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Variational framework and Lewy–Stampacchia type estimates for nonlocal operators on Heisenberg group

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Abstract. The aim of this article is to derive some Lewy–Stampacchia estimates and the existence of solutions for equations driven by a nonlocal integro-differential operator on the Heisenberg group.

Heisenbergin ryhmän paikallistamattomien operaattoreiden variaatiomuotoilu ja Lewyn–Stampacchian tyyppiset arviot

Tiivistelmä. Työn tarkoitus on johtaa Heisenbergin ryhmässä eräitä Lewyn–Stampacchian arvioita sekä paikallistamattoman integraali–differentiaalioperaattorin ohjaamien yhtälöiden ratkaisujen olemassaolo.

1. Introduction

The Heisenberg group $\mathbb{H}^N = \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$, $N \in \mathbb{N}$ is a Lie group, endowed with the following group law

$$(x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' + 2(\langle x', y\rangle - \langle x, y'\rangle))$$

where $x, y, x', y' \in \mathbb{R}^N$. The corresponding Lie algebra of left invariant vector fields is generated by the following vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}.$$

It is straightforward to check that for all $j, k = 1, 2, \cdots, N$,

$$[X_j, X_k] = [Y_j, Y_k] = \left[X_j, \frac{\partial}{\partial t}\right] = \left[Y_j, \frac{\partial}{\partial t}\right] = 0, \quad \text{and} \quad [X_j, Y_k] = -4\delta_{jk} \frac{\partial}{\partial t}. \tag{1.1}$$

These relations (1.1) establish the Heisenberg's canonical commutation relations of quantum mechanics for position and momentum, hence the name Heisenberg group [8].

We define the left translations on $\mathbb{H}^N$ by

$$\tau_\xi \colon \mathbb{H}^N \to \mathbb{H}^N, \quad \tau_\xi(\xi') = \xi \cdot \xi',$$

and the natural $\mathbb{H}$-dilations $\delta_\theta \colon \mathbb{H}^N \to \mathbb{H}^N$ by

$$\delta_\theta(x, y, t) = (\theta x, \theta y, \theta^2 t)$$

for $\theta > 0$. The Jacobian determinant of δ_θ is θ^Q. The number $Q = 2N + 2$ is called the homogeneous dimension of $\mathbb{H}^N$ and it portrays a role equivalent to the topological

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dimension in the Euclidean space. We denote the homogeneous norm on $\mathbb{H}^N$ by

$$|\xi| = |(x,y,t)| = (t^2 + (x^2+y^2)^2)^{1/4}, \quad \text{for all } \xi = (x,y,t) \in \mathbb{H}^N.$$

We shall denote $B_r(\xi)$, the ball of center ξ and radius r. It implies $\tau_\xi(B_r(0)) = B_r(\xi)$ and $\delta_r(B_1(0)) = B_r(0)$.

Recently, many researchers explored the nonlocal operators on Heisenberg group. In [6], Frank, González, Monticelli, and Tan showed that the conformally invariant fractional powers of the sub-Laplacian on the Heisenberg group are given in terms of the scattering operator for an extension problem to the Siegel upper halfspace. Remarkably, this extension problem is different from the one studied, among others, by Caffarelli and Silvestre [4]. In [7], Guidi, Maalaoui, and Martino studied the Palais–Smale sequence of the conformally invariant fractional powers of the sub-Laplacian and proved the existence of solutions. In [9], Liu, Wang, and Xiao discussed the nonnegative solutions of a fractional sub-Laplacian differential inequality on Heisenberg group. In [5], Cinti and Tan established a Liouville-type theorem for a subcritical nonlinear problem, involving a fractional power of the sub-Laplacian in the Heisenberg group. To prove their result authors used the local realization of fractional CR covariant operators, which can be constructed as the Dirichlet-to-Neumann operator of a degenerate elliptic equation as established in [6]. Nonlocal equations with Convolution type nonlinearities had been discussed by Goel and Sreenadh [3]. Their authors established the Brezis–Nirenberg type result for the critical problem. But there is no article which deals a general integro-differential operator over $\mathbb{H}^N$. In this article we consider the following integro-differential operator

$$\mathfrak{L}_{\mathcal{K}} u(\xi) = \frac{1}{2}\int_{\mathbb{H}^N} (u(\xi\eta) + u(\xi\eta^{-1}) - 2u(\xi))\mathcal{K}(\eta)\, d\eta$$

where $\mathcal{K}\colon \mathbb{H}^N \setminus \{0\} \to (0,+\infty)$ be a function with following properties

$$\theta\mathcal{K} \in L^1(\mathbb{H}^N) \quad \text{where} \quad \theta(\xi) = \min\{1, |\xi|^2\}; \tag{1.2}$$

there exists $\mu > 0$ such that for all $\xi \in \mathbb{H}^N \setminus \{0\}$, $\mathcal{K}(\xi) \ge \mu|\xi|^{-(Q+2s)}$, $Q > 2s$;

$$\mathcal{K}(\xi) = \mathcal{K}(\xi^{-1}) \quad \text{for all} \ \ \xi \in \mathbb{H}^N \setminus \{0\}.$$

Employing (1.2), one can easily prove that

$$\mathfrak{L}_{\mathcal{K}} u(\xi) = \int_{\mathbb{H}^N} (u(\eta) - u(\xi))\mathcal{K}(\eta^{-1}\xi)\, d\eta. \tag{1.3}$$

In [11], Roncal and Thangavelu proved the (1.3) with $\mathcal{K} = |\xi|^{-(Q+2s)}$ is the integral representation of fractional sub-Laplacian on the Heisenberg group.

In case of $\mathbb{R}^N$, $\mathfrak{L}_{\mathcal{K}}$ is defined as

$$\mathfrak{L}_{\{\mathcal{K},\mathbb{R}^N\}} u(x) = \frac{1}{2}\int_{\mathbb{R}^N} (u(x+y) + u(x-y) - 2u(x))\mathcal{K}(y)\, dy$$

where $\mathcal{K}\colon \mathbb{R}^N \setminus \{0\} \to (0,+\infty)$ be a function with following properties

$$\theta\mathcal{K} \in L^1(\mathbb{R}^N) \quad \text{where} \quad \theta(x) = \min\{1, |x|^2\},$$

there exists $\mu > 0$ such that for all $x \in \mathbb{R}^N \setminus \{0\}$, $\mathcal{K}(x) \ge \mu|x|^{-(N+2s)}$

$$\mathcal{K}(x) = \mathcal{K}(-x) \quad \text{for all} \ \ x \in \mathbb{R}^N \setminus \{0\}.$$

In recent decade, the subject of nonlocal elliptic equations involving $\mathfrak{L}_{\{\mathcal{K},\mathbb{R}^N\}}$ has gained more popularity because of many applications such as continuum mechanics, game theory and phase transition phenomena. For an extensive survey on integro-differential operators and their applications, one may refer to [1, 10, 13] and references therein.

To proceed further we defined the following space

$$\mathcal{Z} = \{u\colon \mathbb{H}^N \to \mathbb{R}\colon u \in L^2(\Omega), (u(\xi) - u(\eta))\sqrt{\mathcal{K}(\eta^{-1}\xi)} \in L^2(\mathcal{S}, d\xi\, d\eta)\}$$

with $\mathcal{S} = \mathbb{H}^N \setminus (\Omega^c \times \Omega^c)$.

In this article we define an integro-differential operator on $\mathbb{H}^N$ and investigate the Lewy–Stampacchia estimates. Moreover, we proved the existence of solution of a subcritical problem involving the operator $\mathfrak{L}_{\mathcal{K}}$ by establishing the compact embedding of the space $\mathcal{Z}_0$ (see Section 2). In this regard, the results proved in the present article are completely new. The main results proved in this article are the following.

Theorem 1.1. *Let Ω be an bounded extension domain in $\mathbb{H}^N$ and $f \in L^\infty(\Omega)$. Assume $u_0 \in \mathcal{Z} \cap L^\infty(\mathbb{H}^N \setminus \Omega)$, $\phi \in \mathcal{Z}$ with $u_0 \leq \phi$ a.e. in $\mathbb{H}^N$ and $\mathfrak{L}_{\mathcal{K}}\phi \in L^\infty(\Omega)$. If $\mathcal{M}_\phi = \{u \in \mathcal{Z}\colon u = u_0 \text{ in } \Omega^c,\ u \leq \phi \text{ in } \Omega\}$ and $u \in \mathcal{M}_\phi$ is a solution of the variational inequality*

$$\int_{\mathcal{S}} (u(\xi) - u(\eta))((v-u)(\xi) - (v-u)(\eta))\mathcal{K}(\eta^{-1}\xi)\, d\xi\, d\eta \geq \int_\Omega f(v-u)\, d\xi, \tag{1.4}$$

for all $v \in \mathcal{M}_\phi$, then

$$\begin{aligned} 0 &\leq -\int_{\mathcal{S}} (u(\xi) - u(\eta))(\psi(\xi) - \psi(\eta))\mathcal{K}(\eta^{-1}\xi)\, d\xi\, d\eta + \int_\Omega f\psi\, d\xi \\ &\leq \int_\Omega (\mathfrak{L}_{\mathcal{K}}\phi + f)^+\psi\, d\xi \end{aligned} \tag{1.5}$$

for all non-negative functions $\psi \in C_c^\infty(\Omega)$.

Theorem 1.2. *Let f be a Carathéodory function satisfying the following conditions*

$$\begin{cases} \text{there exists } a_1, a_2 > 0 \text{ and } q \in (2, Q^*),\ Q^* = \frac{2Q}{Q-2s} \text{ such that} \\ |f(\xi, l)| \leq a_1 + a_2|l|^{q-1} \text{ a.e. } \xi \in \Omega,\ l \in \mathbb{R}; \\ \lim\limits_{|l|\to 0} \dfrac{f(\xi, l)}{|l|} = 0 \text{ uniformly in } x \in \Omega; \\ \text{there exist } \vartheta > 2 \text{ and } \mathcal{R} > 0 \text{ such that a.e. } x \in \Omega,\ l \in \mathbb{R},\ |l| > \mathcal{R}, \\ 0 < \vartheta F(\xi, l) \leq l f(\xi, l) \text{ where } F \text{ is the primotive of } f. \end{cases} \tag{1.6}$$

Then the following problem

$$(\mathfrak{P})\mathfrak{L}_{\mathcal{K}}u = f(\xi, u) \quad \text{in } \Omega, \quad u = 0 \text{ in } \mathbb{H}^N \setminus \Omega,$$

has a mountain pass type solution which is not identically zero.

In Section 2 we will give the variational framework, fibering map analysis and compactness of Palais–Smale sequences. In Section 3, we give the proof of Theorem 1.1. In Section 4, we give the proof of Theorem 1.2.

2. Preliminaries

In this section, we state some known results required for the variational framework. Let $\mathcal{M}$ is the linear subspace of Lebesgue measurable functions from $\mathbb{H}^N$ to $\mathbb{R}$ with the following property

$$\text{if } u \in \mathcal{M}, \text{ then } u|_\Omega \in L^1(\Omega).$$

Let $A, B \subset \mathcal{M}$ be the sets such that the following product is well defined

$$P\colon A \times B \to L^1(\mathcal{S}, d\xi, d\eta), \quad (a,b) \mapsto P(a,b) := ab.$$

Let $\mathcal{W}$ be the set containing all the non negative constants and the function $u_0 - \phi$, where functions $u_0, \phi \in \mathcal{M}$ such that $u_0 \leq \phi$ a.e. in $\mathbb{H}^N$. Further, we consider a linear subspaces $\tilde{\mathcal{N}_0}$, $\mathcal{N}_0$ such that $\tilde{\mathcal{N}_0} \subset \mathcal{N}_0 \subset \mathcal{M}$ and $\mathcal{N}_0$ satisfying the following property

$$\text{if } v \in \mathcal{N}_0 \ \text{ and } \ w \in \mathcal{W}, \ \text{ then } (v+w)^+ \in \mathcal{N}_0. \tag{2.1}$$

Let $g\colon \mathcal{M} \to A$ and $h\colon \mathcal{M} \to B$ be two well-defined operators and let $\mathcal{J}\colon \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ be the functional defined by

$$\mathfrak{J}(\varphi, \psi) = -\int_{\mathcal{S}} g(\varphi) h(\psi)\, d\xi\, d\eta.$$

Throughout the article we assume the following hypotheses on the functional g, h and $\mathfrak{J}$:

$$\begin{cases} g(u+r) = g(u), \ h(-u) = -h(u) \text{ for all } u \in \mathcal{M} \text{ and } r \in \mathbb{R}; \\ \text{if } u, v \in \mathcal{M} \text{ s.t. } (u-v)^+ \in \mathcal{N}_0 \text{ and } \mathfrak{J}(u, (u-v)^+) \geq \mathfrak{J}(v, (u-v)^+), \\ \text{then } u \leq v \text{ a.e. in } \Omega. \end{cases} \tag{2.2}$$

For $u \in \mathcal{M}$, if there exists $\Upsilon_u \in L^\infty(\Omega)$ such that $\mathfrak{J}(u,v) = \int_\Omega \Upsilon_u v \, dx$ for all $v \in \mathcal{N}_0$ then we denote $\mathfrak{J}(u) := \Upsilon_u$ and $\mathfrak{J}(u) \in L^\infty(\Omega)$. Define the cut-off function

$$D_r(l) = \begin{cases} 0 & \text{if } l \leq 0, \\ l/r & \text{if } 0 < l < r, \\ 1 & \text{if } l \geq r, \end{cases} \tag{2.3}$$

where $r \in (0,1)$. Suppose $\mathfrak{J}(\phi) \in L^\infty(\Omega)$ and $f \in L^\infty(\Omega)$. We assume that there exists $u_r \in \mathcal{M}$ such that $u_r - u_0 \in \mathcal{N}_0$ and satisfies the following

$$\mathfrak{J}(u_r, \varphi) = \int_\Omega [((\mathfrak{J}(\phi) + f)^+)(1 - D_r(\phi - u_r)) - f]\varphi \, d\xi \ \text{ for all } \varphi \in \mathcal{N}_0.$$

Also, if $u_r \to u$ uniformly in $\mathbb{R}^N$ as $r \to 0$ then up to a subsequence, $\mathfrak{J}(u_r, \varphi) \to \mathfrak{J}(u, \varphi)$ for all $\varphi \in \tilde{\mathcal{N}_0}$. Then we have the following theorem from [12]:

Theorem 2.1. *Let u_0, ϕ and $\mathfrak{J}$ satisfy the above assumptions and if $u \in \mathcal{M}$ is such that $u - u_0 \in \mathcal{N}_0$, $u \leq \phi$ a.e. in Ω and is a solution of the following variational inequality*

$$\int_{\mathcal{S}} g(u) h(v-u)\, d\mu \geq \int_\Omega f(v-u)\, d\xi \quad \textit{for all } v \in \{\mathcal{M}\colon v - u_0 \in \mathcal{N}_0\},$$

then

$$0 \leq \mathfrak{J}(u, \varphi) + \int_\Omega f\varphi \, d\xi \leq \int_\Omega (\mathfrak{J}(\phi) + f)^+ \varphi \, d\xi$$

for all $\varphi \in \tilde{\mathcal{N}_0}$, $\varphi \geq 0$ a.e. in Ω.

2.1. Variational Framework for $\mathfrak{L}_\mathcal{K}$. In this section we give the variational setup for the operator $\mathfrak{L}_\mathcal{K}$. Define the following norm on the space $\mathcal{Z}$

$$\|u\|_{\mathcal{Z}} = \|u\|_{L^2(\Omega)} + \left(\int_{\mathcal{S}} |u(\xi) - u(\eta)|^2 \mathcal{K}(\eta^{-1}\xi)\, d\xi\, d\eta\right)^{1/2}$$

Lemma 2.2. *Let $\phi \in C_0^2(\Omega)$. Then $|\phi(\xi) - \phi(\eta)|^2 \mathcal{K}(\eta^{-1}\xi) \in L^1(\mathbb{H}^{2N})$.*

Proof. Since $\phi = 0$ on $\mathbb{H}^N \setminus \Omega$,

$$\begin{aligned}\int_{\mathbb{H}^{2N}} |\phi(\xi)-\phi(\eta)|^2\mathcal{K}(\eta^{-1}\xi)\,d\xi\,d\eta &= \int_{\mathcal{S}} |\phi(\xi)-\phi(\eta)|^2\mathcal{K}(\eta^{-1}\xi)\,d\xi\,d\eta \\ &\le 2\int_{\Omega\times\mathbb{H}^N} |\phi(\xi)-\phi(\eta)|^2\mathcal{K}(\eta^{-1}\xi)\,d\xi\,d\eta.\end{aligned} \tag{2.4}$$

The fact that $\phi \in C_0^2(\Omega)$ implies

$$|\phi(\xi)-\phi(\eta)| \le \|\nabla_{\mathbb{H}}\phi\|_{L^\infty(\mathbb{H}^N)}|\xi-\eta| \quad \text{and} \quad |\phi(\xi)-\phi(\eta)| \le \|\phi\|_{L^\infty(\mathbb{H}^N)}.$$

Hence using the definition of θ, defined in (1.2), we have

$$|\phi(\xi)-\phi(\eta)| \le 2\|\phi\|_{C^1(\mathbb{H}^N)}\min\{|\xi-\eta|,1\} = 2\|\phi\|_{C^1(\mathbb{H}^N)}\sqrt{\theta(\eta^{-1}\xi)} \tag{2.5}$$

From (2.4) and (2.5) and (1.2), we get

$$\begin{aligned}\int_{\mathbb{H}^{2N}} |\phi(\xi)-\phi(\eta)|^2\mathcal{K}(\eta^{-1}\xi)\,d\xi\,d\eta &\le 2\int_{\Omega\times\mathbb{H}^N} |\phi(\xi)-\phi(\eta)|^2\mathcal{K}(\eta^{-1}\xi)\,d\xi\,d\eta \\ &\le 8\|\phi\|^2_{C^1(\mathbb{H}^N)}|\Omega|\int_{\mathbb{H}^N}\theta(\xi)\mathcal{K}(\xi)\,d\xi\,d\eta < \infty.\end{aligned}$$

The proof follows. □

The above Lemma implies $C_0^2(\mathbb{H}^N) \subset \mathcal{Z}$. Now we define the following subspaces

$$\begin{aligned}\mathcal{Z}_0 &= \{u \in \mathcal{Z} \colon u = 0 \text{ a.e. in } \Omega^c\}, \\ \tilde{\mathcal{Z}}_0 &= C_c^\infty(\Omega), \\ A = B &= \big\{u \colon \mathbb{H}^{2N} \to \mathbb{R} \colon u|_{\mathcal{S}} \in L^2(\mathcal{S}, d\xi\,d\eta)\big\}\end{aligned}$$

and P is the usual product between functions. Clearly, $\tilde{\mathcal{Z}}_0 \subseteq \mathcal{Z}_0 \subseteq \mathcal{Z}$ and $u^+ \in \mathcal{Z}$ for all $u \in \mathcal{Z}$.

In [2], Mallick and Adimurthi defined the usual fractional order Sobolev space on $\mathbb{H}^N$ as follows

$$W^{s,2}(\Omega) = \left\{u \colon \mathbb{H}^N \to \mathbb{R} \colon u \in L^2(\Omega) \text{ and } \int_{\Omega\times\Omega}\frac{|u(\xi)-u(\eta)|^2}{|\eta^{-1}\xi|^{Q+2s}}\,d\xi\,d\eta < \infty\right\}$$

where $s \in (0,1)$ and $p > 1$ as $W_0^{s,p}(\Omega)$ is the closure of $C_c^\infty(\Omega)$ with respect to the following norm

$$\|u\|_{W^{s,2}(\Omega)} = \|u\|_{L^2(\Omega)} + \left(\int_{\Omega\times\Omega}\frac{|u(\xi)-u(\eta)|^2}{|\eta^{-1}\xi|^{Q+2s}}\,d\xi\,d\eta\right)^{1/2}.$$

Clearly, $W_0^{s,2}(\mathbb{H}^N) = W^{s,p}(\mathbb{H}^N)$.

Lemma 2.3. *Let $c(\mu) = \max\{1, \mu^{-1/2}\}$. The following assertions holds true:*

(i) *If $v \in \mathcal{Z}$, then $v \in W^{s,2}(\Omega)$ and $\|v\|_{W^{s,2}(\Omega)} \le c(\mu)\|v\|_{\mathcal{Z}}$.*

(ii) *If $v \in \mathcal{Z}_0$ then $v \in W^{s,2}(\mathbb{H}^N)$ and $\|v\|_{W^{s,2}(\Omega)} \le \|v\|_{W^{s,2}(\mathbb{R}^N)} \le c(\mu)\|v\|_{\mathcal{Z}_0}$.*

(iii) *If $v \in \mathcal{Z}_0$ and $Q^* = \frac{2Q}{Q-2s}$, then there exists a positive constant $c(Q,s)$ such that*

$$\|v\|^2_{L^{Q^*}(\Omega)} \le c(Q,s)\int_{\mathbb{H}^{2N}}\frac{|v(\xi)-v(\eta)|^2}{|\eta^{-1}\xi|^{Q+2s}}\,d\xi\,d\eta \le c(Q,s)c^2(\mu)\|v\|^2_{\mathcal{Z}_0}.$$

(iv) *The space $\mathcal{Z}_0$ is a norm linear space endowed with the norm*

$$\|u\|_{\mathcal{Z}_0} = \left(\int_{\mathcal{S}} |u(\xi) - u(\eta)|^2 \mathcal{K}(\eta^{-1}\xi)\, d\xi\, d\eta \right)^{1/2}. \tag{2.6}$$

Moreover, there exists a constant $C > 0$ such that for any $v \in \mathcal{Z}_0$, we have

$$\|v\|_{\mathcal{Z}_0} \leq \|v\|_{\mathcal{Z}} \leq C\|v\|_{\mathcal{Z}_0}. \tag{2.7}$$

Proof. (i) Consider

$$\begin{aligned} \int_{\Omega\times\Omega} \frac{|v(\xi) - v(\eta)|^2}{|\eta^{-1}\xi|^{Q+2s}}\, d\xi\, d\eta &\leq \frac{1}{\mu} \int_{\Omega\times\Omega} |v(\xi) - v(\eta)|^2 \mathcal{K}(\eta^{-1}\xi)\, d\xi\, d\eta \\ &\leq \frac{1}{\mu} \int_{\mathcal{S}} |v(\xi) - v(\eta)|^2 \mathcal{K}(\eta^{-1}\xi)\, d\xi\, d\eta < \infty. \end{aligned}$$

(ii) Using the fact that $v = 0$ a.e. in Ω^c, we get $\|v\|_{L^2(\mathbb{H}^N)} = \|v\|_{L^2(\Omega)}$ and

$$\begin{aligned} \int_{\mathbb{H}^{2N}} \frac{|v(\xi) - v(\eta)|^2}{|\eta^{-1}\xi|^{Q+2s}}\, d\xi\, d\eta &= \int_{\mathcal{S}} \frac{|v(\xi) - v(\eta)|^2}{|\eta^{-1}\xi|^{Q+2s}}\, d\xi\, d\eta \\ &\leq \frac{1}{\mu} \int_{\mathcal{S}} |v(\xi) - v(\eta)|^2 \mathcal{K}(\eta^{-1}\xi)\ d\xi\, d\eta < \infty. \end{aligned}$$

(iii) Let $v \in \mathcal{Z}_0$ then by (ii), $v \in W^{s,2}(\mathbb{H}^N)$. Now using the [2, Theorem 1.1] with $\alpha = 0$, we have

$$\|v\|^2_{L^{Q^*}(\Omega)} \leq c \int_{\mathbb{H}^{2N}} \frac{|v(\xi) - v(\eta)|^2}{|\eta^{-1}\xi|^{Q+2s}}\, d\xi\, d\eta$$

where c depends on Q and s.

(iv) Let $\|v\|_{\mathcal{Z}_0} = 0$ then It implies $v(\xi) = v(\eta)$ a.e in $\mathcal{S}$. Let $v = c \geq 0$ a.e. in $\mathbb{H}^N$ but $v \in \mathcal{Z}_0$ implies $v = 0$ a.e. in Ω^c. That is, $v = 0$ a.e. in $\mathbb{H}^N$. Hence, $\|\cdot\|_{\mathcal{Z}_0}$ is a norm. For (2.7), by definition of $\|\cdot\|_{\mathcal{Z}}$, $\|v\|_{\mathcal{Z}_0} \leq \|v\|_{\mathcal{Z}}$. With the help of Hölder's inequality and (iii), we get

$$\begin{aligned} \|v\|^2_{\mathcal{Z}} &\leq 2\|v\|^2_{L^2(\Omega)} + 2\int_{\mathcal{S}} |v(\xi) - v(\eta)|^2 \mathcal{K}(\eta^{-1}\xi)\, d\xi\, d\eta \\ &\leq 2C_1\|v\|^2_{L^{Q^*}(\Omega)} + 2\int_{\mathcal{S}} |v(\xi) - v(\eta)|^2 \mathcal{K}(\eta^{-1}\xi)\, d\xi\, d\eta \\ &\leq 2cC_1 \int_{\mathbb{H}^{2N}} \frac{|v(\xi) - v(\eta)|^2}{|\eta^{-1}\xi|^{Q+2s}}\, d\xi\, d\eta + 2\int_{\mathcal{S}} |v(\xi) - v(\eta)|^2 \mathcal{K}(\eta^{-1}\xi)\, d\xi\, d\eta \\ &\leq 2\left(\frac{cC_1}{\mu} + 1\right) \int_{\mathcal{S}} |v(\xi) - v(\eta)|^2 \mathcal{K}(\eta^{-1}\xi)\, d\xi\, d\eta = C_2\|v\|^2_{\mathcal{Z}_0}. \end{aligned}$$

This proves the desired result with $C = \sqrt{C_2}$. □

Lemma 2.4. *$\mathcal{Z}_0$ is a Hilbert space endowed with the following inner product*

$$\langle u, v\rangle = \int_{\mathcal{S}} (u(\xi) - u(\eta))(v(\xi) - v(\eta))\mathcal{K}(\eta^{-1}\xi)\, d\xi\, d\eta \quad \textit{for all } u, v \in \mathcal{Z}_0$$

Proof. It is easy to prove that $\langle\cdot,\cdot\rangle$ is an inner product space and lead to the norm defined in (2.6). Now we prove that $\mathcal{Z}_0$ is a complete with respect to norm $\|\cdot\|_{\mathcal{Z}_0}$. Let u_n be a cauchy sequence in $\mathcal{Z}_0$. Hence for any $\varepsilon > 0$ there exists n_ε such that for all $n, m \geq n_\varepsilon$,

$$\|u_n - u_m\|^2_{L^2(\Omega)} \leq \|u_n - u_m\|^2_{\mathcal{Z}} \leq C\|u_n - u_m\|^2_{\mathcal{Z}_0} < \varepsilon. \tag{2.8}$$

Since $L^2(\Omega)$ is a complete space, there exists $u_* \in L^2(\Omega)$ such that $u_n \to u_*$ in $L^2(\Omega)$ as $n \to \infty$. Up to a subsequence(denoted by u_n) such that $u_n \to u_*$ a.e. in $\mathbb{H}^N$. By using Fatou's Lemma, we get

$$\|u_*\|^2_{\mathcal{Z}_0} \le \liminf_{n\to\infty} \|u_n\|^2_{\mathcal{Z}_0} \le \liminf_{n\to\infty} (\|u_n - u_*\|_{\mathcal{Z}_0} + \|u_*\|_{\mathcal{Z}_0})^2 < (1 + \|u_*\|_{\mathcal{Z}_0})^2 < \infty$$

where we used (2.8) with $\varepsilon = 1$. It implies $u_* \in \mathcal{Z}_0$. Using (2.8), with $m \ge n_\varepsilon$, we obtain

$$\|u_m - u_*\|^2_{\mathcal{Z}_0} \le \|u_m - u_*\|^2_{\mathcal{Z}} \le \liminf_{n\to\infty} \|u_m - u_n\|^2_{\mathcal{Z}} \le C \liminf_{n\to\infty} \|u_m - u_n\|^2_{\mathcal{Z}_0} \le C\varepsilon.$$

It implies that $u_m \to u_*$ as $m \to \infty$. Hence, we get the desired result. □

Lemma 2.5. *Let u_n is a bounded sequence in X_0. Then there exists $u_* \in L^r(\mathbb{H}^N)$ such that, up to a subsequence, $u_n \to u_*$ in $L^r(\mathbb{H}^N)$ as $n \to \infty$ for all $r \in [1, Q^*)$.*

Proof. By Lemma 2.3 (ii), $u_n \in W^{s,2}(\Omega)$ and u_n is a bounded sequence in $W^{s,2}(\Omega)$. By using [2, Theorem 1.4], there exists $u_* \in L^r(\Omega)$ such that $u_n \to u_*$ in $L^r(\Omega)$ for any $r \in [1, Q^*)$. Moreover, $u_n = 0$ a.e. in $\mathbb{H}^N \setminus \Omega$ implies $u_* = 0$ a.e. in $\mathbb{H}^N \setminus \Omega$, that is, $u_n \to u_*$ in $L^r(\mathbb{H}^N)$. □

3. Proof of Theorem 1.1

In this section, we gave the proof of Theorem 1.1 using the Theorem 1.2. For $u, v \in \mathcal{Z}$, we define $g(u)(\xi,\eta) = (u(\xi) - u(\eta))\sqrt{\mathcal{K}(\eta^{-1}\xi)} \in A$, and $g(v)(\xi,\eta) = (v(\xi) - v(\eta))\sqrt{\mathcal{K}(\eta^{-1}\xi)} \in A$, and

$$\mathfrak{J}(u, v) = \int_{\mathcal{S}} (u(\xi) - u(\eta))(v(\xi) - v(\eta))\mathcal{K}(\eta^{-1}\xi)\, d\xi\, d\eta.$$

Lemma 3.1. *The following holds:*

(i) *$\mathfrak{J}$ is well defined map.*
(ii) *The assumptions* (2.1) *and* (2.2) *are satisfied.*

Proof. (i) By using the Cauchy–Schwarz inequality and the fact that $u, v \in \mathcal{Z}$, we get

$$\begin{aligned} &2|u(\xi) - u(\eta)||v(\xi) - v(\eta)|\sqrt{\mathcal{K}(\eta^{-1}\xi)}\sqrt{\mathcal{K}(\eta^{-1}\xi)} \\ &\le \left(|u(\xi) - u(\eta)|^2 + |v(\xi) - v(\eta)|^2\right)\mathcal{K}(\eta^{-1}\xi). \end{aligned}$$

(ii) Let $v \in \mathcal{Z}_0$ and $w \in \mathcal{Z}$ with $w \le 0$ a.e. in $\mathbb{H}^N$. The fact that $v + w \in \mathcal{Z}$ implies $(v + w)^+ \in \mathcal{Z}$. Also $v + w(\xi) \le v(\xi) = 0$ a.e. in Ω^c, implies $(v + w)^+ = 0$ in Ω^c. Therefore, $(v + w)^+ \in \mathcal{Z}_0$. It proves (2.1).

By the definition of g and h, one can easily show that $g(u + r) = g(u)$, $h(-u) = -h(u)$ for all $u \in \mathcal{Z}$ and $r \in \mathbb{R}$. Let $u, v \in \mathcal{Z}$ with $(u - v)^+ \in \mathcal{Z}_0$ and $\mathfrak{J}(u, (u - v)^+) \ge \mathfrak{J}(v, (u - v)^+)$. Set $w = u - v$ and $w = w^+ - w^-$. Consider

$$\begin{aligned} &(w(\xi) - w(\eta))(w^+(\xi) - w^+(\eta)) \\ &= (w^+(\xi) - w^+(\eta))^2 + w^-(\xi)w^+(\eta) + w^+(\xi)w^-(\eta). \end{aligned} \tag{3.1}$$

By using (1.2) and (3.1), we have

$$\begin{aligned} 0 &\geq \mathfrak{J}(v,(u-v)^+) - \mathfrak{J}(u,(u-v)^+) \\ &= \int_{\mathcal{S}} (w(\xi)-w(\eta))(w^+(\xi)-w^+(\eta))\mathcal{K}(\eta^{-1}\xi)\, d\xi\, d\eta \\ &= \int_{\mathcal{S}} \left((w^+(\xi)-w^+(\eta))^2 + w^-(\xi)w^+(\eta) + w^+(\xi)w^-(\eta)\right)\mathcal{K}(\eta^{-1}\xi)\, d\xi\, d\eta \geq 0. \end{aligned}$$

It implies $(w^+(\xi)-w^+(\eta))^2 + w^-(\xi)w^+(\eta) + w^+(\xi)w^-(\eta) = 0$ a.e. in $\mathcal{S}$, that is, $w^+(\xi) = w^+(\eta)$ a.e. in $\mathcal{S}$. Let $w^+ = c \geq 0$ a.e. in $\mathbb{H}^N$ but $w^+ \in \mathcal{Z}_0$ implies $c = 0$. Hence $(u-v)^+ = 0$, that is, $u \leq v$ a.e. in $\mathbb{H}^N$. Thus, (2.2) is satisfied. □

We set two functions $u_0 \in \mathcal{Z} \cap L^\infty(\Omega^c)$ and $\phi \in \mathcal{Z}$ with $u_0 \leq \phi$ a.e. in $\mathbb{H}^N$. Assume $\mathfrak{L}_{\mathcal{K}}\phi, f \in L^\infty(\Omega)$. For a.e. $\xi \in \Omega$ and $l \in \mathbb{R}$, define $T = (\mathfrak{L}_{\mathcal{K}}\phi + f)^+ \in L^\infty(\Omega)$ and $w_r(\xi, l) = T(\xi)(1 - D_r(\phi(\xi) - l)) - f(\xi)$ where D_r is defined in (2.3) and $r \in (0,1)$.

Proposition 3.2. *Let $r \in (0,1)$. Then there exists $u_r \in \mathcal{Z}$ such that is a solution to $\mathfrak{L}_{\mathcal{K}} u_r = w_r(\xi, u_r)$ and $u_r = u_0 \in \Omega^c$. That is, for all $v \in \mathcal{Z}_0$*

$$\begin{cases} \displaystyle\int_{\mathcal{S}} (u_r(\xi)-u_r(\eta))(v(\xi)-v(\eta))\mathcal{K}(\eta^{-1}\xi)\, d\xi\, d\eta + \int_\Omega w_r(\xi, u_r(\xi)) v\, d\xi = 0, \\ u_r \in \mathcal{Z}, \quad u_r - u_0 \in \mathcal{Z}_0. \end{cases} \tag{3.2}$$

Proof. Consider the space $\mathcal{Z}_{u_0} = \{u \in \mathcal{Z} \colon u - u_0 \in \mathcal{Z}_0\}$ and the functional $I_r \colon \mathcal{Z}_{u_0} \to \mathbb{R}$ defined as

$$I_r(u) = \frac{1}{2}\int_{\mathcal{S}} |u(\xi)-u(\eta)|^2 \mathcal{K}(\eta^{-1}\xi)\ d\xi d\eta + \int_\Omega W_r(\xi, u_r(\xi))\, d\xi,$$

where W_r is the primitive of w_r. Clearly, by definition of D_r,

$$\begin{aligned} |w_r| &\leq \|T\|_{L^\infty(\Omega)} \|(1 - D_r(\phi(\xi) - l))\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)} \\ &\leq \|T\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)} := \zeta. \end{aligned}$$

It implies $|W_r(\xi, u(\xi)| \leq \zeta |u(\xi)|$. Employing the Young's inequality, Minkowski's inequality, Hölder's inequality, Lemma 2.3(iii) and (1.2), we deduce that

$$\begin{aligned} I_r(u) &\geq \frac{1}{2}\int_{\mathcal{S}} |u(\xi)-u(\eta)|^2 \mathcal{K}(\eta^{-1}\xi)\ d\xi d\eta - \zeta \int_\Omega |u(\xi)|\, d\xi \\ &\geq \frac{1}{2}\int_{\mathcal{S}} |u(\xi)-u(\eta)|^2 \mathcal{K}(\eta^{-1}\xi)\ d\xi d\eta - \frac{\zeta\varepsilon}{2}\|u\|^2_{L^2(\Omega)} - \frac{\zeta}{2\varepsilon}|\Omega| \\ &\geq \frac{\mu}{2}\int_{\mathcal{S}} \frac{|u(\xi)-u(\eta)|^2}{|\eta^{-1}\xi|^{Q+2s}}\, d\xi\, d\eta - \zeta\varepsilon\|u-u_0\|^2_{L^2(\Omega)} - \zeta\varepsilon\|u_0\|^2_{L^2(\Omega)} - \frac{\zeta}{2\varepsilon}|\Omega| \qquad (3.3) \\ &\geq \mu\|u-u_0\|^2_{\mathcal{Z}_0} - \mu\|u_0\|^2_{\mathcal{Z}_0} - \zeta\varepsilon C|\Omega|^{\frac{Q^*-2}{2}}\|u-u_0\|^2_{\mathcal{Z}_0} - \zeta\varepsilon\|u_0\|^2_{L^2(\Omega)} - \frac{\zeta}{2\varepsilon}|\Omega| \\ &= \left(\mu - \zeta\varepsilon C|\Omega|^{\frac{Q^*-2}{2}}\right)\|u-u_0\|^2_{\mathcal{Z}_0} - \mu\|u_0\|^2_{\mathcal{Z}_0} - \zeta\varepsilon\|u_0\|^2_{L^2(\Omega)} - \frac{\zeta}{2\varepsilon}|\Omega|. \end{aligned}$$

Using the fact that $u_0 \in \mathcal{Z}$ and the properties of $\mathcal{K}$, we get $\|u_0\|^2\mathcal{Z}_0, \|u_0\|^2_{L^2(\Omega)} < \infty$. Now choosing $\varepsilon > 0$ such that $\mu - \zeta\varepsilon C|\Omega|^{\frac{Q^*-2}{2}} > 0$, we deduce that

$$I_r(u) \geq -\mu\|u_0\|^2_{\mathcal{Z}_0} - \zeta\varepsilon\|u_0\|^2_{L^2(\Omega)} - \frac{\zeta}{2\varepsilon}|\Omega| > -\infty.$$

It implies that $\inf_{u \in \mathcal{Z}_{u_0}} I_r(u) > -\infty$. Now let $u_n \in \mathcal{Z}_{u_0}$ be the minimizing sequence for I_r then $I_r(u_n) \to \inf_{u \in \mathcal{Z}_{u_0}} I_r(u)$. Thus from (3.3) with $u = u_n$ we get $u_n - u_0$ is a bounded sequence is X_0. From Lemma 2.5, there exists $u^* \in X_0$ such tha, up to a subsequence, $u_n - u_0 \to u^*$ in L^ν for $\nu \in [1, Q^*)$ and $u_n - u_0 \to u^*$ a.e. in $\mathbb{H}^N$. Thus $u^* \in \mathcal{Z}_0$. Define $u_* = u_0 + u^*$. Then $u_* \in \mathcal{Z}_{u_0}$. Now using the continuity of the map $i \mapsto W_r(\xi, i)$ for all $\xi \in \mathbb{H}^N$ and $i \in \mathbb{R}$ and the dominated convergence theorem, we get

$$\lim_{n\to\infty} I_r(u_n) \geq \frac{1}{2}\int_{\mathcal{S}} |u_*(\xi) - u_*(\eta)|^2 \mathcal{K}(\eta^{-1}\xi)\, d\xi\, d\eta + \int_\Omega W_r(\xi, u_*(\xi))\, d\xi = I_r(u_*).$$

Hence $I_r(u_*) = \inf_{u\in\mathcal{Z}_{u_0}} I_r(u)$. It implies (3.2) has a solution. □

Proposition 3.3. *Let $v \in C_0^2(\Omega)$ and $u \in \mathcal{Z} \cap L^\infty(\Omega^c)$. Then*

$$\begin{aligned} &\int_{\mathcal{S}} (u(\xi) - u(\eta))(v(\xi) - v(\eta))\mathcal{K}(\eta^{-1}\xi)\, d\xi\, d\eta \\ &= \int_{\mathbb{H}^{2N}} u(\xi)(2v(\xi) - v(\xi\tilde{\xi}) - v(\xi\tilde{\xi}^{-1}))\mathcal{K}(\tilde{\xi})\, d\xi\, d\tilde{\xi}. \end{aligned} \tag{3.4}$$

Proof. Let

$$\begin{aligned} &\mathfrak{E}_0 = \mathcal{S} = \mathbb{H}^{2N} \setminus (\Omega^c \times \Omega^c), \\ &\mathfrak{E}_\delta = \{(\xi, \eta) \in \mathfrak{E}_0 \colon |\eta^{-1}\xi| \geq \delta\}, \\ &\mathfrak{E}_\delta^+ = \{(\xi, \tilde{\xi}) \in \mathbb{H}^{2N} \colon (\xi, \xi\tilde{\xi}) \in \mathfrak{E}_0 \text{ and } |\tilde{\xi}| \geq \delta\}, \\ &\mathfrak{E}_\delta^- = \{(\xi, \tilde{\xi}) \in \mathbb{H}^{2N} \colon (\xi, \xi\tilde{\xi}^{-1}) \in \mathfrak{E}_0 \text{ and } |\tilde{\xi}| \geq \delta\}. \end{aligned}$$

Claim. $(\xi, \eta) \mapsto u(\xi)(v(\xi) - v(\eta))\mathcal{K}(\eta^{-1}\xi) \in L^1(\mathfrak{E}_\delta\, d\xi\, d\eta)$.

For δ small and $|\eta^{-1}\xi| \geq \delta$, we have $\theta(\eta^{-1}\xi) \geq \delta^2$, and using properties of $\mathcal{K}$ and the fact that $v = 0$ in Ω^c, we deduce that

$$\begin{aligned} &\int_{\mathfrak{E}_\delta} |u(\xi)||v(\xi) - v(\eta)|\mathcal{K}(\eta^{-1}\xi)\, d\xi\, d\eta \\ &= \int_{\{\Omega\times\mathbb{H}^N\}\cap\{|\eta^{-1}\xi|\geq\delta\}} |u(\xi)||v(\xi) - v(\eta)|\mathcal{K}(\eta^{-1}\xi)\, d\xi\, d\eta \\ &\quad + \int_{\{\Omega\times\Omega^c\}\cap\{|\eta^{-1}\xi|\geq\delta\}} |u(\eta)||v(\xi) - v(\eta)|\mathcal{K}(\eta^{-1}\xi)\, d\xi\, d\eta \\ &\leq 2\|v\|_{L^\infty(\mathbb{H}^N)} \bigg(\int_{\{\Omega\times\mathbb{H}^N\}\cap\{|\eta^{-1}\xi|\geq\delta\}} |u(\xi)|\mathcal{K}(\eta^{-1}\xi)\, d\xi\, d\eta \\ &\quad + \int_{\{\Omega\times\Omega^c\}\cap\{|\eta^{-1}\xi|\geq\delta\}} |u(\eta)|\mathcal{K}(\eta^{-1}\xi)\, d\xi\, d\eta\bigg) \\ &\leq \frac{2\|v\|_{L^\infty(\mathbb{H}^N)}}{\delta^2} \bigg(\int_{\{\Omega\times\mathbb{H}^N\}\cap\{|\eta^{-1}\xi|\geq\delta\}} |u(\xi)|\theta(\eta^{-1}\xi)\mathcal{K}(\eta^{-1}\xi)\, d\xi\, d\eta \\ &\quad + \int_{\{\Omega\times\Omega^c\}\cap\{|\eta^{-1}\xi|\geq\delta\}} |u(\eta)|\theta(\eta^{-1}\xi)\mathcal{K}(\eta^{-1}\xi)\, d\xi\, d\eta\bigg) \\ &\leq \frac{2\|v\|_{L^\infty(\mathbb{H}^N)}}{\delta^2} \bigg(\int_{\{\Omega\times\mathbb{H}^N\}} |u(\xi)|\theta(\tilde{\xi})\mathcal{K}(\tilde{\xi})\, d\xi\, d\tilde{\xi} + \|u\|_{L^\infty(\Omega^c)} \int_{\{\Omega\times\Omega^c\}} \theta(\tilde{\xi})\mathcal{K}(\tilde{\xi})\, d\xi\, d\tilde{\xi}\bigg) \\ &\leq \frac{2\|v\|_{L^\infty(\mathbb{H}^N)}}{\delta^2} \big(\|u\|_{L^1(\Omega)} + |\Omega|\|u\|_{L^\infty(\Omega^c)}\big) \int_{\mathbb{H}^N} \theta(\tilde{\xi})\mathcal{K}(\tilde{\xi})\, d\tilde{\xi} < \infty. \end{aligned}$$

Hence the claim. Similarly, $(\xi,\eta)\mapsto u(\eta)(v(\xi)-v(\eta))\mathcal{K}(\eta^{-1}\xi)\in L^1(\mathfrak{E}_\delta)$. It implies that

$$\begin{aligned}
&\int_{\mathfrak{E}_\delta}(u(\xi)-u(\eta))(v(\xi)-v(\eta))\mathcal{K}(\eta^{-1}\xi)\,d\xi\,d\eta\\
&=\int_{\mathfrak{E}_\delta}u(\xi)(v(\xi)-v(\eta))\mathcal{K}(\eta^{-1}\xi)\,d\xi\,d\eta+\int_{\mathfrak{E}_\delta}u(\xi)(v(\xi)-v(\eta))\mathcal{K}(\xi^{-1}\eta)d\xi d\eta\\
&=\int_{\mathfrak{E}_\delta^+}u(\xi)(v(\xi)-v(\xi\tilde\xi))\mathcal{K}(\tilde\xi^{-1})\,d\xi\,d\tilde\xi \qquad (3.5)\\
&\quad+\int_{\mathfrak{E}_\delta^-}u(\xi)(v(\xi)-v(\xi\tilde\xi^{-1}))\mathcal{K}(\tilde\xi^{-1})\,d\xi\,d\tilde\xi\\
&=\int_{\mathfrak{E}_\delta^+}u(\xi)(v(\xi)-v(\xi\tilde\xi))\mathcal{K}(\tilde\xi)\,d\xi\,d\tilde\xi+\int_{\mathfrak{E}_\delta^-}u(\xi)(v(\xi)-v(\xi\tilde\xi^{-1}))\mathcal{K}(\tilde\xi)\,d\xi\,d\tilde\xi
\end{aligned}$$

Notice that if $(\xi,\tilde\xi)\in\mathfrak{E}_\delta^-\setminus E_\delta^+$ then $(\xi,\xi\tilde\xi)\in\Omega^c\times\Omega^c$. It implies that

$$\int_{\mathfrak{E}_\delta^-\setminus\mathfrak{E}_\delta^+}u(\xi)(v(\xi)-v(\xi\tilde\xi))\mathcal{K}(\tilde\xi)\,d\xi\,d\tilde\xi=0$$

Similarly, $\int_{\mathfrak{E}_\delta^+\setminus\mathfrak{E}_\delta^-}u(\xi)(v(\xi)-v(\xi\tilde\xi^{-1}))\mathcal{K}(\tilde\xi)\,d\xi\,d\tilde\xi=0$. Using this with (3.5), we obtain

$$\begin{aligned}
&\int_{\mathfrak{E}_\delta}(u(\xi)-u(\eta))(v(\xi)-v(\eta))\mathcal{K}(\eta^{-1}\xi)\,d\xi\,d\eta\\
&=\int_{\mathfrak{E}_\delta^+\cup\mathfrak{E}_\delta^-}u(\xi)(v(\xi)-v(\xi\tilde\xi))\mathcal{K}(\tilde\xi)\,d\xi\,d\tilde\xi \qquad (3.6)\\
&\quad+\int_{\mathfrak{E}_\delta^+\cup\mathfrak{E}_\delta^-}u(\xi)(v(\xi)-v(\xi\tilde\xi^{-1}))\mathcal{K}(\tilde\xi)\,d\xi\,d\tilde\xi\\
&=\int_{\mathfrak{E}_\delta^+\cup\mathfrak{E}_\delta^-}u(\xi)(2v(\xi)-v(\xi\tilde\xi)-v(\xi\tilde\xi^{-1}))\mathcal{K}(\tilde\xi)\,d\xi\,d\tilde\xi
\end{aligned}$$

We now show that $\mathfrak{V}(\xi,\tilde\xi):=(2v(\xi)-v(\xi\tilde\xi)-v(\xi\tilde\xi^{-1}))\mathcal{K}(\tilde\xi)\in L^1(\mathbb{H}^{2N})$. Notice that $|\mathfrak{V}(\xi,\tilde\xi)|\le 4\|v\|_{L^\infty(\mathbb{H}^N)}\mathcal{K}(\tilde\xi)$ and, by Taylor expansion

$$|\mathfrak{V}(\xi,\tilde\xi)|\le\|D^2v\|_{L^\infty(\mathbb{H}^N)}|\tilde\xi|^2\mathcal{K}(\tilde\xi).$$

It implies that

$$|\mathfrak{V}(\xi,\tilde\xi)|\le 4\|v\|_{C^2(\mathbb{H}^N)}\theta(\tilde\xi)\mathcal{K}(\tilde\xi).$$

Now we first recall the triangle inequality which states that there exists $\underline{c}<1$ such that

$$\underline{c}\big||\xi|-|\eta|\big|\le|\xi\eta|\quad\text{for all }\xi,\eta\in\mathbb{H}^N. \tag{3.7}$$

Choose $R>1$ such that $\Omega\subset B_{\underline{c}R}$. If $\xi\in(B_{2R})^c$, $\xi\tilde\xi\in\Omega$ and $\xi\tilde\xi^{-1}\in\Omega\subset B_{\underline{c}R}$, then by (3.7), we get

$$|\tilde\xi|\ge|\xi|-\frac{1}{\underline{c}}|\xi\tilde\xi|\ge 2R-\frac{1}{\underline{c}}\underline{c}R=R>1. \tag{3.8}$$

Define

$$\mathfrak{R} = \left\{(\xi,\tilde{\xi}) \in \mathbb{H}^{2N} : \xi \in (B_{2R}(0))^c \text{ and } \left(\tilde{\xi} \in B_{\underline{c}R}(\xi^{-1}) \text{ or } \tilde{\xi}^{-1} \in B_{\underline{c}R}(\xi^{-1})\right)\right\},$$
$$\mathfrak{R}_* = \left\{(\xi,\tilde{\xi}) \in \mathbb{H}^{2N} : \xi^{-1} \in B_{\underline{c}R}(\tilde{\xi}^{-1}) \cup B_{\underline{c}R}(\tilde{\xi}) \text{ and } \tilde{\xi} \in (B_1(0))^c\right\}.$$

Let $(\xi,\tilde{\xi}) \in \mathfrak{R}$ then

$$(|\xi| > 2R \text{ and } |\xi\tilde{\xi}| < \underline{c}R) \text{ or } (|\xi| > 2R \text{ and } |\xi\tilde{\xi}^{-1}| < \underline{c}R). \tag{3.9}$$

Taking into account (3.8) and (3.9), we deduce that either $|\tilde{\xi}| > 1$ and $\xi^{-1} \in B_{\underline{c}R}(\tilde{\xi})$ or $|\tilde{\xi}| > 1$ and $\xi^{-1} \in B_{\underline{c}R}(\tilde{\xi}^{-1})$. It implies that $\mathfrak{R} \subset \mathfrak{R}_*$. Furthermore, if $(\xi,\tilde{\xi}) \in \mathfrak{R}_*$ then

$$\mathcal{K}(\tilde{\xi}) = \theta(\tilde{\xi})\mathcal{K}(\tilde{\xi}) \tag{3.10}$$

Now let $(\xi,\tilde{\xi}) \in ((B_{2R}(0))^c \times \mathbb{H}^N) \setminus \mathfrak{R}$. Then $\xi \in (B_{2R}(0))^c$, $\tilde{\xi} \in \mathbb{H}^N$, $(\xi,\tilde{\xi}) \notin \mathfrak{R}$. That is, $\xi \in (B_{2R}(0))^c$, $|\xi\tilde{\xi}| > \underline{c}R$, and $|\xi\tilde{\xi}^{-1}| > \underline{c}R$. As a result

$$v(\xi) = v(\xi\tilde{\xi}) = v(\xi\tilde{\xi}^{-1}) = 0 \quad \text{for all } (\xi,\tilde{\xi}) \in ((B_{2R}(0))^c \times \mathbb{H}^N) \setminus \mathfrak{R}. \tag{3.11}$$

Using (3.10), (3.11) and definition of $\mathcal{K}$, we get

$$\begin{aligned}
&\int_{(B_{2R}(0))^c \times \mathbb{H}^N} |\mathfrak{V}(\xi,\tilde{\xi})| \, d\xi \, d\tilde{\xi} \\
&= \int_{\mathfrak{R}_*} |2v(\xi) - v(\xi\tilde{\xi}) - v(\xi\tilde{\xi}^{-1})|\mathcal{K}(\tilde{\xi}) \, d\xi \, d\tilde{\xi} \\
&\le 4\|v\|_{L^\infty(\mathbb{H}^N)} \int_{\mathfrak{R}_*} \mathcal{K}(\tilde{\xi}) \, d\xi \, d\tilde{\xi} = 4\|v\|_{L^\infty(\mathbb{H}^N)} \int_{\mathfrak{R}_*} \theta(\tilde{\xi})\mathcal{K}(\tilde{\xi}) \, d\xi \, d\tilde{\xi} \\
&\le C(Q,R)\|v\|_{L^\infty(\mathbb{H}^N)} \int_{\mathbb{H}^N} \theta(\tilde{\xi})\mathcal{K}(\tilde{\xi}) \, d\xi \, d\tilde{\xi} = C(Q,R,\mathcal{K}) < \infty.
\end{aligned} \tag{3.12}$$

Consider

$$\begin{aligned}
\int_{\mathbb{H}^{2N}} |\mathfrak{V}(\xi,\tilde{\xi})| \, d\xi \, d\tilde{\xi} &\le \int_{B_{2R} \times \mathbb{H}^N} |\mathfrak{V}(\xi,\tilde{\xi})| \, d\xi \, d\tilde{\xi} + C(Q,R,\mathcal{K}) \\
&\le C(Q,R,\mathcal{K}) + 4\|v\|_{C^2(\mathbb{H}^N)} \int_{B_{2R} \times \mathbb{H}^N} \theta(\tilde{\xi})\mathcal{K}(\tilde{\xi}) \, d\xi \, d\tilde{\xi} \\
&= C(Q,R,\mathcal{K}) + 4\|v\|_{C^2(\mathbb{H}^N)}|B_{2R}| \int_{\mathbb{H}^N} \theta(\tilde{\xi})\mathcal{K}(\tilde{\xi}) \, d\xi \, d\tilde{\xi} < \infty.
\end{aligned} \tag{3.13}$$

Hence we show that $\mathfrak{V} \in L^1(\mathbb{H}^{2N})$. Taking into account Lemma 3.1 (i), (3.13), and passing limit $\delta \to 0$ in (3.6), we get

$$\begin{aligned}
&\int_{\mathfrak{E}_0} (u(\xi) - u(\eta))(v(\xi) - v(\eta))\mathcal{K}(\eta^{-1}\xi) \, d\xi \, d\eta \\
&= \int_{\mathfrak{E}_0^+ \cup \mathfrak{E}_0^-} u(\xi)(2v(\xi) - v(\xi\tilde{\xi}) - v(\xi\tilde{\xi}^{-1}))\mathcal{K}(\tilde{\xi}) \, d\xi \, d\tilde{\xi}.
\end{aligned} \tag{3.14}$$

Using the definition of $\mathfrak{E}_0^+$ and $\mathfrak{E}_0^-$ and the fact that $v = 0$ in $\mathbb{H}^N \setminus \Omega$, we get (3.4). □

Lemma 3.4. *Let $\psi \in C_0^2(\Omega)$ and $u_n \in \mathcal{Z}$ be a sequence converging uniformly to $u_* \in \mathcal{Z}$ as $n \to \infty$ and $u_n - u_* \in \mathcal{Z}_0$. Then*

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathcal{S}}(u_n(\xi)-u_n(\eta))(v(\xi)-v(\eta))\mathcal{K}(\eta^{-1}\xi)\,d\xi\,d\eta\\&=\int_{\mathcal{S}}(u_*(\xi)-u_*(\eta))(v(\xi)-v(\eta))\mathcal{K}(\eta^{-1}\xi)\,d\xi\,d\eta\end{aligned}\tag{3.15}$$

Proof. Using Proposition 3.3 to $u_n - u_*$, for any $v \in C_0^2(\Omega)$, we deduce that

$$\begin{aligned}&\int_{\mathcal{S}}(u_n(\xi)-u_n(\eta))(v(\xi)-v(\eta))\mathcal{K}(\eta^{-1}\xi)\,d\xi\,d\eta\\&\quad-\int_{\mathcal{S}}(u_*(\xi)-u_*(\eta))(v(\xi)-v(\eta))\mathcal{K}(\eta^{-1}\xi)\,d\xi\,d\eta\\&=\int_{\mathcal{S}}((u_n-u_*)(\xi)-(u_n-u_*)(\eta))(v(\xi)-v(\eta))\mathcal{K}(\eta^{-1}\xi)\,d\xi\,d\eta\\&=\int_{\mathcal{S}}(u_n-u_*)(\xi)(2v(\xi)-v(\xi\eta)-v(\xi\eta^{-1}))\mathcal{K}(\eta^{-1}\xi)\,d\xi\,d\eta.\end{aligned}$$

Since $u_n \to u_*$ uniformly in $\mathbb{H}^N$, we get

$$\begin{aligned}&\lim_{n\to\infty}\left|\int_{\mathcal{S}}(u_n-u_*)(\xi)(2v(\xi)-v(\xi\eta)-v(\xi\eta^{-1}))\mathcal{K}(\eta^{-1}\xi)\,d\xi\,d\eta\right|\\&\le \lim\|u_n-u_*\|_{L^\infty(\Omega)}\|\mathfrak{V}\|_{L^1(\mathbb{H}^{2N})}=0\end{aligned}$$

where $\mathfrak{V}$ is defined in Proposition 3.3. Hence (3.15) holds true. □

Proof of Theorem 1.1. Let u be a solution to variational inequality (1.4). In the framework of Theorem 1.1, we now apply Theorem 2.1 and deduce that (1.5) holds.

4. Proof of Theorem 1.2

In this Section, We proove trhe existence of mountain pass solution of the problem. Notice that the problem $(\mathcal{P})$ has a variational structure and the energy functional associated to the problem $(\mathcal{P})$ is given as

$$\mathcal{H}(u)=\frac{1}{2}\int_{\mathcal{S}}|u(\xi)-u(\eta)|^2\mathcal{K}(\eta^{-1}\xi)\,d\xi\,d\eta-\int_\Omega F(\xi,u(\xi))\,d\xi.$$

Observe that $\mathcal{H} \in C^1(\mathcal{Z}_0, \mathbb{R})$ and for $u \in \mathcal{Z}_0$ and for any $v \in \mathcal{Z}_0$,

$$\langle \mathcal{H}'(u),v\rangle=\int_{\mathcal{S}}(u(\xi)-u(\eta))(v(\xi)-v(\eta))\mathcal{K}(\eta^{-1}\xi)\,d\xi\,d\eta-\int_\Omega f(\xi,u(\xi))\,d\xi$$

Remark 4.1. Under the assumptions (1.6), for any $\varepsilon > 0$ there exists $\delta > 0$ such that a.e. $\xi \in \Omega$ and for any $l \in \mathbb{R}$, we have

$$|f(\xi,l)|\le 2\varepsilon|l|+q\delta|l|^{q-1}\quad\text{and}\quad|F(\xi,l)|\le\varepsilon|l|^2+\delta|l|^q.\tag{4.1}$$

Also, there exist two positive functions m and M belonging in $L^\infty(\Omega)$ such that a.e. $\xi \in \Omega$ and for any $l \in \mathbb{R}$,

$$F(\xi,l)\ge m(\xi)|l|^\vartheta-M(\xi).\tag{4.2}$$

Proposition 4.2. *The following holds:*

(i) *There exist $\alpha, \rho > 0$ such that $\mathcal{H}(u) \ge \alpha$ for $\|u\|_{\mathcal{Z}_0} = \rho$.*
(ii) *There exists $e \in \mathcal{Z}_0$ such that $e \ge 0$ a.e. in $\mathbb{H}^N$, $\|e\| > \rho$ and $\mathcal{H}(e) < \beta$.*

Proof. From Hölder's inequality, Lemma 2.3, and (4.1), we deduce that

$$\begin{aligned}\mathcal{H}(u) &\geq \frac{1}{2}\|u\|^2_{\mathcal{Z}_0} - \varepsilon\int_\Omega |u|^2\,d\xi - \delta\int_\Omega |u|^q\,d\xi\\ &\geq \frac{1}{2}\|u\|^2_{\mathcal{Z}_0} - \varepsilon|\Omega|^{(Q^*-2)/Q^*}\|u\|^2_{L^{Q^*}(\Omega)} - \delta|\Omega|^{(Q^*-q)/Q^*}\|u\|^q_{L^{Q^*}(\Omega)}\\ &\geq \left(\frac{1}{2} - \varepsilon|\Omega|^{(Q^*-2)/Q^*}c(Q,s)c^2(\mu)\right)\|u\|^2_{\mathcal{Z}_0} - \delta|\Omega|^{(Q^*-q)/Q^*}(c(Q,s)c^2(\mu))^{q/2}\|u\|^q_{\mathcal{Z}_0}.\end{aligned}$$

Choose ε such that $\varepsilon|\Omega|^{(Q^*-2)/Q^*}c(Q,s)c^2(\mu) < \frac{1}{2}$. It implies that

$$\mathcal{H}(u) \geq \beta\|u\|^2_{\mathcal{Z}_0} - \kappa\|u\|^q_{\mathcal{Z}_0}.$$

Since $q > 2$, so we can choose $\alpha, \rho > 0$ such that $\mathcal{J}_a(u) \geq \alpha$ for $\|u\|_{\mathcal{Z}_0} = \rho$.
(ii) Let $u \in \mathcal{Z}_0$ and $j > 0$. From 4.2, we obtain

$$\begin{aligned}\mathcal{H}(ju) &= \frac{j^2}{2}\|u\|^2_{\mathcal{Z}_0} - \int_\Omega F(\xi, ju(\xi))\,dx\\ &\leq \frac{j^2}{2}\|u\|^2_{\mathcal{Z}_0} - j^\vartheta\int_\Omega m(\xi)|u(\xi)|^\vartheta\,d\xi + \int_\Omega M(\xi)\,d\xi \to -\infty\end{aligned}$$

as $j \to \infty$. Let $e = ju$ for j large then (ii) follows. □

Proposition 4.3. *Let u_n be a sequence in $\mathcal{Z}_0$ such that $\mathcal{H}(u_n) \to c$ and $\|\mathcal{H}'(u_n)\| \to 0$ as $n \to \infty$. Then there exists $u_* \in X_0$ such that, up to subsequence, $u_n \to u_*$ in $\mathcal{Z}_0$.*

Proof. First we prove that u_n is a bounded sequence in $\mathcal{Z}_0$. Using (4.1) with $\varepsilon = 1$, we get

$$\begin{aligned}&\left|\int_{\Omega\cap\{|u_n|\leq\mathcal{R}\}}\left(F(\xi, u_n(\xi)) - \frac{1}{\vartheta}f(\xi, u_n(\xi))u_n(\xi)\right)d\xi\right|\\ &\leq \left(\mathcal{R}^2 + \delta\mathcal{R}^q + \frac{2}{\vartheta}\mathcal{R} + \frac{q}{\vartheta}\delta\mathcal{R}^{q-1}\right)|\Omega|.\end{aligned} \tag{4.3}$$

Furthermore from the assumptions on the sequence $\mathcal{Z}_0$, we can choose $\tilde{c} > 0$ such that

$$|\mathcal{H}(u_n)| \leq \tilde{c} \text{ and } \left|\left\langle \mathcal{H}'(u_n), \frac{u_n}{\|u_n\|_{\mathcal{Z}_0}}\right\rangle\right| \leq \tilde{c} \tag{4.4}$$

Taking into account (1.6), (4.3), and (4.4), we conclude

$$\begin{aligned}\tilde{c}(1 + \|u\|_{\mathcal{Z}_0}) &\geq \mathcal{H}(u_n) - \frac{1}{\vartheta}\langle\mathcal{H}'(u_n), u_n\rangle\\ &\geq \left(\frac{1}{2} - \frac{1}{\vartheta}\right)\|u\|^2_{\mathcal{Z}_0} - \int_{\Omega\cap\{|u_n|\leq\mathcal{R}\}}\left(F(\xi, u_n(\xi)) - \frac{1}{\vartheta}f(\xi, u_n(\xi))u_n(\xi)\right)d\xi\\ &\geq \left(\frac{1}{2} - \frac{1}{\vartheta}\right)\|u\|^2_{\mathcal{Z}_0} - c_1.\end{aligned}$$

It implies that u_n is a bounded sequence in $\mathcal{Z}_0$. Since $\mathcal{Z}_0$ is a Hilbert space, then up to a subsequence, there exists $u_* \in \mathcal{Z}_0$ such that $u_n \rightharpoonup u_*$ weakly in $\mathcal{Z}_0$. From Lemma 2.5, we have

$$\begin{aligned}&u_n \to u_* \text{ in } L^q(\mathbb{H}^N);\ u_n \to u_* \text{ a.e in } \mathbb{H}^N;\\ &\text{there exists } w \in L^q(\mathbb{H}^N) \text{ such that } |u_n(\xi)| \leq w(\xi) \text{ a.e. in } \mathbb{H}^N \text{ and } n \in \mathbb{N}\end{aligned} \tag{4.5}$$

Taking into account (1.6), (4.5), and dominated convergence theorem, we get

$$\begin{aligned} &\int_\Omega f(\xi,u_n(\xi))u_n(\xi)\,d\xi \to \int_\Omega f(\xi,u_*(\xi))u_*(\xi)\,d\xi;\\ &\int_\Omega f(\xi,u_n(\xi))u_*(\xi)\,d\xi \to \int_\Omega f(\xi,u_*(\xi))u_*(\xi)\,d\xi \end{aligned} \tag{4.6}$$

Taking into account the fact that $\|\mathcal{H}'(u_n)\| \to 0$ as $n\to\infty$ and (4.6), we deduce that

$$\begin{aligned} &\|u_n\|^2_{\mathcal{Z}_0} \to \int_\Omega f(\xi,u_*(\xi))u_*(\xi)\,d\xi;\\ &\langle u_n,u_*\rangle_{\mathcal{Z}_0} - \int_\Omega f(\xi,u_n(\xi))u_*(\xi)\,d\xi = \langle \mathcal{H}'(u_n),u_*\rangle \quad \to 0 \text{ as } n\to\infty. \end{aligned} \tag{4.7}$$

From (4.5), (4.6), and (4.7), we get

$$\|u_n\|^2_{\mathcal{Z}_0} \to \|u_n\|^2_{\mathcal{Z}_0} \text{ as } n\to\infty.$$

Hence $\|u_n-u_*\|^2_{\mathcal{Z}_0} = \|u_n\|^2_{\mathcal{Z}_0} + \|u_*\|^2_{\mathcal{Z}_0} - 2\langle u_n,u_*\rangle_{\mathcal{Z}_0} \to 0$ as $n\to\infty$. This completes the proof. □

Proof of Theorem 1.2. Using Mountain Pass Theorem along with Propositions 4.2 and 4.3, there exists a critical point $u_0 \in \mathcal{Z}_0$. Also, $\mathcal{H}(u_0) \geq \alpha > 0 = \mathcal{H}(0)$. It implies $u \not\equiv 0$.

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