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Boundary points of angular type form a set of zero harmonic measure

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Abstract. This note addresses a problem of Dvoretzky concerning the harmonic measure of the set of boundary points of a domain in Euclidean space that are of angular type.

Randpunkter av vinkeltyp utgör en mängd med harmoniskt mått noll

Sammanfattning. Denna korta artikel behandlar ett problem av Dvoretzky angående det harmoniska måttet hos mängden av randpunkter till ett område i ett Euklidiskt rum som är av vinkeltyp.

1. Introduction

Let D be a domain in the complex plane $\mathbb{C}$. A boundary point z_0 of D is called *angular* if there exists $\varepsilon > 0$ such that each component of the open set $\{z \in D\colon |z - z_0| < \varepsilon\}$ which has z_0 as a boundary point is contained in an angle of vertex z_0 and aperture less than π. The set $A(D)$ of all angular boundary points of D may have positive 1-dimensional measure, as can be seen if we form a domain D_0 by deleting from the upper half-plane a "comb" of vertical unit line segments standing on a closed nowhere dense subset E of the real axis that has positive length. Indeed, we can replicate this construction at lower scales, as follows. For each interval (a, b) of $\mathbb{R}\backslash E$ we can remove from D_0 a comb of horizontal line segments L of length $(b-a)/3$ with left endpoints on the line segment $(a, a+i)$, then remove another comb of vertical line segments ending at one of the line segments L, and so on.

Nevertheless, Dvoretzky remarked that a non-trivial argument based on Brownian paths may be used to show that $A(D)$ always has zero harmonic measure for D. In Problem 7.15 of [5], published originally in 1974 and restated recently in [6], he asked for a more direct proof of this result, and whether:

(i) such a result holds when angles less than π are replaced by more general approach regions;
(ii) a corresponding result holds in higher dimensions.

We provide below a self-contained argument that establishes the result for angular boundary points of plane domains and that, in combination with results of Dahlberg [4] and Burdzy and Williams [2], readily yields an extension to more general approach regions in all dimensions.

A typical point x of $\mathbb{R}^N$ $(N \geq 2)$ will be denoted by (x', x_N), where $x' \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$. The open ball in $\mathbb{R}^N$ of centre x and radius r will be denoted by $B_N(x, r)$, and we write $B_N = B_N(0, 1)$. For each $\kappa > 0$ let Φ_κ denote the collection

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of all functions $\phi\colon \overline{B}_{N-1} \to \mathbb{R}$ such that $\phi(0') = 0$,

$$|\phi(x') - \phi(y')| \le \kappa \|x' - y'\| \quad \text{for all } x', y' \in \overline{B}_{N-1}$$

and

$$\int_{B_{N-1}} \frac{\phi^-(x')}{\|x'\|^N}\, dt < \infty, \quad \int_{B_{N-1}} \frac{\phi^+(x')}{\|x'\|^N}\, dt = \infty, \tag{1}$$

where $\phi^+ = \max\{\phi, 0\}$ and $\phi^- = \max\{-\phi, 0\}$. Further, let

$$V_\phi = \{(x', x_N) \in B_N \colon x_N > \phi(x')\} \quad (\phi \in \Phi_\kappa, \kappa > 0).$$

Let Ω be a domain in $\mathbb{R}^N$. We say that a boundary point x_0 of Ω is of *angular type* if there exist $\kappa > 0$ and $\varepsilon \in (0, 1]$ such that each component of the open set $\Omega \cap B_N(x_0, \varepsilon)$ which has x_0 as a boundary point is contained in $T(V_\phi)$ for some $\phi \in \Phi_\kappa$ and some isometry T that maps 0 to x_0. The set of all boundary points of Ω that are of angular type is denoted by $AT(\Omega)$. When $N = 2$ any angular boundary point is clearly of angular type. In the result below we assume that Ω has non-polar complement when $N = 2$ in order for harmonic measure to exist.

Theorem 1. *If $\Omega \subset \mathbb{R}^N$ is a domain, then the set $AT(\Omega)$ has zero harmonic measure for Ω.*

2. Proof of Theorem 1

The significance of the collection Φ_κ for this problem lies in the following result. It was proved originally by Burdzy and Williams [2] using probabilistic methods. Carroll [3] subsequently provided an alternative analytic argument.

Theorem A. *Let $\phi \in \Phi_\kappa$ for some $\kappa > 0$ and let $\varepsilon \in (0, 1]$. If h is a positive harmonic function on $V_\phi \cap B_N(0, \varepsilon)$ that continuously vanishes on $\partial V_\phi \cap B_N(0, \varepsilon)$, then $h(0', \delta)/\delta \to 0$ as $\delta \to 0+$.*

We will also need the following straightforward lemma about harmonic measure.

Lemma 2. *Suppose that Ω is a domain, W is an open set, and L is compactly contained in $\partial\Omega \cap W$. Then L has zero harmonic measure for Ω if it has zero harmonic measure for each component U of $\Omega \cap W$.*

Proof of Lemma. Let H_g^ω denote the (PWB) solution to the Dirichlet problem on an open set ω with Borel boundary data g. If we define $f = H_{\chi_L}^\Omega$ on $\Omega \cap \partial W$ and $f = \chi_L$ on $\partial\Omega \cap \overline{W}$, then $H_f^{\Omega \cap W} = H_{\chi_L}^\Omega$ on $\Omega \cap W$ by Theorem 6.3.6 of [1]. Since $H_f^{\Omega \cap W} = H_f^U$ in any component U of $\Omega \cap W$, it follows from our hypothesis that $H_f^{\Omega \cap W} = H_{f\chi_\Omega}^{\Omega \cap W}$, and so $H_{\chi_L}^\Omega$ continuously vanishes at each regular boundary point of Ω. Hence $H_{\chi_L}^\Omega \equiv 0$, as required. □

Proof of Theorem 1. For each $m \in \mathbb{N}$ we define the set

$$A_m = \{x_0 \in \partial\Omega \colon \text{each component of } \Omega \cap B_N(x_0, m^{-1}) \text{ that has } x_0 \text{ as a boundary point is contained in } T(V_\phi) \text{ for some } \phi \text{ in } \Phi_m \text{ and some isometry } T \text{ that maps 0 to } x_0\}.$$

The sequence (A_m) is increasing and

$$\textstyle\bigcup_m A_m = AT(\Omega). \tag{2}$$

We now fix $m \in \mathbb{N}$, choose $y_0 \in \partial\Omega$, and then choose a component U of $\Omega \cap B_N(y_0,(2m)^{-1})$ such that $\overline{U} \cap \overline{B}_N(y_0,(4m)^{-1}) \neq \emptyset$, if such a component exists. Let

$$E = A_m \cap \partial U \cap \overline{B}_N(y_0,(4m)^{-1}). \tag{3}$$

Claim. *The set E has zero harmonic measure for U.*

Proof of Claim. We need only consider the case where E is non-empty. For each $y \in E$ we have

$$U \subset \Omega \cap B_N(y_0,(2m)^{-1}) \subset \Omega \cap B_N(y,m^{-1}),$$

so there is a component U_y of $\Omega \cap B_N(y,m^{-1})$ that contains U. Since $y \in A_m \cap \partial U_y$, we can choose $\phi_y \in \Phi_m$ and an isometry T_y such that $T_y(0) = y$ and $U_y \subset T_y(V_{\phi_y})$. It follows that

$$U \subset \bigcap_{y\in E} T_y(V_{\phi_y}).$$

Now let K denote the cap of centre $(0',1)$ and half-angle $(\tan^{-1} m^{-1})/2$ in the sphere ∂B_N. We define

$$E_K = \{y \in E \colon T_y(0',1) - y \in K\}$$

and

$$V = B_N(y_0,(2m)^{-1}) \cap \left(\bigcap\nolimits_{y\in E_K} T_y(V_{\phi_y})\right).$$

Clearly $U \subset V$. Since cones of half-angle $\tan^{-1} m^{-1}$ and vertical axis lie above and below each point on the graph of any function ϕ in Φ_m, cones of half-angle $(\tan^{-1} m^{-1})/2$ and vertical axis must sit above and below each point of the set

$$B_N(y_0,(2m)^{-1}) \cap T_y(\partial V_{\phi_y}) \quad (y \in E_K).$$

Thus each of the above sets is contained in the graph of a Lipschitz function defined on $B_{N-1}(y_0',(2m)^{-1})$ with Lipschitz constant bounded by $\cot((\tan^{-1} m^{-1})/2)$, and the same must therefore also be true of $\partial V \cap B_N(y_0,(2m)^{-1})$. Also, $E_K \subset \partial U \cap \partial V$.

Let $z_0 \in U$ and $G_V(z_0,\cdot)$ denote the Green function for V with pole at z_0. It follows from Theorem A and Harnack's inequalities that

$$\frac{G_V(z_0,(x',x_N+\delta))}{\delta} \to 0 \quad (\delta \to 0+, (x',x_N) \in E_K).$$

Thus, at any point of E_K where the normal derivative of $G_V(z_0,\cdot)$ with respect to ∂V exists, this derivative must be 0. It follows from Theorem 3 of Dahlberg [4] that E_K has zero harmonic measure for V. Since $U \subset V$, the set E_K also has zero harmonic measure for U. Using a change of axes we can apply the same reasoning with K replaced by any other spherical cap J of half-angle $(\tan^{-1} m^{-1})/2$. Since we can write E as a finite union of sets of the form E_J, and we know that each of these sets has zero harmonic measure for U, the claim is now proved. □

We have established the above claim for any component U of the open set $\Omega \cap B_N(y_0,(2m)^{-1})$ which satisfies $\overline{U} \cap \overline{B}_N(y_0,(4m)^{-1}) \neq \emptyset$. On the other hand, the set E in (3) trivially has zero harmonic measure for U when U is a component of $\Omega \cap B_N(y_0,(2m)^{-1})$ such that $\overline{U} \cap \overline{B}_N(y_0,(4m)^{-1}) = \emptyset$. Hence, by applying Lemma 2 with $W = B_N(y_0,(2m)^{-1})$, we see that the set $A_m \cap B_N(y_0,(4m)^{-1})$ has zero harmonic measure for Ω. Since y_0 was an arbitrary point of $\partial\Omega$, it follows that A_m has zero harmonic measure for Ω. Finally, the result follows from (2) on considering arbitrarily large values of m. □

Remark. If $N = 2$ and we merely wish to show that the set of angular points $A(\Omega)$ has zero harmonic measure for Ω, then we do not need to appeal to Theorem A or the above result of Dahlberg. For, in this case, we only consider functions $\phi\colon [-1,1] \to \mathbb{R}$ of the form $\phi(t) = \alpha\,|t|$ $(\alpha > 0)$. Using the notation in the proof above, the set $\partial V \cap B_N(y_0,(2m)^{-1})$ is then the graph of a convex function ψ, the one-sided derivatives of which satisfy $\psi'_- < \psi'_+$ at each point of E_K. It follows that E_K is countable, and so certainly has zero harmonic measure (for any domain).

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