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Multiplicity and concentration of solutions to a fractional p-Laplace problem with exponential growth

Nguyen Van Thin

Abstract. In this paper, we study the Schrödinger equation involving the $\frac{N}{s}$-fractional Laplacian:

$$\varepsilon^N(-\Delta)^s_{N/s}u + V(x)|u|^{\frac{N}{s}-2}u = f(u) \quad \text{in } \mathbb{R}^N,$$

where ε is a positive parameter, $N = ps$, $s \in (0,1)$. The nonlinear function f has exponential growth and the potential function V is a continuous function satisfying some suitable conditions. Our problem lacks compactness. By using the Ljusternik–Schnirelmann theory, we obtain the existence, multiplicity and concentration of nontrivial nonnegative solutions for small values of the parameter.

Eksponentiaalisesti kasvavan murtoasteisen p-Laplacen ongelman ratkaisuiden monilukuisuus ja kasautuminen

Tiivistelmä. Tässä työssä tutkimme seuraavaa $\frac{N}{s}$-murtoasteisen Laplacen operaattorin sisältävää Schrödingerin yhtälöä

$$\varepsilon^N(-\Delta)^s_{N/s}u + V(x)|u|^{\frac{N}{s}-2}u = f(u) \quad \text{avaruudessa } \mathbb{R}^N,$$

missä ε on positiivinen parametri, $N = ps$ ja $s \in (0,1)$. Epälineaarinen funktio f kasvaa eksponentiaalisesti, ja potentiaali V on sopivat ehdot toteuttava jatkuva funktio. Tältä ongelmalta puuttuu kompaktisuusominaisuuksia. Ljusternikin–Schnirelmannin teorian avulla osoitamme pienillä parametriarvoilla epätriviaalien ei-negatiivisten ratkaisujen olemassaolon, monilukuisuuden ja kasautumisen.

1. Introduction and main results

In this paper, we first study the existence and concentration of nontrivial nonnegative solutions for the fractional $\frac{N}{s}$-Laplace Schrödinger equation

$$\varepsilon^N(-\Delta)^s_{N/s}u(x) + V(x)|u|^{\frac{N}{s}-2}u = f(u) \quad \text{in } \mathbb{R}^N, \tag{1.1} \qquad (P_\varepsilon)$$

where ε is small positive parameter, $0 < s < 1, 2 \le p < +\infty$, $N = ps$, the potential V is bounded below by $V_0 > 0$, the nonlinearity f has exponential critical growth, and $(-\Delta)^s_p$ is the fractional p-Laplace operator which may be defined along a function $\varphi \in C_0^\infty(\mathbb{R}^N)$ (up to a normalization constant) as

$$(-\Delta)^s_p\varphi(x) = 2\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dy$$

for $x \in \mathbb{R}^N$, where $B_\varepsilon(x)$ is a ball with center x and radius ε.

In order to study the problem (1.1), we need some assumptions on V and f as follows:

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(V) $V\colon \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying

$$V_\infty = \liminf_{|x|\to\infty} V(x) > V_0 = \inf_{x\in\mathbb{R}^N} V(x) > 0,$$

where $V_\infty < \infty$ or $V_\infty = \infty$. This kind of a hypothesis was introduced by Rabinowitz in [34].

(f_1) The nonlinearity $f \in C^1(\mathbb{R})$ satisfies $f(t) = 0$ for all $t \in (-\infty, 0]$, $f(t) > 0$ for all $t > 0$ and there exist constants $\alpha_0 \in (0, \alpha_*)$, $b_1, b_2 > 0$ such that for any $t \in \mathbb{R}$,

$$|f'(t)| \le b_1|t|^{p-2} + b_2|t|^{p-2}\Phi_{N,s}(\alpha_0|t|^{N/(N-s)}),$$

where $\Phi_{N,s}(y) = e^y - \sum_{i=0}^{j_p-2} \frac{y^j}{j!}$, $j_p = \min\{j \in \mathbb{N}\colon j \ge p\}$ and α_* is given in the Lemma 1.

(f_2) There exists $\mu > \frac{N}{s}$ such that

$$f(t)t - \mu F(t) \ge 0$$

for all $t \in \mathbb{R}$, where $F(t) = \int_0^\tau f(\tau)d\tau$.

(f_3)

$$\lim_{t\to 0^+} \frac{f'(t)}{t^{\frac{N}{s}-2}} = 0.$$

(f_4) There exists $\gamma_1 > 0$ large enough such that $F(t) \ge \gamma_1|t|^\mu$ for all $t \ge 0$.

(f_5) $\frac{f(t)}{t^{p-1}}$ is a strictly increasing function of $t \ge 0$.

Remark 1. (i) From the condition (f_3), we have

$$\lim_{t\to 0^+} \frac{f(t)}{t^{\frac{N}{s}-1}} = \lim_{t\to 0^+} \frac{F(t)}{t^{\frac{N}{s}}} = 0.$$

(ii) The condition (f_5) implies that the function $\frac{1}{p}f(t)t - F(t)$ is an increasing function of $t \ge 0$. Indeed, we have

$$\left(\frac{1}{p}f(t)t - F(t)\right)'_t = \frac{f'(t)t - (p-1)f(t)}{p} > 0$$

for all $t > 0$ due to the condition (f_5).

(iii) The condition (f_5) leads to that $f(t) > 0$ for all $t > 0$. Indeed, we have $\frac{f(t)}{t^{p-1}} > \lim_{t\to 0^+} \frac{f(t)}{t^{p-1}} = 0$ for all $t > 0$. Then $f(t) > 0$ for all $t > 0$.

(iv) We have that $f(t)t$ is an increasing function on $(0, +\infty)$. From the condition (f_5), we have

$$\frac{f(t_1)t_1}{t_1^p} < \frac{f(t_2)t_2}{t_2^p}$$

for all $0 < t_1 < t_2$. Then

$$f(t_1)t_1 < \left(\frac{t_2}{t_1}\right)^p f(t_1)t_1 < f(t_2)t_2.$$

We get the claim.

In 2019, Miyagaki and Pucci [28] have studied the nonlocal Kirchhoff problem with critical Trudinger–Moser nonlinearity

$$-M(\iint\limits_{\mathbb{R}^{2N}} |u(x) - u(y)|^2 H(x-y)\,dx\,dy + \int_{\mathbb{R}} V(x)|u|^2\,dx)(\mathbb{L}_K u + V(x)u) = P(x)f(u)$$

in $\mathbb{R}$, where H satisfies the two following conditions:

(h_1) $mH \in L^1(\mathbb{R}^N)$, where $m(x) = \min\{|x|^2, 1\}$;

(h_2) there exists $k_0 > 0$ such that $H(x) \geq k_0|x|^{-2}$ for any $x \in \mathbb{R}^N \setminus \{0\}$.

The Kirchhoff function $M\colon [0,+\infty) \to [0,+\infty)$ is continuous and satisfies the conditions:

(M_1) For any $\tau > 0$, there exists $\kappa = \kappa(\tau)$ such that $M(t) \geq \kappa$ for all $t \geq \tau$.
(M_2) There exists $\theta \geq 1$ such that $tM(t) \leq \theta\mathcal{M}(t)$ for all $t \in [0,+\infty)$, where $\mathcal{M}(t) = \int_0^t M(\tau)\,d\tau$.
(M_3) The function $\theta\mathcal{M}(t) - tM(t)$ in nondecreasing on $[0,+\infty)$.

The nonlinear function f satisfies the subcritical exponential growth or critical exponential growth, V and P satisfy some following conditions:

(i) The potentials V and P are continuous and strictly positive in $\mathbb{R}$;
(ii) If $\{A_n\}$ is a sequence of Borel sets of $\mathbb{R}$, with $|A_n| \leq R$ for all $n \in \mathbb{N}$ and some $R > 0$, then
$$\lim_{r\to\infty} \int_{A_n \cap B_r^c(0)} P(x)\,dx = 0, \quad \text{uniformly with respect to } n \in \mathbb{N},$$
where $B_r^c(0)$ is the complement of the closed interval $B_r = [-r, r]$.
(iii) The potential P is in $L^\infty(\mathbb{R})$ and there exists $C_0 > 0$ such that $V(x) \geq C_0$ for all $x \in \mathbb{R}$.

In the work of Miyagaki and Pucci, the potential is bounded from below by a positive constant. In order to study their problem, they need the nonlinear function with the form $P(x)f(u)$, where P and V satisfy the conditions (i) to (iii). With these conditions, they get the compact embedding from the solution space E into the Lebesgue space with weight $L_P^q(\mathbb{R}^N)$, $q \in (2,+\infty)$.

This paper was motivated by some work that have appeared very recently on the fractional p-Laplace Schrödinger equation with the form

$$\varepsilon^{ps}(-\Delta)_p^s u(x) + V(x)|u|^{p-2}u = f(u) \quad \text{in } \mathbb{R}^N,\ N > ps, \tag{1.2}$$

[8, 10, 7, 14, 9] and the work of Miyagaki and Pucci [28]. When $p = 2$, the equation (1.2) becomes a fractional Schrödinger equation of the type

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

which has been widely investigated in the last decade [3, 4, 5, 22, 37, 23, 36, 26, 35] and references therein. The study of (1.3) is strongly motivated by the search of standing waves solutions for the heat fractional Schrödinger equation

$$i\varepsilon\frac{\partial\psi}{\partial t} = (-\Delta)^s\psi + (V(x)+E)\psi - f(\psi) \quad \text{for } (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{1.4}$$

whose solutions have the form $\psi(x,t) = u(x)e^{-\frac{iEt}{\varepsilon}}$, where E is a constant. The equation (1.4) is a fundamental equation of the fractional Quantum Mechanics.

When $s \to 1$, the equation (1.2) becomes

$$-\varepsilon^N\Delta_N u + V(x)|u|^{N-2}u = f(u) \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

which arises in applications when $\varepsilon = 1$, such as image processing, non-Newtonian fluids and pseudo-plastic fluids. We refer the reader to [11, 13] for more details. In [34] Rabinowitz used variational methods to prove the existence of positive solutions to (1.5) for ε sufficiently small by assuming condition (V) and $p = 2$. Later Wang [40] showed that these solutions concentrate at global minimum points of $V(x)$ as $\varepsilon \to 0$. Denote $M = \{x \in \mathbb{R}^N\colon V(x) = V_0\}$ and

$$M_\delta = \{x \in \mathbb{R}^N\colon \operatorname{dist}(x,M) \leq \delta\} \quad \text{for } \delta > 0.$$

Using Lusternik–Schnirelmann category, Alves and Figueiredo [2] showed that problem (1.5) has at least $\mathrm{cat}_{M_\delta}(M)$ positive solutions for small ε when $f \in C^1(\mathbb{R})$ satisfies the following conditions:

(c_1) $f(t) = 0$ for all $t \in (-\infty, 0]$ and f has critical growth at both $+\infty$ and $-\infty$, that is there exists $\alpha_0 > 0$ such that

$$\lim_{|s|\to+\infty} \frac{f(s)}{e^{\alpha|s|^{N/(N-1)}}} = 0 \text{ for } \alpha > \alpha_0$$

and

$$\lim_{|s|\to+\infty} \frac{f(s)}{e^{\alpha|s|^{N/(N-1)}}} = \infty \text{ for all } \alpha < \alpha_0.$$

(c_2) $\lim_{s\to 0} \frac{|f'(s)|}{|s|^{N-2}} = 0$ and there exists $C > 0$ such that

$$|f'(s)| \leq C\exp(\alpha_N s^{N/(N-1)})$$

for all $s \geq 0$, where $\alpha_N = N w_N^{1/(N-1)}$ and w_N is the $(N-1)$-dimensional measure of the $(N-1)$-sphere.

(c_3) There exist constants $p > N$ and $\mu > 0$ large enough such that $f(s) \geq \mu s^{p-1}$ for all $s \geq 0$.

(c_4) There exists $C_1 > 0$ and $\sigma \geq N$ such that

$$f'(s)s - (N-1)f(s) \geq C_1 s^\sigma$$

for all $s \geq 0$.

(c_5) There exists $\theta > N$ such that

$$0 < \theta F(s) = \theta \int_0^s f(t)dt \leq sf(s)$$

for all $s > 0$.

(c_6) The function $\frac{f(s)}{s^{N-1}}$ is strictly increasing in $(0, +\infty)$.

For more results about existence of solution to the problem (1.5), we refer the reader to [2] and references therein.

When $s = \frac{1}{2}$ and $N = 1$, Alves, Do Ó and Miyagaki [1] studied the concentration of solutions to the problem (1.1) with the following assumptions:

$(V)'$ V is bounded function and locally Hölder continuous and there exists $V_0 > 0$ such that
 (i) $V(x) \geq V_0$ for all $x \in \mathbb{R}^N$,
 (ii) There exists a bounded interval $\Lambda \subset \mathbb{R}$ such that

$$V_0 \equiv \inf_{\Lambda} \Lambda(x) < \min_{\partial\Lambda} V(x).$$

$(f_1)'$ $f \colon \mathbb{R} \to \mathbb{R}^+$ is a C^1 function with $f(t) = 0$ if $t \leq 0$.

$(f_2)'$ $f(t) = o(t)$ near original.

$(f_3)'$ $\frac{f(t)}{t}$ is an increasing function in $\mathbb{R}^+$.

$(f_4)'$ There exist a constant $p > 2$ and a suitable constant $C_p > 0$ such that

$$f(t) \geq C_p t^{p-1} \quad \text{for all } t > 0.$$

For more results on Trundinger–Moser inequality and its applications, we refer the readers to [16, 17, 18, 15, 24, 31, 39, 42, 27, 43, 20, 19].

Before starting our results, we recall some useful notations. Suppose that $N = ps$ in our paper. The fractional Sobolev space $W^{s,p}$ is defined by

$$W^{s,p}(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) \colon [u]_{s,p} < \infty\},$$

where $[u]_{s,p}$ denotes the Gagliardo seminorm, that is

$$[u]_{s,p} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{2N}} \, dx \, dy \right)^{1/p}.$$

$W^{s,p}(\mathbb{R}^N)$ is a uniformly convex Banach space (similar to [32]) with norm

$$\|u\| = \left(\|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{s,p} \right)^{1/p}.$$

Given $\eta > 0$, another norm on $W^{s,p}(\mathbb{R}^N)$ is given by

$$\|u\|_\eta = \left(\eta \|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{s,p} \right)^{1/p}.$$

Then $\| \cdot \|$ and $\| \cdot \|_\eta$ are two equivalent norms on $W^{s,p}(\mathbb{R}^N)$. For each $\varepsilon > 0$, let W_ε denote the completion of $C_0^\infty(\mathbb{R}^N)$, with respect to the norm

$$\|u\|_{W_\varepsilon} = \left([u]^p_{s,p} + \|u\|^p_{p,V,\varepsilon} \right)^{1/p}, \quad \|u\|^p_{p,V,\varepsilon} = \int_{\mathbb{R}^N} V(\varepsilon x) |u(x)|^p \, dx.$$

Then W_ε is a uniformly convex Banach space (similar to [32, Lemma 10]), and then W_ε is a reflexive space. By the condition (V) and [29, Theorem 6.9], we have that the embedding from W_ε into $L^\nu(\mathbb{R}^N)$ is continuous for any $\nu \in [\frac{N}{s}, +\infty)$. Then there exists a best constant $S_{\nu,\varepsilon} > 0$ for all $\nu \in [\frac{N}{s}, +\infty)$:

$$S_{\nu,\varepsilon} = \inf_{u \neq 0, u \in W_\varepsilon} \frac{\|u\|_{W_\varepsilon}}{\|u\|_{L^\nu(\mathbb{R}^N)}}.$$

This implies

$$\|u\|_{L^\nu(\mathbb{R}^N)} \leq S^{-1}_{\nu,\varepsilon} \|u\|_{W_\varepsilon} \quad \text{for all } u \in W_\varepsilon. \tag{1.6}$$

By [29, Theorem 6.9], we have that the embedding from $W^{s,N/s}(\mathbb{R}^N)$ into $L^\nu(\mathbb{R}^N)$ is continuous for any $\nu \in [\frac{N}{s}, +\infty)$, and there exists a best constant $A_{\nu,\eta} > 0$ for all $\nu \in [\frac{N}{s}, +\infty)$ as follows:

$$A_{\nu,\eta} = \inf_{u \neq 0, u \in W^{s,N/s}(\mathbb{R}^N)} \frac{\|u\|_\eta}{\|u\|_{L^\nu(\mathbb{R}^N)}}.$$

This implies

$$\|u\|_{L^\nu(\mathbb{R}^N)} \leq A^{-1}_{\nu,\eta} \|u\|_\eta \quad \text{for all } u \in W^{s,N/s}(\mathbb{R}^N). \tag{1.7}$$

We denote by $\operatorname{cat}_B(A)$ the category of A with respect to B, namely the least integer k such that $A \subset A_1 \cup \cdots \cup A_k$, where A_i $(i = 1, \ldots, k)$ is closed and contractible in B. We set $\operatorname{cat}_B(\emptyset) = 0$ and $\operatorname{cat}_B(A) = +\infty$ if there is no integer with the above property. We refer the reader to [41] for more details on Ljusternik–Schnirelmann theory. Now, we state the main result in this paper.

Theorem 2. *Let (V) and (f_1)–(f_5) hold. Then for any $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that problem (P_ε) has at least $\operatorname{cat}_{M_\delta}(M)$ nontrivial nonnegative weak solutions for any $0 < \varepsilon < \varepsilon_\delta$. Moreover, if u_ε denotes one of these solutions and η_ε is its global maximum, then*

$$\lim_{\varepsilon \to 0^+} V(\eta_\varepsilon) = V_0.$$

Remark 3. Comparing Theorem 2 with Theorem 1.1 [2], we do not need the condition of the form (c_4). Therefore, when $s \to 1^-$, we get an improvement on the result of Alves and Figueiredo [2].

Remark 4. We use the Nehari manifold, variational method, concentration compactness principle and Ljusternik–Schnirelmann theory to prove the main result. There are some difficulties in proving our theorem. The first difficulty is that the nonlinearity f has exponential critical growth. The second is that since $N = ps$, we lack the fractional Sobolev embedding. Comparing our work with the work of Alves and Figueiredo [2], we meet several new difficulties in employing the methods to deal with our problem due to the nonlocal property of equation (1.1). We recommend the readers Lemma 4, Lemma 8, Lemma 10 and Lemma 12, and more results in our paper for this comment. Comparing our work with the work of Miyagaki and Pucci [28], we only have a continuous embedding since our nonlinear function does not contain the function P as in (i) to (iii). Hence, the method of Miyagaki and Pucci [28] is not enough to solve our problem.

The paper is organized as follows. In Section 2, we study the autonomous associated problem. In Section 3, we study the auxiliary problem. We prove the Palais–Smale condition for the energy functional and provide some tools which are useful to establish a multiplicity result. This allows us to show that the auxiliary problem has multiple solutions. In Section 4, we prove the existence of ground state solutions to the auxiliary problem. Finally, in Section 5, we complete the paper with the proof of Theorem 2.

2. Autonomous problem

In this section, we study the autonomous problem associated to (1.1):

$$(-\Delta)^s_{N/s} u + \eta |u|^{\frac{N}{s}-2} u = f(u) \quad \text{in } \mathbb{R}^N, \qquad (P_\eta) \tag{2.1}$$

where $\eta > 0$ is a constant.

We denote by $J_\eta\colon W^{s,N/s}(\mathbb{R}^N) \to \mathbb{R}$ the corresponding energy functional for problem (2.1)

$$J_\eta(u) = \frac{1}{p}\|u\|^p_\eta - \int_{\mathbb{R}^N} F(u)\, dx.$$

From the condition (f_3), there exist $\tau > 0$ and $\delta > 0$ such that for all $|t| \leq \delta$, we have

$$|f'(t)| \leq \tau |t|^{\frac{N}{s}-2}. \tag{2.2}$$

Moreover, from the conditions (f_1) and that f' is a continuous function, for each $q \geq \frac{N}{s}$, we can find a constant $C = C(q, \delta) > 0$ such that

$$|f'(t)| \leq C|t|^{q-2} \Phi_{N,s}(\alpha_0 |t|^{N/(N-s)}) \tag{2.3}$$

for all $|t| \geq \delta$. Combining (2.2) and (2.3), we get

$$|f'(t)| \leq \tau |t|^{\frac{N}{s}-2} + C|t|^{q-2} \Phi_{N,s}(\alpha_0 |t|^{N/(N-s)}) \tag{2.4}$$

for all $t \geq 0$. Then we obtain

$$|f(t)| \leq \int_0^t |f'(s)|\, ds \leq \tau |t|^{\frac{N}{s}-1} + C|t|^{q-1} \Phi_{N,s}(\alpha_0 |t|^{N/(N-s)}) \tag{2.5}$$

and

$$|F(t)| \leq \int_0^t |f(s)|\, ds \leq \tau |t|^{\frac{N}{s}} + C|t|^{q} \Phi_{N,s}(\alpha_0 |t|^{N/(N-s)}) \tag{2.6}$$

for all $t \geq 0$.

In order to prove the result in this section, we need the following result:

Lemma 1. [43] *Let $s \in (0,1)$ and $sp = N$. Then for every $0 \le \alpha < \alpha_* \le \alpha^*_{s,N}$, the following inequality holds:*

$$\sup_{u \in W^{s,p}(\mathbb{R}^N), \|u\|_{W^{s,p}(\mathbb{R}^N)} \le 1} \int_{\mathbb{R}^N} \Phi_{N,s}(\alpha |u|^{N/(N-s)})\, dx < +\infty,$$

where $\Phi_{N,s}(t) = e^t - \sum_{i=0}^{j_p - 2} \frac{t^j}{j!}$, $j_p = \min\{j \in \mathbb{N} \colon j \ge p\}$. Moreover, for $\alpha > \alpha^*_{s,N}$,

$$\sup_{u \in W^{s,p}(\mathbb{R}^N), \|u\|_{W^{s,p}(\mathbb{R}^N)} \le 1} \int_{\mathbb{R}^N} \Phi_{N,s}(\alpha |u|^{N/(N-s)})\, dx = +\infty.$$

Remark 5. From Lemma 1, if we use the norm $\|\cdot\|_\eta$ on $W^{s,N/s}(\mathbb{R}^N)$, then we have

$$(\max\{1,\eta\})^{-1/p} \|u\|_\eta \le \|u\|_{W^{s,p}(\mathbb{R}^N)} \le (\min\{1,\eta\})^{-1/p} \|u\|_\eta.$$

We get

$$\sup_{u \in W^{s,p}(\mathbb{R}^N), \|u\|_\eta \le (\min\{1,\eta\})^{s/N}} \int_{\mathbb{R}^N} \Phi_{N,s}(\alpha |u|^{N/(N-s)})\, dx < +\infty$$

for all $0 \le \alpha < \alpha_* \le \alpha^*_{s,N}$.

Using Lemma 1 and noticing that $C_0^\infty(\mathbb{R}^N)$ is a dense subspace of $W^{s,p}(\mathbb{R}^N)$, we see that J_η is well defined on $W^{s,N/s}(\mathbb{R}^N)$. Furthermore, we have

$$\begin{aligned} < J_\eta'(u), \varphi > = & \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{\frac{N}{s}-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{2N}}\, dx\, dy \\ & + \eta \int_{\mathbb{R}^N} |u|^{\frac{N}{s}-2} u \varphi\, dx - \int_{\mathbb{R}^N} f(u)\varphi\, dx. \end{aligned}$$

Lemma 2. *Suppose that (f_1) and (f_3) hold. Then there exist positive constants t_0, ρ_0 such that $J_\eta(u) \ge \rho_0$ for all $u \in W^{s,N/s}(\mathbb{R}^N)$, with $\|u\|_{W^{s,N/s}(\mathbb{R}^N)} = t_0$.*

Proof. From (2.6), for some $q > \frac{N}{s}$, we have

$$|F(t)| \le \tau |t|^{N/s} + C|t|^q \Phi_{N,s}(\alpha_0 |t|^{N/(N-s)})$$

for all $t \in \mathbb{R}$. Then we get

$$\begin{aligned} J_\eta(u) &= \frac{s}{N}\|u\|_\eta^{N/s} - \int_{\mathbb{R}^N} F(u)\, dx \\ &\ge \frac{s}{N}\|u\|_\eta^{N/s} - \tau \int_{\mathbb{R}^N} |u|^{N/s}\, dx - C \int_{\mathbb{R}^N} |u|^q \Phi_{N,s}(\alpha_0 |u|^{N/(N-s)})\, dx. \end{aligned} \tag{2.7}$$

Using Hölder's inequality, we have

$$\int_{\mathbb{R}^N} |u|^q \Phi_{N,s}(\alpha_0 |u|^{N/(N-s)})\, dx \le \left(\int_{\mathbb{R}^N} \left(\Phi_{N,s}(\alpha_0 |u|^{N/(N-s)})\right)^t dx \right)^{1/t} \|u\|^q_{L^{qt'}(\mathbb{R}^N)}, \tag{2.8}$$

where $t > 1, t' > 1$ such that $\frac{1}{t} + \frac{1}{t'} = 1$. By Lemma 2.3 [25], for any $\mathfrak{b} > t$, there exists a constant $C(\mathfrak{b}) > 0$ such that

$$\left(\Phi_{N,s}(\alpha_0 |u|^{N/(N-s)})\right)^t \le C(\mathfrak{b}) \Phi_{N,s}(\mathfrak{b}\alpha_0 |u|^{N/(N-s)}) \tag{2.9}$$

on $\mathbb{R}^N$. Denoting $\mathfrak{d} = \min\{1, \eta\}$, we get

$$\int_{\mathbb{R}^N} \left(\Phi_{N,s}(\alpha_0|u|^{N/(N-s)})\right)^t \leq C(\mathfrak{b}) \int_{\mathbb{R}^N} \Phi_{N,s}(\mathfrak{b}\alpha_0|u|^{N/(N-s)})\, dx$$

$$= C(\mathfrak{b}) \int_{\mathbb{R}^N} \Phi_{N,s}(\mathfrak{b}\alpha_0\mathfrak{d}^{-s/(N-s)}\|u\|_\eta^{N/(N-s)}|\mathfrak{d}^{s/N}u/\|u\|_\eta|^{N/(N-s)})\, dx. \tag{2.10}$$

When $\|u\|_\eta$ is small enough and $\mathfrak{b}$ near t, we have

$$\mathfrak{b}\alpha_0\mathfrak{d}^{-s/(N-s)}\|u\|_\eta^{N/(N-s)} \leq \beta_* < \alpha_*. \tag{2.11}$$

By Remark 5, (2.10) and (2.11), there exists a constant $D > 0$ such that

$$\left(\int_{\mathbb{R}^N} \left(\Phi_{N,s}(\alpha_0|u|^{N/(N-s)})\right)^t dx\right)^{1/t} \leq D.$$

Since the embedding from $W_{s,N/s}(\mathbb{R}^N) \to L^{qt'}(\mathbb{R}^N)$ is continuous, we get

$$\int_{\mathbb{R}^N} |u|^q\Phi_{N,s}(\alpha_0|u|^{N/(N-s)})\, dx \leq DA_{qt',\eta}^{-q}\|u\|_\eta^q < +\infty. \tag{2.12}$$

From (1.7), we have

$$\|u\|_{L^{N/s}(\mathbb{R}^N} \leq A_{N/s,\eta}^{-1}\|u\|_\eta \quad \text{for all } u \in W^{s,N/s}(\mathbb{R}^N). \tag{2.13}$$

Hence, combining (2.7), (2.12) and (2.13), we obtain

$$J_\eta(u) \geq \frac{s}{N}\|u\|_\eta^{N/s} - \tau A_{N/s,\eta}^{-N/s}\|u\|_\eta^{N/s} - CDA_{qt',\eta}^{-q}\|u\|_\eta^q$$

$$= \|u\|_\eta^{N/s}\Big[\Big(\frac{s}{N} - \tau A_{N/s,\eta}^{-N/s}\Big) - CDA_{qt',\eta}^{-q}\|u\|_\eta^{q-\frac{N}{s}}\Big]. \tag{2.14}$$

We see that $\frac{s}{N} - \tau A_{N/s,\eta}^{-N/s} > 0$ for τ small enough. Let

$$h(t) = \frac{s}{N} - \tau A_{N/s,\eta}^{-N/s} - CDA_{qt',\eta}^{-q}t^{q-\frac{N}{s}}, \quad t \geq 0.$$

We now prove that there exists small $t_0 > 0$ satisfying $h(t_0) \geq \frac{1}{2}(\frac{s}{N} - \tau A_{N/s,\eta}^{-N/s})$. We see that h is continuous on $[0, +\infty)$ and $\lim_{t\to 0^+} h(t) = \frac{s}{N} - \tau A_{N/s,\eta}^{-N/s}$. Then there exists t_0 such that $h(t) \geq \frac{s}{N} - \tau A_{N/s,\eta}^{-N/s} - \varepsilon_1$ for all $0 \leq t \leq t_0$ and t_0 is small enough such that $\|u\|_\eta = t_0$ satisfies (2.11). If we choose $\varepsilon_1 = \frac{1}{2}(\frac{s}{N} - \tau A_{N/s,\eta}^{-N/s})$, we have

$$h(t) \geq \frac{1}{2}\left(\frac{s}{N} - \tau A_{N/s,\eta}^{-N/s}\right)$$

for all $0 \leq t \leq t_0$. Especially,

$$h(t_0) \geq \frac{1}{2}\left(\frac{s}{N} - \tau A_{N/s,\eta}^{-N/s}\right). \tag{2.15}$$

From (2.14) and (2.15), for $\|u\|_\eta = t_0$, we have

$$J_\eta(u) \geq \frac{t_0^{N/s}}{2} \cdot \left(\frac{s}{N} - \tau A_{N/s,\eta}^{-N/s}\right) = \rho_0. \qquad \square$$

Lemma 3. *Suppose that (f_4) holds. Then there exists a function $v \in C_0^\infty(\mathbb{R}^N)$ with $\|v\|_\eta > t_0$, such that $J_\eta(v) < 0$, where $t_0 > 0$ is the number given in Lemma 2.*

Proof. For all $u \in C_0^\infty(\mathbb{R}^N)$ with $\|u\|_\eta = 1$, from (f_4) and all $t > 0$, we obtain

$$J_\eta(tu) = \frac{st^{N/s}}{N}\|u\|_\eta^{N/s} - \int_{\mathbb{R}^N} F(tu)\,dx \le \frac{st^{N/s}}{N}\|u\|_\eta^{N/s} - \gamma_1 t^\mu \int_{\mathbb{R}^N} |u(x)|^\mu\,dx.$$

By (1.7), for all $\nu \in [\frac{N}{s}, \mu)$, we have

$$0 < \frac{1}{A_{\nu,\eta} + \varepsilon} = \frac{\|u\|_\eta}{A_{\nu,\eta} + \varepsilon} \le \|u\|_{L^\nu(\mathbb{R}^N)} \le A_{\nu,\eta}^{-1}\|u\|_\eta = A_{\nu,\eta}^{-1} < +\infty,$$

where $\varepsilon > 0$. Since $\mu > \frac{N}{s}$, $t^{N/s}$ has growth smaller than t^μ as $t \to +\infty$, then we have $J_\eta(tu) \to -\infty$ as $t \to +\infty$. Taking $v = \rho_1 u, \rho_1 > t_0 > 0$ large enough, we have $J_\eta(v) < 0, \|v\|_\eta > t_0$. □

From Lemma 2, Lemma 3 and a version of Mountain Pass Theorem without the Palais–Smale condition, we get a sequence $\{u_n\} \subset W^{s,N/s}(\mathbb{R}^N)$ such that

$$J_\eta(u_n) \to c_\eta \quad \text{and} \quad J'_\eta(u_n) \to 0 \quad \text{as } n \to \infty,$$

where the level c_η is characterized by

$$c_\eta = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\eta(\gamma(t))$$

and $\Gamma = \{\gamma \in C([0,1], W^{s,N/s}(\mathbb{R}^N))\colon \gamma(0) = 0, J_\eta(\gamma(1)) < 0\}$.

Lemma 4. *Let $\{u_n\}$ be $(PS)_{c_\eta}$ sequence for J_η. Then*

(i) *there exists a constant C_{γ_1} such that $\rho_0 \le c_\eta \le C_{\gamma_1}$,*

(ii) *$u_n \to u$ weakly in $W^{s,N/s}(\mathbb{R}^N)$ and $J'_\eta(u) = 0$.*

Proof. We choose a function $w \in W^{s,N/s}(\mathbb{R}^N) \setminus \{0\}$ such that $\|w\|_{L^\mu(\mathbb{R}^N)} = 1$ and $\|w\|_\eta = A_{\mu,\eta}$. This means

$$\frac{\|w\|_\eta}{\|w\|_{L^\mu(\mathbb{R}^N)}} = A_{\mu,\eta} = \inf_{u\in W^{s,N/s}(\mathbb{R}^N), u\neq 0} \frac{\|u\|_\eta}{\|u\|_{L^\mu(\mathbb{R}^N)}}.$$

Then, from $A_{N/s,\eta} \le \frac{\|w\|_\eta}{\|w\|_{L^{N/s}(\mathbb{R}^N)}}$, we get $\|w\|_{L^{N/s}(\mathbb{R}^N)} \le A_{N/s,\eta}^{-1}$. We see that

$$c_\eta \le \max_{t\ge0} J_\eta(tw) \le \max_{t\ge0}\Big\{\frac{st^{N/s}|}{N}\|w\|_\eta{}^{N/s} - \gamma_1 t^\mu \int_{\mathbb{R}^N} |w(x)|^\mu\,dx\Big\}$$

$$= \max_{t\ge0}\Big\{\frac{sA_{\mu,\eta}^{N/s} t^{N/s}}{N} - \gamma_1 t^\mu\Big\}. \tag{2.16}$$

Set $g(t) = \frac{sA_{\mu,\eta}^{N/s}}{N} t^{N/s} - \gamma_1 t^\mu$ on $[0, +\infty)$. We easily get

$$g(t) \le g(\theta_{\gamma_1}) = C_{\gamma_1}$$

on $[0, +\infty)$, where

$$\theta_{\gamma_1} = \left(\frac{A_{\mu,\eta}^{N/s}}{\gamma_1\mu}\right)^{s/(\mu s-N)}.$$

Computing directly, we get

$$C_{\gamma_1} = g(\theta_{\gamma_1}) = \theta_{\gamma_1}^{N/s}\left[\frac{sA_{\mu,\eta}^{N/s}}{N} - \gamma_1\theta_{\gamma_1}^{\frac{\mu s-N}{s}}\right]$$

$$= \left(\frac{A_{\mu,\eta}^{N/s}}{\gamma_1\mu}\right)^{N/(\mu s-N)}\left(\frac{s}{N} - \frac{1}{\mu}\right) A_{\mu,\eta}^{N/s}. \tag{2.17}$$

We see that $\lim_{\gamma_1\to+\infty}\theta_{\gamma_1} = 0$ gives $\lim_{\gamma_1\to+\infty} g(\theta_{\gamma_1}) = 0$. Therefore, the Mountain Pass level c is small enough when γ_1 is large enough, which will be used later. Combine Lemma 2 and (2.16), we get $\rho_0 \le c_\eta \le C_{\gamma_1}$.

Note that $\{u_n\}$ is a (PS) sequence with level $c_\eta \in \mathbb{R}$ in $W^{s,N/s}(\mathbb{R}^N)$. This means

$$J_\eta(u_n) \to c_\eta \quad \text{and} \quad \sup_{\|\varphi\|_\eta=1} |\langle J'_\eta(u_n), \varphi\rangle| \to 0 \tag{2.18}$$

as $n\to\infty$. We show that the sequence $\{u_n\}$ is bounded in $W^{s,N/s}(\mathbb{R}^N)$. From (2.18), we have

$$\left\langle J'_\eta(u_n), \frac{u_n}{\|u_n\|_\eta}\right\rangle = o_n(1) \quad \text{and} \quad J_\eta(u_n) = c_\eta + o_n(1)$$

when n large enough. This implies

$$J_\eta(u_n) - \frac{1}{\mu}\langle J'_\eta(u_n), u_n\rangle = c_\eta + o_n(1) + o_n(1)\|u_n\|_\eta, \tag{2.19}$$

where μ is a parameter in the condition (f_2). We have

$$\begin{aligned}
&J_\eta(u_n) - \frac{1}{\mu}\langle J'_\eta(u_n), u_n\rangle\\
&= \frac{s}{N}\|u_n\|_\eta^{N/s} - \int_{\mathbb{R}^N} F(u_n)\,dx - \frac{1}{\mu}\Big[\|u_n\|_\eta^{N/s} - \int_{\mathbb{R}^N} f(u_n)u_n\,dx\Big]\\
&= \left(\frac{s}{N} - \frac{1}{\mu}\right)\|u_n\|_\eta^{N/s} + \frac{1}{\mu}\int_{\mathbb{R}^N} f(u_n)u_n - \mu F(u_n)\,dx.
\end{aligned}$$

Therefore, we have

$$J_\eta(u_n) - \frac{1}{\mu}\langle J'_\eta(u_n), u_n\rangle \ge \Big(\frac{s}{N} - \frac{1}{\mu}\Big)\|u_n\|_\eta^{N/s}. \tag{2.20}$$

Combining (2.19) and (2.20), we get

$$\Big(\frac{s}{N} - \frac{1}{\mu}\Big)\|u_n\|_\eta^{N/s} \le c_\eta + o_n(1) + o_n(1)\|u_n\|_\eta. \tag{2.21}$$

From (2.21), we conclude that the sequence $\{u_n\}$ is bounded in $W^{s,N/s}(\mathbb{R}^N)$. Since

$$J_\eta(u_n) - \frac{1}{\mu}\langle J'_\eta(u_n), u_n\rangle \to c_\eta$$

as $n\to\infty$, then

$$\limsup_{n\to\infty}\|u_n\|_\eta^{N/s} \le \frac{c_\eta}{\frac{s}{N} - \frac{1}{\mu}} \le \frac{C_{\gamma_1}}{\frac{s}{N} - \frac{1}{\mu}}. \tag{2.22}$$

Going if necessary to a subsequence, for any $q \ge \frac{N}{s}$, we have

$$\begin{aligned}
&u_n \to u \text{ weakly in } W^{s,N/s}(\mathbb{R}^N), u_n \to u \text{ in } L^q_{\mathrm{loc}}(\mathbb{R}^N)\\
&u_n \to u \text{ in } \mathbb{R}^N \text{ outside a set with measure zero.}
\end{aligned}$$

Using the Trudinger–Moser inequality, Vitali's theorem and by arguments as in [12, Lemma 5], we can prove that $J'_\eta(u) = 0$. For convenience of the readers, we give a detailed proof here. We need to prove that $\langle J'_\eta(u), \varphi\rangle = 0$ for all $\varphi \in W^{s,N/s}(\mathbb{R}^N)$. First we show that

$$\int_{\mathbb{R}^N} f(u_n)\varphi\,dx \to \int_{\mathbb{R}^N} f(u)\varphi\,dx \tag{2.23}$$

for all $\varphi \in W^{s,N/s}(\mathbb{R}^N)$. From (2.5) for $q = \frac{N}{s}$, we have

$$\begin{aligned}&\int_{\mathbb{R}^N} |f(u_n)\varphi|\,dx \\ &\le b_1 \int_{\mathbb{R}^N} |u_n|^{p-1}|\varphi|\,dx + b_2 \int_{\mathbb{R}^N} |u_n|^{p-1}\Phi_{N,s}(\alpha_0|u_n|^{N/(N-s)}))|\varphi|\,dx\end{aligned} \tag{2.24}$$

for some constants $b_1 > 0, b_2 > 0$. Using Hölder's inequality, we get

$$\int_{\mathbb{R}^N} |u_n|^{p-1}|\varphi|\,dx \le \|u_n\|^{p-1}_{L^p(\mathbb{R}^N)}\|\varphi\|_{L^p(\mathbb{R}^N)} < +\infty \tag{2.25}$$

due to the boundedness of sequence $\{u_n\}$ in $W^{s,N/s}(\mathbb{R}^N)$ and the continuity of the embedding of $W^{s,N/s}(\mathbb{R}^N)$ into $L^{N/s}(\mathbb{R}^N)$. Using Hölder's inequality for $q \ge \frac{N}{s} = p$, $q_* \ge \frac{p}{p-1}$, $q^{'} > 1$ near 1, noting that $\frac{1}{q} + \frac{1}{q'} + \frac{1}{q_*} = 1$, we deduce

$$\begin{aligned}&\int_{\mathbb{R}^N} |u_n|^{p-1}\Phi_{N,s}(\alpha_0|u_n|^{N/(N-s)})|\varphi|\,dx \\ &\le \left(\int_{\mathbb{R}^N} |u_n|^{q_*(p-1)}\,dx\right)^{1/q_*}\left(\int_{\mathbb{R}^N}(\Phi_{N,s}(\alpha_0|u_n|^{N/(N-s)}))^{q'}\,dx\right)^{1/q'}\left(\int_{\mathbb{R}^N}|\varphi|^q\,dx\right)^{1/q} \\ &= \left(\int_{\mathbb{R}^N}(\Phi_{N,s}(\alpha_0|u_n|^{N/(N-s)}))^{q'}\,dx\right)^{1/q'} \|u_n\|^{p-1}_{L^{q_*(p-1)}(\mathbb{R}^N)}\|\varphi\|_{L^q(\mathbb{R}^N)}.\end{aligned} \tag{2.26}$$

By [25, Lemma 2.3], choosing $\mathfrak{c} > q' > 1$, $\mathfrak{c}$ near q', there exists a constant $C(\mathfrak{c}) > 0$ such that

$$(\Phi_{N,s}(\alpha_0|u|^{N/(N-s)}))^{q'} \le C(\mathfrak{c})\Phi_{N,s}(\mathfrak{c}\alpha_0|u|^{N/(N-s)}) \tag{2.27}$$

for all $u \in W^{s,N/s}(\mathbb{R}^N)$. By (2.17) and (2.22), we see that

$$\limsup_{n\to\infty} \|u_n\|^{N/s}_\eta \le \left(\frac{A^{N/s}_{\mu,\eta}}{\gamma_1\mu}\right)^{N/(\mu s-N)}\left(\frac{s}{N} - \frac{1}{\mu}\right)A^{N/s}_{\mu,\eta}\frac{1}{\frac{s}{N} - \frac{1}{\mu}}. \tag{2.28}$$

Combining (2.28) and (2.11), we have

$$\mathfrak{c}.\alpha_0\mathfrak{d}^{-s/(N-s)}\sup_n \|u_n\|^{N/(N-s)}_\eta < \alpha_* \tag{2.29}$$

when $\gamma_1 \ge \gamma_0$, where γ_0 satisfies

$$\mathfrak{c}\alpha_0\mathfrak{d}^{-s/(N-s)}\left(\frac{A^{N/s}_{\mu,\eta}}{\gamma_0\mu}\right)^{\frac{sN}{(\mu s-N)(N-s)}} A^{N/(N-s)}_{\mu,\eta} < \alpha_*. \tag{2.30}$$

Then, applying Remark 5, we deduce

$$\begin{aligned}&\sup_n \int_{\mathbb{R}^N} \Phi_{N,s}(\mathfrak{c}\alpha_0|u_n|^{N/(N-s)})\,dx \\ &= \sup_n \int_{\mathbb{R}^N} \Phi_{N,s}(\mathfrak{c}\alpha_0\mathfrak{d}^{-s/(N-s)}\|u_n\|^{N/(N-s)}_\eta(\mathfrak{d}^{s/N}u/\|u_n\|_\eta)^{N/(N-s)})\,dx < +\infty.\end{aligned} \tag{2.31}$$

From (2.26) and (2.31), we get

$$\int_{\mathbb{R}^N} |u_n|^{p-1}\Phi_{N,s}(\alpha_0|u_n|^{N/(N-s)})|\varphi|\,dx < +\infty. \tag{2.32}$$

Combining (2.25) and (2.32), we obtain

$$\int_{\mathbb{R}^N} |f(u_n)\varphi|\,dx < +\infty, \tag{2.33}$$

i.e. $f(u_n)\varphi \in L^1(\mathbb{R}^N)$ for all n. Then there exists a constant $\kappa > 0$ such that $|f(u_n)\varphi| \le \kappa$ for all $n \in \mathbb{N}$. For any $\varepsilon > 0$, there exists $\delta = \frac{\varepsilon}{\kappa}$ such that for all measurable sets $E \subset \mathbb{R}^N$ such that $|E| < \delta$, we have

$$\int_E |f(u_n)\varphi|\,dx \le \kappa|E| < \varepsilon.$$

This means that $\{f(u_n)\varphi\}$ is equi-integrable. Clearly, $f(u_n)\varphi \to f(u)\varphi$ almost everywhere on $\mathbb{R}^N$. Since $\varphi \in W^{s,N/s}(\mathbb{R}^N)$ and $W^{s,N/s}(\mathbb{R}^N)$ is continuously embedded into $L^q(\mathbb{R}^N)$ ($q \ge \frac{N}{s}$), then $\|\varphi\|_{L^q(\mathbb{R}^N)} \le A_{q,\eta}^{-1}\|\varphi\|_\eta < +\infty$. Then there exists $R > 0$ such that

$$\int_{\mathbb{R}^N\setminus B_R(0)} |\varphi|^{N/s}\,dx < \varepsilon^{N/s} \quad \text{and} \quad \int_{\mathbb{R}^N\setminus B_R(0)} |\varphi|^q\,dx < \varepsilon^q. \tag{2.34}$$

By arguments as (2.25), (2.26), (2.31) and combining with (2.34), we only integrate on $\mathbb{R}^N \setminus B_R(0)$ and get

$$\int_{\mathbb{R}^N\setminus B_R(0)} |f(u_n)\varphi|\,dx \le \kappa_*\varepsilon,$$

where κ_* is a suitable constant. Therefore, all conditions of Vitali's theorem are satisfied and (2.23) is proved. Similarly, we have

$$\int_{\mathbb{R}^N} |u_n|^{\frac{N}{s}-2}u_n\varphi\,dx \to \int_{\mathbb{R}^N} |u|^{\frac{N}{s}-2}u\varphi\,dx \tag{2.35}$$

as $n \to \infty$. Finally, we prove that

$$\begin{aligned} &\int_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{2N}}\,dx\,dy \\ &\to \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{2N}}\,dx\,dy. \end{aligned} \tag{2.36}$$

Using Hölder's inequality, we see that

$$\begin{aligned} &\int_{\mathbb{R}^{2N}} \left| \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{2N}} \right| dx\,dy \\ &\le \left(\int_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{N/s}}{|x-y|^{2N}}\,dx\,dy \right)^{(\frac{N}{s}-1)/(N/s)} \\ &\quad \cdot \left(\int_{\mathbb{R}^{2N}} \frac{|\varphi(x)-\varphi(y)|^{N/s}}{|x-y|^{2N}}\,dx\,dy \right)^{1/(N/s)} \\ &\le \|u_n\|_\eta^{\frac{N}{s}-1}\|\varphi\|_\eta < +\infty. \end{aligned} \tag{2.37}$$

Hence

$$\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{2N}} \in L^1(\mathbb{R}^{2N})$$

for all n, and there exists a constant $K > 0$ such that

$$\left| \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{2N}} \right| \le K$$

for all $(x,y) \in \mathbb{R}^{2N}$ outside a set with measure zero. For any $\varepsilon > 0$, there exists $\delta = \frac{\varepsilon}{K}$ such that for all measurable sets $E \subset \mathbb{R}^{2N}$ such that $|E| < \delta$, we have

$$\int_E \left| \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x-y|^{2N}} \right| dx\,dy \le K|E| < \varepsilon.$$

Hence $\left\{ \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{2N}} \right\}$ is equi-integrable on $\mathbb{R}^{2N}$. Clearly,

$$\frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x-y|^{2N}} \to \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{2N}}$$

almost everywhere on $\mathbb{R}^{2N}$. Since $\varphi \in W^{s,N/s}(\mathbb{R}^N)$, then there exists $R > 0$ such that

$$\int_{\mathbb{R}^{2N} \setminus \mathcal{B}_R(0)} \frac{|\varphi(x) - \varphi(y)|^{N/s}}{|x-y|^{2N}} dx\,dy < \varepsilon^{N/s},$$

where $\mathcal{B}_R(0)$ is a ball in $\mathbb{R}^{2N}$ with center 0 and radius R. By arguments as (2.37) and as only integrating on $\mathbb{R}^{2N} \setminus \mathcal{B}_R(0)$, $\{u_n\}$ is a bounded sequence in $W^{s,N/s}(\mathbb{R}^N)$, there exists a suitable constant $K_* > 0$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^{2N} \setminus \mathcal{B}_R(0)} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x-y|^{2N}} dx\,dy \\
&\le \left(\int_{\mathbb{R}^{2N} \setminus \mathcal{B}_R(0)} \frac{|u_n(x) - u_n(y)|^{N/s}}{|x-y|^{2N}} dx\,dy \right)^{(\frac{N}{s}-1)/(N/s)} \\
&\quad \cdot \left(\int_{\mathbb{R}^{2N} \setminus \mathcal{B}_R(0)} \frac{|\varphi(x) - \varphi(y)|^{N/s}}{|x-y|^{2N}} dx\,dy \right)^{1/(N/s)} < K_* \varepsilon.
\end{aligned}$$

Therefore all conditions of Vitali's theorem hold and we get (2.36). Combining (2.18), (2.23), (2.35) and (2.36), we get $\langle J'_\eta(u), \varphi \rangle = 0$ for all $\varphi \in W^{s,N/s}(\mathbb{R}^N)$. Hence, $J'_\eta(u) = 0$ on $(W^{s,N/s}(\mathbb{R}^N))^*$ which is a dual space of $W^{s,N/s}(\mathbb{R}^N)$. □

The following result is the version of Lions's result:

Lemma 5. [12] *If $\{u_n\}$ is a bounded sequence in $W^{s,N/s}(\mathbb{R}^N)$ and*

$$\lim_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n(x)|^{N/s} dx = 0$$

for some $R > 0$, then $u_n \to 0$ strongly in $L^q(\mathbb{R}^N)$ for all $q \in (\frac{N}{s}, +\infty)$.

Lemma 6. [12] *Let $\{u_n\}$ be a sequence in $W^{s,N/s}(\mathbb{R}^N)$ converging weakly to 0 with $\limsup_{n\to\infty} \|u_n\|_\eta^{N/(N-s)} \le \frac{\beta_* \mathfrak{d}^{s/(N-s)}}{\mathfrak{c}\alpha_0}$, where $\mathfrak{c} > 1$ is a suitable constant and assume that (f_1) holds and $\lim_{t\to 0^+} \frac{f(t)}{t^{\frac{N}{s}-1}} = 0$. If there exists $R > 0$ such that $\liminf_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^{N/s} dx = 0$, it follows that*

$$\int_{\mathbb{R}^N} f(u_n)u_n \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} F(u_n) \to 0.$$

Proposition 1. [12] *Assume that the conditions (f_1)–(f_5) are satisfied. Then problem (2.1) admits a nontrivial nonnegative weak solution.*

3. The auxiliary problem

Using the change of variable $x \mapsto \varepsilon x$, the problem (P_ε) is equivalent to the following problem

$$(2.38)\qquad (-\Delta)_p^s u + V(\varepsilon x)|u|^{p-2}u = f(u). \qquad (P_\varepsilon^*)$$

Definition 6. We say that $u \in W_\varepsilon$ is a weak solution of problem (2.38) if

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{\frac{N}{s}-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{2N}}\,dx\,dy + \int_{\mathbb{R}^N} V(\varepsilon x)|u(x)|^{\frac{N}{s}-2}u(x)\varphi(x)\,dx = \int_{\mathbb{R}^N} f(u(x))\varphi(x)\,dx$$

for any $\varphi \in W_\varepsilon$.

In order to study the equation (2.38), we consider the energy functional $I_\varepsilon \colon W_\varepsilon \to \mathbb{R}$ given by

$$I_\varepsilon(u) = \frac{1}{p}\|u\|_{W_\varepsilon}^p - \int_{\mathbb{R}^N} F(u)\,dx.$$

By the condition (f_1), I_ε is well defined on W_ε, $I_\varepsilon \in C^2(W_\varepsilon, \mathbb{R})$ and its critical points are weak solutions of problem (2.38). Associated to I_ε, we consider the Nehari manifold $\mathcal{N}_\varepsilon$ given by

$$\mathcal{N}_\varepsilon = \{u \in W_\varepsilon \setminus \{0\} \colon \langle I_\varepsilon'(u), u\rangle = 0\},$$

where

$$\langle I_\varepsilon'(u), \varphi\rangle = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dx\,dy + \int_{\mathbb{R}^N} V(\varepsilon x)|u|^{p-2}u\varphi\,dx - \int_{\mathbb{R}^N} f(u)\varphi\,dx$$

for any $u, \varphi \in W_\varepsilon$.

Proposition 2. *There exists $r_* > 0$ such that*

$$\|u\|_{W_\varepsilon} \ge r_* > 0 \quad \textit{for all } u \in \mathcal{N}_\varepsilon.$$

Proof. We easily get the inequality

$$(2.39)\qquad \|u\|_{W^{s,p}(\mathbb{R}^N)} \le \min\{1, V_0\}^{-1/p}\|u\|_{W_\varepsilon}.$$

Then from Lemma 1 and (2.39), we have

$$(2.40)\qquad \sup_{u\in W_\varepsilon, \|u\|_{W_\varepsilon} \le (\min\{1,V_0\})^{s/N}} \int_{\mathbb{R}^N} \Phi_{N,s}(\alpha|u|^{N/(N-s)})\,dx < +\infty$$

for all $0 \le \alpha < \alpha_* \le \alpha_{s,N}^*$. From the condition (f_1) and (f_3), for any $\varepsilon_* > 0$ and $q > \frac{N}{s}$, there exists $C_{q,\varepsilon_*} > 0$ such that

$$(2.41)\qquad |f(t)t| \le \varepsilon_*|t|^{N/s} + C_{q,\varepsilon_*}|t|^q\Phi_{N,s}(\alpha_0|t|^{N/(N-s)})$$

for all $t \ge 0$. Using inequality (2.40) and by arguments as Lemma 2, there exists a constant $C(q, \varepsilon_*)$ such that

$$(2.42)\qquad \int_{\mathbb{R}^N} f(u)u\,dx \le \varepsilon_* S_{N/s,\varepsilon}^{-N/s}\|u\|_{W_\varepsilon}^{N/s} + C(q,\varepsilon_*)\|u\|_{W_\varepsilon}^q$$

for some $q > \frac{N}{s}$ and all $u \in \mathcal{N}_\varepsilon$ and $\|u\|_{W_\varepsilon}$ is small enough. Assume that by contradiction, there exists a sequence $\{u_n\} \subset \mathcal{N}_\varepsilon$ such that $\|u_n\|_{W_\varepsilon} \to 0$ as $n \to \infty$.

Hence (2.42) holds for $u = u_n$ when n is large enough. From the definition of $\mathcal{N}_\varepsilon$, we get

$$\|u_n\|_{W_\varepsilon}^{N/s} = \int_{\mathbb{R}^N} f(u_n)u_n\,dx \le \varepsilon_* S_{N/s,\varepsilon}^{-N/s}\|u_n\|_{W_\varepsilon}^{N/s} + C(q,\varepsilon_*)\|u_n\|_{W_\varepsilon}^{q}.$$

Divide both sides of above inequality by $\|u_n\|_{W_\varepsilon}^{N/s}$ and take $n \to \infty$, we get a contradiction when ε_* is small enough. Therefore, there exists $r_* > 0$ such that $\|u\|_{W_\varepsilon} \ge r_* > 0$ for all $u \in \mathcal{N}_\varepsilon$. □

Lemma 7. *The functional I_ε satisfies the following conditions:*

(i) *There exists $\alpha > 0, \rho > 0$ such that $I_\varepsilon(u) \ge \alpha$ for all $u \in W_\varepsilon$ with $\|u\|_{W_\varepsilon} = \rho$.*
(ii) *There exists $e \in W_\varepsilon$ with $\|e\|_{W_\varepsilon} > \rho$ such that $I_\varepsilon(e) < 0$.*

Proof. Lemma 7 is proved similarly to Lemma 2 and Lemma 3 using the inequality (2.40). We omit the details. □

From Lemma 7 and a version of Mountain Pass Theorem, there exists a $(PS)_{c_\varepsilon}$ sequence $\{u_n\} \subset W_\varepsilon$, that is,

$$I_\varepsilon(u_n) \to c_\varepsilon \quad \text{and} \quad I_\varepsilon'(u_n) \to 0,$$

where

$$c_\varepsilon = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\varepsilon(\gamma(t))$$

and $\Gamma = \{\gamma \in C([0,1], W_\varepsilon) \colon \gamma(0) = 0, I_\varepsilon(\gamma(1)) < 0\}$.

The following result is proved in [12] but the original idea comes from [34].

Proposition 3. *We have $c_\varepsilon = \inf_{u\in W_\varepsilon\setminus\{0\}} \sup_{t\ge 0} I_\varepsilon(tu) = \inf_{u\in\mathcal{N}_\varepsilon} I_\varepsilon(u)$.*

Lemma 8. *Let $\{u_n\}$ be a bounded sequence in W_ε satisfying*

$$\limsup_{n\to\infty} \|u_n\|_{W_\varepsilon}^{N/(N-s)} < \frac{\beta_* \mathfrak{d}_*^{s/(N-s)}}{2^{N/(N-s)}\mathfrak{c}\alpha_0},$$

where $\mathfrak{d}_* = \min\{1, V_0\}$, $\mathfrak{c} > 1$ is a suitable constant and assume that (f_1) and (f_3) hold. Up to a subsequence, we may suppose that $u_n \to u$ weakly in W_ε and $u_n(x) \to u(x)$ everywhere in $\mathbb{R}^N$. Then it follows that

(i) $\lim_{n\to\infty}\int_{\mathbb{R}^N} |F(v_n + u) - F(v_n) - F(u)|\,dx = 0$, *where $v_n = u_n - u$.*
(ii) *For any $r > 1$ such that $r(\frac{N}{s} - 1) \ge \frac{N}{s}$, we have*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |f(v_n + u) - f(v_n) - f(u)|^r\,dx = 0.$$

Proof. From the condition (f_1), we have

$$|f(t)| \le b_1|t|^{p-1} + b_2|t|^{p-1}\Phi_{N,s}(\alpha_0|t|^{N/(N-s)}), \tag{2.43}$$

$$|f(t)t| \le b_1|t|^{N/s} + b_2|t|^{p}\Phi_{N,s}(\alpha_0|t|^{N/(N-s)}), \tag{2.44}$$

$$|F(t)| \le b_1|t|^{N/s} + b_2|t|^{p}\Phi_{N,s}(\alpha_0|t|^{N/(N-s)}) \tag{2.45}$$

for all $t \in \mathbb{R}$ and some constants $b_1 > 0, b_2 > 0$. We begin remarking that

$$F(v_n + u) - F(v_n) = f(v_n + tu)u, \quad \text{where } t \in [0,1]. \tag{2.46}$$

Combining (2.43) and (2.46), we get

$$\begin{aligned}&|F(v_n+u)-F(v_n)|\\&\le b_1|v_n+tu|^{\frac{N}{s}-1}|u|+b_2|v_n+tu|^{\frac{N}{s}-1}|u|\Phi_{N,s}(\alpha_0|v_n+tu|^{N/(N-s)})\\&\le b_1(|v_n|+|u|)^{\frac{N}{s}-1}|u|+b_2(|v_n|+|u|)^{\frac{N}{s}-1}|u|\Phi_{N,s}(\alpha_0(|v_n|+|u|)^{N/(N-s)})\\&\le 2^{\frac{N}{s}-1}b_1(|v_n|^{\frac{N}{s}-1}+|u|^{\frac{N}{s}-1})|u|\\&\quad+b_2(|v_n|+|u|)^{\frac{N}{s}-1}|u|\Phi_{N,s}(\alpha_0(|v_n|+|u|)^{N/(N-s)}).\end{aligned}\tag{2.47}$$

Now, we prove

$$|F(v_n+u)-F(v_n)|\in L^1(\mathbb{R}^N).\tag{2.48}$$

By the Brezis–Lieb Lemma, we have

$$\|u_n-u\|_{W_\varepsilon}^p=\|u_n\|_{W_\varepsilon}^p-\|u\|_{W_\varepsilon}^p+o_n(1)\le\|u_n\|_{W_\varepsilon}^p+o_n(1)$$

as $n\to\infty$. Thus,

$$\limsup_{n\to\infty}\|u_n-u\|_{W_\varepsilon}^p\le\sup_{n\in\mathbb{N}}\|u_n\|_{W_\varepsilon}^p<\left(\frac{\beta_*}{2^{N/(N-s)}\mathfrak{c}\alpha_0}\right)^{(N-s)/s}\mathfrak{d}_*.$$

Therefore, there exists n_0 such that

$$\sup_{n\ge n_0}\|u_n-u\|_{W_\varepsilon}^p<\left(\frac{\beta_*}{2^{N/(N-s)}\mathfrak{c}\alpha_0}\right)^{(N-s)/s}\mathfrak{d}_*.\tag{2.49}$$

By Fatou's lemma, we have

$$\|u\|_{W_\varepsilon}^p\le\liminf_{n\to\infty}\|u_n\|^p<\left(\frac{\beta_*}{2^{N/(N-s)}\mathfrak{c}\alpha_0}\right)^{(N-s)/s}\mathfrak{d}_*.\tag{2.50}$$

Using Hölder's inequality, we get

$$\int_{\mathbb{R}^N}|v_n|^{\frac{N}{s}-1}|u|\,dx\le\|v_n\|_{L^{N/s}(\mathbb{R}^N)}^{\frac{N}{s}-1}\|u\|_{L^{N/s}}(\mathbb{R}^N).\tag{2.51}$$

and for $t>\frac{N}{s},t'>1,t_*(p-1)\ge p$ such that $\frac{1}{t_*}+\frac{1}{t}+\frac{1}{t'}=1$, we get

$$\begin{aligned}&\int_{\mathbb{R}^N}(|v_n|+|u|)^{p-1}|u|\Phi_{N,s}(\alpha_0(|v_n|+|u|)^{N/(N-s)})\,dx\\&\le\||v_n|+|u|\|_{L^{t_*(p-1)}(\mathbb{R}^N)}^{p-1}\|u\|_{L^t(\mathbb{R}^N)}\left(\int_{\mathbb{R}^N}(\Phi_{N,s}(\alpha_0(|v_n|+|u|)^{N/(N-s)}))^{t'}dx\right)^{1/t'}.\end{aligned}\tag{2.52}$$

Then by [25, Lemma 2.3], for any $\mathfrak{c}>t'$, there exists a constant $C(\mathfrak{c})>0$ such that

$$\left(\Phi_{N,s}(\alpha_0(|v_n|+|u|)^{N/(N-s)})\right)^{t'}\le C(\mathfrak{c})\Phi_{N,s}(\mathfrak{c}\alpha_0(|v_n|+|u|)^{N/(N-s)})\tag{2.53}$$

on $\mathbb{R}^N$. Noting that $\mathfrak{d}_*=\min\{1,V_0\}$, we get

$$\begin{aligned}\int_{\mathbb{R}^N}\left(\Phi_{N,s}(\alpha_0(|v_n|+|u|)^{N/(N-s)})\right)^t&\le C(\mathfrak{c})\int_{\mathbb{R}^N}\Phi_{N,s}(\mathfrak{b}\alpha_0(|v_n|+|u|)^{N/(N-s)})\,dx\\&=C(\mathfrak{c})\int_{\mathbb{R}^N}\Phi_{N,s}\Bigg(\mathfrak{c}\alpha_0\mathfrak{d}_*^{-s/(N-s)}\||v_n|+|u|\|_{W_\varepsilon}^{N/(N-s)}\\&\qquad\cdot\mathfrak{d}_*^{s/N}\frac{|v_n|+|u|}{|\||v_n|+|u|\|_{W_\varepsilon}|^{N/(N-s)}}\Bigg)dx.\end{aligned}\tag{2.54}$$

When $\mathfrak{c}$ is near t', combining (2.49) and (2.50), we have

$$\mathfrak{c}\alpha_0\mathfrak{d}_*^{-s/(N-s)}\||v_n|+|u|\|_{W_\varepsilon}^{N/(N-s)} \le \beta_* < \alpha_*, \tag{2.55}$$

by (2.40), (2.54) and (2.55), there exists a constant $D > 0$ such that

$$\left(\int_{\mathbb{R}^N} \left(\Phi_{N,s}(\alpha_0(|v_n|+|u|)^{N/(N-s)})\right)^{t'} dx\right)^{1/t'} \le D. \tag{2.56}$$

Since the embedding sequence from $W_\varepsilon \to W^{s,N/s}(\mathbb{R}^N) \to L^{\mathfrak{q}}(\mathbb{R}^N)$ is continuous for all $\mathfrak{q} \in [\frac{N}{s}, +\infty)$, we get

$$\begin{aligned}&\int_{\mathbb{R}^N} (|v_n|+|u|)^{p-1}|u|\Phi_{N,s}(\alpha_0(|v_n|+|u|)^{N/(N-s)})\,dx\\ &\le D2^{p-1}S_{t,\varepsilon}^{-1}\|u\|_{W_\varepsilon}(\|v_n\|_{L^{t*(p-1)}(\mathbb{R}^N)}^{p-1} + \|u\|_{L^{t*(p-1)}(\mathbb{R}^N)}^{p-1}) < +\infty.\end{aligned} \tag{2.57}$$

Combining (2.49), (2.51) and (2.57), we get the claim (2.48). By arguments as in [39, Lemma 7], for any $u \in W_\varepsilon$ and $\alpha > 0$, we have

$$\int_{\mathbb{R}^N} \Phi_{N,s}(\alpha|u|^{N/(N-s)})\,dx < +\infty. \tag{2.58}$$

Using Hölder's inequality for $t > \frac{N}{s}$ and $t' > 1$ such that $\frac{1}{t}+\frac{1}{t'} = 1$, (2.58), and [25, Lemma 2.3], it is easy to get

$$\begin{aligned}\int_{\mathbb{R}^N} |F(u)|\,dx &\le b_1\|u\|_{L^{N/s}(\mathbb{R}^N)}^{N/s} + b_2\|u\|_{L^{pt}(\mathbb{R}^N)}^{p}\left(\int_{\mathbb{R}^N} (\Phi_{N,s}(\alpha_0|u|^{N/(N-s)}))^{t'}dx\right)^{1/t'}\\ &< +\infty.\end{aligned} \tag{2.59}$$

From (2.48) and (2.59), we obtain $|F(v_n+u)-F(v_n)-F(u)| \in L^1(\mathbb{R}^N)$ for all n large enough. Then there exists a constant $\kappa > 0$ such that $|F(v_n+u)-F(v_n)-F(u)| \le \kappa$ on $\mathbb{R}^N$ outside a set with measure zero. From $u_n(x) \to u(x)$ almost everywhere on $\mathbb{R}^N$, we see that

$$F(v_n+u)-F(v_n)-F(u) \to 0 \quad \text{as } n \to \infty \tag{2.60}$$

outside a set with measure zero. For any $\varepsilon > 0$, there exists $\delta = \frac{\varepsilon}{\kappa} > 0$ such that for all $U \subset \mathbb{R}^N$ with $|U| < \delta$, we have

$$\int_U |F(v_n+u)-F(v_n)-F(u)|\,dx \le \kappa|U| = \varepsilon \tag{2.61}$$

for all n large enough. Since $u \in L^{N/s}(\mathbb{R}^N)$, $u \in L^t(\mathbb{R}^N)$ and the embedding from $W_\varepsilon \to L^{\mathfrak{q}}(\mathbb{R}^N)$ is continuous for all $\mathfrak{q} \ge \frac{N}{s}$, then there exists $R > 0$ such that

$$\int_{\mathbb{R}^N\setminus B_R(0)} |u|^{N/s}\,dx < \varepsilon^p \quad \text{and} \quad \int_{\mathbb{R}^N\setminus B_R(0)} |u|^{pt}\,dx < \varepsilon^{pt}. \tag{2.62}$$

Combining (2.47), (2.51), (2.52), (2.56) and (2.59), only integrating on $\mathbb{R}^N \setminus B_R(0)$ gives a constant $\kappa_* > 0$ such that

$$\int_{\mathbb{R}^N\setminus B_R(0)} |F(v_n+u)-F(v_n)-F(u)|\,dx \le \kappa_*\varepsilon. \tag{2.63}$$

Combining (2.60), (2.61) and (2.63) and using Vitali's theorem, we get (i).

For any $\varepsilon > 0$, from the conditions (f_1) and (f_3), there exists $C(\varepsilon) > 0$ such that

$$|f(t)| \le \varepsilon|t|^{\frac{N}{s}-1} + C(\varepsilon)|t|^{p-1}\Phi_{N,s}(\alpha_0|t|^{N/(N-s)}).$$

We have

$$|f(v_n+u)-f(v_n)-f(u)|^r \le 3^r(|f(v_n+u)|^r+|f(v_n)|^r+|f(u)|^r).$$

Then statement (ii) is proved similarly as (i). We omit the details. □

Lemma 9. *Let* $\{u_n\}\subset W_\varepsilon$ *be a* $(PS)_d$ *sequence for* I_ε *such that* $u_n\to 0$ *weakly in* W_ε *verifying* $\limsup_{n\to\infty}\|u_n\|_{W_\varepsilon}^{N/(N-s)} \le \frac{\beta_*\mathfrak{d}^{s/(N-s)}}{\mathfrak{c}\alpha_0}$, *where* $\mathfrak{c}>1$ *is a suitable constant. Then we have either:*

(i) $u_n\to 0$ *in* W_ε *or*

(ii) *there exists a sequence* $\{y_n\}\subset\mathbb{R}^N$ *and constants* $R>0,\beta>0$ *such that*

$$\liminf_{n\to\infty}\int_{B_R(y_n)}|u_n|^{N/s}\,dx\ge\beta>0.$$

Proof. Suppose that (ii) does not occur. First from the condition (f_2), any $(PS)_d$ sequence of I_ε must be bounded in W_ε. Then by arguments as in Lemma 5, we have $u_n\to 0$ in $L^q(\mathbb{R}^N)$ for $q\in(\frac{N}{s},+\infty)$. By arguments as in Lemma 6, from the conditions (f_1) and (f_3), we have $\lim_{n\to\infty}\int_{\mathbb{R}^N} f(u_n)u_n dx=0$. Recalling that $< I_\varepsilon'(u_n),u_n >\to 0$ as $n\to\infty$, then we get $u_n\to 0$ strongly in W_ε. The proof of Lemma 9 is complete. □

Lemma 10. *Suppose that* $V_\infty<+\infty$. *Let* $\{v_n\}\subset W_\varepsilon$ *be a* $(PS)_d$ *sequence converging weakly to* 0 *satisfying* $\limsup_{n\to\infty}\|v_n\|_{W_\varepsilon}^{N/(N-s)}\le\frac{\beta_*\mathfrak{d}_*{}^{s/(N-s)}}{\mathfrak{c}\alpha_0}$, *where* $\mathfrak{c}>1$ *is a suitable constant. If* $v_n\not\to 0$ *in* W_ε, *then* $d\ge c_{V_\infty}$, *where* c_{V_∞} *is the maximum level of energy function associated to the problem* (P_{V_∞}).

Proof. Let $(t_n)\subset(0,+\infty)$ be a sequence such that $(t_nv_n)\subset\mathcal{N}_{V_\infty}$. We start by showing the following claim.

Claim 1. *The sequence* $\{t_n\}$ *satisfies* $\limsup_{n\to\infty}t_n\le 1$. Indeed, suppose that the above claim does not hold. Then there exist $\delta>0$ and a subsequence still denoted by (t_n) such that

$$t_n\ge 1+\delta\quad\text{for all } n\in\mathbb{N}. \tag{2.64}$$

By the condition (f_2), $\{v_n\}$ is bounded sequence in W_ε, and then $< I_\varepsilon'(v_n),v_n >= o_n(1)$ as $n\to\infty$. This means that

$$\|v_n\|_{W_\varepsilon}^p=\int_{\mathbb{R}^N}f(v_n)v_n\,dx+o_n(1).$$

Moreover, recalling that $(t_nv_n)\subset\mathcal{N}_{V_\infty}$, we get

$$t_n^p\|v_n\|_{V_\infty}^p=\int_{\mathbb{R}^N}f(t_nv_n)t_nv_n\,dx.$$

The above equalities imply that

$$\int_{\mathbb{R}^N}\left[\frac{f(t_nv_n)v_n}{t_n^{p-1}}-f(v_n)v_n\right]=\int_{\mathbb{R}^N}[V_\infty-V(\varepsilon x)]|v_n|^p\,dx+o_n(1). \tag{2.65}$$

Given any $\xi>0$, from the condition (V), there exists $R=R(\varepsilon)>0$ such that

$$V(\varepsilon x)\ge V_\infty-\xi\quad\text{for any } |x|\ge R. \tag{2.66}$$

Since $\{v_n\}$ is a bounded sequence in W_ε and the embedding from $W_\varepsilon\to L^{N/s}(\mathbb{R}^N)$ is continuous, then there exists $C>0$ such $\|v_n\|_{L^{N/s}(\mathbb{R}^N)}\le C$. From $v_n\to 0$ in

$L^{N/s}(B_R(0))$, V is a continuous function and (2.66), there exists a suitable constant $C_* > 0$ such that

$$\begin{aligned}&\int_{\mathbb{R}^N}[V_\infty - V(\varepsilon x)]|v_n|^p\,dx\\ &= \int_{B_R(0)}[V_\infty - V(\varepsilon x)]|v_n|^p\,dx + \int_{\mathbb{R}^N\setminus B_R(0)}[V_\infty - V(\varepsilon x)]|v_n|^p\,dx \le C_*\xi.\end{aligned} \tag{2.67}$$

Because $v_n \not\to 0$ in W_ε, by Lemma 9, there exist a sequence $\{y_n\} \subset \mathbb{R}^N$ and $R_* > 0$, $\beta > 0$ such that

$$\int_{B_{R_*}(y_n)} |v_n|^{N/s}\,dx \ge \beta > 0. \tag{2.68}$$

Note that $\{v_n\}$ is a $(PS)_d$ sequence of I_ε. We denote $v_n^-(x) = \min\{v_n(x), 0\}$ and $v_n^+(x) = \max\{v_n(x), 0\}$. Since $\{v_n\}$ is bounded in W_ε, then $\{v_n^-\}$ is also bounded in W_ε, and we have $< I'_\varepsilon(v_n), v_n^- >\to 0$ as $n \to \infty$. We see that

$$\begin{aligned}\langle I'_\varepsilon(v_n), v_n^-\rangle = &\int_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))(v_n^-(x) - v_n^-(y))}{|x-y|^{2N}}\\ &+ \int_{\mathbb{R}^N} V(\varepsilon x)|v_n|^{p-2}v_n v_n^-\,dx - \int_{\mathbb{R}^N} f(v_n)v_n^-\,dx.\end{aligned} \tag{2.69}$$

We denote $\Omega_n^+ = \operatorname{supp}(v_n^+)$, $\Omega_n^- = \operatorname{supp}(v_n^-), \Omega_n = \operatorname{supp}(v_n)$. Then we get

$$\begin{aligned}&\int_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))(v_n^-(x) - v_n^-(y))}{|x-y|^{2N}}\,dx\,dy\\
&= \int_{(\Omega_n^+\cup\Omega_n^-\cup\Omega_n^c)\times(\Omega_n^+\cup\Omega_n^-\cup\Omega_n^c)} \frac{|(v_n^+(x) - v_n^+(y) + (v_n^-(x) - v_n^-(y)|^{p-2}}{|x-y|^{2N}}\\
&\quad\times (v_n^+(x) - v_n^+(y) + v_n^-(x) - v_n^-(y))(v_n^-(x) - v_n^-(y))\,dx\,dy\\
&= -\int_{\Omega_n^+\times\Omega_n^-} \frac{|v_n^+(x) - v_n^-(y)|^{p-1}v_n^-(y)}{|x-y|^{2N}}\,dx\,dy\\
&\quad - \int_{\Omega_n^-\times\Omega_n^+} \frac{|v_n^-(x) - v_n^+(y)|^{p-1}v_n^-(x)}{|x-y|^{2N}}\,dx\,dy\\
&\quad + \int_{\Omega_n^-\times\Omega_n^-} \frac{|v_n^-(x) - v_n^-(y)|^{p}}{|x-y|^{2N}}\,dx\,dy + \int_{\Omega_n^c\times\Omega_n^-} \frac{|v_n^-(y)|^{p}}{|x-y|^{2N}}\,dx\,dy\\
&\quad + \int_{\Omega_n^-\times\Omega_n^c} \frac{|v_n^-(x)|}{|x-y|^{N+ps}}\,dx\,dy,\end{aligned} \tag{2.70}$$

where $\Omega^c = \mathbb{R}^N \setminus \Omega$ for some set $\Omega \subset \mathbb{R}^N$. Note that $\Omega_n^{-c} = \Omega_n^+ \cup \Omega_n^c$ and

$$\begin{aligned}\int_{\mathbb{R}^{2N}} \frac{|v_n^-(x) - v_n^-(y)|^{p}}{|x-y|^{2N}}\,dx\,dy &= \int_{\Omega_n^-\times\Omega_n^-} \frac{|v_n^-(x) - v_n^-(y)|^{p}}{|x-y|^{2N}}\,dx\,dy\\
&+ \int_{\Omega_n^-\times\Omega_n^{-c}} \frac{|v_n^-(x) - v_n^-(y)|^{p}}{|x-y|^{2N}}\,dx\,dy + \int_{\Omega_n^{-c}\times\Omega_n^-} \frac{|v_n^-(x) - v_n^-(y)|^{p}}{|x-y|^{2N}}\,dx\,dy\end{aligned}$$

$$= \int_{\Omega_n^- \times \Omega_n^-} \frac{|v_n^-(x) - v_n^-(y)|^p}{|x-y|^{2N}} \, dx \, dy + \int_{\Omega_n^- \times \Omega_n^+} \frac{|v_n^-(x) - v_n^-(y)|^p}{|x-y|^{2N}} \, dx \, dy$$
$$+ \int_{\Omega_n^- \times {\Omega_n}^c} \frac{|v_n^-(x) - v_n^-(y)|^p}{|x-y|^{2N}} \, dx \, dy + \int_{\Omega_n^+ \times \Omega_n^-} \frac{|v_n^-(x) - v_n^-(y)|^p}{|x-y|^{2N}} \, dx \, dy$$
$$+ \int_{\Omega_n^c \times \Omega_n^-} \frac{|v_n^-(x) - v_n^-(y)|^p}{|x-y|^{2N}} \, dx \, dy.$$

Hence, we get

$$\int_{\mathbb{R}^{2N}} \frac{|v_n^-(x) - v_n^-(y)|^p}{|x-y|^{2N}} \, dx \, dy = \int_{\Omega_n^- \times \Omega_n^-} \frac{|v_n^-(x) - v_n^-(y)|^p}{|x-y|^{2N}} \, dx \, dy$$
$$+ \int_{\Omega_n^- \times \Omega_n^+} \frac{|v_n^-(x)|^p}{|x-y|^{2N}} \, dx \, dy + \int_{\Omega_n^- \times {\Omega_n}^c} \frac{|v_n^-(x)|^p}{|x-y|^{2N}} \, dx \, dy$$
$$+ \int_{\Omega_n^+ \times \Omega_n^-} \frac{|v_n^-(y)|^p}{|x-y|^{2N}} \, dx \, dy + \int_{\Omega_n^c \times \Omega_n^-} \frac{|v_n^-(y)|^p}{|x-y|^{2N}} \, dx \, dy. \tag{2.71}$$

We have

$$- \int_{\Omega_n^+ \times \Omega_n^-} \frac{|v_n^+(x) - v_n^-(y)|^{p-1} v_n^-(y)}{|x-y|^{2N}} \, dx \, dy - \int_{\Omega_n^+ \times \Omega_n^-} \frac{|v_n^-(y)|^p}{|x-y|^{2N}} \, dx \, dy$$
$$= \int_{\Omega_n^+ \times \Omega_n^-} \frac{(|1 - \frac{v_n^+(x)}{v_n^-(y)}|^p - 1)|v_n^-(y)|^p}{|x-y|^{2N}} \, dx \, dy > 0. \tag{2.72}$$

Similary, we can get

$$- \int_{\Omega_n^- \times \Omega_n^+} \frac{|v_n^-(x) - v_n^+(y)|^{p-1} v_n^-(x)}{|x-y|^{2N}} \, dx \, dy - \int_{\Omega_n^- \times \Omega_n^+} \frac{|v_n^-(x)|^p}{|x-y|^{2N}} \, dx \, dy > 0. \tag{2.73}$$

From (2.70) to (2.73), we deduce

$$\int_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))(v_n^-(x) - v_n^-(y))}{|x-y|^{2N}} \, dx \, dy$$
$$\geq \int_{\mathbb{R}^{2N}} \frac{|v_n^-(x) - v_n^-(y)|^p}{|x-y|^{2N}} \, dx \, dy. \tag{2.74}$$

Combining (2.69) and (2.73) and noting that $f(t) = 0$ for $t \in (-\infty, 0]$, we get

$$\|v_n^-\|_{W_\varepsilon} \to 0$$

as $n \to \infty$ or equivalent $v_n^- \to 0$ in W_ε. Note that W_ε is continuously embedded into $W^{s,N/s}(\mathbb{R}^N)$, then $v_n^- \to 0$ in $W^{s,N/s}(\mathbb{R}^N)$. We denote $\overline{v}_n(x) = v_n(x + y_n)$. Since $\|.\|_{V_0}$ is invariant under the translations, we get

$$\|\overline{v}_n\|_{V_0}^p = \|v_n\|_{V_0}^p \leq [v_n]_{s,p}^p + \int_{\mathbb{R}^N} V(\varepsilon x)|v_n|^p = \|v_n\|_{W_\varepsilon}^p.$$

Then $\{\overline{v}_n\}$ is a bounded sequence in $W^{s,N/s}(\mathbb{R}^N)$. Up to a subsequence, we may assume that there exists $\overline{v} \in W^{s,N/s}(\mathbb{R}^N)$ such that $\overline{v}_n \to \overline{v}$ weakly in $W^{s,N/s}(\mathbb{R}^N)$ and $\overline{v}_n(x) \to \overline{v}(x)$ on $\mathbb{R}^N$. By arguments as above and Fatou's lemma, we get

$$\|\overline{v}^-\|_{V_0} \leq \liminf_{n\to\infty} \|\overline{v}_n^-\|_{V_0} \leq \liminf_{n\to\infty} \|v_n^-\|_{W_\varepsilon} = 0.$$

Then $\overline{v} = \overline{v}^+$. From (2.68) and $\overline{v}_n \to \overline{v}$ in $L^{N/s}(B_{R_*}(0))$, we deduce $\int_{B_{R_*}(0)} |\overline{v}|^{N/s} dx \geq \beta/2 > 0$. Therefore, there exists a subset $\Omega \subset B_{R_*}(0) \subset \mathbb{R}^N$ such that $|\Omega| > 0$ and

$\overline{v}(x) > a > 0$ for all $x \in \Omega$, where $a > 0$ is a suitable constant. Since $\overline{v}_n \to \overline{v}$ on $L^q(B_{R_*}(0))$ for all $q \in [\frac{N}{s}, +\infty)$, we can assume $\overline{v}_n(x) \to \overline{v}(x)$ on Ω and then $\overline{v}_n(x) > \frac{a}{2} > 0$ for all $x \in \Omega$ and n large enough. From (2.65) and (2.67), we obtain

$$\begin{aligned}\int_{\operatorname{supp}(\overline{v}_n^+)} [\frac{f(t_n\overline{v}_n)\overline{v}_n}{t_n^{p-1}} - f(\overline{v}_n)\overline{v}_n]\,dx &= \int_{\mathbb{R}^N} [\frac{f(t_n\overline{v}_n)\overline{v}_n}{t_n^{p-1}} - f(\overline{v}_n)\overline{v}_n]\,dx \\ &= \int_{\mathbb{R}^N} [\frac{f(t_n v_n)v_n}{t_n^{p-1}} - f(v_n)v_n]\,dx \le C_*\xi \end{aligned} \tag{2.75}$$

for any $\xi > 0$. Using the condition (f_5), Fatou's lemma, (2.64), (2.75) and $f(t) = 0$ for all $t \in (-\infty, 0]$, we get

$$\begin{aligned} 0 &< \int_\Omega [\frac{f((1+\delta)\overline{v})\overline{v}}{(1+\delta)^{p-1}} - f(\overline{v})\overline{v}]\,dx = \int_\Omega [\frac{f((1+\delta)\overline{v})}{((1+\delta)\overline{v})^{p-1}} - \frac{f(\overline{v})}{\overline{v}^{p-1}}]\overline{v}^p\,dx \\ &\le \liminf_{n\to\infty} \int_\Omega [\frac{f((1+\delta)\overline{v}_n)}{((1+\delta)\overline{v}_n)^{p-1}} - \frac{f(\overline{v}_n)}{\overline{v}_n^{p-1}}]\overline{v}_n^p\,dx \le \liminf_{n\to\infty} \int_\Omega [\frac{f(t_n\overline{v}_n)}{(t_n\overline{v}_n)^{p-1}} - \frac{f(\overline{v}_n)}{\overline{v}_n^{p-1}}]\overline{v}_n^p\,dx \\ &= \liminf_{n\to\infty} \int_\Omega [\frac{f(t_n\overline{v}_n)\overline{v}_n}{t_n^{p-1}} - f(\overline{v}_n)\overline{v}_n]\,dx \le \liminf_{n\to\infty} \int_{\operatorname{supp}(\overline{v}_n^+)} [\frac{f(t_n\overline{v}_n)\overline{v}_n}{t_n^{p-1}} - f(\overline{v}_n)\overline{v}_n]\,dx \\ &\le C_*\xi \end{aligned}$$

for any $\xi > 0$ and n large enough. This is a contradiction. Then Claim 1 is proved. Now, we will consider the following cases:

Case 1. $\limsup_{n\to\infty} t_n = 1$. Then there exists a subsequence, still denoted by t_n such that $t_n \to 1$. Recalling that $I_\varepsilon(v_n) \to d$ as $n \to \infty$, and noting that $J_{V_\infty}(t_n v_n) \ge c_{V_\infty}$, we have

$$\begin{aligned} d + o_n(1) = I_\varepsilon(v_n) &= I_\varepsilon(v_n) - J_{V_\infty}(t_n v_n) + J_{V_\infty}(t_n v_n) \\ &\ge I_\varepsilon(v_n) - J_{V_\infty}(t_n v_n) + c_{V_\infty}. \end{aligned} \tag{2.76}$$

Let us compute

$$\begin{aligned} I_\varepsilon(v_n) - J_{V_\infty}(t_n v_n) &= \frac{1-t_n^p}{p}[v_n]_{s,p}^p + \frac{1}{p}\int_{\mathbb{R}^N} (V(\varepsilon x) - t_n^p V_\infty)|v_n|^p\,dx \\ &\quad + \int_{\mathbb{R}^N} (F(t_n v_n) - F(v_n))\,dx. \end{aligned} \tag{2.77}$$

Using the condition (V), (2.66), $v_n \to 0$ in $L^{N/s}(B_R(0))$, $t_n \to 1$ and

$$V(\varepsilon x) - t_n^p V_\infty = (V(\varepsilon x) - V_\infty) + (1-t_n^p)V_\infty \ge -\xi + (1-t_n^p)V_\infty \quad \text{for all } |x| \ge R,$$

we get

$$\begin{aligned} &\int_{\mathbb{R}^N} (V(\varepsilon x) - t_n^p V_\infty)|v_n|^p\,dx \\ &= \int_{B_R(0)} (V(\varepsilon x) - t_n^p V_\infty)|v_n|^p\,dx + \int_{\mathbb{R}^N\setminus B_R(0)} (V(\varepsilon x) - t_n^p V_\infty)|v_n|^p\,dx \\ &\ge (V_0 - t_n^p V_\infty)\int_{B_R(0)} |v_n|^p\,dx - \xi \int_{\mathbb{R}^N\setminus B_R(0)} |v_n|^p\,dx \\ &\quad + V_\infty(1-t_n^p)\int_{\mathbb{R}^N\setminus B_R(0)} |v_n|^p\,dx \ge o_n(1) - \xi C^*, \end{aligned} \tag{2.78}$$

where C^* is a suitable constant. Since $\{v_n\}$ is a bounded sequence in W_ε, then

$$\lim_{n\to\infty} \frac{(1-t_n^p)}{p}[v_n]_{s,p}^p = 0. \tag{2.79}$$

From the condition $\limsup_{n\to\infty} \|v_n\|_{W_\varepsilon}^{N/(N-s)} \leq \frac{\beta_* \mathfrak{d}_*{}^{s/(N-s)}}{\mathfrak{c}\alpha_0}$, by arguments in Lemma 8 and noting that $\Phi_{N,s}(t)$ is an increasing function on $[0, +\infty)$, it is easy to get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (F(t_n v_n) - F(v_n))dx = 0. \tag{2.80}$$

Combining (2.76)–(2.80), we obtain

$$d + o_n(1) \geq c_{V_\infty} - C^*\xi + o_n(1).$$

Taking the limit in the above inequality, we get $d \geq c_{V_\infty}$.

Case 2. $\limsup_{n\to\infty} t_n = t_0 < 1$. There exists a subsequence, still denoted by $\{t_n\}$ such that $t_n \to t_0$ (< 1) and then $t_n < 1$ for all $n \in \mathbb{N}$. We see that

$$d + o_n(1) = I_\varepsilon(v_n) - \frac{1}{p}\langle I_\varepsilon'(v_n), v_n\rangle = \int_{\mathbb{R}^N} \left(\frac{1}{p} f(v_n)v_n - F(v_n)\right) dx. \tag{2.81}$$

Noting that $t_n v_n \in \mathcal{N}_{V_\infty}$, by the condition (f_5) and (2.81), we get

$$\begin{aligned} c_{V_\infty} \leq J_{V_\infty}(t_n v_n) &= J_{V_\infty}(t_n v_n) - \frac{1}{p}\langle J_{V_\infty}'(t_n v_n), t_n v_n\rangle \\ &= \int_{\mathbb{R}^N} \left(\frac{1}{p} f(t_n v_n)t_n v_n - F(t_n v_n)\right) dx \\ &\leq \int_{\mathbb{R}^N} \left(\frac{1}{p} f(v_n)v_n - F(v_n)\right) dx = d + o_n(1). \end{aligned}$$

Taking the limit of the above inequality as $n \to \infty$, we get $d \geq c_{V_\infty}$. □

Lemma 11. *Let u_n be a $(PS)_c$ for I_ε satisfying*

$$\limsup_{n\to\infty} \|u_n\|_{W_\varepsilon}^{N/(N-s)} \leq \frac{\beta_* \mathfrak{d}_*{}^{s/(N-s)}}{2^{N/(N-s)}\mathfrak{c}\alpha_0},$$

where $\mathfrak{c} > 1$ is a suitable constant. Assume that $c < c_{V_\infty}$ when $V_\infty < \infty$ or $c \in \mathbb{R}$ if $V_\infty = +\infty$. Then $\{u_n\}$ has a convergent subsequence in W_ε.

Proof. First, we consider the case $V_\infty < +\infty$. From the condition (f_2), we see that $\{u_n\}$ is a bounded sequence in W_ε. Then, up to a subsequence, we may assume that

$$u_n \to u \text{ weakly in } W_\varepsilon, \tag{2.82}$$

$$u_n \to u \text{ in } L^q_{\text{loc}}(\mathbb{R}^N) \text{ for any } q \in [\tfrac{N}{s}, +\infty), \tag{2.83}$$

$$u_n \to u \text{ a.e. in } \mathbb{R}^N. \tag{2.84}$$

By arguments as in Lemma 4, we have $I_\varepsilon'(u) = 0$. Set $v_n = u_n - u$. Using the Brezis–Lieb Lemma and Lemma 8, we get

$$\begin{aligned} I_\varepsilon(v_n) &= \frac{\|u_n\|_{W_\varepsilon}^p}{p} - \frac{\|u\|_{W_\varepsilon}^p}{p} - \int_{\mathbb{R}^N} F(u_n)\,dx + \int_{\mathbb{R}^N} F(u)\,dx + o_n(1) \\ &= I_\varepsilon(u_n) - I_\varepsilon(u) + o_n(1) \\ &= c - I_\varepsilon(u) + o_n(1) := d + o_n(1). \end{aligned} \tag{2.85}$$

By [8, Lemma 2.6], we have

$$\int_{\mathbb{R}^{2N}} |\mathcal{A}(u_n) - \mathcal{A}(v_n) - \mathcal{A}(u)|^{p'}\, dx\, dy = o_n(1), \tag{2.86}$$

where $p' = \frac{p}{p-1}$ is the conjugate exponent of p and

$$\mathcal{A}(v) := \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))}{|x-y|^{\frac{2N}{p'}}}$$

for all $v \in W_\varepsilon$. Noting that V is a bounded function, then by [8, Lemma 3.3], we can see that

$$\int_{\mathbb{R}^N} V(\varepsilon x)\| v_n|^{p-2}v_n - |u_n|^{p-2}u_n + |u|^{p-2}u|^{p'}\, dx = o_n(1). \tag{2.87}$$

Hence, using Hölder's inequality, for any $\varphi \in W_\varepsilon$, we have

$$\begin{aligned} &|\langle I_\varepsilon'(v_n) - I_\varepsilon'(u_n) + I_\varepsilon'(u), \varphi\rangle| \\ &\leq \left(\int_{\mathbb{R}^{2N}} |\mathcal{A}(u_n) - \mathcal{A}(v_n) - \mathcal{A}(u)|^{p'}\, dx\, dy\right)^{\frac{1}{p'}} [\varphi]_{s,p}^p \\ &\quad + \left(\int_{\mathbb{R}^N} V(\varepsilon x)\| v_n|^{p-2}v_n - |u_n|^{p-2}u_n + |u|^{p-2}u|^{p'}\, dx\right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}^N} V(\varepsilon x)|\varphi|^p\, dx\right)^{\frac{1}{p}} \\ &\quad + \left(\int_{\mathbb{R}^N} |f(v_n) - f(u_n) + f(u)|^{p'}\, dx\right)^{\frac{1}{p'}} \|\varphi\|_{L^p(\mathbb{R}^N)}. \end{aligned} \tag{2.88}$$

Since the embeddings $W_\varepsilon \hookrightarrow W^{s,p}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ are continuous, then from (2.86)–(2.88) and Lemma 8, we deduce $I_\varepsilon'(v_n) \to 0$ in W_ε^*. By the condition (f_2), we have

$$I_\varepsilon(u) = I_\varepsilon(u) - \frac{1}{p}\langle I_\varepsilon'(u), u\rangle = \int_{\mathbb{R}^N} \left[\frac{1}{p} f(u)u - F(u)\right] dx \geq 0. \tag{2.89}$$

Combining (2.85) and (2.89), we obtain

$$d \leq c < c_{V_\infty}$$

which together Lemma 10 gives $v_n \to 0$ in W_ε. That is $u_n \to u$ strongly in W_ε.

Next, we consider the case $V_\infty = +\infty$. Then V is a coercive function on $\mathbb{R}^N$: $V(\varepsilon x) \to +\infty$ as $|x| \to +\infty$ for each $\varepsilon > 0$. Therefore meas$(\{x \in \mathbb{R}^N : V(\varepsilon x) \leq c\}) < +\infty$ for any $c > 0$. By arguments as in [39, Lemma 5], we have that the embedding from $W_\varepsilon \hookrightarrow L^q(\mathbb{R}^N)$ is compact for any $q \in [\frac{N}{s}, +\infty)$. Then $v_n \to 0$ in $L^q(\mathbb{R}^N)$ for any $q \in [\frac{N}{s}, +\infty)$. From the condition (f_1) and by arguments as in Lemma 8, we easily get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(v_n)v_n\, dx = 0. \tag{2.90}$$

Furthermore, using Vitali's theorem and by arguments as in Lemma 8, we deduce

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |f(u_n)u_n - f(v_n)v_n - f(u)u|\, dx = 0. \tag{2.91}$$

Combining (2.90), (2.91) and the fact that $\langle I_\varepsilon'(u), u\rangle = 0$, we get

$$\lim_{n\to\infty} \|u_n\|_{W_\varepsilon} = \|u\|_{W_\varepsilon}.$$

This implies that $u_n \to u$ in W_ε due to the Brezis–Lieb Lemma. □

Lemma 12. *Let $\{u_n\}$ be a $(PS)_c$ sequence for I_ε restricted to $\mathcal{N}_\varepsilon$ with*

$$\limsup_{n\to\infty} \|u_n\|_{W_\varepsilon}^{N/(N-s)} \le \frac{\beta_* \mathfrak{d}_*{}^{s/(N-s)}}{\mathfrak{c}\alpha_0}, \tag{2.92}$$

where $\mathfrak{c} > 1$ is a suitable constant. Assume that $c < c_{V_\infty}$ when $V_\infty < \infty$ or $c \in \mathbb{R}$ if $V_\infty = +\infty$. Then $\{u_n\}$ has a convergent subsequence in W_ε.

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence of I_ε restricted to $\mathcal{N}_\varepsilon$, that is,

$$I_\varepsilon(u_n) \to c, \quad (I_\varepsilon|_{\mathcal{N}_\varepsilon})' \to 0 \quad \text{as } n \to +\infty.$$

Noting that $\langle I_\varepsilon^{'}(u_n), u_n\rangle = 0$ and using the condition (f_2), we have that $\{u_n\}$ is a bounded sequence in W_ε. From Proposition 2, we have

$$\|u_n\|_{W_\varepsilon} \ge r_* > 0. \tag{2.93}$$

Then up to a subsequence, we can assume that

$$\lim_{n\to\infty} \|u_n\|_{W_\varepsilon} = l, \quad r_* \le l \le \sup_n \|u_n\|_{W_\varepsilon} < +\infty \tag{2.94}$$

and

$$\begin{aligned} &u_n \to u \text{ weaklyin } W_\varepsilon,\\ &u_n \to u \text{ in } L^q_{\text{loc}}(\mathbb{R}^N) \text{ for any } q \in [\tfrac{N}{s}, +\infty),\\ &u_n \to u \text{ a.e. in } \mathbb{R}^N. \end{aligned}$$

Since $\{u_n\} \subset \mathcal{N}_\varepsilon$, then we have

$$\int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{2N}}\, dx\, dy + \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^p\, dx = \int_{\mathbb{R}^N} f(u_n)u_n\, dx.$$

First we consider the case $V_\infty = +\infty$. Then up to a subsequence, we assume that $u_n \to u$ strongly in $L^q(\mathbb{R}^N)$ for all $q \in [\frac{N}{s}, +\infty)$. By the assumption (2.92), (2.4) for $q = \frac{N}{s}$ and using Vitali's theorem as in Lemma 8, we get

$$\begin{aligned} &\lim_{n\to\infty} \int_{\mathbb{R}^N} f(u_n)u_n\, dx = \int_{\mathbb{R}^N} f(u)u\, dx \quad \text{and}\\ &\lim_{n\to\infty} \int_{\mathbb{R}^N} f'(u_n)u_n^2\, dx = \int_{\mathbb{R}^N} f'(u)u^2\, dx. \end{aligned} \tag{2.95}$$

Combining (2.94) and (2.95), we obtain

$$0 < l = \int_{\mathbb{R}^N} f(u)u\, dx = \int_{\{x\in\mathbb{R}^N :\, u(x)>0\}} f(u^+)u^+\, dx. \tag{2.96}$$

Since $f(t) > 0$ for all $t > 0$, then (2.96) implies that $u^+ \not\equiv 0$. Conversely, we get $l = 0$ which is a contradiction. By the method of Lagrange multipliers, there exists a real sequence $\{\lambda_n\} \subset \mathbb{R}$ such that

$$I_\varepsilon^{'}(u_n) = \lambda_n K_\varepsilon^{'}(u_n) + o_n(1), \tag{2.97}$$

where $K_\varepsilon \colon W_\varepsilon \to \mathbb{R}$ is given by

$$K_\varepsilon(u) = \langle I_\varepsilon^{'}(u), u\rangle = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{2N}}\, dx\, dy + \int_{\mathbb{R}^{2N}} V(\varepsilon x)|u|^p\, dx - \int_{\mathbb{R}^N} f(u)u\, dx.$$

Consequently, we have

$$\begin{aligned}\langle K_\varepsilon^{'}(u_n), u_n\rangle &= p\int_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{2N}}\,dx\,dy + p\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^p\,dx\\ &\quad - \int_{\mathbb{R}^N} f(u_n)u_n\,dx - \int_{\mathbb{R}^N} f'(u_n)u_n^2\,dx\\ &= \int_{\mathbb{R}^N} ((p-1)f(u_n)u_n - f'(u_n)u_n^2)\,dx\\ &= \int_{\{x\in\mathbb{R}^N:\,u_n(x)>0\}} ((p-1)f(u_n^+)u_n^+ - f'(u_n^+)(u_n^+)^2)\,dx. \end{aligned} \tag{2.98}$$

From Remark 1 (ii), we have $f'(t)t-(p-1)f(t) > 0$ for all $t > 0$, then $(p-1)f(u_n^+)u_n^+ - f'(u_n^+)(u_n^+)^2 < 0$ for all n. The equality (2.98) implies that $\sup_{n\in\mathbb{R}^N}\langle K_\varepsilon^{'}(u_n), u_n\rangle \le 0$. If $\sup_{n\in\mathbb{R}^N}\langle K_\varepsilon^{'}(u_n), u_n\rangle = 0$. Then up to a subsequence, we suppose that $\lim_{n\to\infty}\langle K_\varepsilon^{'}(u_n), u_n\rangle = 0$. Then from (2.95) and (2.98), it holds

$$\int_{\mathbb{R}^N} ((p-1)f(u)u - f'(u)u^2)\,dx = 0.$$

Thus we get

$$\int_{\{x\in\mathbb{R}^N:\,u(x)>0\}} ((p-1)f(u^+)u^+ - f'(u^+)u^{+2})\,dx = 0. \tag{2.99}$$

Using again Remark 1 (ii), we have

$$(p-1)f(u^+)u^+ - f'(u^+)u^{+2} < 0 \tag{2.100}$$

for all $x\in\Omega=\{x\in\mathbb{R}^N\colon u(x)>0\}$. Hence, we get a contradiction from (2.99) and (2.100).

Next we investigate the case $V_\infty < +\infty$. If $u\not\equiv 0$, then there exists $t\in(0,+\infty)$ such that $tu\in\mathcal{N}_{V_0}$. This means that

$$t^p\|u\|_{V_0}^p = \int_{\mathbb{R}^N} f(tu)(tu)\,dx. \tag{2.101}$$

By the property of $\mathcal{N}_{V_0}$, there exists $r>0$ such that $\|u\|_{V_0}>0$, then (2.101) implies $u^+\not\equiv 0$ and there exists $\varepsilon_0>0$ such that $u(x)\ge\varepsilon_0>0$ on $\Omega\subset\mathbb{R}^N$ with $|\Omega|>0$. From (2.98), we have

$$\begin{aligned}-\langle K_\varepsilon^{'}(u_n), u_n\rangle &= -p\int_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{2N}}\,dx\,dy - p\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^p\,dx\\ &\quad + \int_{\mathbb{R}^N} f(u_n)u_n\,dx + \int_{\mathbb{R}^N} f'(u_n)u_n^2\,dx\\ &= \int_{\mathbb{R}^N} (f'(u_n)u_n^2 - (p-1)f(u_n)u_n)\,dx \end{aligned} \tag{2.102}$$

By arguments as in the first case, we can assume that $\lim_{n\to\infty}\langle K_\varepsilon'(u_n), u_n\rangle = 0$. Using Fatou's lemma and (2.102), we get

$$
\begin{aligned}
0 &\geq \liminf_{n\to\infty}\int_{\mathbb{R}^N}(f'(u_n)u_n^2-(p-1)f(u_n)u_n)\,dx\\
&\geq \int_{\mathbb{R}^N}(f'(u)u^2-(p-1)f(u)u)\,dx\\
&= \int_{\{x\in\mathbb{R}^N:\,u(x)>0\}}(f'(u)u^2-(p-1)f(u)u)\,dx\\
&\geq \int_{\Omega}(f'(u)u^2-(p-1)f(u)u)\,dx>0 \qquad (2.103)
\end{aligned}
$$

due to the condition (f_5). This is a contradiction.

Finally if $u\equiv 0$, then (2.93) implies that $u_n \not\to 0$ in W_ε. By arguments as in Lemma 9, we can show that there exists a sequence $\{y_n\}\subset\mathbb{R}^N$ and constants $R>0, \beta>0$ such that

$$
\liminf_{n\to\infty}\int_{B_R(y_n)}|u_n|^{N/s}\,dx \geq \beta>0. \qquad (2.104)
$$

We denote $v_n(x)=u_n(x+y_n)$. Since the norm in $W^{s,N/s}(\mathbb{R}^N)$ and the integrals are invariant under translations, we have $\|v_n\|_{V_0}=\|u_n\|_{V_0}\leq\|u_n\|_{W_\varepsilon}$. Therefore $\{v_n\}$ is a bounded sequence in $W^{s,N/s}(\mathbb{R}^N)$. Then up to a subsequence, we have

$$
\begin{aligned}
&v_n\to v \text{ weakly in } W^{s,N/s}(\mathbb{R}^N),\\
&v_n\to v \text{ in } L^q_{\mathrm{loc}}(\mathbb{R}^N) \text{ for any } q\in[\tfrac{N}{s},+\infty),\\
&v_n\to v \text{ a.e. in } \mathbb{R}^N.
\end{aligned}
$$

From (2.104), we obtain $v\not\equiv 0$. We see that

$$
\begin{aligned}
-\langle K_\varepsilon'(u_n),u_n\rangle &= \int_{\mathbb{R}^N}(f'(u_n)u_n^2-(p-1)f(u_n)u_n)\,dx\\
&= \int_{\mathbb{R}^N}(f'(v_n)v_n^2-(p-1)f(v_n)v_n)\,dx. \qquad (2.105)
\end{aligned}
$$

Now, we repeat the method as in case $u\not\equiv 0$ and get a contradiction.

In conclusion, we get $\sup_{n\in\mathbb{N}} < K_\varepsilon'(u_n), u_n >< 0$, and (2.97) implies $\lambda_n=o_n(1)$ as $n\to\infty$. Therefore, $\{u_n\}$ is a $(PS)_c$ sequence of I_ε and Lemma 12 is obtained from Lemma 11. □

Corollary 1. *The critical points of I_ε on $\mathcal{N}_\varepsilon$ are critical points of I_ε in W_ε.*

Proof. The key idea is to show $< K_\varepsilon(u), u >< 0$ for all $u\in\mathcal{N}_\varepsilon$. This follows by using similar arguments as in Lemma 12. We refer the readers to [21, Proposition 2.1] for a proof of this kind of a lemma. □

4. Existence of a ground state solution

In this section, we prove the existence of a ground state solution for problem (P_ε^*). That is a critical point u_ε of I_ε satisfying $I_\varepsilon(u_\varepsilon)=c_\varepsilon$. We consider the energy function

$$
J_{V_0}(u)=\frac{1}{p}\left(\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{2N}}+\int_{\mathbb{R}^N}V_0|u|^p\,dx\right)-\int_{\mathbb{R}^N}F(u)\,dx
$$

of problem (P_{V_0}). We recall that c_{V_0} is the minimax level related to J_{V_0} and $\mathcal{N}_{V_0}$ is the Nehari manifold related to J_{V_0}, given by

$$\mathcal{N}_{V_0} = \left\{ u \in W^{s,N/s}(\mathbb{R}^N) \setminus \{0\} \colon \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{2N}} + \int_{\mathbb{R}^N} V_0 |u|^p \, dx = \int_{\mathbb{R}^N} f(u)u \, dx \right\}.$$

Now we state the main result of this section:

Theorem 7. *Assume that (f_1)–(f_5) and (V) hold. Then there exists $\overline{\varepsilon} > 0$ such that (P^*_ε) has a ground state solution for all $0 < \varepsilon < \overline{\varepsilon}$.*

Proof. We will prove that there exists $\overline{\varepsilon} > 0$ such that $c_\varepsilon < c_{V_0}$ for all $\varepsilon \in (0, \overline{\varepsilon})$. Since $c_{V_0} < c_{V_\infty}$ when $V_\infty < +\infty$, Lemma 11 implies that I_ε satisfies the $(PS)_{c_\varepsilon}$ condition. Further, combining that result with Lemma 7, I_ε has a critical point at level c_ε. Next, without the loss of generality, we assume that $V(0) = V_0$.

Let $\phi \in C_0^\infty(\mathbb{R}^N, [0,1])$ be such that $\phi \equiv 1$ on $B_1(0)$ and $\phi \equiv 0$ on $\mathbb{R}^N \setminus B_2(0)$. For each $r > 0$, let us define $v_r(x) = \phi(\frac{x}{r}) w(x)$, where w is a ground state solution of the problem (P_{V_0}) given in Proposition 1. For each v_r, there exists $t_{\varepsilon,r} > 0$ such that $t_{\varepsilon,r} v_r \in \mathcal{N}_\varepsilon$, and we have

$$\begin{aligned} c_\varepsilon \le I_\varepsilon(t_{\varepsilon,r} v_r) = \frac{t^p_{\varepsilon,r}}{p} \int_{\mathbb{R}^{2N}} \frac{|v_r(x)-v_r(y)|^p}{|x-y|^{2N}} \, dx\, dy + \frac{t^p_{\varepsilon,r}}{p} \int_{\mathbb{R}^N} V(\varepsilon x) |v_r(x)|^p \, dx \\ - \int_{\mathbb{R}^N} F(t_{\varepsilon,r} v_r) \, dx. \end{aligned}$$

For any $u \in \mathcal{N}_\varepsilon$, we have

$$\|u\|^p_{W_\varepsilon} = \int_{\mathbb{R}^N} f(u)u \, dx.$$

Then we get

$$I_\varepsilon(u)|_{N_\varepsilon} = \frac{1}{p}\|u\|^p_{W_\varepsilon} - \int_{\mathbb{R}^N} F(u)\, dx = \int_{\mathbb{R}^N} \left(\frac{1}{p} f(u)u - F(u) \right) dx \ge 0. \tag{3.1}$$

From (3.1), we see that the sequence $\{t_{\varepsilon,r}\}$ must be bounded as $\varepsilon \to 0^+$ for each $r > 0$. Indeed, if $t_{\varepsilon,r} \to +\infty$ as $\varepsilon \to 0^+$ for fixed r, then using the condition (f_4), we have

$$I_\varepsilon(t_{\varepsilon,r} v_r) \ge \frac{t^p_{\varepsilon,r}}{p} \|v_r\|^p_{W_\varepsilon} - \gamma_1 t^\mu_{\varepsilon,r} \|v_r\|^\mu_{L^\mu(\mathbb{R}^N)} \to -\infty,$$

which is a contradiction with (3.1). Thus, we can assume that $t_{\varepsilon,r} \to t_r$ as $\varepsilon \to 0^+$. Since v_r has compact support, then we get

$$\begin{aligned} \limsup_{\varepsilon \to 0^+} c_\varepsilon &\le \frac{t^p_r}{p} \int_{\mathbb{R}^{2N}} \frac{|v_r(x)-v_r(y)|^p}{|x-y|^{2N}} \, dx\, dy + \frac{t^p_r}{p} \int_{\mathbb{R}^N} V_0 |v_r|^p \, dx - \int_{\mathbb{R}^N} F(t_r v_r)\, dx \\ &= J_{V_0}(t_r v_r) \end{aligned}$$

via Vitali's theorem or Lebesgue's dominated convergence theorem. Noting that $t_r v_r, w \in \mathcal{N}_{V_0}$ and $v_r \to w$ in $W^{s,N/s}(\mathbb{R}^N)$ as $r \to +\infty$ (see [8, Lemma 2.2]) and using Remark 1 (IV), we can prove that $t_r \to 1$ as $t \to \infty$. Then we get

$$\limsup_{\varepsilon \to 0^+} c_\varepsilon \le \lim_{r \to +\infty} J_{V_0}(t_r v_r) = J_{V_0}(w) = c_{V_0}.$$

By arguments as in Lemma 4, we get $c_{V_0} \le C_{\gamma_1}$. Thus, if we choose γ_1 large enough as in (2.30) for d_* instead of $\mathfrak{d}_*$, and $C_{\gamma_1} < c_{V_\infty}$ if $V_\infty < +\infty$, then for any $(PS)_{c_\varepsilon}$

sequence (u_n) for I_ε, we get

$$\limsup_{n\to\infty} \|u_n\|_{W_\varepsilon}^{N/(N-s)} \le \frac{\beta_* \mathfrak{d}_*^{s/(N-s)}}{2^{N/(N-s)}\mathfrak{c}\alpha_0}.$$

Now, the result of this theorem comes from Lemma 11. □

5. Multiplicity of solutions to (P_ε^*)

In this section, we show the existence of multiple weak solutions and study the behavior of their maximum points related with the set M. The main result of this section is equivalent to Theorem 2 and it is stated as follows:

Theorem 8. *Assume that (f_1)–(f_5) and (V) hold. Then for any $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that (P_ε^*) has at least $\operatorname{cat}_{M_\delta}(M)$ nontrival nonnegative solutions, for any $0 < \varepsilon < \varepsilon_\delta$. Moreover, if u_ε denotes one of these solutions and z_ε is its global maximum, then*

$$\lim_{\varepsilon\to 0^+} V(\varepsilon z_\varepsilon) = V_0.$$

Proof. Let $\delta > 0$ be a fixed and w be a ground state solution of problem (P_{V_0}). It means that $J_{V_0}(w) = c_{V_0}$ and $J'_{V_0}(w) = 0$. Let η be a smooth nonincreasing cut-off function in $[0, +\infty)$ such that $\eta(s) = 1$ if $0 \le s \le \frac{\delta}{2}$ and $\eta(s) = 0$ if $s \ge \delta$. For any $y \in M$, we denote

$$\psi_{\varepsilon,y} = \eta(|\varepsilon x - y|) w\left(\frac{\varepsilon x - y}{\varepsilon}\right)$$

and $\Phi_\varepsilon \colon M \to \mathcal{N}_\varepsilon$ which is defined by $\Phi_\varepsilon(y) = t_\varepsilon \psi_{\varepsilon,y}$, where $t_\varepsilon > 0$ satisfies

$$\max_{t\ge 0} I_\varepsilon(t\psi_{\varepsilon,y}) = I_\varepsilon(t_\varepsilon \psi_{\varepsilon,y}).$$

From the construction, $\Phi_\varepsilon(y)$ has compact support for any $y \in M$.

Lemma 13. *The function Φ_ε satisfies the following limit*

$$\lim_{\varepsilon\to 0^+} I_\varepsilon(\Phi_\varepsilon(y)) = c_{V_0} \quad \text{uniformly in } y \in M.$$

Proof. Suppose that the statement of Lemma 13 does not hold. Then there exists $\delta_0 > 0$, $\{y_n\} \subset M$ and $\varepsilon_n \to 0$ such that

$$|I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - c_{V_0}| \ge \delta_0. \tag{4.1}$$

By Lemma 2.2 [8], we have

$$\lim_{n\to\infty} \|\psi_{\varepsilon_n,y_n}\|_{W_{\varepsilon_n}}^p = \|w\|_{W_{V_0}}^p. \tag{4.2}$$

Since $< I'_{\varepsilon_n}(t_{\varepsilon_n}\psi_{\varepsilon_n,y_n}), t_{\varepsilon_n}\psi_{\varepsilon_n,y_n} >= 0$, using the change of variable $z = \frac{\varepsilon_n x - y_n}{\varepsilon_n}$ we get

$$\begin{aligned} \|t_\varepsilon \psi_{\varepsilon_n,y_n}\|_{W_{\varepsilon_n}}^p &= \int_{\mathbb{R}^N} f(t_\varepsilon \psi_{\varepsilon_n,y_n}) t_{\varepsilon_n}\psi_{\varepsilon_n,y_n}\, dx \\ &= \int_{\mathbb{R}^N} f(t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z)) t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z)\, dz. \end{aligned} \tag{4.3}$$

Now we prove that $t_{\varepsilon_n} \to 1$. First we show that $t_{\varepsilon_n} \to t_0 < +\infty$. Conversely if $t_{\varepsilon_n} \to +\infty$, from (4.3) we have

$$\|\psi_{\varepsilon_n,y_n}\|_{W_{\varepsilon_n}}^p \ge \int_{|z|\le \frac{\delta}{2\varepsilon_n}} \frac{f(t_{\varepsilon_n} w(z)) w(z)}{t_{\varepsilon_n}^{p-1}}\, dz. \tag{4.4}$$

From the condition (f_2) and (f_4), we have $f(t) \geq \gamma_1 \mu |t|^{\mu-1}$ for all $t \geq 0$. Combining that property and (4.4), we deduce

$$\|\psi_{\varepsilon_n,y_n}\|^p_{W_{\varepsilon_n}} \geq \int_{|z|\leq \frac{\delta}{2\varepsilon_n}} \frac{f(t_{\varepsilon_n} w(z))w(z)}{t^{p-1}_{\varepsilon_n}}\, dz \geq \gamma_1 \mu t^{\mu-p}_{\varepsilon_n} \int_{|z|<\frac{\delta}{2\varepsilon_n}} w^\mu \, dx \to +\infty$$

as $n \to \infty$. This is a contradiction with (4.2). Therefore, up to a subsequence, we may assume that $t_{\varepsilon_n} \to t_0 \geq 0$ as $n \to \infty$. If $t_0 = 0$, from $t_{\varepsilon_n}\psi_{\varepsilon_n,y_n} \in \mathcal{N}_{\varepsilon_n}$, by Lemma 2, there exists $r_* > 0$ such that for n large enough, we have

$$\|t_{\varepsilon_n}\psi_{\varepsilon_n,y_n}\|_{W_{\varepsilon_n}} \geq r_* > 0.$$

This is contradiction since $t_{\varepsilon_n} \to 0$ and $\|\psi_{\varepsilon_n,y_n}\|_{W_{\varepsilon_n}} \to \|w\|_{W_{V_0}} > 0$ as $n \to \infty$. Now we prove that $t_0 = 1$. From (4.3), using Lebesgue Dominated Convergence Theorem, we have

$$\|w\|^p_{W_{V_0}} = \int_{\mathbb{R}^N} \frac{f(t_0 w)w}{t_0^{p-1}}\, dx.$$

Note that $w \in \mathcal{N}_{V_0}$, then Remark 1 implies $t_0 = 1$. Still using Lebesgue Dominated Convergence Theorem or Vitali's theorem, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(t_\varepsilon \psi_{\varepsilon_n,y_n}(x))\, dx = \int_{\mathbb{R}^N} F(w)\, dx.$$

Hence, we obtain

$$\begin{aligned}\lim_{n\to\infty} I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) &= \lim_{n\to\infty} \Big[\frac{t^p_{\varepsilon_n}}{p}\|\psi_{\varepsilon_n,y_n}\|^p_{W_{\varepsilon_n}} - \int_{\mathbb{R}^N} F(t_{\varepsilon_n}\psi_{\varepsilon_n,y_n})\, dx\Big] \\ &= \frac{\|w\|^p_{W_{V_0}}}{p} - \int_{\mathbb{R}^N} F(w)\, dx = J_{V_0}(w) = c_{V_0}\end{aligned}$$

which contradicts with (4.1). □

For any $\delta > 0$, let $\rho = \rho(\delta) > 0$ be such that $M_\delta \subset B_\rho(0)$. Let $\chi\colon \mathbb{R}^N \to \mathbb{R}^N$ be define as

$$\chi(x) = \begin{cases} x & \text{if } |x| < \rho, \\ \frac{\rho x}{|x|} & \text{if } |x| \geq \rho. \end{cases}$$

Next, we define the barycenter map $\beta_\varepsilon\colon \mathcal{N}_\varepsilon \to \mathbb{R}^N$ given by

$$\beta_\varepsilon(u) = \frac{\int_{\mathbb{R}^N} \chi(\varepsilon x)|u(x)|^p\, dx}{\int_{\mathbb{R}^N} |u|^p\, dx}.$$

Lemma 14. *The functional Φ_ε satisfies the following limit*

$$\lim_{\varepsilon\to 0^+} \beta_\varepsilon(\Phi_\varepsilon(y)) = y \quad \textit{uniformly in } y \in M. \tag{4.5}$$

Proof. Lemma 14 is proved similarly as [8, Lemma 3.13]. For convenience, we prove it here. Suppose that the statement of Lemma 14 does not hold. Then there exists $\delta_0 > 0$, $\{y_n\} \subset M$ and $\varepsilon_n \to 0$ such that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \geq \delta_0. \tag{4.6}$$

From the definition of $\Phi_{\varepsilon_n}(y_n)$, β_{ε_n}, ψ and using the change of variable $z = \frac{\varepsilon_n x - y_n}{\varepsilon_n}$, we have

$$\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = y_n + \frac{\int_{\mathbb{R}^N}[\chi(\varepsilon_n z + y_n) - y_n]|\eta(|\varepsilon_n z|)w(z)|^p\, dz}{\int_{\mathbb{R}^N} |\eta(|\varepsilon_n z|)w(z)|^p\, dz}.$$

Since $\{y_n\} \subset M \subset B_\rho(0)$, we get

$$\lim_{n\to\infty} |\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| = 0$$

via Lebesgue Dominated Convergence Theorem, which contradicts with (4.6). $\square$

Lemma 15. *Let $\varepsilon_n \to 0^+$ and $\{u_n\} \subset \mathcal{N}_{\varepsilon_n}$ be such that $I_{\varepsilon_n}(u_n) \to c_{V_0}$. Then there exists $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that the translation sequence $v_n(x) = u_n(x+\tilde{y}_n)$ has a subsequence which converges in $W^{s,N/s}(\mathbb{R}^N)$. Moreover, up to a subsequence, $\{y_n\}\colon y_n = \varepsilon\tilde{y}_n \to y \in M$.*

Proof. Since $< I'_{\varepsilon_n}(u_n), u_n >= 0$ and $I_{\varepsilon_n}(u_n) \to c_{V_0}$, then we have

$$\begin{aligned} I_{\varepsilon_n}(u_n) &= I_{\varepsilon_n}(u_n) - \frac{1}{\mu}\langle I'_{\varepsilon_n}(u_n), u_n\rangle \\ &= \left(\frac{s}{N} - \frac{1}{\mu}\right)\|u_n\|^p_{W_{\varepsilon_n}} + \frac{1}{\mu}\int_{\mathbb{R}^N}(f(u_n)u_n - F(u_n))\,dx \geq \left(\frac{s}{N} - \frac{1}{\mu}\right)\|u_n\|^p_{W_{\varepsilon_n}}. \end{aligned}$$

Thus, there exists a constant $C = \left(\frac{c_{V_0}}{\frac{s}{N}-\frac{1}{\mu}}\right)^{s/N}$ such that $\limsup_{n\to\infty}\|u_n\|_{W_{\varepsilon_n}} \leq C$. Since W_{ε_n} is continuously embedded into $W^{s,N/s}(\mathbb{R}^N)$ and (2.39), we get $\{u_n\}$ is a bounded sequence in $W^{s,N/s}(\mathbb{R}^N)$. Now, we show that there exist a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^N$ and constants $r > 0, \beta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_r(\tilde{y}_n)} |u_n|^{N/s}\,dx \geq \beta > 0. \tag{4.7}$$

Indeed, if (4.7) is false, then for any $r > 0$, we have

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_r(y)} |u_n|^{N/s}\,dx = 0.$$

By Lemma 5, we have $u_n \to 0$ strongly in $L^q(\mathbb{R}^N)$ for any $q \in (\frac{N}{s}, +\infty)$. If we take γ_1 large enough, by the method of Lemma 4, we get

$$\limsup_{n\to\infty}\|u_n\|_{V_0}^{N/(N-s)} \leq \frac{\beta_*\mathfrak{d}_*^{s/(N-s)}}{\mathfrak{c}\alpha_0},$$

for a suitable constant $\mathfrak{c} > 1$ and near 1. Applying Lemma 6, we deduce

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} f(u_n)u_n\,dx = 0.$$

Combining this result and $u_n \in \mathcal{N}_{\varepsilon_n}$, we obtain $\|u_n\|_{W_{\varepsilon_n}} \to 0$ as $n \to \infty$. It is a contradiction with Proposition 2. Therefore, (4.7) holds. Let us define $v_n := u_n(x+\tilde{y}_n)$. Since the $\|\cdot\|_{V_0}$ is invariant under translations, then $\{v_n\}$ is a bounded sequence in $W^{s,N/s}(\mathbb{R}^N)$. Thus up to a subsequence, we can assume that there exists $v \in W^{s,N/s}(\mathbb{R}^N)$ such that $v_n \to v$ weakly in $W^{s,N/s}(\mathbb{R}^N)$ and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$ and $v_n \to v$ in $L^q_{\text{loc}}(\mathbb{R}^N)$ for any $q \in [\frac{N}{s}, +\infty)$. From this result and (4.7), we get $v \not\equiv 0$. Let $t_n > 0$ such that $w_n = t_nv_n \in \mathcal{N}_{V_0}$ and we set $y_n := \varepsilon_n\tilde{y}_n$. Thus, using the change of the variable $z = x + \tilde{y}_n$, $V(\varepsilon_n(x+\tilde{y}_n)) \geq V_0$ and the invariance under translations, we can see that

$$\begin{aligned} c_{V_0} \leq J_{V_0}(w_n) &\leq \frac{1}{p}[w_n]^p_{s,p} + \frac{1}{p}\int_{\mathbb{R}^N} V(\varepsilon_nx + y_n)|w_n|^p\,dx - \int_{\mathbb{R}^N} F(w_n)\,dx \\ &= I_{\varepsilon_n}(t_nu_n) \leq I_{\varepsilon_n}(u_n) \leq c_{V_0} + o_n(1). \end{aligned}$$

Then we get $J_{V_0}(w_n) \to c_{V_0}$. Since $\{w_n\} \subset \mathcal{N}_{V_0}$, using the condition (f_2), there exists a constant $K > 0$ such that $\|w_n\|_{V_0} \le K$ for all n. We have $v_n \not\to 0$ strongly in $W^{s,N/s}(\mathbb{R}^N)$. Indeed, if $v_n \to 0$ in $W^{s,N/s}(\mathbb{R}^N)$, then $v_n \to 0$ weakly in $W^{s,N/s}(\mathbb{R}^N)$, which contradicts with $v_n \to v \not\equiv 0$ weakly in $W^{s,N/s}(\mathbb{R}^N)$. Hence, there exists $\alpha > 0$ such that $\|v_n\|_{V_0} \ge \alpha > 0$ for all n. Consequently, we have

$$t_n\alpha \le \|t_n v_n\|_{V_0} = \|w_n\|_{V_0} \le K,$$

which yields $t_n \le \frac{K}{\alpha}$ for all $n \in \mathbb{N}$. Therefore, up to a subsequence, we can assume that $t_n \to t_0 \ge 0$. We prove that $t_0 > 0$. If $t_0 = 0$, then $w_n \to 0$ strongly in $W^{s,N/s}(\mathbb{R}^N)$, which implies that $J_{V_0}(w_n) \to 0$. It contradicts with $c_{V_0} > 0$. Up to a subsequence, we suppose that $w_n \to w := t_0 v \not\equiv 0$ weakly in $W^{s,N/s}(\mathbb{R}^N)$ and $w_n(x) \to w(x)$ a.e. on $\mathbb{R}^N$. By arguments as in Lemma 4, we can get $J'_{V_0}(w) = 0$. Now we prove

$$\lim_{n\to\infty} \|w_n\|^p_{V_0} = \|w\|^p_{V_0}. \tag{4.8}$$

Using Brezis–Lieb's lemma and (4.8), we obtain $w_n \to w$ strongly in $W^{s,N/s}(\mathbb{R}^N)$. By Fatou's lemma, we have

$$\|w\|^p_{V_0} \le \liminf_{n\to\infty} \|w_n\|^p_{V_0}. \tag{4.9}$$

Assume by contradiction that

$$\|w\|^p_{V_0} < \limsup_{n\to\infty} \|w_n\|^p_{V_0}.$$

Note that

$$\begin{aligned} c_{V_0} + o_n(1) &= J_{V_0}(w_n) - \frac{1}{\mu}\langle J'_{V_0}(w_n), w_n\rangle \\ &= \left(\frac{1}{p} - \frac{1}{\mu}\right)\|w_n\|^p_{V_0} + \int_{\mathbb{R}^N}\left[\frac{1}{\mu}f(w_n)w_n - F(w_n)\right]dx. \end{aligned}$$

Using the condition (f_2), and Fatou's lemma, we get

$$\begin{aligned} c_{V_0} &\ge \left(\frac{1}{p} - \frac{1}{\mu}\right)\limsup_{n\to\infty}\|w_n\|^p_{V_0} + \liminf_{n\to\infty}\int_{\mathbb{R}^N}\left[\frac{1}{\mu}f(w_n)w_n - F(w_n)\right]dx \\ &> \left(\frac{1}{p} - \frac{1}{\mu}\right)\|w\|^p_{V_0} + \int_{\mathbb{R}^N}\left[\frac{1}{\mu}f(w)w - F(w)\right]dx \\ &= J_{V_0}(w) - \frac{1}{\mu}\langle J'_{V_0}(w), w\rangle = J_{V_0}(w) \ge c_{V_0}, \end{aligned}$$

which is a contradiction. Then

$$\|w\|^p_{V_0} \ge \limsup_{n\to\infty}\|w_n\|^p_{V_0}. \tag{4.10}$$

Combining (4.9) and (4.10), we get (4.8). Since $t_n \to t_0$ as $n \to \infty$, then $v_n \to v$ in $W^{s,N/s}$ as $n \to \infty$. Now we prove that $\{y_n\}$ has a subsequence such that $y_n \to y \in M$. Indeed, if $\{y_n\}$ is not bounded, there exists a subsequence, still denoted by $\{y_n\}$, such that $|y_n| \to +\infty$. First, we consider the case $V_\infty = \infty$. Using the fact that $\{u_n\} \subset \mathcal{N}_{\varepsilon_n}$ and a change of variable $z = x + \tilde{y}_n$, we can see that

$$\begin{aligned} \int_{|x|\le R\varepsilon_n^{-1}} V(\varepsilon_n x + y_n)|v_n|^p\,dx &\le [v_n]^p_{s,p} + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)|v_n|^p\,dx \\ &= \|u_n\|^p_{W_{\varepsilon_n}} = \int_{\mathbb{R}^N} f(u_n)u_n\,dx = \int_{\mathbb{R}^N} f(v_n)v_n\,dx. \end{aligned}$$

Apply Fatou's lemma and Lebesgue Dominated Convergence Theorem or Vitali's theorem and $v_n \to v$ in $W^{s,N/s}(\mathbb{R}^N)$, we deduce that

$$\begin{aligned}+\infty &= \liminf_{n\to\infty} \int_{|x|\le R\varepsilon_n^{-1}} V(\varepsilon_n x + y_n)|v_n|^p\,dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)|v_n|^p\,dx \\ &\le \lim_{n\to\infty} \int_{\mathbb{R}^N} f(v_n)v_n\,dx = \int_{\mathbb{R}^N} f(v)v\,dx < +\infty,\end{aligned}$$

which gives a contradiction. Next, we consider the case $V_\infty < +\infty$. From the fact that $w_n \to w$ strongly in $W^{s,N/s}(\mathbb{R}^N)$ and the condition (V), using the change of variable $z = x + \tilde{y}_n$, we have

$$\begin{aligned}c_{V_0} &= J_{V_0}(w) < J_{V_\infty}(w) \\ &\le \liminf_{n\to\infty} \left[\frac{1}{p}\left([w_n]_{s,p}^p + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)|w_n|^p\,dx\right) - \int_{\mathbb{R}^N} F(w_n)\,dx\right] \\ &= \liminf_{n\to\infty} \left[\frac{t_n^p}{p}[u_n]_{s,p}^p + \frac{t_n^p}{p}\int_{\mathbb{R}^N} V(\varepsilon_n z)|u_n|^p\,dz - \int_{\mathbb{R}^N} F(t_n u_n)\,dz\right] \\ &= \liminf_{n\to\infty} I_{\varepsilon_n}(t_n u_n) \le \liminf_{n\to\infty} I_{\varepsilon_n}(u_n) = c_{V_0},\end{aligned} \tag{4.11}$$

which is absurd. Then $\{y_n\}$ must be a bounded sequence. Up to a subsequence, we can assume that $y_n \to y$. If $y \notin M$, then $V_0 < V(y)$. By an argument as in (4.11), we get a contradiction. Hence $y \in M$. □

Let $\mathbb{R}^+ \to \mathbb{R}^+$ be a positive function such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ and let

$$\tilde{\mathcal{N}}_\varepsilon = \{u \in \mathcal{N}_\varepsilon \colon I_\varepsilon(u) \le c_{V_0} + h(\varepsilon)\}.$$

By Lemma 14, we have $h(\varepsilon) = |I_\varepsilon(\Phi_\varepsilon(y)) - c_{V_0}| \to 0$ as $\varepsilon \to 0^+$. Hence $\Phi_\varepsilon(y) \in \mathcal{N}_\varepsilon$ and $\tilde{\mathcal{N}}_\varepsilon \ne \emptyset$ for any $\varepsilon > 0$. Moreover, we have the following result:

Lemma 16. *For any $\delta > 0$, it holds that*

$$\lim_{\varepsilon\to 0^+} \sup_{u\in\tilde{\mathcal{N}}_\varepsilon} \operatorname{dist}(\beta_\varepsilon(u), M_\delta) = 0.$$

Proof. Lemma 16 is proved similarly as [8, Lemma 3.14]. For convenience, we prove it here. Let $\varepsilon_n \to 0$ as $n \to \infty$. By the definition of supremum, there exists $\{u_n\} \subset \tilde{\mathcal{N}}_{\varepsilon_n}$ such that

$$\sup_{u\in\tilde{\mathcal{N}}_{\varepsilon_n}} \inf_{y\in M_\delta} |\beta_{\varepsilon_n}(u) - y| = \inf_{y\in M_\delta} |\beta_{\varepsilon_n}(u_n) - y| + o_n(1).$$

Therefore, it suffices that there exists $\{y_n\} \subset M_\delta$ such that

$$\lim_{n\to\infty} |\beta_{\varepsilon_n}(u_n) - y_n| = 0. \tag{4.12}$$

Noting that $\{u_n\} \subset \tilde{\mathcal{N}}_{\varepsilon_n} \subset \mathcal{N}_{\varepsilon_n}$, we deduce that

$$c_{V_0} \le c_{\varepsilon_n} \le I_{\varepsilon_n}(u_n) \le c_{V_0} + h(\varepsilon_n),$$

which leads to $I_{\varepsilon_n}(u_n) \to c_{V_0}$. By Lemma 15, there exists a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that $y_n = \varepsilon_n \tilde{y}_n \in M_\delta$ for all n large enough. We have

$$\beta_{\varepsilon_n}(u_n) = y_n + \frac{\int_{\mathbb{R}^N} [\chi(\varepsilon_n z + y_n) - y_n]|u_n|^p(z + \tilde{y}_n)\,dz}{\int_{\mathbb{R}^N} |u_n|^p(z + \tilde{y}_n)\,dz}.$$

Since $u_n(x + \tilde{y}_n)$ converges strongly in $W^{s,N/s}(\mathbb{R}^N)$ and $\varepsilon_n z + y_n \to y \in M$, we can get $\beta_{\varepsilon_n}(u_n) = y_n + o_n(1)$ via Lebesgue Dominated Convergence Theorem. Therefore (4.12) holds. □

Lemma 17. *Assume that (V) and (f_1)–(f_5) hold and let v_n be a nontrivial nonnegative solution of the following problem*

$$(-\Delta)^s_{N/s} v_n + V_n(x)|v_n|^{\frac{N}{s}-2} v_n = f(v_n) \quad \text{in } \mathbb{R}^N, \tag{4.13}$$

where $V_n(x) = V(\varepsilon_n x + \varepsilon_n \tilde{y}_n)$ and $\varepsilon_n \tilde{y}_n \to y \in M$. If $\{v_n\}$ is a bounded sequence in $W^{s,N/s}(\mathbb{R}^N)$ verifying

$$\limsup_{n\to\infty} \|v_n\|_{V_0}^{N/(N-s)} \leq \frac{\beta_* \mathfrak{d}_*{}^{s/(N-s)}}{\mathfrak{c}\alpha_0},$$

where $\mathfrak{c} > 1$ is a suitable constant and $v_n \to v$ strongly in $W^{s,N/s}(\mathbb{R}^N)$, then $v_n \in L^\infty(\mathbb{R}^N)$ and there exists $C > 0$ such that $\|v_n\|_{L^\infty(\mathbb{R}^N)} \leq C$ for all $n \in \mathbb{N}$. Furthermore

$$\lim_{|x|\to+\infty} v_n(x) = 0 \quad \text{uniformly in } n.$$

Proof. For any $L > 0$ and $\beta > 1$, let us to consider the function $\gamma(t) = t(\min\{t, L\})^{p(\beta-1)}$ and

$$\gamma(v_n) = \gamma_{L,\beta}(v_n) = v_n v_{L,n}^{p(\beta-1)} \in W_\varepsilon, v_{L,n} = \min\{v_n, L\}.$$

Set

$$\Lambda(t) = \frac{|t|^p}{p} \quad \text{and} \quad \Gamma(t) = \int_0^t (\gamma'(t))^{\frac{1}{p}}\, d\tau.$$

Then we have [8]

$$\Lambda'(a-b)(\gamma(a)-\gamma(b)) \geq |\Gamma(a)-\Gamma(b)|^p \quad \text{for any } a, b \in \mathbb{R}. \tag{4.14}$$

From (4.14), we get

$$\begin{aligned} &|\Gamma(v_n(x)) - \Gamma(v_n(y))|^p \\ &\leq |v_n(x)-v_n(y)|^{p-2}(v_n(x)-v_n(y))((v_n v_{L,n}^{p(\beta-1)})(x) - (v_n v_{L,n}^{p(\beta-1)})(y)). \end{aligned} \tag{4.15}$$

Therefore, taking $\gamma(v_n) = v_n v_{L,n}^{p(\beta-1)}$ as a test function in (4.13) and combining with (4.15), we have

$$\begin{aligned} &[\Gamma(v_n)]^p_{s,p} + \int_{\mathbb{R}^N} V_n(x)|v_n|^p v_{L,n}^{p(\beta-1)}\, dx \\ &\leq \int_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^{p-2}(v_n(x)-v_n(y))((v_n v_{L,n}^{p(\beta-1)})(x)-(v_n v_{L,n}^{p(\beta-1)})(y))}{|x-y|^{2N}}\, dx\, dy \\ &\quad + \int_{\mathbb{R}^N} V_n(x)|v_n|^p v_{L,n}^{p(\beta-1)}\, dx = \int_{\mathbb{R}^N} f(v_n) v_n v_{L,n}^{p(\beta-1)}\, dx. \end{aligned} \tag{4.16}$$

Using (4.14), we have $v_n v_{L,n}^{\beta-1} \geq |\Gamma(v_n)|$. Since $\Gamma(v_n) \geq \frac{1}{\beta} v_n v_{L,n}^{\beta-1}$ and the embedding from $W^{s,N/s}(\mathbb{R}^N) \to L^{N^*}(\mathbb{R}^N)$ $(N^* > \frac{N}{s})$ is continuous, then there exists a suitable constant $S_* > 0$ such that

$$\|\Gamma(v_n)\|^p_{V_0/2} \geq S_* \|\Gamma(v_n)\|^p_{L^{N^*}(\mathbb{R}^N)} \geq \frac{1}{\beta^p} S_* \|v_n v_{L,n}^{\beta-1}\|^p_{L^{N^*}(\mathbb{R}^N)}. \tag{4.17}$$

From the condition (f_1) and (f_3), for any $\varepsilon > 0$, there exist $C(\varepsilon) > 0$ such that

$$|f(t)| \leq \varepsilon |t|^{p-1} + C(\varepsilon)|t|^{p-1} \Phi_{N,s}(\alpha_0 |t|^{N/(N-s)})$$

for all $t \in \mathbb{R}$. Then we obtain

$$\begin{aligned}&\frac{1}{\beta^p} S_* \|v_n v_{L,n}^{\beta-1}\|^p_{L^{N^*}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} V_n(x)|v_n|^p v_{L,n}^{p(\beta-1)}\,dx \\ &\le \varepsilon \int_{\mathbb{R}^N} |v_n v_{L,n}^{\beta-1}|^p\,dx + C(\varepsilon)\int_{\mathbb{R}^N} \Phi_{N,s}(\alpha_0|v_n|^{N/(N-s)})|v_n v_{L,n}^{\beta-1}|^p\,dx. \end{aligned} \tag{4.18}$$

Choose $0 < \varepsilon < V_0/2$, then (4.18) implies

$$\begin{aligned}&\frac{1}{\beta^p} S_* \|v_n v_{L,n}^{\beta-1}\|^p_{L^{N^*}(\mathbb{R}^N)} \\ &\le C(\varepsilon)\left(\int_{\mathbb{R}^N} (\Phi_{N,s}(\alpha_0|v_n|^{N/(N-s)}))^{q'}\,dx\right)^{\frac{1}{q'}} \left(\int_{\mathbb{R}^N} |v_n v_{L,n}^{\beta-1}|^{qp}\,dx\right)^{\frac{1}{q}}.\end{aligned}$$

Using the Trudinger–Moser inequality in $W^{s,N/s}(\mathbb{R}^N)$ with $q >> \frac{N}{s}$ such that $N^{**} = qp < N^*$, $q' > 1$ and q' near 1, then there exists a constant $D > 0$ such that

$$\|v_n v_{L,n}^{\beta-1}\|^p_{L^{N^*}(\mathbb{R}^N)} \le D\beta^p \|v_n v_{L,n}^{\beta-1}\|^p_{L^{qp}(\mathbb{R}^N)}.$$

Letting $L \to +\infty$ in the above inequality, we deduce

$$\|v_n\|_{L^{N^*\beta}} \le D^{\frac{1}{p\beta}} \beta^{\frac{1}{\beta}} \|v_n\|_{L^{N^{**}\beta}(\mathbb{R}^N)}. \tag{4.19}$$

Now, we set $\beta = \frac{N^*}{N^{**}} > 1$. Then $\beta^2 N^{**} = \beta N^*$ and (4.19) holds with β replaced by β^2. Therefore, we obtain

$$\begin{aligned}\|v_n\|_{L^{N^*\beta^2}} &\le D^{\frac{1}{p\beta^2}} \beta^{\frac{2}{\beta^2}} \|v_n\|_{L^{N^{**}\beta^2}(\mathbb{R}^N)} = D^{\frac{1}{p\beta^2}} \beta^{\frac{2}{\beta^2}} \|v_n\|_{L^{N^*\beta}(\mathbb{R}^N)} \\ &\le D^{\frac{1}{p}(\frac{1}{\beta}+\frac{1}{\beta^2})} \beta^{\frac{1}{\beta}+\frac{2}{\beta^2}} \|v_n\|_{L^{N^{**}\beta}(\mathbb{R}^N)}.\end{aligned} \tag{4.20}$$

Iterating this process as in (4.20), we can infer that for any positive integer m,

$$\|v_n\|_{L^{N^*\beta^m}} \le D^{\sum_{j=1}^m \frac{1}{p\beta^j}} \beta^{\sum_{j=1}^m j\beta^{-j}} \|v_n\|_{L^{N^{**}\beta}(\mathbb{R}^N)}. \tag{4.21}$$

Taking the limit in (4.21) as $m \to \infty$, we get

$$\|v_n\|_{L^\infty(\mathbb{R}^N)} \le C$$

for all n, where $C = D^{\sum_{j=1}^\infty \frac{1}{p\beta^j}} \beta^{\sum_{j=1}^\infty j\beta^{-j}} \sup_{n\in\mathbb{N}} \|v_n\|_{L^{N^{**}\beta}(\mathbb{R}^N)} < +\infty$. □

Now, we give the proof of Theorem 8. We fix $\varepsilon > 0$ small enough. Then by Lemma 13 and Lemma 16, we have that $\beta_\varepsilon \circ \Phi_\varepsilon$ is homotopic to the inclusion map $\mathrm{id}\colon M \to M_\delta$. Then we get

$$\mathrm{cat}_{\tilde{\mathcal{N}}_\varepsilon}(\tilde{\mathcal{N}}_\varepsilon) \ge \mathrm{cat}_{M_\delta}(M).$$

Since the functional I_ε satisfies the $(PS)_c$ condition for $c \in (c_{V_0}, c_{V_0} + h(\varepsilon))$, then by Lusternik–Schnirelmann theory of critical points, see Willem [41], I_ε has at least $\mathrm{cat}_{M_\delta}(M)$ critical points on $\mathcal{N}_\varepsilon$. By Corollary 1, I_ε has at least $\mathrm{cat}_{M_\delta}(M)$ critical points in W_ε.

Let u_{ε_n} is a solution of problem (P_{ε_n}). Then $v_n(x) = u_{\varepsilon_n}(x + \tilde{y}_n)$ is a solution of the equation (4.13). Moreover, up to a subsequence, we may assume that $v_n \to v$ strongly in $W^{s,N/s}(\mathbb{R}^N)$ and $y_n = \varepsilon_n \tilde{y}_n \to y \in M$. Next, we prove that there exists $\delta > 0$ such that $\|v_n\|_{L^\infty(\mathbb{R}^N)} \ge \delta$ for all n large enough. Indeed, by Lemma 15 (see (4.7)), we have

$$0 < \frac{\beta}{2} \le \int_{B_r(0)} |v_n|^{N/s}\,dx \le |B_r(0)| \|v_n\|_{L^\infty(\mathbb{R}^N)} \tag{4.22}$$

for all n large enough. Here, we choose $\delta = \frac{\beta}{2}$. Since $v_n \to v$ strongly in $W^{s,N/s}(\mathbb{R}^N)$, then we have $\lim_{|x|\to\infty} v_n(x) = 0$ uniformly in $n \in \mathbb{N}$. We denote by p_n the global maximum of v_n. Then by Lemma 17 and (4.22), there exists $R > 0$ such that $|p_n| \leq R$ for all $n \in \mathbb{N}$. Therefore, the maximum point of u_{ε_n} is given by $z_{\varepsilon_n} = p_n + \tilde{y}_n$ and $\varepsilon_n z_{\varepsilon_n} \to y \in M$. By the continuity of V, we get $V(\varepsilon_n z_{\varepsilon_n}) \to V(y) = V_0$ as $n \to \infty$.

If u_ε is a nontrivial nonnegative solution of problem (P_ε^*), then $w_\varepsilon(x) = u_\varepsilon(x/\varepsilon)$ is a nontrivial nonnegative solution of (P_ε). Thus the maximum points η_ε and z_ε of w_ε and u_ε respectively, satisfy $\eta_\varepsilon = \varepsilon z_\varepsilon$. We deduce

$$\lim_{\varepsilon\to 0^+} V(\eta_\varepsilon) = \lim_{n\to\infty} V(\varepsilon_n z_{\varepsilon_n}) = V_0. \qquad \square$$

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Nguyen Van Thin
Thai Nguyen University of Education, Department of Mathematics
Luong Ngoc Quyen Street, Thai Nguyen City, Thai Nguyen, Viet Nam
and Thang Long University, Thang Long Institute of Mathematics and Applied Sciences
Nghiem Xuan Yem, Hoang Mai, Hanoi, Viet Nam
thinmath@gmail.com, thinnv@tnue.edu.vn