\leq \max_{t \geq 0} J_\eta(tw) \leq \max_{t \geq 0} \left\{ \frac{st^{N/s}}{N} \|w\|_\eta^{N/s} - \gamma_1 t^\mu \int_{\mathbb{R}^N} |w(x)|^\mu \, dx \right\} \\ (2.16) \quad &= \max_{t \geq 0} \left\{ \frac{sA_{\mu,\eta}^{N/s} t^{N/s}}{N} - \gamma_1 t^\mu \right\}. \end{aligned}$$

Set  $g(t) = \frac{sA_{\mu,\eta}^{N/s}}{N} t^{N/s} - \gamma_1 t^\mu$  on  $[0, +\infty)$ . We are easy to get

$$g(t) \leq g(\theta_{\gamma_1}) = C_{\gamma_1}$$

on  $[0, +\infty)$ , where

$$\theta_{\gamma_1} = \left( \frac{A_{\mu,\eta}^{N/s}}{\gamma_1 \mu} \right)^{s/(\mu s - N)}.$$

Compute directly, we get

$$\begin{aligned}
 C_{\gamma_1} &= g(\theta_{\gamma_1}) = \theta_{\gamma_1}^{N/s} \left[ \frac{sA_{\mu,\eta}^{N/s}}{N} - \gamma_1 \theta_{\gamma_1}^{\frac{\mu s - N}{s}} \right] \\
 (2.17) \quad &= \left( \frac{A_{\mu,\eta}^{N/s}}{\gamma_1 \mu} \right)^{N/(\mu s - N)} \left( \frac{s}{N} - \frac{1}{\mu} \right) A_{\mu,\eta}^{N/s}.
 \end{aligned}$$

We see that  $\lim_{\gamma_1 \rightarrow +\infty} \theta_{\gamma_1} = 0$ , then  $\lim_{\gamma_1 \rightarrow +\infty} g(\theta_{\gamma_1}) = 0$ . Therefore, the Mountain Pass level  $c$  is small enough when  $\gamma_1$  is large enough, which will be used later. Combine Lemma 2 and (2.16), we get  $\rho_0 \leq c_\eta \leq C_{\gamma_1}$ .

Note that  $\{u_n\}$  is a (PS) sequence with level  $c_\eta \in \mathbb{R}$  in  $W^{s,N/s}(\mathbb{R}^N)$ . This means

$$(2.18) \quad J_\eta(u_n) \rightarrow c_\eta \quad \text{and} \quad \sup_{\|\varphi\|_\eta=1} |\langle J'_\eta(u_n), \varphi \rangle| \rightarrow 0$$

as  $n \rightarrow \infty$ . We show that the sequence  $\{u_n\}$  is bounded in  $W^{s,N/s}(\mathbb{R}^N)$ . From (2.18), we have

$$\left\langle J'_\eta(u_n), \frac{u_n}{\|u_n\|_\eta} \right\rangle = o_n(1) \quad \text{and} \quad J_\eta(u_n) = c_\eta + o_n(1)$$

when  $n$  large enough. It implies

$$(2.19) \quad J_\eta(u_n) - \frac{1}{\mu} \langle J'_\eta(u_n), u_n \rangle = c_\eta + o_n(1) + o_n(1) \|u_n\|_\eta,$$

where  $\mu$  is a parameter in the condition  $(f_2)$ . We have

$$\begin{aligned}
 &J_\eta(u_n) - \frac{1}{\mu} \langle J'_\eta(u_n), u_n \rangle \\
 &= \frac{s}{N} \|u_n\|_\eta^{N/s} - \int_{\mathbb{R}^N} F(u_n) dx - \frac{1}{\mu} \left[ \|u_n\|_\eta^{N/s} - \int_{\mathbb{R}^N} f(u_n) u_n dx \right] \\
 &= \left( \frac{s}{N} - \frac{1}{\mu} \right) \|u_n\|_\eta^{N/s} + \frac{1}{\mu} \int_{\mathbb{R}^N} f(u_n) u_n - \mu F(u_n) dx.
 \end{aligned}$$

Therefore, we have

$$(2.20) \quad J_\eta(u_n) - \frac{1}{\mu} \langle J'_\eta(u_n), u_n \rangle \geq \left( \frac{s}{N} - \frac{1}{\mu} \right) \|u_n\|_\eta^{N/s}.$$

Combine (2.19) and (2.20), we get

$$(2.21) \quad \left( \frac{s}{N} - \frac{1}{\mu} \right) \|u_n\|_\eta^{N/s} \leq c_\eta + o_n(1) + o_n(1) \|u_n\|_\eta.$$

From (2.21), we conclude that the sequence  $\{u_n\}$  is bounded in  $W^{s,N/s}(\mathbb{R}^N)$ . Since

$$J_\eta(u_n) - \frac{1}{\mu} \langle J'_\eta(u_n), u_n \rangle \rightarrow c_\eta$$

as  $n \rightarrow \infty$ , then

$$(2.22) \quad \limsup_{n \rightarrow \infty} \|u_n\|_\eta^{N/s} \leq \frac{c_\eta}{\frac{s}{N} - \frac{1}{\mu}} \leq \frac{C_{\gamma_1}}{\frac{s}{N} - \frac{1}{\mu}}.$$

Going if necessary to a subsequence, for any  $q \geq \frac{N}{s}$ , we have

$$\begin{aligned}
 &u_n \rightarrow u \text{ weakly in } W^{s,N/s}(\mathbb{R}^N), u_n \rightarrow u \text{ in } L^q_{\text{loc}}(\mathbb{R}^N) \\
 &u_n \rightarrow u \text{ in } \mathbb{R}^N \text{ outside a set with measure zero.}
 \end{aligned}$$

Using Trudinger–Moser inequality, Vitali’s theorem and by arguments as [12, Lemma 5], we can prove that  $J'_\eta(u) = 0$ . However, for convenience to the readers, we give a detailed proof at here. We need prove that  $\langle J'_\eta(u), \varphi \rangle = 0$  for all  $\varphi \in W^{s,N/s}(\mathbb{R}^N)$ . First we show that

$$(2.23) \quad \int_{\mathbb{R}^N} f(u_n)\varphi \, dx \rightarrow \int_{\mathbb{R}^N} f(u)\varphi \, dx$$

for all  $\varphi \in W^{s,N/s}(\mathbb{R}^N)$ . From (2.5) for  $q = \frac{N}{s}$ , we have

$$(2.24) \quad \begin{aligned} & \int_{\mathbb{R}^N} |f(u_n)\varphi| \, dx \\ & \leq b_1 \int_{\mathbb{R}^N} |u_n|^{p-1}|\varphi| \, dx + b_2 \int_{\mathbb{R}^N} |u_n|^{p-1}\Phi_{N,s}(\alpha_0|u_n|^{N/(N-s)})|\varphi| \, dx \end{aligned}$$

for some constants  $b_1 > 0, b_2 > 0$ . Using Hölder inequality, we get

$$(2.25) \quad \int_{\mathbb{R}^N} |u_n|^{p-1}|\varphi| \, dx \leq \|u_n\|_{L^p(\mathbb{R}^N)}^{p-1} \|\varphi\|_{L^p(\mathbb{R}^N)} < +\infty$$

due to the bounded of sequence  $\{u_n\}$  in  $W^{s,N/s}(\mathbb{R}^N)$  and the continuous of embedding  $W^{s,N/s}(\mathbb{R}^N)$  into  $L^{N/s}(\mathbb{R}^N)$ . Still using Hölder inequality for  $q \geq \frac{N}{s} = p, q_* \geq \frac{p}{p-1}, q' > 1$  near 1, note that  $\frac{1}{q} + \frac{1}{q'} + \frac{1}{q_*} = 1$ , we deduce

$$(2.26) \quad \begin{aligned} & \int_{\mathbb{R}^N} |u_n|^{p-1}\Phi_{N,s}(\alpha_0|u_n|^{N/(N-s)})|\varphi| \, dx \\ & \leq \left( \int_{\mathbb{R}^N} |u_n|^{q_*(p-1)} \, dx \right)^{1/q_*} \left( \int_{\mathbb{R}^N} (\Phi_{N,s}(\alpha_0|u_n|^{N/(N-s)}))^{q'} \, dx \right)^{1/q'} \left( \int_{\mathbb{R}^N} |\varphi|^q \, dx \right)^{1/q} \\ & = \left( \int_{\mathbb{R}^N} (\Phi_{N,s}(\alpha_0|u_n|^{N/(N-s)}))^{q'} \, dx \right)^{1/q'} \|u_n\|_{L^{q_*(p-1)}(\mathbb{R}^N)}^{p-1} \|\varphi\|_{L^q(\mathbb{R}^N)}. \end{aligned}$$

By [25, Lemma 2.3], choose  $\mathbf{c} > q' > 1$ ,  $\mathbf{c}$  is near  $q'$ , there exist a constant  $C(\mathbf{c}) > 0$  such that

$$(2.27) \quad (\Phi_{N,s}(\alpha_0|u|^{N/(N-s)}))^{q'} \leq C(\mathbf{c})\Phi_{N,s}(\mathbf{c}\alpha_0|u|^{N/(N-s)})$$

for all  $u \in W^{s,N/s}(\mathbb{R}^N)$ . By (2.17) and (2.22), we see that

$$(2.28) \quad \limsup_{n \rightarrow \infty} \|u_n\|_\eta^{N/s} \leq \left( \frac{A_{\mu,\eta}^{N/s}}{\gamma_1\mu} \right)^{N/(\mu s - N)} \left( \frac{s}{N} - \frac{1}{\mu} \right) A_{\mu,\eta}^{N/s} \frac{1}{\frac{s}{N} - \frac{1}{\mu}}.$$

Combining (2.28) and (2.11), we have

$$(2.29) \quad \mathbf{c}\alpha_0\mathfrak{D}^{-s/(N-s)} \sup_n \|u_n\|_\eta^{N/(N-s)} < \alpha_*$$

when  $\gamma_1 \geq \gamma_0$ , where  $\gamma_0$  satisfies

$$(2.30) \quad \mathbf{c}\alpha_0\mathfrak{D}^{-s/(N-s)} \left( \frac{A_{\mu,\eta}^{N/s}}{\gamma_0\mu} \right)^{\frac{sN}{(\mu s - N)(N-s)}} A_{\mu,\eta}^{N/(N-s)} < \alpha_*.$$

Then, applying to Remark 5, we deduce

$$(2.31) \quad \begin{aligned} & \sup_n \int_{\mathbb{R}^N} \Phi_{N,s}(\mathbf{c}\alpha_0|u_n|^{N/(N-s)}) \, dx \\ & = \sup_n \int_{\mathbb{R}^N} \Phi_{N,s}(\mathbf{c}\alpha_0\mathfrak{D}^{-s/(N-s)} \|u_n\|_\eta^{N/(N-s)} (\mathfrak{D}^{s/N} u / \|u_n\|_\eta)^{N/(N-s)}) \, dx < +\infty. \end{aligned}$$

From (2.26) and (2.31), we get

$$(2.32) \quad \int_{\mathbb{R}^N} |u_n|^{p-1} \Phi_{N,s}(\alpha_0 |u_n|^{N/(N-s)}) |\varphi| dx < +\infty.$$

Combining (2.25) and (2.32), we obtain

$$(2.33) \quad \int_{\mathbb{R}^N} |f(u_n)\varphi| dx < +\infty,$$

hence  $f(u_n)\varphi \in L^1(\mathbb{R}^N)$  for all  $n$ . Then there exists a constant  $\kappa > 0$  such that  $|f(u_n)\varphi| \leq \kappa$  for all  $n \in \mathbb{N}$ . For any  $\varepsilon > 0$ , there exists  $\delta = \frac{\varepsilon}{\kappa}$  such that for all measurable set  $E \subset \mathbb{R}^N$  such that  $|E| < \delta$ , we have

$$\int_E |f(u_n)\varphi| dx \leq \kappa |E| < \varepsilon.$$

It means that  $\{f(u_n)\varphi\}$  is equi-integrable. Clearly,  $f(u_n)\varphi \rightarrow f(u)\varphi$  almost everywhere on  $\mathbb{R}^N$ . Since  $\varphi \in W^{s,N/s}(\mathbb{R}^N)$  and  $W^{s,N/s}(\mathbb{R}^N)$  is continuously embedded into  $L^q(\mathbb{R}^N)$  ( $q \geq \frac{N}{s}$ ), then  $\|\varphi\|_{L^q(\mathbb{R}^N)} \leq A_{q,\eta}^{-1} \|\varphi\|_\eta < +\infty$ . Then there exists  $R > 0$  such that

$$(2.34) \quad \int_{\mathbb{R}^N \setminus B_R(0)} |\varphi|^{N/s} dx < \varepsilon^{N/s} \quad \text{and} \quad \int_{\mathbb{R}^N \setminus B_R(0)} |\varphi|^q dx < \varepsilon^q.$$

By arguments as (2.25), (2.26), (2.31) and combine with (2.34), we only take integral on  $\mathbb{R}^N \setminus B_R(0)$  and get

$$\int_{\mathbb{R}^N \setminus B_R(0)} |f(u_n)\varphi| dx \leq \kappa_* \varepsilon,$$

where  $\kappa_*$  is a suitable constant. Therefore, all conditions of Vitali's theorem are satisfied and (2.23) is proved. Similarly, we have

$$(2.35) \quad \int_{\mathbb{R}^N} |u_n|^{\frac{N}{s}-2} u_n \varphi dx \rightarrow \int_{\mathbb{R}^N} |u|^{\frac{N}{s}-2} u \varphi dx$$

as  $n \rightarrow \infty$ . Finally, we prove that

$$(2.36) \quad \begin{aligned} & \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2} (u_n(x) - u_n(y)) (\varphi(x) - \varphi(y))}{|x - y|^{2N}} dx dy \\ & \rightarrow \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x - y|^{2N}} dx dy. \end{aligned}$$

Using Hölder inequality, we see

$$(2.37) \quad \begin{aligned} & \int_{\mathbb{R}^{2N}} \left| \frac{|u_n(x) - u_n(y)|^{p-2} (u_n(x) - u_n(y)) (\varphi(x) - \varphi(y))}{|x - y|^{2N}} \right| dx dy \\ & \leq \left( \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{N/s}}{|x - y|^{2N}} dx dy \right)^{(\frac{N}{s}-1)/(N/s)} \\ & \quad \cdot \left( \int_{\mathbb{R}^{2N}} \frac{|\varphi(x) - \varphi(y)|^{N/s}}{|x - y|^{2N}} dx dy \right)^{1/(N/s)} \\ & \leq \|u_n\|_\eta^{\frac{N}{s}-1} \|\varphi\|_\eta < +\infty. \end{aligned}$$

Hence

$$\frac{|u_n(x) - u_n(y)|^{p-2} (u_n(x) - u_n(y)) (\varphi(x) - \varphi(y))}{|x - y|^{2N}} \in L^1(\mathbb{R}^{2N})$$

for all  $n$ , and there exists a constant  $K > 0$  such that

$$\left| \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{2N}} \right| \leq K$$

for all  $(x, y) \in \mathbb{R}^{2N}$  outside a set with measure zero. For any  $\varepsilon > 0$ , there exists  $\delta = \frac{\varepsilon}{K}$  such that for all measurable set  $E \subset \mathbb{R}^{2N}$  such that  $|E| < \delta$ , we have

$$\int_E \left| \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{2N}} \right| dx dy \leq K|E| < \varepsilon.$$

It means that  $\left\{ \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{2N}} \right\}$  is equi-integrable on  $\mathbb{R}^{2N}$ . Clearly,

$$\begin{aligned} & \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{2N}} \\ & \rightarrow \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{2N}} \end{aligned}$$

almost everywhere on  $\mathbb{R}^{2N}$ . Since  $\varphi \in W^{s, N/s}(\mathbb{R}^N)$ , then there exists  $R > 0$  such that

$$\int_{\mathbb{R}^{2N} \setminus \mathcal{B}_R(0)} \frac{|\varphi(x) - \varphi(y)|^{N/s}}{|x - y|^{2N}} dx dy < \varepsilon^{N/s},$$

where  $\mathcal{B}_R(0)$  is a ball in  $\mathbb{R}^{2N}$  with center 0 and radius  $R$ . By arguments as (2.37) and we only take integral on  $\mathbb{R}^{2N} \setminus \mathcal{B}_R(0)$ , note that  $\{u_n\}$  is a bounded sequence in  $W^{s, N/s}(\mathbb{R}^N)$ , then there exists a suitable constant  $K_* > 0$  such that

$$\begin{aligned} & \int_{\mathbb{R}^{2N} \setminus \mathcal{B}_R(0)} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{2N}} dx dy \\ & \leq \left( \int_{\mathbb{R}^{2N} \setminus \mathcal{B}_R(0)} \frac{|u_n(x) - u_n(y)|^{N/s}}{|x - y|^{2N}} dx dy \right)^{(\frac{N}{s}-1)/(N/s)} \\ & \quad \cdot \left( \int_{\mathbb{R}^{2N} \setminus \mathcal{B}_R(0)} \frac{|\varphi(x) - \varphi(y)|^{N/s}}{|x - y|^{2N}} dx dy \right)^{1/(N/s)} < K_* \varepsilon. \end{aligned}$$

Therefore all conditions of Vitali's theorem hold and we get (2.36). Combine (2.18), (2.23), (2.35) and (2.36), we get  $\langle J'_\eta(u), \varphi \rangle = 0$  for all  $\varphi \in W^{s, N/s}(\mathbb{R}^N)$ . Hence,  $J'_\eta(u) = 0$  on  $(W^{s, N/s}(\mathbb{R}^N))^*$  which is a dual space of  $W^{s, N/s}(\mathbb{R}^N)$ .  $\square$

The following result is the version of Lions's result:

**Lemma 5.** [12] *If  $\{u_n\}$  is a bounded sequence in  $W^{s, N/s}(\mathbb{R}^N)$  and*

$$\limsup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n(x)|^{N/s} dx = 0$$

for some  $R > 0$ , then  $u_n \rightarrow 0$  strongly in  $L^q(\mathbb{R}^N)$  for all  $q \in (\frac{N}{s}, +\infty)$ .

**Lemma 6.** [12] *Let  $\{u_n\}$  be a sequence in  $W^{s, N/s}(\mathbb{R}^N)$  convergence weakly to 0 verifying  $\limsup_{n \rightarrow \infty} \|u_n\|_\eta^{N/(N-s)} \leq \frac{\beta_* \mathfrak{d}^{s/(N-s)}}{\mathfrak{c} \alpha_0}$ , where  $\mathfrak{c} > 1$  is a suitable constant and assume that  $(f_1)$  holds and  $\lim_{t \rightarrow 0^+} \frac{f(t)}{t^{\frac{N}{s}-1}} = 0$ . If there exists  $R > 0$  such that  $\liminf_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^{N/s} dx = 0$ , it follows that*

$$\int_{\mathbb{R}^N} f(u_n)u_n \rightarrow 0 \quad \text{and} \quad \int_{\mathbb{R}^N} F(u_n) \rightarrow 0.$$

**Proposition 1.** [12] *Assume that the conditions  $(f_1)$ – $(f_5)$  satisfies. Then problem (2.1) admits a nontrivial nonnegative weak solution.*

### 3. The auxiliary problem

Using the change variable  $x \mapsto \varepsilon x$ , the problem  $(P_\varepsilon)$  is equivalent to the following problem

$$(2.38) \quad (-\Delta)_p^s u + V(\varepsilon x)|u|^{p-2}u = f(u). \tag{P_\varepsilon^*}$$

**Definition 6.** We say that  $u \in W_\varepsilon$  is a weak solution of problem (2.38) if

$$\begin{aligned} & \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{\frac{N}{s}-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{2N}} dx dy \\ & + \int_{\mathbb{R}^N} V(\varepsilon x)|u(x)|^{\frac{N}{s}-2}u(x)\varphi(x) dx = \int_{\mathbb{R}^N} f(u(x))\varphi(x) dx \end{aligned}$$

for any  $\varphi \in W_\varepsilon$ .

In order to study the equation (2.38), we consider the energy functional  $I_\varepsilon : W_\varepsilon \rightarrow \mathbb{R}$  given by

$$I_\varepsilon(u) = \frac{1}{p} \|u\|_{W_\varepsilon}^p - \int_{\mathbb{R}^N} F(u) dx.$$

By the condition  $(f_1)$ ,  $I_\varepsilon$  is well defined on  $W_\varepsilon$ ,  $I_\varepsilon \in C^2(W_\varepsilon, \mathbb{R})$  and its critical points are weak solution of problem (2.38). Associated to  $I_\varepsilon$ , we consider the Nehari manifold  $\mathcal{N}_\varepsilon$  given by

$$\mathcal{N}_\varepsilon = \{u \in W_\varepsilon \setminus \{0\} : \langle I'_\varepsilon(u), u \rangle = 0\},$$

where

$$\begin{aligned} \langle I'_\varepsilon(u), \varphi \rangle &= \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+ps}} dx dy \\ &+ \int_{\mathbb{R}^N} V(\varepsilon x)|u|^{p-2}u\varphi dx - \int_{\mathbb{R}^N} f(u)\varphi dx \end{aligned}$$

for any  $u, \varphi \in W_\varepsilon$ .

**Proposition 2.** *There exists  $r_* > 0$  such that*

$$\|u\|_{W_\varepsilon} \geq r_* > 0 \quad \text{for all } u \in \mathcal{N}_\varepsilon.$$

*Proof.* We are easy to get the inequality

$$(2.39) \quad \|u\|_{W^{s,p}(\mathbb{R}^N)} \leq \min\{1, V_0\}^{-1/p} \|u\|_{W_\varepsilon}.$$

Then from Lemma 1 and (2.39), we have

$$(2.40) \quad \sup_{u \in W_\varepsilon, \|u\|_{W_\varepsilon} \leq (\min\{1, V_0\})^{s/N}} \int_{\mathbb{R}^N} \Phi_{N,s}(\alpha |u|^{N/(N-s)}) dx < +\infty$$

for all  $0 \leq \alpha < \alpha_* \leq \alpha_{s,N}^*$ . From the condition  $(f_1)$  and  $(f_3)$ , for any  $\varepsilon_* > 0$  and  $q > \frac{N}{s}$ , there exists  $C_{q,\varepsilon_*} > 0$  such that

$$(2.41) \quad |f(t)t| \leq \varepsilon_* |t|^{N/s} + C_{q,\varepsilon_*} |t|^q \Phi_{N,s}(\alpha_0 |t|^{N/(N-s)})$$

for all  $t \geq 0$ . Using inequality (2.40) and by arguments as Lemma 2, there exists a constant  $C(q, \varepsilon_*)$  such that

$$(2.42) \quad \int_{\mathbb{R}^N} f(u)u \, dx \leq \varepsilon_* S_{N/s, \varepsilon}^{-N/s} \|u\|_{W_\varepsilon}^{N/s} + C(q, \varepsilon_*) \|u\|_{W_\varepsilon}^q$$

for some  $q > \frac{N}{s}$  and all  $u \in \mathcal{N}_\varepsilon$  and  $\|u\|_{W_\varepsilon}$  is small enough. Assume that by contradiction, there exists a sequence  $\{u_n\} \subset \mathcal{N}_\varepsilon$  such that  $\|u_n\|_{W_\varepsilon} \rightarrow 0$  as  $n \rightarrow \infty$ . Hence (2.42) holds for  $u = u_n$  when  $n$  is large enough. From the definition of  $\mathcal{N}_\varepsilon$ , we get

$$\|u_n\|_{W_\varepsilon}^{N/s} = \int_{\mathbb{R}^N} f(u_n)u_n \, dx \leq \varepsilon_* S_{N/s, \varepsilon}^{-N/s} \|u_n\|_{W_\varepsilon}^{N/s} + C(q, \varepsilon_*) \|u_n\|_{W_\varepsilon}^q.$$

Divide both sides of above inequality to  $\|u_n\|_{W_\varepsilon}^{N/s}$  and take  $n \rightarrow \infty$ , we get a contradiction when  $\varepsilon_*$  is small enough. Therefore, there exists  $r_* > 0$  such that  $\|u\|_{W_\varepsilon} \geq r_* > 0$  for all  $u \in \mathcal{N}_\varepsilon$ .  $\square$

**Lemma 7.** *The functional  $I_\varepsilon$  satisfies the following conditions:*

- (i) *There exists  $\alpha > 0, \rho > 0$  such that  $I_\varepsilon(u) \geq \alpha$  for all  $u \in W_\varepsilon$  with  $\|u\|_{W_\varepsilon} = \rho$ .*
- (ii) *There exists  $e \in W_\varepsilon$  with  $\|e\|_{W_\varepsilon} > \rho$  such that  $I_\varepsilon(e) < 0$ .*

*Proof.* Lemma 7 is proved similarly Lemma 2 and Lemma 3 and using the inequality (2.40). We omit the details.  $\square$

From Lemma 7 and the version of Mountain Pass Theorem, there exists a  $(PS)_{c_\varepsilon}$  sequence  $\{u_n\} \subset W_\varepsilon$ , that is,

$$I_\varepsilon(u_n) \rightarrow c_\varepsilon \quad \text{and} \quad I'_\varepsilon(u_n) \rightarrow 0,$$

where

$$c_\varepsilon = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\varepsilon(\gamma(t))$$

and  $\Gamma = \{\gamma \in C([0, 1], W_\varepsilon) : \gamma(0) = 0, I_\varepsilon(\gamma(1)) < 0\}$ .

The following result is proved in [12] which the original idea comes from [34].

**Proposition 3.** *We have  $c_\varepsilon = \inf_{u \in W_\varepsilon \setminus \{0\}} \sup_{t \geq 0} I_\varepsilon(tu) = \inf_{u \in \mathcal{N}_\varepsilon} I_\varepsilon(u)$ .*

**Lemma 8.** *Let  $\{u_n\}$  be a bounded sequence in  $W_\varepsilon$  satisfying*

$$\limsup_{n \rightarrow \infty} \|u_n\|_{W_\varepsilon}^{N/(N-s)} < \frac{\beta_* \mathfrak{d}_*^{s/(N-s)}}{2^{N/(N-s)} \mathfrak{c} \alpha_0},$$

where  $\mathfrak{d}_* = \min\{1, V_0\}$ ,  $\mathfrak{c} > 1$  is a suitable constant and assume that  $(f_1)$  and  $(f_3)$  hold. Up to a subsequence, we may suppose that  $u_n \rightarrow u$  weakly in  $W_\varepsilon$  and  $u_n(x) \rightarrow u(x)$  everywhere in  $\mathbb{R}^N$ . Then it follows that

- (i)  $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |F(v_n + u) - F(v_n) - F(u)| \, dx = 0$ , where  $v_n = u_n - u$ .
- (ii) For any  $r > 1$  such that  $r(\frac{N}{s} - 1) \geq \frac{N}{s}$ , we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |f(v_n + u) - f(v_n) - f(u)|^r \, dx = 0.$$

*Proof.* From the condition  $(f_1)$ , we have

$$(2.43) \quad |f(t)| \leq b_1 |t|^{p-1} + b_2 |t|^{p-1} \Phi_{N,s}(\alpha_0 |t|^{N/(N-s)}),$$

$$(2.44) \quad |f(t)t| \leq b_1 |t|^{N/s} + b_2 |t|^p \Phi_{N,s}(\alpha_0 |t|^{N/(N-s)}),$$

$$(2.45) \quad |F(t)| \leq b_1 |t|^{N/s} + b_2 |t|^p \Phi_{N,s}(\alpha_0 |t|^{N/(N-s)})$$

for all  $t \in \mathbb{R}$  and some constants  $b_1 > 0, b_2 > 0$ . We begin remarking that

$$(2.46) \quad F(v_n + u) - F(v_n) = f(v_n + tu)u, \quad \text{where } t \in [0, 1].$$

Combine (2.43) and (2.46), we get

$$(2.47) \quad \begin{aligned} & |F(v_n + u) - F(v_n)| \\ & \leq b_1 |v_n + tu|^{\frac{N}{s}-1} |u| + b_2 |v_n + tu|^{\frac{N}{s}-1} |u| \Phi_{N,s}(\alpha_0 |v_n + tu|^{N/(N-s)}) \\ & \leq b_1 (|v_n| + |u|)^{\frac{N}{s}-1} |u| + b_2 (|v_n| + |u|)^{\frac{N}{s}-1} |u| \Phi_{N,s}(\alpha_0 (|v_n| + |u|)^{N/(N-s)}) \\ & \leq 2^{\frac{N}{s}-1} b_1 (|v_n|^{\frac{N}{s}-1} + |u|^{\frac{N}{s}-1}) |u| \\ & \quad + b_2 (|v_n| + |u|)^{\frac{N}{s}-1} |u| \Phi_{N,s}(\alpha_0 (|v_n| + |u|)^{N/(N-s)}). \end{aligned}$$

Now, we prove

$$(2.48) \quad |F(v_n + u) - F(v_n)| \in L^1(\mathbb{R}^N).$$

By Brezis–Lieb Lemma, we have

$$\|u_n - u\|_{W_\varepsilon}^p = \|u_n\|_{W_\varepsilon}^p - \|u\|_{W_\varepsilon}^p + o_n(1) \leq \|u_n\|_{W_\varepsilon}^p + o_n(1)$$

as  $n \rightarrow \infty$ . Thus,

$$\limsup_{n \rightarrow \infty} \|u_n - u\|_{W_\varepsilon}^p \leq \sup_{n \in \mathbb{N}} \|u_n\|_{W_\varepsilon}^p < \left( \frac{\beta_*}{2^{N/(N-s)} \mathbf{c} \alpha_0} \right)^{(N-s)/s} \mathfrak{d}_*.$$

Therefore, there exists  $n_0$  such that

$$(2.49) \quad \sup_{n \geq n_0} \|u_n - u\|_{W_\varepsilon}^p < \left( \frac{\beta_*}{2^{N/(N-s)} \mathbf{c} \alpha_0} \right)^{(N-s)/s} \mathfrak{d}_*.$$

By Fatou’s lemma, we have

$$(2.50) \quad \|u\|_{W_\varepsilon}^p \leq \liminf_{n \rightarrow \infty} \|u_n\|^p < \left( \frac{\beta_*}{2^{N/(N-s)} \mathbf{c} \alpha_0} \right)^{(N-s)/s} \mathfrak{d}_*.$$

Using Hölder inequality, we get

$$(2.51) \quad \int_{\mathbb{R}^N} |v_n|^{\frac{N}{s}-1} |u| \, dx \leq \|v_n\|_{L^{N/s}(\mathbb{R}^N)}^{\frac{N}{s}-1} \|u\|_{L^{N/s}(\mathbb{R}^N)},$$

and for  $t > \frac{N}{s}, t' > 1, t_*(p-1) \geq p$  such that  $\frac{1}{t_*} + \frac{1}{t} + \frac{1}{t'} = 1$ , we get

$$(2.52) \quad \begin{aligned} & \int_{\mathbb{R}^N} (|v_n| + |u|)^{p-1} |u| \Phi_{N,s}(\alpha_0 (|v_n| + |u|)^{N/(N-s)}) \, dx \\ & \leq \| |v_n| + |u| \|_{L^{t_*(p-1)}(\mathbb{R}^N)}^{p-1} \|u\|_{L^t(\mathbb{R}^N)} \left( \int_{\mathbb{R}^N} (\Phi_{N,s}(\alpha_0 (|v_n| + |u|)^{N/(N-s)}))^{t'} \, dx \right)^{1/t'}. \end{aligned}$$

Then by [25, Lemma 2.3], for any  $\mathbf{c} > t'$ , there exist a constant  $C(\mathbf{c}) > 0$  such that

$$(2.53) \quad (\Phi_{N,s}(\alpha_0 (|v_n| + |u|)^{N/(N-s)}))^{t'} \leq C(\mathbf{c}) \Phi_{N,s}(\mathbf{c} \alpha_0 (|v_n| + |u|)^{N/(N-s)})$$

on  $\mathbb{R}^N$ . Note that  $\mathfrak{d}_* = \min\{1, V_0\}$ , we get

$$(2.54) \quad \begin{aligned} & \int_{\mathbb{R}^N} (\Phi_{N,s}(\alpha_0 (|v_n| + |u|)^{N/(N-s)}))^t \\ & \leq C(\mathbf{c}) \int_{\mathbb{R}^N} \Phi_{N,s}(\mathbf{b} \alpha_0 (|v_n| + |u|)^{N/(N-s)}) \, dx \end{aligned}$$



$$= C(\mathbf{c}) \int_{\mathbb{R}^N} \Phi_{N,s} \left( \mathbf{c} \alpha_0 \mathfrak{D}_*^{-s/(N-s)} \| |v_n| + |u| \|_{W_\varepsilon}^{N/(N-s)} \mathfrak{D}_*^{s/N} \frac{|v_n| + |u|}{\| |v_n| + |u| \|_{W_\varepsilon}^{N/(N-s)}} \right) dx.$$

When  $\mathbf{c}$  near  $t'$ , combine (2.49) and (2.50), we have

$$(2.55) \quad \mathbf{c} \alpha_0 \mathfrak{D}_*^{-s/(N-s)} \| |v_n| + |u| \|_{W_\varepsilon}^{N/(N-s)} \leq \beta_* < \alpha_*,$$

by (2.40), (2.54) and (2.55), there exists a constant  $D > 0$  such that

$$(2.56) \quad \left( \int_{\mathbb{R}^N} (\Phi_{N,s}(\alpha_0(|v_n| + |u|)^{N/(N-s)}))^{t'} dx \right)^{1/t'} \leq D.$$

Since the embedding sequence from  $W_\varepsilon \rightarrow W^{s,N/s}(\mathbb{R}^N) \rightarrow L^q(\mathbb{R}^N)$  is continuous for all  $q \in [\frac{N}{s}, +\infty)$ , we get

$$(2.57) \quad \int_{\mathbb{R}^N} (|v_n| + |u|)^{p-1} |u| \Phi_{N,s}(\alpha_0(|v_n| + |u|)^{N/(N-s)}) dx \leq D 2^{p-1} S_{t,\varepsilon}^{-1} \|u\|_{W_\varepsilon} (\|v_n\|_{L^{t*(p-1)}(\mathbb{R}^N)}^{p-1} + \|u\|_{L^{t*(p-1)}(\mathbb{R}^N)}^{p-1}) < +\infty.$$

Combine (2.49), (2.51) and (2.57), we get the claim (2.48). By arguments as [39, Lemma 7], for any  $u \in W_\varepsilon$  and  $\alpha > 0$ , we have

$$(2.58) \quad \int_{\mathbb{R}^N} \Phi_{N,s}(\alpha |u|^{N/(N-s)}) dx < +\infty.$$

Using Hölder inequality for  $t > \frac{N}{s}$  and  $t' > 1$  such that  $\frac{1}{t} + \frac{1}{t'} = 1$ , (2.58), and [25, Lemma 2.3], it is easy to get

$$(2.59) \quad \int_{\mathbb{R}^N} |F(u)| dx \leq b_1 \|u\|_{L^{N/s}(\mathbb{R}^N)}^{N/s} + b_2 \|u\|_{L^{pt}(\mathbb{R}^N)}^p \left( \int_{\mathbb{R}^N} (\Phi_{N,s}(\alpha_0 |u|^{N/(N-s)}))^{t'} dx \right)^{1/t'} < +\infty.$$

From (2.48) and (2.59), we obtain  $|F(v_n + u) - F(v_n) - F(u)| \in L^1(\mathbb{R}^N)$  for all  $n$  large enough. Then there exists a constant  $\kappa > 0$  such that  $|F(v_n + u) - F(v_n) - F(u)| \leq \kappa$  on  $\mathbb{R}^N$  outside a set with measure zero. From  $u_n(x) \rightarrow u(x)$  almost every where on  $\mathbb{R}^N$ , we see that

$$(2.60) \quad F(v_n + u) - F(v_n) - F(u) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

outside a set with measure zero. For any  $\varepsilon > 0$ , there exists  $\delta = \frac{\varepsilon}{\kappa} > 0$  such that for all  $U \subset \mathbb{R}^N$  with  $|U| < \delta$ , we have

$$(2.61) \quad \int_U |F(v_n + u) - F(v_n) - F(u)| dx \leq \kappa |U| = \varepsilon$$

for all  $n$  large enough. Since  $u \in L^{N/s}(\mathbb{R}^N)$  and  $u \in L^t(\mathbb{R}^N)$  due to the embedding from  $W_\varepsilon \rightarrow L^q(\mathbb{R}^N)$  is continuous for all  $q \geq \frac{N}{s}$ , then there exists  $R > 0$  such that

$$(2.62) \quad \int_{\mathbb{R}^N \setminus B_R(0)} |u|^{N/s} dx < \varepsilon^p \quad \text{and} \quad \int_{\mathbb{R}^N \setminus B_R(0)} |u|^{pt} dx < \varepsilon^{pt}.$$

Combine (2.47), (2.51), (2.52), (2.56) and (2.59), we only take the integral on  $\mathbb{R}^N \setminus B_R(0)$ , there exists a constant  $\kappa_* > 0$  such that

$$(2.63) \quad \int_{\mathbb{R}^N \setminus B_R(0)} |F(v_n + u) - F(v_n) - F(u)| dx \leq \kappa_* \varepsilon.$$

Combine (2.60), (2.61) and (2.63) and using Vitali's theorem, we get (i).

For any  $\varepsilon > 0$ , from the conditions  $(f_1)$  and  $(f_3)$ , there exists  $C(\varepsilon) > 0$  such that

$$|f(t)| \leq \varepsilon |t|^{\frac{N}{s}-1} + C(\varepsilon) |t|^{p-1} \Phi_{N,s}(\alpha_0 |t|^{N/(N-s)}).$$

We have

$$|f(v_n + u) - f(v_n) - f(u)|^r \leq 3^r (|f(v_n + u)|^r + |f(v_n)|^r + |f(u)|^r).$$

Then statement (ii) is proved similarly as (i). We omit the details. □

**Lemma 9.** *Let  $\{u_n\} \subset W_\varepsilon$  be a  $(PS)_d$  sequence for  $I_\varepsilon$  such that  $u_n \rightarrow 0$  weakly in  $W_\varepsilon$  verifying  $\limsup_{n \rightarrow \infty} \|u_n\|_{W_\varepsilon}^{N/(N-s)} \leq \frac{\beta_* d_*^{s/(N-s)}}{c\alpha_0}$ , where  $\mathbf{c} > 1$  is a suitable constant. Then we have either:*

- (i)  $u_n \rightarrow 0$  in  $W_\varepsilon$  or
- (ii) there exists a sequence  $\{y_n\} \subset \mathbb{R}^N$  and constants  $R > 0, \beta > 0$  such that

$$\liminf_{n \rightarrow \infty} \int_{B_R(y_n)} |u_n|^{N/s} dx \geq \beta > 0.$$

*Proof.* Suppose that (ii) does not occur. First from the condition  $(f_2)$ , any  $(PS)_d$  sequence of  $I_\varepsilon$  must be bounded in  $W_\varepsilon$ . Then by arguments as Lemma 5, we have  $u_n \rightarrow 0$  in  $L^q(\mathbb{R}^N)$  for  $q \in (\frac{N}{s}, +\infty)$ . By arguments as Lemma 6, from the conditions  $(f_1)$  and  $(f_3)$ , we have  $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} f(u_n)u_n dx = 0$ . Recalling that  $\langle I'_\varepsilon(u_n), u_n \rangle \rightarrow 0$  as  $n \rightarrow \infty$ , then we get  $u_n \rightarrow 0$  strongly in  $W_\varepsilon$ . The proof of Lemma 9 is completed. □

**Lemma 10.** *Suppose that  $V_\infty < +\infty$ . Let  $\{v_n\} \subset W_\varepsilon$  be a  $(PS)_d$  sequence converging weakly to 0 satisfying  $\limsup_{n \rightarrow \infty} \|v_n\|_{W_\varepsilon}^{N/(N-s)} \leq \frac{\beta_* d_*^{s/(N-s)}}{c\alpha_0}$ , where  $\mathbf{c} > 1$  is a suitable constant. If  $v_n \not\rightarrow 0$  in  $W_\varepsilon$ , then  $d \geq c_{V_\infty}$ , where  $c_{V_\infty}$  is the maximum level of energy function associated to the problem  $(P_{V_\infty})$ .*

*Proof.* Let  $(t_n) \subset (0, +\infty)$  be a sequence such that  $(t_n v_n) \subset \mathcal{N}_{V_\infty}$ . We start by showing the following claim.

**Claim 1.** *The sequence  $\{t_n\}$  satisfies  $\limsup_{n \rightarrow \infty} t_n \leq 1$ . Indeed, suppose that the above claim does not hold, then there exist  $\delta > 0$  and a subsequence still denoted by  $(t_n)$  such that*

$$(2.64) \quad t_n \geq 1 + \delta \text{ for all } n \in \mathbb{N}.$$

By the condition  $(f_2)$ ,  $\{v_n\}$  is bounded sequence in  $W_\varepsilon$ , and then  $\langle I'_\varepsilon(v_n), v_n \rangle = o_n(1)$  as  $n \rightarrow \infty$ . It means that

$$\|v_n\|_{W_\varepsilon}^p = \int_{\mathbb{R}^N} f(v_n)v_n dx + o_n(1).$$

Moreover, recalling that  $(t_n v_n) \subset \mathcal{N}_{V_\infty}$ , we get

$$t_n^p \|v_n\|_{V_\infty}^p = \int_{\mathbb{R}^N} f(t_n v_n)t_n v_n dx.$$

Two above equalities imply that

$$(2.65) \quad \int_{\mathbb{R}^N} \left[ \frac{f(t_n v_n)v_n}{t_n^{p-1}} - f(v_n)v_n \right] = \int_{\mathbb{R}^N} [V_\infty - V(\varepsilon x)]|v_n|^p dx + o_n(1).$$

Given any  $\xi > 0$ , from the condition  $(V)$ , there exists  $R = R(\varepsilon) > 0$  such that

$$(2.66) \quad V(\varepsilon x) \geq V_\infty - \xi \text{ for any } |x| \geq R.$$

Since  $\{v_n\}$  is a bounded sequence in  $W_\varepsilon$  and the embedding from  $W_\varepsilon \rightarrow L^{N/s}(\mathbb{R}^N)$  is continuous, then there exists  $C > 0$  such  $\|v_n\|_{L^{N/s}(\mathbb{R}^N)} \leq C$ . From  $v_n \rightarrow 0$  in  $L^{N/s}(B_R(0))$ ,  $V$  is a continuous function and (2.66), there exists a suitable constant  $C_* > 0$  such that

$$(2.67) \quad \begin{aligned} & \int_{\mathbb{R}^N} [V_\infty - V(\varepsilon x)]|v_n|^p dx \\ &= \int_{B_R(0)} [V_\infty - V(\varepsilon x)]|v_n|^p dx + \int_{\mathbb{R}^N \setminus B_R(0)} [V_\infty - V(\varepsilon x)]|v_n|^p dx \leq C_* \xi. \end{aligned}$$

Because  $v_n \not\rightarrow 0$  in  $W_\varepsilon$ , by Lemma 9, there exist a sequence  $\{y_n\} \subset \mathbb{R}^N$  and  $R_* > 0$ ,  $\beta > 0$  such that

$$(2.68) \quad \int_{B_{R_*}(y_n)} |v_n|^{N/s} dx \geq \beta > 0.$$

Note that  $\{v_n\}$  is a  $(PS)_d$  sequence of  $I_\varepsilon$ . We denote  $v_n^-(x) = \min\{v_n(x), 0\}$  and  $v_n^+(x) = \max\{v_n(x), 0\}$ . Since  $\{v_n\}$  is bounded in  $W_\varepsilon$ , then  $\{v_n^-\}$  is also bounded in  $W_\varepsilon$ , and we have  $\langle I'_\varepsilon(v_n), v_n^- \rangle \rightarrow 0$  as  $n \rightarrow \infty$ . We see that

$$(2.69) \quad \begin{aligned} \langle I'_\varepsilon(v_n), v_n^- \rangle &= \int_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^{p-2} (v_n(x) - v_n(y))(v_n^-(x) - v_n^-(y))}{|x - y|^{2N}} \\ &+ \int_{\mathbb{R}^N} V(\varepsilon x)|v_n|^{p-2} v_n v_n^- dx - \int_{\mathbb{R}^N} f(v_n)v_n^- dx. \end{aligned}$$

We denote  $\Omega_n^+ = \text{supp}(v_n^+)$ ,  $\Omega_n^- = \text{supp}(v_n^-)$ ,  $\Omega_n = \text{supp}(v_n)$ , then we get

$$(2.70) \quad \begin{aligned} & \int_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^{p-2} (v_n(x) - v_n(y))(v_n^-(x) - v_n^-(y))}{|x - y|^{2N}} dx dy \\ &= \int_{(\Omega_n^+ \cup \Omega_n^- \cup \Omega_n^c) \times (\Omega_n^+ \cup \Omega_n^- \cup \Omega_n^c)} \frac{|(v_n^+(x) - v_n^+(y) + (v_n^-(x) - v_n^-(y)))|^{p-2}}{|x - y|^{2N}} \\ &\quad \times (v_n^+(x) - v_n^+(y) + v_n^-(x) - v_n^-(y))(v_n^-(x) - v_n^-(y)) dx dy \\ &= - \int_{\Omega_n^+ \times \Omega_n^-} \frac{|v_n^+(x) - v_n^-(y)|^{p-1} v_n^-(y)}{|x - y|^{2N}} dx dy \\ &\quad - \int_{\Omega_n^- \times \Omega_n^+} \frac{|v_n^-(x) - v_n^+(y)|^{p-1} v_n^+(x)}{|x - y|^{2N}} dx dy \\ &\quad + \int_{\Omega_n^- \times \Omega_n^-} \frac{|v_n^-(x) - v_n^-(y)|^p}{|x - y|^{2N}} dx dy + \int_{\Omega_n^c \times \Omega_n^-} \frac{|v_n^-(y)|^p}{|x - y|^{2N}} dx dy \\ &\quad + \int_{\Omega_n^- \times \Omega_n^c} \frac{|v_n^-(x)|}{|x - y|^{N+ps}} dx dy, \end{aligned}$$

where  $\Omega^c = \mathbb{R}^N \setminus \Omega$  for some set  $\Omega \subset \mathbb{R}^N$ . Note that  $\Omega_n^{-c} = \Omega_n^+ \cup \Omega_n^c$  and

$$\begin{aligned} & \int_{\mathbb{R}^{2N}} \frac{|v_n^-(x) - v_n^-(y)|^p}{|x - y|^{2N}} dx dy = \int_{\Omega_n^- \times \Omega_n^-} \frac{|v_n^-(x) - v_n^-(y)|^p}{|x - y|^{2N}} dx dy \\ & + \int_{\Omega_n^- \times \Omega_n^{-c}} \frac{|v_n^-(x) - v_n^-(y)|^p}{|x - y|^{2N}} dx dy + \int_{\Omega_n^{-c} \times \Omega_n^-} \frac{|v_n^-(x) - v_n^-(y)|^p}{|x - y|^{2N}} dx dy \\ & = \int_{\Omega_n^- \times \Omega_n^-} \frac{|v_n^-(x) - v_n^-(y)|^p}{|x - y|^{2N}} dx dy + \int_{\Omega_n^- \times \Omega_n^+} \frac{|v_n^-(x) - v_n^-(y)|^p}{|x - y|^{2N}} dx dy \\ & + \int_{\Omega_n^- \times \Omega_n^c} \frac{|v_n^-(x) - v_n^-(y)|^p}{|x - y|^{2N}} dx dy + \int_{\Omega_n^+ \times \Omega_n^-} \frac{|v_n^-(x) - v_n^-(y)|^p}{|x - y|^{2N}} dx dy \\ & + \int_{\Omega_n^c \times \Omega_n^-} \frac{|v_n^-(x) - v_n^-(y)|^p}{|x - y|^{2N}} dx dy. \end{aligned}$$

Hence, we get

$$\begin{aligned} & \int_{\mathbb{R}^{2N}} \frac{|v_n^-(x) - v_n^-(y)|^p}{|x - y|^{2N}} dx dy = \int_{\Omega_n^- \times \Omega_n^-} \frac{|v_n^-(x) - v_n^-(y)|^p}{|x - y|^{2N}} dx dy \\ & + \int_{\Omega_n^- \times \Omega_n^+} \frac{|v_n^-(x)|^p}{|x - y|^{2N}} dx dy + \int_{\Omega_n^- \times \Omega_n^c} \frac{|v_n^-(x)|^p}{|x - y|^{2N}} dx dy \\ (2.71) \quad & + \int_{\Omega_n^+ \times \Omega_n^-} \frac{|v_n^-(y)|^p}{|x - y|^{2N}} dx dy + \int_{\Omega_n^c \times \Omega_n^-} \frac{|v_n^-(y)|^p}{|x - y|^{2N}} dx dy. \end{aligned}$$

We have

$$\begin{aligned} & - \int_{\Omega_n^+ \times \Omega_n^-} \frac{|v_n^+(x) - v_n^-(y)|^{p-1} v_n^-(y)}{|x - y|^{2N}} dx dy - \int_{\Omega_n^+ \times \Omega_n^-} \frac{|v_n^-(y)|^p}{|x - y|^{2N}} dx dy \\ (2.72) \quad & = \int_{\Omega_n^+ \times \Omega_n^-} \frac{(|1 - \frac{v_n^+(x)}{v_n^-(y)}|^p - 1) |v_n^-(y)|^p}{|x - y|^{2N}} dx dy > 0. \end{aligned}$$

Similarly, we can get

$$(2.73) \quad - \int_{\Omega_n^- \times \Omega_n^+} \frac{|v_n^-(x) - v_n^+(y)|^{p-1} v_n^-(x)}{|x - y|^{2N}} dx dy - \int_{\Omega_n^- \times \Omega_n^+} \frac{|v_n^-(x)|^p}{|x - y|^{2N}} dx dy > 0.$$

From (2.70) to (2.73), we deduce

$$\begin{aligned} & \int_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^{p-2} (v_n(x) - v_n(y)) (v_n^-(x) - v_n^-(y))}{|x - y|^{2N}} dx dy \\ (2.74) \quad & \geq \int_{\mathbb{R}^{2N}} \frac{|v_n^-(x) - v_n^-(y)|^p}{|x - y|^{2N}} dx dy. \end{aligned}$$

Combine (2.69) and (2.73) and note that  $f(t) = 0$  for all  $t \in (-\infty, 0]$ , we get

$$\|v_n^-\|_{W_\varepsilon} \rightarrow 0$$

as  $n \rightarrow \infty$  or equivalent  $v_n^- \rightarrow 0$  in  $W_\varepsilon$ . Note that  $W_\varepsilon$  is continuously embedded into  $W^{s,N/s}(\mathbb{R}^N)$ , then  $v_n^- \rightarrow 0$  in  $W^{s,N/s}(\mathbb{R}^N)$ . We denote  $\bar{v}_n(x) = v_n(x + y_n)$ . Since  $\|\cdot\|_{V_0}$  is invariant by the translation, we get

$$\|\bar{v}_n\|_{V_0}^p = \|v_n\|_{V_0}^p \leq [v_n]_{s,p}^p + \int_{\mathbb{R}^N} V(\varepsilon x) |v_n|^p = \|v_n\|_{W_\varepsilon}^p.$$

Then  $\{\bar{v}_n\}$  is a bounded sequence in  $W^{s,N/s}(\mathbb{R}^N)$ , up to a subsequence, we may assume that there exists  $\bar{v} \in W^{s,N/s}(\mathbb{R}^N)$  such that  $\bar{v}_n \rightarrow \bar{v}$  weakly in  $W^{s,N/s}(\mathbb{R}^N)$  and  $\bar{v}_n(x) \rightarrow \bar{v}(x)$  on  $\mathbb{R}^N$ . By arguments as above and Fatou's lemma, we get

$$\|\bar{v}^-\|_{V_0} \leq \liminf_{n \rightarrow \infty} \|\bar{v}_n^-\|_{V_0} \leq \liminf_{n \rightarrow \infty} \|v_n^-\|_{W_\varepsilon} = 0.$$

Then  $\bar{v} = \bar{v}^+$ . From (2.68) and  $\bar{v}_n \rightarrow \bar{v}$  in  $L^{N/s}(B_{R_*}(0))$ , we deduce  $\int_{B_{R_*}(0)} |\bar{v}|^{N/s} dx \geq \beta/2 > 0$ . Therefore, there exists a subset  $\Omega \subset B_{R_*}(0) \subset \mathbb{R}^N$  such that  $|\Omega| > 0$  and  $\bar{v}(x) > a > 0$  for all  $x \in \Omega$ , where  $a > 0$  is a suitable constant. Since  $\bar{v}_n \rightarrow \bar{v}$  on  $L^q(B_{R_*}(0))$  for all  $q \in [\frac{N}{s}, +\infty)$ , we can assume  $\bar{v}_n(x) \rightarrow \bar{v}(x)$  on  $\Omega$  and then  $\bar{v}_n(x) > \frac{a}{2} > 0$  for all  $x \in \Omega$  and  $n$  large enough. From (2.65) and (2.67), we obtain

$$\begin{aligned} \int_{\text{supp}(\bar{v}_n^+)} \left[ \frac{f(t_n \bar{v}_n) \bar{v}_n}{t_n^{p-1}} - f(\bar{v}_n) \bar{v}_n \right] dx &= \int_{\mathbb{R}^N} \left[ \frac{f(t_n \bar{v}_n) \bar{v}_n}{t_n^{p-1}} - f(\bar{v}_n) \bar{v}_n \right] dx \\ (2.75) \qquad \qquad \qquad &= \int_{\mathbb{R}^N} \left[ \frac{f(t_n v_n) v_n}{t_n^{p-1}} - f(v_n) v_n \right] dx \leq C_* \xi \end{aligned}$$

for any  $\xi > 0$ . Using the condition  $(f_5)$ , Fatou's lemma, (2.64), (2.75) and  $f(t) = 0$  for all  $t \in (-\infty, 0]$ , we get

$$\begin{aligned} 0 &< \int_{\Omega} \left[ \frac{f((1+\delta)\bar{v})\bar{v}}{(1+\delta)^{p-1}} - f(\bar{v})\bar{v} \right] dx = \int_{\Omega} \left[ \frac{f((1+\delta)\bar{v})}{((1+\delta)\bar{v})^{p-1}} - \frac{f(\bar{v})}{\bar{v}^{p-1}} \right] \bar{v}^p dx \\ &\leq \liminf_{n \rightarrow \infty} \int_{\Omega} \left[ \frac{f((1+\delta)\bar{v}_n)}{((1+\delta)\bar{v}_n)^{p-1}} - \frac{f(\bar{v}_n)}{\bar{v}_n^{p-1}} \right] \bar{v}_n^p dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} \left[ \frac{f(t_n \bar{v}_n)}{(t_n \bar{v}_n)^{p-1}} - \frac{f(\bar{v}_n)}{\bar{v}_n^{p-1}} \right] \bar{v}_n^p dx \\ &= \liminf_{n \rightarrow \infty} \int_{\Omega} \left[ \frac{f(t_n \bar{v}_n) \bar{v}_n}{t_n^{p-1}} - f(\bar{v}_n) \bar{v}_n \right] dx \leq \liminf_{n \rightarrow \infty} \int_{\text{supp}(\bar{v}_n^+)} \left[ \frac{f(t_n \bar{v}_n) \bar{v}_n}{t_n^{p-1}} - f(\bar{v}_n) \bar{v}_n \right] dx \\ &\leq C_* \xi \end{aligned}$$

for any  $\xi > 0$  and  $n$  large enough. It is a contradiction. Then Claim 1 is proved. Now, we will consider the following cases:

Case 1.  $\limsup_{n \rightarrow \infty} t_n = 1$ . Then there exists a subsequence, still denoted by  $t_n$  such that  $t_n \rightarrow 1$ . Recalling that  $I_\varepsilon(v_n) \rightarrow d$  as  $n \rightarrow \infty$ , and note that  $J_{V_\infty}(t_n v_n) \geq c_{V_\infty}$ , we have

$$\begin{aligned} d + o_n(1) &= I_\varepsilon(v_n) = I_\varepsilon(v_n) - J_{V_\infty}(t_n v_n) + J_{V_\infty}(t_n v_n) \\ (2.76) \qquad \qquad &\geq I_\varepsilon(v_n) - J_{V_\infty}(t_n v_n) + c_{V_\infty}. \end{aligned}$$

Let us compute

$$\begin{aligned} I_\varepsilon(v_n) - J_{V_\infty}(t_n v_n) &= \frac{1 - t_n^p}{p} [v_n]_{s,p}^p + \frac{1}{p} \int_{\mathbb{R}^N} (V(\varepsilon x) - t_n^p V_\infty) |v_n|^p dx \\ (2.77) \qquad \qquad \qquad &+ \int_{\mathbb{R}^N} (F(t_n v_n) - F(v_n)) dx. \end{aligned}$$

Using the condition  $(V)$ , (2.66),  $v_n \rightarrow 0$  in  $L^{N/s}(B_R(0))$ ,  $t_n \rightarrow 1$  and

$$V(\varepsilon x) - t_n^p V_\infty = (V(\varepsilon x) - V_\infty) + (1 - t_n^p) V_\infty \geq -\xi + (1 - t_n^p) V_\infty \quad \text{for all } |x| \geq R,$$

we get

$$\begin{aligned}
 & \int_{\mathbb{R}^N} (V(\varepsilon x) - t_n^p V_\infty) |v_n|^p dx \\
 &= \int_{B_R(0)} (V(\varepsilon x) - t_n^p V_\infty) |v_n|^p dx + \int_{\mathbb{R}^N \setminus B_R(0)} (V(\varepsilon x) - t_n^p V_\infty) |v_n|^p dx \\
 &\geq (V_0 - t_n^p V_\infty) \int_{B_R(0)} |v_n|^p dx - \xi \int_{\mathbb{R}^N \setminus B_R(0)} |v_n|^p dx \\
 (2.78) \quad &+ V_\infty(1 - t_n^p) \int_{\mathbb{R}^N \setminus B_R(0)} |v_n|^p dx \geq o_n(1) - \xi C^*,
 \end{aligned}$$

where  $C^*$  is a suitable constant. Since  $\{v_n\}$  is a bounded sequence in  $W_\varepsilon$ , then

$$(2.79) \quad \lim_{n \rightarrow \infty} \frac{(1 - t_n^p)}{p} [v_n]_{s,p}^p = 0.$$

From the condition  $\limsup_{n \rightarrow \infty} \|v_n\|_{W_\varepsilon}^{N/(N-s)} \leq \frac{\beta_* \mathfrak{D}_*^{s/(N-s)}}{\mathfrak{c}\alpha_0}$ , by arguments Lemma 8 and note that  $\Phi_{N,s}(t)$  is an increasing function on  $[0, +\infty)$ , we are easy to get

$$(2.80) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (F(t_n v_n) - F(v_n)) dx = 0.$$

Combine (2.76)–(2.80), we obtain

$$d + o_n(1) \geq c_{V_\infty} - C^* \xi + o_n(1).$$

Taking the limit of above inequality, we get  $d \geq c_{V_\infty}$ .

Case 2.  $\limsup_{n \rightarrow \infty} t_n = t_0 < 1$ . Then exists a subsequence, still denoted by  $\{t_n\}$  such that  $t_n \rightarrow t_0 (< 1)$  and then  $t_n < 1$  for all  $n \in \mathbb{N}$ . We see that

$$(2.81) \quad d + o_n(1) = I_\varepsilon(v_n) - \frac{1}{p} \langle I'_\varepsilon(v_n), v_n \rangle = \int_{\mathbb{R}^N} \left( \frac{1}{p} f(v_n) v_n - F(v_n) \right) dx.$$

Note that  $t_n v_n \in \mathcal{N}_{V_\infty}$ , by the condition  $(f_5)$  and (2.81), we get

$$\begin{aligned}
 c_{V_\infty} &\leq J_{V_\infty}(t_n v_n) = J_{V_\infty}(t_n v_n) - \frac{1}{p} \langle J'_{V_\infty}(t_n v_n), t_n v_n \rangle \\
 &= \int_{\mathbb{R}^N} \left( \frac{1}{p} f(t_n v_n) t_n v_n - F(t_n v_n) \right) dx \\
 &\leq \int_{\mathbb{R}^N} \left( \frac{1}{p} f(v_n) v_n - F(v_n) \right) dx = d + o_n(1).
 \end{aligned}$$

Taking the limit of above inequality as  $n \rightarrow \infty$ , we get  $d \geq c_{V_\infty}$ . □

**Lemma 11.** *Let  $u_n$  be a  $(PS)_c$  for  $I_\varepsilon$  satisfying*

$$\limsup_{n \rightarrow \infty} \|u_n\|_{W_\varepsilon}^{N/(N-s)} \leq \frac{\beta_* \mathfrak{D}_*^{s/(N-s)}}{2^{N/(N-s)} \mathfrak{c}\alpha_0},$$

where  $\mathfrak{c} > 1$  is a suitable constant. Assume that  $c < c_{V_\infty}$  when  $V_\infty < \infty$  or  $c \in \mathbb{R}$  if  $V_\infty = +\infty$ . Then  $\{u_n\}$  has a convergent subsequence in  $W_\varepsilon$ .

*Proof.* First, we consider the case  $V_\infty < +\infty$ . From the condition  $(f_2)$ , we see that  $\{u_n\}$  is a bounded sequence in  $W_\varepsilon$ . Then, up to a subsequence, we may assume

that

$$(2.82) \quad u_n \rightarrow u \text{ weakly in } W_\varepsilon,$$

$$(2.83) \quad u_n \rightarrow u \text{ in } L^q_{\text{loc}}(\mathbb{R}^N) \text{ for any } q \in [\frac{N}{s}, +\infty),$$

$$(2.84) \quad u_n \rightarrow u \text{ a.e. in } \mathbb{R}^N.$$

By arguments as Lemma 4, we have  $I'_\varepsilon(u) = 0$ . Set  $v_n = u_n - u$ . Using Brezis–Lieb Lemma and Lemma 8, we get

$$\begin{aligned} I_\varepsilon(v_n) &= \frac{\|u_n\|_{W_\varepsilon}^p}{p} - \frac{\|u\|_{W_\varepsilon}^p}{p} - \int_{\mathbb{R}^N} F(u_n) dx + \int_{\mathbb{R}^N} F(u) dx + o_n(1) \\ &= I_\varepsilon(u_n) - I_\varepsilon(u) + o_n(1) \\ (2.85) \quad &= c - I_\varepsilon(u) + o_n(1) := d + o_n(1). \end{aligned}$$

By [8, Lemma 2.6], we have

$$(2.86) \quad \int_{\mathbb{R}^{2N}} |\mathcal{A}(u_n) - \mathcal{A}(v_n) - \mathcal{A}(u)|^{p'} dx dy = o_n(1),$$

where  $p' = \frac{p}{p-1}$  is the conjugate exponent of  $p$  and

$$\mathcal{A}(v) := \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))}{|x - y|^{\frac{2N}{p'}}$$

for all  $v \in W_\varepsilon$ . Note that  $V$  is a bounded function, then by [8, Lemma 3.3], we can see that

$$(2.87) \quad \int_{\mathbb{R}^N} V(\varepsilon x) (|v_n|^{p-2}v_n - |u_n|^{p-2}u_n + |u|^{p-2}u)^{p'} dx = o_n(1).$$

Hence, using Hölder inequality, for any  $\varphi \in W_\varepsilon$ , we have

$$\begin{aligned} &|\langle I'_\varepsilon(v_n) - I'_\varepsilon(u_n) + I'_\varepsilon(u), \varphi \rangle| \\ &\leq \left( \int_{\mathbb{R}^{2N}} |\mathcal{A}(u_n) - \mathcal{A}(v_n) - \mathcal{A}(u)|^{p'} dx dy \right)^{\frac{1}{p'}} [\varphi]_{s,p}^p \\ &\quad + \left( \int_{\mathbb{R}^N} V(\varepsilon x) (|v_n|^{p-2}v_n - |u_n|^{p-2}u_n + |u|^{p-2}u)^{p'} dx \right)^{\frac{1}{p'}} \left( \int_{\mathbb{R}^N} V(\varepsilon x) |\varphi|^p dx \right)^{\frac{1}{p}} \\ (2.88) \quad &+ \left( \int_{\mathbb{R}^N} |f(v_n) - f(u_n) + f(u)|^{p'} dx \right)^{\frac{1}{p'}} \|\varphi\|_{L^p(\mathbb{R}^N)}. \end{aligned}$$

Since the embeddings  $W_\varepsilon \hookrightarrow W^{s,p}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$  are continuous, then from (2.86)–(2.88) and Lemma 8, we deduce  $I'_\varepsilon(v_n) \rightarrow 0$  in  $W_\varepsilon^*$ . By the condition  $(f_2)$ , we have

$$(2.89) \quad I_\varepsilon(u) = I_\varepsilon(u) - \frac{1}{p} \langle I'_\varepsilon(u), u \rangle = \int_{\mathbb{R}^N} \left[ \frac{1}{p} f(u)u - F(u) \right] dx \geq 0.$$

Combine (2.85) and (2.89), we obtain

$$d \leq c < c_{V_\infty}$$

which together Lemma 10 gives  $v_n \rightarrow 0$  in  $W_\varepsilon$ . That is  $u_n \rightarrow u$  strongly in  $W_\varepsilon$ .

Next, we consider the case  $V_\infty = +\infty$ , then  $V$  is a coercive function on  $\mathbb{R}^N$ . It means that  $V(\varepsilon x) \rightarrow +\infty$  as  $|x| \rightarrow +\infty$  for each  $\varepsilon > 0$ . Therefore  $\text{meas}(\{x \in \mathbb{R}^N : V(\varepsilon x) \leq c\}) < +\infty$  for any  $c > 0$ . By arguments as [39, Lemma 5], we have the embedding from  $W_\varepsilon \hookrightarrow L^q(\mathbb{R}^N)$  is compact for any  $q \in [\frac{N}{s}, +\infty)$ . Then  $v_n \rightarrow 0$  in

$L^q(\mathbb{R}^N)$  for any  $q \in [\frac{N}{s}, +\infty)$ . From the condition  $(f_1)$  and by arguments as Lemma 8, we are easy to get

$$(2.90) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} f(v_n)v_n \, dx = 0.$$

Furthermore, using Vitali's theorem and by arguments as Lemma 8, we deduce

$$(2.91) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |f(u_n)u_n - f(v_n)v_n - f(u)u| \, dx = 0.$$

Combine (2.90), (2.91) and the fact that  $\langle I'_\varepsilon(u), u \rangle = 0$ , we get

$$\lim_{n \rightarrow \infty} \|u_n\|_{W_\varepsilon} = \|u\|_{W_\varepsilon}.$$

It implies that  $u_n \rightarrow u$  in  $W_\varepsilon$  due to Brezis–Lieb Lemma.  $\square$

**Lemma 12.** *Let  $\{u_n\}$  be a  $(PS)_c$  sequence for  $I_\varepsilon$  restricted to  $\mathcal{N}_\varepsilon$  with*

$$(2.92) \quad \limsup_{n \rightarrow \infty} \|u_n\|_{W_\varepsilon}^{N/(N-s)} \leq \frac{\beta_* \mathfrak{D}_*^{s/(N-s)}}{\mathfrak{c}\alpha_0},$$

where  $\mathfrak{c} > 1$  is a suitable constant. Assume that  $c < c_{V_\infty}$  when  $V_\infty < \infty$  or  $c \in \mathbb{R}$  if  $V_\infty = +\infty$ . Then  $\{u_n\}$  has a convergent subsequence in  $W_\varepsilon$ .

*Proof.* Let  $\{u_n\}$  be a  $(PS)_c$  sequence of  $I_\varepsilon$  restricted to  $\mathcal{N}_\varepsilon$ , that is,

$$I_\varepsilon(u_n) \rightarrow c, \quad (I_\varepsilon|_{\mathcal{N}_\varepsilon})' \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Note that  $\langle I'_\varepsilon(u_n), u_n \rangle = 0$  and using the condition  $(f_2)$ , we have  $\{u_n\}$  is a bounded sequence in  $W_\varepsilon$ . From Proposition 2, we have

$$(2.93) \quad \|u_n\|_{W_\varepsilon} \geq r_* > 0.$$

Then up to a subsequence, we can assume that

$$(2.94) \quad \lim_{n \rightarrow \infty} \|u_n\|_{W_\varepsilon} = l, \quad r_* \leq l \leq \sup_n \|u_n\|_{W_\varepsilon} < +\infty$$

and

$$\begin{aligned} u_n &\rightarrow u \text{ weakly in } W_\varepsilon, \\ u_n &\rightarrow u \text{ in } L^q_{\text{loc}}(\mathbb{R}^N) \text{ for any } q \in [\frac{N}{s}, +\infty), \\ u_n &\rightarrow u \text{ a.e. in } \mathbb{R}^N. \end{aligned}$$

Since  $\{u_n\} \subset \mathcal{N}_\varepsilon$ , then we have

$$\int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{2N}} \, dx \, dy + \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^p \, dx = \int_{\mathbb{R}^N} f(u_n)u_n \, dx.$$

First we consider the case  $V_\infty = +\infty$ , then up to a subsequence, we assume that  $u_n \rightarrow u$  strong in  $L^q(\mathbb{R}^N)$  for all  $q \in [\frac{N}{s}, +\infty)$ . By the assumption (2.92), (2.4) for  $q = \frac{N}{s}$  and using Vitali's theorem as in Lemma 8, we get

$$(2.95) \quad \begin{aligned} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} f(u_n)u_n \, dx &= \int_{\mathbb{R}^N} f(u)u \, dx \quad \text{and} \\ \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} f'(u_n)u_n^2 \, dx &= \int_{\mathbb{R}^N} f'(u)u^2 \, dx. \end{aligned}$$



Combine (2.94) and (2.95), we obtain

$$(2.96) \quad 0 < l = \int_{\mathbb{R}^N} f(u)u \, dx = \int_{\{x \in \mathbb{R}^N : u(x) > 0\}} f(u^+)u^+ \, dx.$$

We know that  $f(t) > 0$  for all  $t > 0$ , then (2.96) implies that  $u^+ \not\equiv 0$ . Conversely, we get  $l = 0$  which is a contradiction. By the method of Lagrange multipliers, there exists a real sequence  $\{\lambda_n\} \subset \mathbb{R}$  such that

$$(2.97) \quad I'_\varepsilon(u_n) = \lambda_n K'_\varepsilon(u_n) + o_n(1),$$

where  $K_\varepsilon : W_\varepsilon \rightarrow \mathbb{R}$  is given by

$$K_\varepsilon(u) = \langle I'_\varepsilon(u), u \rangle = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{2N}} \, dx \, dy + \int_{\mathbb{R}^{2N}} V(\varepsilon x)|u|^p \, dx - \int_{\mathbb{R}^N} f(u)u \, dx.$$

Consequently, we have

$$(2.98) \quad \begin{aligned} \langle K'_\varepsilon(u_n), u_n \rangle &= p \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{2N}} \, dx \, dy + p \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^p \, dx \\ &\quad - \int_{\mathbb{R}^N} f(u_n)u_n \, dx - \int_{\mathbb{R}^N} f'(u_n)u_n^2 \, dx \\ &= \int_{\mathbb{R}^N} ((p - 1)f(u_n)u_n - f'(u_n)u_n^2) \, dx \\ &= \int_{\{x \in \mathbb{R}^N : u_n(x) > 0\}} ((p - 1)f(u_n^+)u_n^+ - f'(u_n^+)(u_n^+)^2) \, dx. \end{aligned}$$

From Remark 1 (ii), we have  $f'(t)t - (p - 1)f(t) > 0$  for all  $t > 0$ , then  $(p - 1)f(u_n^+)u_n^+ - f'(u_n^+)(u_n^+)^2 < 0$  for all  $n$ . The equality (2.98) implies that  $\sup_{n \in \mathbb{R}^N} \langle K'_\varepsilon(u_n), u_n \rangle \leq 0$ . If  $\sup_{n \in \mathbb{R}^N} \langle K'_\varepsilon(u_n), u_n \rangle = 0$ . Then up to a subsequence, we suppose that  $\lim_{n \rightarrow \infty} \langle K'_\varepsilon(u_n), u_n \rangle = 0$ , then from (2.95) and (2.98), it holds

$$\int_{\mathbb{R}^N} ((p - 1)f(u)u - f'(u)u^2) \, dx = 0.$$

Thus we get

$$(2.99) \quad \int_{\{x \in \mathbb{R}^N : u(x) > 0\}} ((p - 1)f(u^+)u^+ - f'(u^+)u^{+2}) \, dx = 0.$$

Using again Remark 1 (ii), we have

$$(2.100) \quad (p - 1)f(u^+)u^+ - f'(u^+)u^{+2} < 0$$

for all  $x \in \Omega = \{x \in \mathbb{R}^N : u(x) > 0\}$ . Hence, we get a contradiction from (2.99) and (2.100).

Next we investigate the case  $V_\infty < +\infty$ . If  $u \not\equiv 0$ , then there exists  $t \in (0, +\infty)$  such that  $tu \in \mathcal{N}_{V_0}$ . It means that

$$(2.101) \quad t^p \|u\|_{V_0}^p = \int_{\mathbb{R}^N} f(tu)(tu) \, dx.$$

By the property of  $\mathcal{N}_{V_0}$ , there exists  $r > 0$  such that  $\|u\|_{V_0} > 0$ , then (2.101) implies  $u^+ \not\equiv 0$  and there exists  $\varepsilon_0 > 0$  such that  $u(x) \geq \varepsilon_0 > 0$  on  $\Omega \subset \mathbb{R}^N$  with  $|\Omega| > 0$ .

From (2.98), we have

$$\begin{aligned}
 -\langle K'_\varepsilon(u_n), u_n \rangle &= -p \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{2N}} dx dy - p \int_{\mathbb{R}^N} V(\varepsilon x) |u_n|^p dx \\
 &\quad + \int_{\mathbb{R}^N} f(u_n) u_n dx + \int_{\mathbb{R}^N} f'(u_n) u_n^2 dx \\
 (2.102) \qquad &= \int_{\mathbb{R}^N} (f'(u_n) u_n^2 - (p-1) f(u_n) u_n) dx
 \end{aligned}$$

By arguments as first case, we can assume that  $\lim_{n \rightarrow \infty} \langle K'_\varepsilon(u_n), u_n \rangle = 0$ . Using Fatou's lemma and (2.102), we get

$$\begin{aligned}
 0 &\geq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} (f'(u_n) u_n^2 - (p-1) f(u_n) u_n) dx \\
 &\geq \int_{\mathbb{R}^N} (f'(u) u^2 - (p-1) f(u) u) dx \\
 &= \int_{\{x \in \mathbb{R}^N : u(x) > 0\}} (f'(u) u^2 - (p-1) f(u) u) dx \\
 (2.103) \qquad &\geq \int_{\Omega} (f'(u) u^2 - (p-1) f(u) u) dx > 0
 \end{aligned}$$

due to the condition  $(f_5)$ . It is a contradiction.

Finally if  $u \equiv 0$ , then (2.93) implies that  $u_n \not\rightarrow 0$  in  $W_\varepsilon$ . By arguments as Lemma 9, we can show that there exists a sequence  $\{y_n\} \subset \mathbb{R}^N$  and constants  $R > 0, \beta > 0$  such that

$$(2.104) \qquad \liminf_{n \rightarrow \infty} \int_{B_R(y_n)} |u_n|^{N/s} dx \geq \beta > 0.$$

We denote  $v_n(x) = u_n(x + y_n)$ . Since the norm in  $W^{s, N/s}(\mathbb{R}^N)$  and the integrals are invariant with the translation, we have  $\|v_n\|_{V_0} = \|u_n\|_{V_0} \leq \|u_n\|_{W_\varepsilon}$ . Therefore  $\{v_n\}$  is a bounded sequence in  $W^{s, N/s}(\mathbb{R}^N)$ , then up to a subsequence, we have

$$\begin{aligned}
 v_n &\rightarrow v \text{ weakly in } W^{s, N/s}(\mathbb{R}^N), \\
 v_n &\rightarrow v \text{ in } L^q_{\text{loc}}(\mathbb{R}^N) \text{ for any } q \in [\frac{N}{s}, +\infty), \\
 v_n &\rightarrow v \text{ a.e. in } \mathbb{R}^N.
 \end{aligned}$$

From (2.104), we obtain  $v \neq 0$ . We see that

$$\begin{aligned}
 -\langle K'_\varepsilon(u_n), u_n \rangle &= \int_{\mathbb{R}^N} (f'(u_n) u_n^2 - (p-1) f(u_n) u_n) dx \\
 (2.105) \qquad &= \int_{\mathbb{R}^N} (f'(v_n) v_n^2 - (p-1) f(v_n) v_n) dx.
 \end{aligned}$$

Now, we repeat the method as in case  $u \neq 0$  and get a contradiction.

In conclusion, we get  $\sup_{n \in \mathbb{N}} \langle K'_\varepsilon(u_n), u_n \rangle < 0$ , and (2.97) implies  $\lambda_n = o_n(1)$  as  $n \rightarrow \infty$ . Therefore,  $\{u_n\}$  is a  $(PS)_c$  sequence of  $I_\varepsilon$  and Lemma 12 is obtained from Lemma 11.  $\square$

**Corollary 1.** *The critical points of  $I_\varepsilon$  on  $\mathcal{N}_\varepsilon$  are critical points of  $I_\varepsilon$  in  $W_\varepsilon$ .*

*Proof.* The key idea is to  $\langle K_\varepsilon(u), u \rangle < 0$  for all  $u \in \mathcal{N}_\varepsilon$ . The proof follows by using similar arguments Lemma 12. We omit at here. We refer the readers to [21, Proposition 2.1] for a proof of this kind lemma.  $\square$

### 4. Existence of a ground state solution

In this section, we prove the existence of a ground state solution for problem  $(P_\varepsilon^*)$ . That is a critical point  $u_\varepsilon$  of  $I_\varepsilon$  satisfying  $I_\varepsilon(u_\varepsilon) = c_\varepsilon$ . In the next, we consider the energy function

$$J_{V_0}(u) = \frac{1}{p} \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{2N}} + \int_{\mathbb{R}^N} V_0|u|^p dx \right) - \int_{\mathbb{R}^N} F(u) dx$$

of problem  $(P_{V_0})$ . We recall that  $c_{V_0}$  is the minimax level related to  $J_{V_0}$  and  $\mathcal{N}_{V_0}$  is the Nehari manifold related to  $J_{V_0}$  is given by

$$\mathcal{N}_{V_0} = \left\{ u \in W^{s,N/s}(\mathbb{R}^N) \setminus \{0\} : \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{2N}} + \int_{\mathbb{R}^N} V_0|u|^p dx = \int_{\mathbb{R}^N} f(u)u dx \right\}.$$

Now we state the main result of this section:

**Theorem 7.** *Assume that  $(f_1)$ – $(f_5)$  and  $(V)$  hold. Then there exists  $\bar{\varepsilon} > 0$  such that  $(P_\varepsilon^*)$  has a ground state solution for all  $0 < \varepsilon < \bar{\varepsilon}$ .*

*Proof.* We will prove that there exists  $\bar{\varepsilon} > 0$  such that  $c_\varepsilon < c_{V_0}$  for all  $\varepsilon \in (0, \bar{\varepsilon})$ . Since  $c_{V_0} < c_{V_\infty}$  when  $V_\infty < +\infty$ , Lemma 11 implies that  $I_\varepsilon$  satisfies the  $(PS)_{c_\varepsilon}$  condition. Further, combine that result with Lemma 7,  $I_\varepsilon$  has a critical point at level  $c_\varepsilon$ . Next, without the loss of generality, we assume that  $V(0) = V_0$ .

Let  $\phi \in C_0^\infty(\mathbb{R}^N, [0, 1])$  be such that  $\phi \equiv 1$  on  $B_1(0)$  and  $\phi \equiv 0$  on  $\mathbb{R}^N \setminus B_2(0)$ . For each  $r > 0$ , let us define  $v_r(x) = \phi(\frac{x}{r})w(x)$ , where  $w$  is a ground state solution of the problem  $(P_{V_0})$  given in Proposition 1. For each  $v_r$ , there exists  $t_{\varepsilon,r} > 0$  such that  $t_{\varepsilon,r}v_r \in \mathcal{N}_\varepsilon$ , and we have

$$\begin{aligned} c_\varepsilon \leq I_\varepsilon(t_{\varepsilon,r}v_r) &= \frac{t_{\varepsilon,r}^p}{p} \int_{\mathbb{R}^{2N}} \frac{|v_r(x) - v_r(y)|^p}{|x - y|^{2N}} dx dy + \frac{t_{\varepsilon,r}^p}{p} \int_{\mathbb{R}^N} V(\varepsilon x)|v_r(x)|^p dx \\ &\quad - \int_{\mathbb{R}^N} F(t_{\varepsilon,r}v_r) dx. \end{aligned}$$

For any  $u \in \mathcal{N}_\varepsilon$ , we have

$$\|u\|_{W_\varepsilon}^p = \int_{\mathbb{R}^N} f(u)u dx.$$

Then we get

$$(3.1) \quad I_\varepsilon(u)|_{\mathcal{N}_\varepsilon} = \frac{1}{p} \|u\|_{W_\varepsilon}^p - \int_{\mathbb{R}^N} F(u) dx = \int_{\mathbb{R}^N} \left( \frac{1}{p} f(u)u - F(u) \right) dx \geq 0.$$

From (3.1), we see that the sequence  $\{t_{\varepsilon,r}\}$  must be bounded as  $\varepsilon \rightarrow 0^+$  for each  $r > 0$ . Indeed, if  $t_{\varepsilon,r} \rightarrow +\infty$  as  $\varepsilon \rightarrow 0^+$  for fixed  $r$ , then using the condition  $(f_4)$ , we have

$$I_\varepsilon(t_{\varepsilon,r}v_r) \geq \frac{t_{\varepsilon,r}^p}{p} \|v_r\|_{W_\varepsilon}^p - \gamma_1 t_{\varepsilon,r}^\mu \|v_r\|_{L^\mu(\mathbb{R}^N)}^\mu \rightarrow -\infty,$$

which is a contradiction with (3.1). Thus, we can assume that  $t_{\varepsilon,r} \rightarrow t_r$  as  $\varepsilon \rightarrow 0^+$ . Since  $v_r$  has compact support, then we get

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0^+} c_\varepsilon &\leq \frac{t_r^p}{p} \int_{\mathbb{R}^{2N}} \frac{|v_r(x) - v_r(y)|^p}{|x - y|^{2N}} dx dy + \frac{t_r^p}{p} \int_{\mathbb{R}^N} V_0|v_r|^p dx - \int_{\mathbb{R}^N} F(t_r v_r) dx \\ &= J_{V_0}(t_r v_r) \end{aligned}$$

via to Vitali’s theorem or Lebesgue’s dominated convergence theorem. Note that  $t_r v_r, w \in \mathcal{N}_{V_0}$  and  $v_r \rightarrow w$  in  $W^{s, N/s}(\mathbb{R}^N)$  as  $r \rightarrow +\infty$  (see [8, Lemma 2.2]) and using Remark 1 (IV), we can prove that  $t_r \rightarrow 1$  as  $t \rightarrow \infty$ . Then we get

$$\limsup_{\varepsilon \rightarrow 0^+} c_\varepsilon \leq \lim_{r \rightarrow +\infty} J_{V_0}(t_r v_r) = J_{V_0}(w) = c_{V_0}.$$

By arguments as Lemma 4, we get  $c_{V_0} \leq C_{\gamma_1}$ . Thus, if we choose  $\gamma_1$  large enough as in (2.30) for  $d_*$  instead of  $\mathfrak{d}_*$ , and  $C_{\gamma_1} < c_{V_\infty}$  if  $V_\infty < +\infty$ , then for any  $(PS)_{c_\varepsilon}$  sequence  $(u_n)$  for  $I_\varepsilon$ , we get

$$\limsup_{n \rightarrow \infty} \|u_n\|_{W_\varepsilon}^{N/(N-s)} \leq \frac{\beta_* \mathfrak{d}_*^{s/(N-s)}}{2^{N/(N-s)} c \alpha_0}.$$

Now, the result of this theorem comes from Lemma 11. □

### 5. Multiplicity of solutions to $(P_\varepsilon^*)$

In this section, we show that the existence of multiple weak solutions and study the behavior of its maximum points related with the set  $M$ . The main result of this section is equivalent to Theorem 2 and it is stated as follows:

**Theorem 8.** *Assume that  $(f_1)$ – $(f_5)$  and  $(V)$  hold. Then for any  $\delta > 0$ , there exists  $\varepsilon_\delta > 0$  such that  $(P_\varepsilon^*)$  has at least  $\text{cat}_{M_\delta}(M)$  nontrivial nonnegative solutions, for any  $0 < \varepsilon < \varepsilon_\delta$ . Moreover, if  $u_\varepsilon$  denotes one of these solutions and  $z_\varepsilon$  is its global maximum, then*

$$\lim_{\varepsilon \rightarrow 0^+} V(\varepsilon z_\varepsilon) = V_0.$$

*Proof.* Let  $\delta > 0$  be a fixed and  $w$  be a ground state solution of problem  $(P_{V_0})$ . It means that  $J_{V_0}(w) = c_{V_0}$  and  $J'_{V_0}(w) = 0$ . Let  $\eta$  be a smooth nonincreasing cut-off function in  $[0, +\infty)$  such that  $\eta(s) = 1$  if  $0 \leq s \leq \frac{\delta}{2}$  and  $\eta(s) = 0$  if  $s \geq \delta$ . For any  $y \in M$ , we denote

$$\psi_{\varepsilon, y} = \eta(|\varepsilon x - y|) w \left( \frac{\varepsilon x - y}{\varepsilon} \right)$$

and  $\Phi_\varepsilon : M \rightarrow \mathcal{N}_\varepsilon$  which is defined by  $\Phi_\varepsilon(y) = t_\varepsilon \psi_{\varepsilon, y}$ , where  $t_\varepsilon > 0$  satisfies

$$\max_{t \geq 0} I_\varepsilon(t \psi_{\varepsilon, y}) = I_\varepsilon(t_\varepsilon \psi_{\varepsilon, y}).$$

From the construction,  $\Phi_\varepsilon(y)$  has compact support for any  $y \in M$ .

**Lemma 13.** *The function  $\Phi_\varepsilon$  satisfies the following limit*

$$\lim_{\varepsilon \rightarrow 0^+} I_\varepsilon(\Phi_\varepsilon(y)) = c_{V_0} \quad \text{uniformly in } y \in M.$$

*Proof.* Suppose that the statement of Lemma 13 does not hold, then there exists  $\delta_0 > 0$ ,  $\{y_n\} \subset M$  and  $\varepsilon_n \rightarrow 0$  such that

$$(4.1) \quad |I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - c_{V_0}| \geq \delta_0.$$

By Lemma 2.2 [8], we have

$$(4.2) \quad \lim_{n \rightarrow \infty} \|\psi_{\varepsilon_n, y_n}\|_{W_{\varepsilon_n}}^p = \|w\|_{W_{V_0}}^p.$$

Since  $\langle I'_{\varepsilon_n}(t_{\varepsilon_n}\psi_{\varepsilon_n,y_n}), t_{\varepsilon_n}\psi_{\varepsilon_n,y_n} \rangle = 0$ , using the change of variable  $z = \frac{\varepsilon_n x - y_n}{\varepsilon_n}$ , then we get

$$\begin{aligned} \|t_{\varepsilon_n}\psi_{\varepsilon_n,y_n}\|_{W_{\varepsilon_n}}^p &= \int_{\mathbb{R}^N} f(t_{\varepsilon_n}\psi_{\varepsilon_n,y_n})t_{\varepsilon_n}\psi_{\varepsilon_n,y_n} \, dx \\ (4.3) \qquad \qquad \qquad &= \int_{\mathbb{R}^N} f(t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z))t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z) \, dz. \end{aligned}$$

Now we prove that  $t_{\varepsilon_n} \rightarrow 1$ . First we show that  $t_{\varepsilon_n} \rightarrow t_0 < +\infty$ . Conversely if  $t_{\varepsilon_n} \rightarrow +\infty$ , from (4.3) we have

$$(4.4) \qquad \qquad \qquad \|\psi_{\varepsilon_n,y_n}\|_{W_{\varepsilon_n}}^p \geq \int_{|z| \leq \frac{\delta}{2\varepsilon_n}} \frac{f(t_{\varepsilon_n}w(z))w(z)}{t_{\varepsilon_n}^{p-1}} \, dz.$$

From the condition  $(f_2)$  and  $(f_4)$ , we have  $f(t) \geq \gamma_1\mu|t|^{\mu-1}$  for all  $t \geq 0$ . Combine that property and (4.4), we deduce

$$\|\psi_{\varepsilon_n,y_n}\|_{W_{\varepsilon_n}}^p \geq \int_{|z| \leq \frac{\delta}{2\varepsilon_n}} \frac{f(t_{\varepsilon_n}w(z))w(z)}{t_{\varepsilon_n}^{p-1}} \, dz \geq \gamma_1\mu t_{\varepsilon_n}^{\mu-p} \int_{|z| < \frac{\delta}{2\varepsilon_n}} w^\mu \, dx \rightarrow +\infty$$

as  $n \rightarrow \infty$ . It is a contradiction with (4.2). Therefore, up to a subsequence, we may assume that  $t_{\varepsilon_n} \rightarrow t_0 \geq 0$  as  $n \rightarrow \infty$ . If  $t_0 = 0$ , from  $t_{\varepsilon_n}\psi_{\varepsilon_n,y_n} \in \mathcal{N}_{\varepsilon_n}$ , by Lemma 2, there exists  $r_* > 0$  such that for  $n$  large enough, we have

$$\|t_{\varepsilon_n}\psi_{\varepsilon_n,y_n}\|_{W_{\varepsilon_n}} \geq r_* > 0.$$

It is contradiction since  $t_{\varepsilon_n} \rightarrow 0$  and  $\|\psi_{\varepsilon_n,y_n}\|_{W_{\varepsilon_n}} \rightarrow \|w\|_{W_{V_0}} > 0$  as  $n \rightarrow \infty$ . Now we prove that  $t_0 = 1$ . From (4.3), using Lebesgue Dominated Convergence Theorem, we have

$$\|w\|_{W_{V_0}}^p = \int_{\mathbb{R}^N} \frac{f(t_0w)w}{t_0^{p-1}} \, dx.$$

Note that  $w \in \mathcal{N}_{V_0}$ , then Remark 1 implies  $t_0 = 1$ . Still using Lebesgue Dominated convergence theorem or Vitali's theorem, we get

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} F(t_{\varepsilon_n}\psi_{\varepsilon_n,y_n}(x)) \, dx = \int_{\mathbb{R}^N} F(w) \, dx.$$

Hence, we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) &= \lim_{n \rightarrow \infty} \left[ \frac{t_{\varepsilon_n}^p}{p} \|\psi_{\varepsilon_n,y_n}\|_{W_{\varepsilon_n}}^p - \int_{\mathbb{R}^N} F(t_{\varepsilon_n}\psi_{\varepsilon_n,y_n}) \, dx \right] \\ &= \frac{\|w\|_{W_{V_0}}^p}{p} - \int_{\mathbb{R}^N} F(w) \, dx = J_{V_0}(w) = c_{V_0} \end{aligned}$$

which contradicts with (4.1). □

For any  $\delta > 0$ , let  $\rho = \rho(\delta) > 0$  be such that  $M_\delta \subset B_\rho(0)$ . Let  $\chi: \mathbb{R}^N \rightarrow \mathbb{R}^N$  be define as

$$\chi(x) = \begin{cases} x & \text{if } |x| < \rho, \\ \frac{\rho x}{|x|} & \text{if } |x| \geq \rho. \end{cases}$$

Next, we define the barycenter map  $\beta_\varepsilon: \mathcal{N}_\varepsilon \rightarrow \mathbb{R}^N$  given by

$$\beta_\varepsilon(u) = \frac{\int_{\mathbb{R}^N} \chi(\varepsilon x)|u(x)|^p \, dx}{\int_{\mathbb{R}^N} |u|^p \, dx}.$$

**Lemma 14.** *The functional  $\Phi_\varepsilon$  satisfies the following limit*

$$(4.5) \quad \lim_{\varepsilon \rightarrow 0^+} \beta_\varepsilon(\Phi_\varepsilon(y)) = y \quad \text{uniformly in } y \in M.$$

*Proof.* Lemma 14 is proved similarly as [8, Lemma 3.13]. For convenience, we prove it at here. Suppose that the statement of Lemma 14 doesnot hold, then there exists  $\delta_0 > 0$ ,  $\{y_n\} \subset M$  and  $\varepsilon_n \rightarrow 0$  such that

$$(4.6) \quad |\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \geq \delta_0.$$

From the definition of  $\Phi_{\varepsilon_n}(y_n)$ ,  $\beta_{\varepsilon_n}$ ,  $\psi$  and using the change of variable  $z = \frac{\varepsilon_n x - y_n}{\varepsilon_n}$ , we have

$$\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = y_n + \frac{\int_{\mathbb{R}^N} [\chi(\varepsilon_n z + y_n) - y_n] |\eta(|\varepsilon_n z|) w(z)|^p dz}{\int_{\mathbb{R}^N} |\eta(|\varepsilon_n z|) w(z)|^p dz}.$$

Since  $\{y_n\} \subset M \subset B_\rho(0)$ , we get

$$\lim_{n \rightarrow \infty} |\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| = 0$$

via to Lebesgue Dominated Convergence Theorem, which contradicts with (4.6).  $\square$

**Lemma 15.** *Let  $\varepsilon_n \rightarrow 0^+$  and  $\{u_n\} \subset \mathcal{N}_{\varepsilon_n}$  be such that  $I_{\varepsilon_n}(u_n) \rightarrow c_{V_0}$ . Then there exists  $\{\tilde{y}_n\} \subset \mathbb{R}^N$  such that the translation sequence  $v_n(x) = u_n(x + \tilde{y}_n)$  has a subsequence which converges in  $W^{s,N/s}(\mathbb{R}^N)$ . Moreover, up to a subsequence,  $\{y_n\}: y_n = \varepsilon \tilde{y}_n \rightarrow y \in M$ .*

*Proof.* Since  $\langle I'_{\varepsilon_n}(u_n), u_n \rangle = 0$  and  $I_{\varepsilon_n}(u_n) \rightarrow c_{V_0}$ , then we have

$$\begin{aligned} I_{\varepsilon_n}(u_n) &= I_{\varepsilon_n}(u_n) - \frac{1}{\mu} \langle I'_{\varepsilon_n}(u_n), u_n \rangle \\ &= \left( \frac{s}{N} - \frac{1}{\mu} \right) \|u_n\|_{W_{\varepsilon_n}}^p + \frac{1}{\mu} \int_{\mathbb{R}^N} (f(u_n)u_n - F(u_n)) dx \geq \left( \frac{s}{N} - \frac{1}{\mu} \right) \|u_n\|_{W_{\varepsilon_n}}^p. \end{aligned}$$

Thus, there exists a constant  $C = \left( \frac{c_{V_0}}{\frac{s}{N} - \frac{1}{\mu}} \right)^{s/N}$  such that  $\limsup_{n \rightarrow \infty} \|u_n\|_{W_{\varepsilon_n}} \leq C$ .

Since  $W_{\varepsilon_n}$  is continuously embedded into  $W^{s,N/s}(\mathbb{R}^N)$  and (2.39), we get  $\{u_n\}$  is a bounded sequence  $W^{s,N/s}(\mathbb{R}^N)$ . Now, we show that there exist a sequence  $\{\tilde{y}_n\} \subset \mathbb{R}^N$  and constants  $r > 0, \beta > 0$  such that

$$(4.7) \quad \liminf_{n \rightarrow \infty} \int_{B_r(\tilde{y}_n)} |u_n|^{N/s} dx \geq \beta > 0.$$

Indeed, if (4.7) is false, then for any  $r > 0$ , we have

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_r(y)} |u_n|^{N/s} dx = 0.$$

By Lemma 5, we have  $u_n \rightarrow 0$  strongly in  $L^q(\mathbb{R}^N)$  for any  $q \in (\frac{N}{s}, +\infty)$ . If we take  $\gamma_1$  large enough as the method of Lemma 4, we get

$$\limsup_{n \rightarrow \infty} \|u_n\|_{V_0}^{N/(N-s)} \leq \frac{\beta_* \mathfrak{d}_*^{s/(N-s)}}{\mathfrak{c} \alpha_0},$$

for a suitable constant  $\mathfrak{c} > 1$  and near 1. Apply Lemma 6, we deduce

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} f(u_n)u_n dx = 0.$$

Combine that result and  $u_n \in \mathcal{N}_{\varepsilon_n}$ , we obtain  $\|u_n\|_{W_{\varepsilon_n}} \rightarrow 0$  as  $n \rightarrow \infty$ . It is a contradiction with Proposition 2. Therefore, (4.7) holds. Let us define  $v_n := u_n(x + \tilde{y}_n)$ . Since the  $\|\cdot\|_{V_0}$  is invariant with the translation, then  $\{v_n\}$  is a bounded sequence in  $W^{s,N/s}(\mathbb{R}^N)$ , thus up to a subsequence, we can assume that there exists  $v \in W^{s,N/s}(\mathbb{R}^N)$  such that  $v_n \rightarrow v$  weakly in  $W^{s,N/s}(\mathbb{R}^N)$  and  $v_n(x) \rightarrow v(x)$  a.e. in  $\mathbb{R}^N$  and  $v_n \rightarrow v$  in  $L^q_{\text{loc}}(\mathbb{R}^N)$  for any  $q \in [\frac{N}{s}, +\infty)$ . From that result and (4.7), we get  $v \neq 0$ . Let  $t_n > 0$  such that  $w_n = t_n v_n \in \mathcal{N}_{V_0}$  and we set  $y_n := \varepsilon_n \tilde{y}_n$ . Thus, using the change of the variable  $z = x + \tilde{y}_n$ ,  $V(\varepsilon_n(x + \tilde{y}_n)) \geq V_0$  and the invariance by translation, we can see that

$$\begin{aligned} c_{V_0} \leq J_{V_0}(w_n) &\leq \frac{1}{p}[w_n]_{s,p}^p + \frac{1}{p} \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)|w_n|^p dx - \int_{\mathbb{R}^N} F(w_n) dx \\ &= I_{\varepsilon_n}(t_n u_n) \leq I_{\varepsilon_n}(u_n) \leq c_{V_0} + o_n(1). \end{aligned}$$

Then we get  $J_{V_0}(w_n) \rightarrow c_{V_0}$ . Since  $\{w_n\} \subset \mathcal{N}_{V_0}$ , using the condition  $(f_2)$ , there exists a constant  $K > 0$  such that  $\|w_n\|_{V_0} \leq K$  for all  $n$ . We have  $v_n \not\rightarrow 0$  strongly in  $W^{s,N/s}(\mathbb{R}^N)$ . Indeed, if  $v_n \rightarrow 0$  in  $W^{s,N/s}(\mathbb{R}^N)$ , then  $v_n \rightarrow 0$  weakly in  $W^{s,N/s}(\mathbb{R}^N)$ , it contradicts with  $v_n \rightarrow v \neq 0$  weakly in  $W^{s,N/s}(\mathbb{R}^N)$ . Hence, there exists  $\alpha > 0$  such that  $\|v_n\|_{V_0} \geq \alpha > 0$  for all  $n$ . Consequently, we have

$$t_n \alpha \leq \|t_n v_n\|_{V_0} = \|w_n\|_{V_0} \leq K,$$

which yields  $t_n \leq \frac{K}{\alpha}$  for all  $n \in \mathbb{N}$ . Therefore, up to a subsequence, we can assume that  $t_n \rightarrow t_0 \geq 0$ . We prove that  $t_0 > 0$ . If  $t_0 = 0$ , then  $w_n \rightarrow 0$  strong in  $W^{s,N/s}(\mathbb{R}^N)$ , it implies that  $J_{V_0}(w_n) \rightarrow 0$ . It contradicts with  $c_{V_0} > 0$ . Up to a subsequence, we suppose that  $w_n \rightarrow w := t_0 v \neq 0$  weakly in  $W^{s,N/s}(\mathbb{R}^N)$  and  $w_n(x) \rightarrow w(x)$  a.e. on  $\mathbb{R}^N$ . By arguments as Lemma 4, we can get  $J'_{V_0}(w) = 0$ . Now we prove that

$$(4.8) \quad \lim_{n \rightarrow \infty} \|w_n\|_{V_0}^p = \|w\|_{V_0}^p.$$

Using Brezis–Lieb’s lemma and (4.8), we obtain  $w_n \rightarrow w$  strong in  $W^{s,N/s}(\mathbb{R}^N)$ . By Fatou’s lemma, we have

$$(4.9) \quad \|w\|_{V_0}^p \leq \liminf_{n \rightarrow \infty} \|w_n\|_{V_0}^p$$

Assume that by contradiction that

$$\|w\|_{V_0}^p < \limsup_{n \rightarrow \infty} \|w_n\|_{V_0}^p.$$

Note that

$$\begin{aligned} c_{V_0} + o_n(1) &= J_{V_0}(w_n) - \frac{1}{\mu} \langle J'_{V_0}(w_n), w_n \rangle \\ &= \left( \frac{1}{p} - \frac{1}{\mu} \right) \|w_n\|_{V_0}^p + \int_{\mathbb{R}^N} \left[ \frac{1}{\mu} f(w_n) w_n - F(w_n) \right] dx. \end{aligned}$$

Using the condition  $(f_2)$ , and Fatou’s lemma, we get

$$\begin{aligned} c_{V_0} &\geq \left( \frac{1}{p} - \frac{1}{\mu} \right) \limsup_{n \rightarrow \infty} \|w_n\|_{V_0}^p + \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} \left[ \frac{1}{\mu} f(w_n) w_n - F(w_n) \right] dx \\ &> \left( \frac{1}{p} - \frac{1}{\mu} \right) \|w\|_{V_0}^p + \int_{\mathbb{R}^N} \left[ \frac{1}{\mu} f(w) w - F(w) \right] dx \\ &= J_{V_0}(w) - \frac{1}{\mu} \langle J'_{V_0}(w), w \rangle = J_{V_0}(w) \geq c_{V_0}, \end{aligned}$$

which is a contradiction. Then

$$(4.10) \quad \|w\|_{V_0}^p \geq \limsup_{n \rightarrow \infty} \|w_n\|_{V_0}^p.$$

Combine (4.9) and (4.10), we get (4.8). Since  $t_n \rightarrow t_0$  as  $n \rightarrow \infty$ , then  $v_n \rightarrow v$  in  $W^{s,N/s}$  as  $n \rightarrow \infty$ . Now we prove that  $\{y_n\}$  has a subsequence such that  $y_n \rightarrow y \in M$ . Indeed, if  $\{y_n\}$  is not bounded, that is there exists a subsequence, still denoted by  $\{y_n\}$ , such that  $|y_n| \rightarrow +\infty$ . First, we consider the case  $V_\infty = \infty$ . Using the fact that  $\{u_n\} \subset \mathcal{N}_{\varepsilon_n}$  and a change of variable  $z = x + \tilde{y}_n$ , we can see that

$$\begin{aligned} \int_{|x| \leq R\varepsilon_n^{-1}} V(\varepsilon_n x + y_n) |v_n|^p dx &\leq [v_n]_{s,p}^p + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) |v_n|^p dx \\ &= \|u_n\|_{W_{\varepsilon_n}}^p = \int_{\mathbb{R}^N} f(u_n) u_n dx = \int_{\mathbb{R}^N} f(v_n) v_n dx. \end{aligned}$$

Apply Fatou's lemma and Lebesgue Dominated Convergence Theorem or Vitali's theorem and  $v_n \rightarrow v$  in  $W^{s,N/s}(\mathbb{R}^N)$ , we deduce that

$$\begin{aligned} +\infty &= \liminf_{n \rightarrow \infty} \int_{|x| \leq R\varepsilon_n^{-1}} V(\varepsilon_n x + y_n) |v_n|^p dx \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) |v_n|^p dx \\ &\leq \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} f(v_n) v_n dx = \int_{\mathbb{R}^N} f(v) v dx < +\infty, \end{aligned}$$

which gives a contradiction. Next, we consider the case  $V_\infty < +\infty$ . From the fact that  $w_n \rightarrow w$  strongly in  $W^{s,N/s}(\mathbb{R}^N)$  and the condition (V), using the change of variable  $z = x + \tilde{y}_n$ , we have

$$\begin{aligned} c_{V_0} &= J_{V_0}(w) < J_{V_\infty}(w) \\ &\leq \liminf_{n \rightarrow \infty} \left[ \frac{1}{p} \left( [w_n]_{s,p}^p + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) |w_n|^p dx \right) - \int_{\mathbb{R}^N} F(w_n) dx \right] \\ &= \liminf_{n \rightarrow \infty} \left[ \frac{t_n^p}{p} [u_n]_{s,p}^p + \frac{t_n^p}{p} \int_{\mathbb{R}^N} V(\varepsilon_n z) |u_n|^p dz - \int_{\mathbb{R}^N} F(t_n u_n) dz \right] \\ (4.11) \quad &= \liminf_{n \rightarrow \infty} I_{\varepsilon_n}(t_n u_n) \leq \liminf_{n \rightarrow \infty} I_{\varepsilon_n}(u_n) = c_{V_0}, \end{aligned}$$

which is an absurd. Then  $\{y_n\}$  must be a bounded sequence. Up to a subsequence, we can assume that  $y_n \rightarrow y$ . If  $y \notin M$ , then  $V_0 < V(y)$ . By argument as (4.11), we get a contradiction. Hence  $y \in M$ .  $\square$

Let  $\mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a positive function such that  $h(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0^+$  and let

$$\tilde{\mathcal{N}}_\varepsilon = \{u \in \mathcal{N}_\varepsilon : I_\varepsilon(u) \leq c_{V_0} + h(\varepsilon)\}.$$

By Lemma 14, we have  $h(\varepsilon) = |I_\varepsilon(\Phi_\varepsilon(y)) - c_{V_0}| \rightarrow 0$  as  $\varepsilon \rightarrow 0^+$ . Hence  $\Phi_\varepsilon(y) \in \mathcal{N}_\varepsilon$  and  $\tilde{\mathcal{N}}_\varepsilon \neq \emptyset$  for any  $\varepsilon > 0$ . Moreover, we have the following result:

**Lemma 16.** *For any  $\delta > 0$ , it holds that*

$$\lim_{\varepsilon \rightarrow 0^+} \sup_{u \in \tilde{\mathcal{N}}_\varepsilon} \text{dist}(\beta_\varepsilon(u), M_\delta) = 0.$$

*Proof.* Lemma 16 is proved similarly as [8, Lemma 3.14]. For convenience, we prove it at here. Let  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ . By the definition of supremum, there exists  $\{u_n\} \subset \tilde{\mathcal{N}}_{\varepsilon_n}$  such that

$$\sup_{u \in \tilde{\mathcal{N}}_{\varepsilon_n}} \inf_{y \in M_\delta} |\beta_{\varepsilon_n}(u) - y| = \inf_{y \in M_\delta} |\beta_{\varepsilon_n}(u_n) - y| + o_n(1).$$



Therefore, it suffices that there exists  $\{y_n\} \subset M_\delta$  such that

$$(4.12) \quad \lim_{n \rightarrow \infty} |\beta_{\varepsilon_n}(u_n) - y_n| = 0.$$

Note that  $\{u_n\} \subset \tilde{\mathcal{N}}_{\varepsilon_n} \subset \mathcal{N}_{\varepsilon_n}$ , we deduce that

$$c_{V_0} \leq c_{\varepsilon_n} \leq I_{\varepsilon_n}(u_n) \leq c_{V_0} + h(\varepsilon_n),$$

which leads to that  $I_{\varepsilon_n}(u_n) \rightarrow c_{V_0}$ . By Lemma 15, there exists a sequence  $\{\tilde{y}_n\} \subset \mathbb{R}^N$  such that  $y_n = \varepsilon_n \tilde{y}_n \in M_\delta$  for all  $n$  large enough. We have

$$\beta_{\varepsilon_n}(u_n) = y_n + \frac{\int_{\mathbb{R}^N} [\chi(\varepsilon_n z + y_n) - y_n] |u_n|^p(z + \tilde{y}_n) dz}{\int_{\mathbb{R}^N} |u_n|^p(z + \tilde{y}_n) dz}.$$

Since  $u_n(x + \tilde{y}_n)$  converges strongly in  $W^{s, N/s}(\mathbb{R}^N)$  and  $\varepsilon_n z + y_n \rightarrow y \in M$ , we can get  $\beta_{\varepsilon_n}(u_n) = y_n + o_n(1)$  via to Lebesgue Dominated Convergence Theorem. Therefore (4.12) holds.  $\square$

**Lemma 17.** Assume that (V) and  $(f_1)$ – $(f_5)$  hold and let  $v_n$  be a nontrivial nonnegative solution of the following problem

$$(4.13) \quad (-\Delta)_{N/s}^s v_n + V_n(x) |v_n|^{\frac{N}{s}-2} v_n = f(v_n) \quad \text{in } \mathbb{R}^N,$$

where  $V_n(x) = V(\varepsilon_n x + \varepsilon_n \tilde{y}_n)$  and  $\varepsilon_n \tilde{y}_n \rightarrow y \in M$ . If  $\{v_n\}$  is a bounded sequence in  $W^{s, N/s}(\mathbb{R}^N)$  verifying

$$\limsup_{n \rightarrow \infty} \|v_n\|_{V_0}^{N/(N-s)} \leq \frac{\beta_* \mathfrak{D}_*^{s/(N-s)}}{\mathfrak{c} \alpha_0},$$

where  $\mathfrak{c} > 1$  is a suitable constant and  $v_n \rightarrow v$  strong in  $W^{s, N/s}(\mathbb{R}^N)$ , then  $v_n \in L^\infty(\mathbb{R}^N)$  and there exists  $C > 0$  such that  $\|v_n\|_{L^\infty(\mathbb{R}^N)} \leq C$  for all  $n \in \mathbb{N}$ . Furthermore

$$\lim_{|x| \rightarrow +\infty} v_n(x) = 0 \quad \text{uniformly in } n.$$

*Proof.* For any  $L > 0$  and  $\beta > 1$ , let us to consider the function  $\gamma(t) = t(\min\{t, L\})^{p(\beta-1)}$  and

$$\gamma(v_n) = \gamma_{L, \beta}(v_n) = v_n v_{L, n}^{p(\beta-1)} \in W_\varepsilon, v_{L, n} = \min\{v_n, L\}.$$

Set

$$\Lambda(t) = \frac{|t|^p}{p} \quad \text{and} \quad \Gamma(t) = \int_0^t (\gamma'(t))^{1/p} d\tau.$$

Then we have [8]

$$(4.14) \quad \Lambda'(a-b)(\gamma(a) - \gamma(b)) \geq |\Gamma(a) - \Gamma(b)|^p \quad \text{for any } a, b \in \mathbb{R}.$$

From (4.14), we get

$$(4.15) \quad \begin{aligned} & |\Gamma(v_n(x)) - \Gamma(v_n(y))|^p \\ & \leq |v_n(x) - v_n(y)|^{p-2} (v_n(x) - v_n(y)) ((v_n v_{L, n}^{p(\beta-1)})(x) - (v_n v_{L, n}^{p(\beta-1)})(y)). \end{aligned}$$

Therefore, taking  $\gamma(v_n) = v_n v_{L,n}^{p(\beta-1)}$  as a test function in (4.13) and combine with (4.15), we have

$$\begin{aligned}
 & [\Gamma(v_n)]_{s,p}^p + \int_{\mathbb{R}^N} V_n(x) |v_n|^p v_{L,n}^{p(\beta-1)} dx \\
 & \leq \int_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^{p-2} (v_n(x) - v_n(y)) ((v_n v_{L,n}^{p(\beta-1)})(x) - (v_n v_{L,n}^{p(\beta-1)})(y))}{|x - y|^{2N}} dx dy \\
 (4.16) \quad & + \int_{\mathbb{R}^N} V_n(x) |v_n|^p v_{L,n}^{p(\beta-1)} dx = \int_{\mathbb{R}^N} f(v_n) v_n v_{L,n}^{p(\beta-1)} dx.
 \end{aligned}$$

Using (4.14), we have  $v_n v_{L,n}^{\beta-1} \geq |\Gamma(v_n)|$ . Since  $\Gamma(v_n) \geq \frac{1}{\beta} v_n v_{L,n}^{\beta-1}$  and the embedding from  $W^{s,N/s}(\mathbb{R}^N) \rightarrow L^{N^*}(\mathbb{R}^N)$  ( $N^* > \frac{N}{s}$ ) is continuous, then there exists a suitable constant  $S_* > 0$  such that

$$(4.17) \quad \|\Gamma(v_n)\|_{V_0/2}^p \geq S_* \|\Gamma(v_n)\|_{L^{N^*}(\mathbb{R}^N)}^p \geq \frac{1}{\beta^p} S_* \|v_n v_{L,n}^{\beta-1}\|_{L^{N^*}(\mathbb{R}^N)}^p.$$

From the condition  $(f_1)$  and  $(f_3)$ , for any  $\varepsilon > 0$ , there exist  $C(\varepsilon) > 0$  such that

$$|f(t)| \leq \varepsilon |t|^{p-1} + C(\varepsilon) |t|^{p-1} \Phi_{N,s}(\alpha_0 |t|^{N/(N-s)})$$

for all  $t \in \mathbb{R}$ . Then we obtain

$$\begin{aligned}
 & \frac{1}{\beta^p} S_* \|v_n v_{L,n}^{\beta-1}\|_{L^{N^*}(\mathbb{R}^N)}^p + \int_{\mathbb{R}^N} V_n(x) |v_n|^p v_{L,n}^{p(\beta-1)} dx \\
 (4.18) \quad & \leq \varepsilon \int_{\mathbb{R}^N} |v_n v_{L,n}^{\beta-1}|^p dx + C(\varepsilon) \int_{\mathbb{R}^N} \Phi_{N,s}(\alpha_0 |v_n|^{N/(N-s)}) |v_n v_{L,n}^{\beta-1}|^p dx.
 \end{aligned}$$

Choose  $0 < \varepsilon < V_0/2$ , then (4.18) implies

$$\begin{aligned}
 & \frac{1}{\beta^p} S_* \|v_n v_{L,n}^{\beta-1}\|_{L^{N^*}(\mathbb{R}^N)}^p \\
 & \leq C(\varepsilon) \left( \int_{\mathbb{R}^N} (\Phi_{N,s}(\alpha_0 |v_n|^{N/(N-s)}))^{q'} dx \right)^{\frac{1}{q'}} \left( \int_{\mathbb{R}^N} |v_n v_{L,n}^{\beta-1}|^{qp} dx \right)^{\frac{1}{q}}.
 \end{aligned}$$

Using Trudinger–Moser inequality in  $W^{s,N/s}(\mathbb{R}^N)$  with  $q \gg \frac{N}{s}$  such that  $N^{**} = qp < N^*$ ,  $q' > 1$  and  $q'$  near 1, then there exists a constant  $D > 0$  such that

$$\|v_n v_{L,n}^{\beta-1}\|_{L^{N^*}(\mathbb{R}^N)}^p \leq D \beta^p \|v_n v_{L,n}^{\beta-1}\|_{L^{qp}(\mathbb{R}^N)}^p.$$

Let  $L \rightarrow +\infty$  in above inequality, we deduce

$$(4.19) \quad \|v_n\|_{L^{N^* \beta}} \leq D^{\frac{1}{p\beta}} \beta^{\frac{1}{\beta}} \|v_n\|_{L^{N^{**} \beta}(\mathbb{R}^N)}.$$

Now, we set  $\beta = \frac{N^*}{N^{**}} > 1$ . Then  $\beta^2 N^{**} = \beta N^*$  and (4.19) holds with  $\beta$  replaced by  $\beta^2$ . Therefore, we obtain

$$\begin{aligned}
 & \|v_n\|_{L^{N^* \beta^2}} \leq D^{\frac{1}{p\beta^2}} \beta^{\frac{2}{\beta^2}} \|v_n\|_{L^{N^{**} \beta^2}(\mathbb{R}^N)} = D^{\frac{1}{p\beta^2}} \beta^{\frac{2}{\beta^2}} \|v_n\|_{L^{N^* \beta}(\mathbb{R}^N)} \\
 (4.20) \quad & \leq D^{\frac{1}{p}(\frac{1}{\beta} + \frac{1}{\beta^2})} \beta^{\frac{1}{\beta} + \frac{2}{\beta^2}} \|v_n\|_{L^{N^{**} \beta}(\mathbb{R}^N)}.
 \end{aligned}$$

Iterating this process as in (4.20), we can infer that for any positive integer  $m$ ,

$$(4.21) \quad \|v_n\|_{L^{N^* \beta^m}} \leq D^{\sum_{j=1}^m \frac{1}{p\beta^j}} \beta^{\sum_{j=1}^m j\beta^{-j}} \|v_n\|_{L^{N^{**} \beta}(\mathbb{R}^N)}.$$

Taking the limit in (4.21) as  $m \rightarrow \infty$ , we get

$$\|v_n\|_{L^\infty(\mathbb{R}^N)} \leq C$$

for all  $n$ , where  $C = D^{\sum_{j=1}^{\infty} \frac{1}{p\beta^j}} \beta^{\sum_{j=1}^{\infty} j\beta^{-j}} \sup_{n \in \mathbb{N}} \|v_n\|_{L^{N^{**\beta}}(\mathbb{R}^N)} < +\infty$ . □

Now, we give the proof of Theorem 8. We fix  $\varepsilon > 0$  small enough. Then by Lemma 13 and Lemma 16, we have  $\beta_\varepsilon \circ \Phi_\varepsilon$  is homotopic to inclusion map  $\text{id}: M \rightarrow M_\delta$ . Then we get

$$\text{cat}_{\tilde{\mathcal{N}}_\varepsilon}(\tilde{\mathcal{N}}_\varepsilon) \geq \text{cat}_{M_\delta}(M).$$

Since the functional  $I_\varepsilon$  satisfies the  $(PS)_c$  condition for  $c \in (c_{V_0}, c_{V_0} + h(\varepsilon))$ , then by Lusternik–Schnirelmann theory of critical points, see Willem [41],  $I_\varepsilon$  has at least  $\text{cat}_{M_\delta}(M)$  critical points on  $\mathcal{N}_\varepsilon$ . By Corollary 1,  $I_\varepsilon$  has at least  $\text{cat}_{M_\delta}(M)$  critical points in  $W_\varepsilon$ .

Let  $u_{\varepsilon_n}$  is a solution of problem  $(P_{\varepsilon_n})$ , then  $v_n(x) = u_{\varepsilon_n}(x + \tilde{y}_n)$  is a solution of the equation (4.13). Moreover, up to a subsequence, we may assume that  $v_n \rightarrow v$  strong in  $W^{s,N/s}(\mathbb{R}^N)$  and  $y_n = \varepsilon_n \tilde{y}_n \rightarrow y \in M$ . Next, we prove that there exists  $\delta > 0$  such that  $\|v_n\|_{L^\infty(\mathbb{R}^N)} \geq \delta$  for all  $n$  large enough. Indeed, by Lemma 15 (see (4.7)), we have

$$(4.22) \quad 0 < \frac{\beta}{2} \leq \int_{B_r(0)} |v_n|^{N/s} dx \leq |B_r(0)| \|v_n\|_{L^\infty(\mathbb{R}^N)}$$

for all  $n$  large enough. Here, we choose  $\delta = \frac{\beta}{2}$ . Since  $v_n \rightarrow v$  strong in  $W^{s,N/s}(\mathbb{R}^N)$ , then we have  $\lim_{|x| \rightarrow \infty} v_n(x) = 0$  uniformly in  $n \in \mathbb{N}$ . We denote  $p_n$  by the global maximum of  $v_n$ , then by Lemma 17 and (4.22), there exists  $R > 0$  such that  $|p_n| \leq R$  for all  $n \in \mathbb{N}$ . Therefore, the maximum point of  $u_{\varepsilon_n}$  is given by  $z_{\varepsilon_n} = p_n + \tilde{y}_n$  and  $\varepsilon_n z_{\varepsilon_n} \rightarrow y \in M$ . By the continuity of  $V$ , we get  $V(\varepsilon_n z_{\varepsilon_n}) \rightarrow V(y) = V_0$  as  $n \rightarrow \infty$ .

If  $u_\varepsilon$  is a nontrivial nonnegative solution of problem  $(P_\varepsilon^*)$ , then  $w_\varepsilon(x) = u_\varepsilon(x/\varepsilon)$  is a nontrivial nonnegative solution of  $(P_\varepsilon)$ . Thus the maximum points  $\eta_\varepsilon$  and  $z_\varepsilon$  of  $w_\varepsilon$  and  $u_\varepsilon$  respectively, satisfy  $\eta_\varepsilon = \varepsilon z_\varepsilon$ . We deduce

$$\lim_{\varepsilon \rightarrow 0^+} V(\eta_\varepsilon) = \lim_{n \rightarrow \infty} V(\varepsilon_n z_{\varepsilon_n}) = V_0. \quad \square$$

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