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Pointwise inequalities for Sobolev functions on generalized cuspidal domains

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In memory of Prof. Jan Malý

Abstract. Let $\Omega \subset \mathbb{R}^{n-1}$ be a bounded star-shaped domain and Ω_ψ be an outward cuspidal domain with base domain Ω. We prove that for $1 < p \leq \infty$, $W^{1,p}(\Omega_\psi) = M^{1,p}(\Omega_\psi)$ if and only if $W^{1,p}(\Omega) = M^{1,p}(\Omega)$.

Sobolevin funktioiden pisteittäisiä epäyhtälöitä yleistetyissä kärkialueissa

Tiivistelmä. Olkoon $\Omega \subset \mathbb{R}^{n-1}$ rajoitettu tähtimäinen alue ja Ω_ψ ulkoneva kärkialue, jonka kanta-alue on Ω. Arvoilla $1 < p \leq \infty$ osoitamme, että $W^{1,p}(\Omega_\psi) = M^{1,p}(\Omega_\psi)$ jos ja vain jos $W^{1,p}(\Omega) = M^{1,p}(\Omega)$.

1. Introduction

A Sobolev function u on $\mathbb{R}^n$ satisfies the pointwise inequality

$$|u(z_1) - u(z_2)| \leq |z_1 - z_2|(CM(|\nabla u|)(z_1) + CM(|\nabla u|)(z_2))$$

at every Lebesgue points of u, where $M(|\nabla u|)$ is the Hardy–Littlewood maximal function of $|\nabla u|$, see [1, 2, 5, 10]. Motivated by this fact, Hajłasz defined the so-called Hajłasz–Sobolev space $M^{1,p}(U)$ which consists of all $u \in L^p(U)$ with a nonnegative $g \in L^p(U)$ such that for every $z_1, z_2 \in U \setminus E$ with $|E| = 0$, we have

$$|u(z_1) - u(z_2)| \leq |z_1 - z_2|(g(z_1) + g(z_2)),$$

where $U \subset \mathbb{R}^n$ is a domain. For all domains U and any $p \in [1, \infty]$, one always has $M^{1,p}(U) \subset W^{1,p}(U)$. Furthermore, when $p = 1$, the inclusion is strict, see [5, 9]. Since $M^{1,p}(U) = W^{1,p}(U)$ implies that every $u \in W^{1,p}(U)$ supports a global Poincaré inequality on a bounded domain U, see [5], it is a natural question to ask

For which domains $U \subset \mathbb{R}^n$ do we have $M^{1,p}(U) = W^{1,p}(U)$?

In this note, we concentrate on a class of generalized outward cuspidal domains. Let $\Omega \subset \mathbb{R}^{n-1}$ be a bounded star-shaped domain with a star-center $x_o \in \Omega$. Let $\psi\colon (0, 1] \to (0, \infty)$ be a left continuous and non-decreasing function. We consider the

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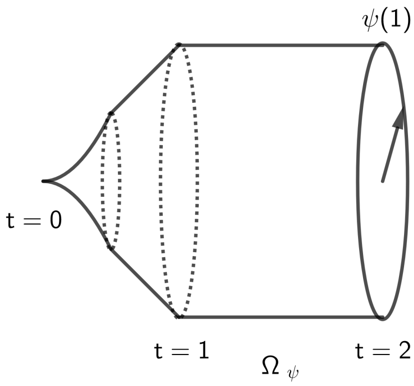
outward cuspidal domain of the form

$$\begin{aligned}\Omega_\psi := &\left\{(t,x)\in(0,1)\times\mathbb{R}^{n-1}\colon \frac{x-x_o}{\psi(t)}+x_o\in\Omega\right\}\\ &\cup\left\{(t,x)\in[1,2)\times\mathbb{R}^{n-1}\colon \frac{x-x_o}{\psi(1)}+x_o\in\Omega\right\},\end{aligned}\tag{1.1}$$



We require ψ is left-continuous just to ensure that Ω_ψ is open. The $(n-1)$-dimensional bounded star-shaped domain Ω is called the base domain of the outward cuspidal domain Ω_ψ. The star-shapeness is necessary to guarantee that Ω_ψ is always a domain for every left continuous and increasing function ψ. One can easily see that Ω_ψ is also star-shaped.

In our main theorem, we will show that the classical Sobolev space coincides with the Hajłasz–Sobolev space on an outward cuspidal domain Ω_ψ if and only if these two function spaces coincide on the base domain Ω. This theorem implies that even on a very irregular domain $U\subset\mathbb{R}^n$, we can have $M^{1,p}(U)=W^{1,p}(U)$, for example see Corollary 1.4 below. Hence, it is difficult, or maybe even impossible to give a characterization to those domains where the classical Sobolev space and Hajłasz-Sobolev spaces coincide.

Theorem 1.2. *Let $\Omega\subset\mathbb{R}^{n-1}$ be a bounded star-shaped domain and $\psi\colon(0,1]\to(0,\infty)$ be a left continuous and increasing function. Define the corresponding cuspidal domain Ω_ψ as in* (1.1). *Then for $1<p\le\infty$, $W^{1,p}(\Omega_\psi)=M^{1,p}(\Omega_\psi)$ if and only if $W^{1,p}(\Omega)=M^{1,p}(\Omega)$.*

By [5], $M^{1,p}(U)$ can always be embedded into $W^{1,p}(U)$, for arbitrary $1\le p\le\infty$ and an arbitrary domain $U\subset\mathbb{R}^n$. Hence, if one can show they coincide as sets, the classical *Open Mapping Theorem* will imply that the corresponding norms of a fixed element are comparable up to a uniform constant. In [12], Romanov showed $W^{1,p}(\Omega_\psi)=M^{1,p}(\Omega_\psi)$ if $\Omega\subset\mathbb{R}^{n-1}$ is a unit ball, $\psi(t)=t^s$ with $s>1$ and $p>\frac{1+(n-1)s}{n}$. The main result in [3] told us that the above restriction on p is superfluous. To be more precise, the authors showed if $\Omega\subset\mathbb{R}^{n-1}$ is an unit ball, $M^{1,p}(\Omega_\psi)=W^{1,p}(\Omega_\psi)$ for arbitrary $1<p\le\infty$ and an arbitrary left continuous and increasing function ψ. Since the unit ball is a bounded star-shaped domain for which $W^{1,p}(B^{n-1}(0,1))=M^{1,p}(B^{n-1}(0,1))$ for every $1<p\le\infty$, the result in [3] is a special case of Theorem 1.2 here.

A domain $U\subset\mathbb{R}^n$ is called a $W^{1,p}$-extension domain for $1\le p\le\infty$, if for every $u\in W^{1,p}(U)$, there exists an extension function $E(u)\in W^{1,p}(\mathbb{R}^n)$ with $E(u)\big|_\Omega\equiv u$

and

$$\|E(u)\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}$$

for a positive constant C independent of u. By the classical result due to Hajłasz [5], if $U \subset \mathbb{R}^n$ is a $W^{1,p}$-extension domain for $1 < p \leq \infty$, then $W^{1,p}(U) = M^{1,p}(U)$. Hence, we have the following corollary to Theorem 1.2.

Corollary 1.3. *Let $\Omega \subset \mathbb{R}^{n-1}$ be a bounded star-shaped $W^{1,p}$-extension domain for $1 < p \leq \infty$ and $\psi\colon (0,1] \to (0,\infty)$ be a left continuous and increasing function. Then we have $W^{1,p}(\Omega_\psi) = M^{1,p}(\Omega_\psi)$, where Ω_ψ is the corresponding outward cuspidal domain defined in (1.1).*

Our theorem enables a large class of new examples. In particular, the following corollary shows how the cuspidal construction can be iterated to give examples which have different cuspidal sigularities in every coordinate directions. We thank Sylvester Eriksson-Bique for pointing out this application.

Corollary 1.4. *Let $I \subset \mathbb{R}$ be an interval which contains 0 and $\{\psi_i\colon (0,1] \to (0,\infty)\}_{i=1}^n$ be a class of left continuous and increasing function. Set $\Omega_1 := I$ and define a sequence of outward cuspidal domains $\{\Omega_i \subset \mathbb{R}^i\}_{i=2}^n$ inductively by setting*

$$\Omega_i := \left\{(t,x) \in (0,1) \times \mathbb{R}^{i-1}\colon \frac{x - x_o^{i-1}}{\psi_i(t)} + x_o^{i-1} \in \Omega_{i-1}\right\}$$
$$\cup \left\{(t,x) \in [1,2) \times \mathbb{R}^{i-1}\colon \frac{x - x_o^{i-1}}{\psi_i(1)} + x_o^{i-1} \in \Omega_{i-1}\right\},$$

where $x_o^{i-1} \subset \Omega_{i-1}$ is a star center. Then $W^{1,p}(\Omega_n) = M^{1,p}(\Omega_n)$ for every $1 < p \leq \infty$.

Proof. By induction, for every $i \in \{1, 2, \cdots, n-1\}$, Ω_i is a star-shaped domain with $W^{1,p}(\Omega_i) = M^{1,p}(\Omega_i)$. The conclusion follows now directly from Theorem 1.2. □

2. Definitions and preliminaries

In what follows, $U \subset \mathbb{R}^n$ is always a domain and $\Omega \subset \mathbb{R}^n$ is always a bounded star-shaped domain. We denote $C^\infty(\overline{U})$ to be the restriction of $C^\infty(\mathbb{R}^n)$ on $\overline{U}$ by setting

$$C^\infty(\overline{U}) := \{u\big|_{\overline{U}}\colon u \in C^\infty(\mathbb{R}^n)\}.$$

For a measurable subset $E \subset \mathbb{R}^n$, χ_E is the corresponding characteristic function and $|E|$ means the n-dimensional Hausdorff measure of E. Typically, c or C will be constants that depend on various parameters and may vary even on the same line of inequalities. The Euclidean distance between points x, y in the Euclidean space $\mathbb{R}^n$ is denoted by $|x - y|$. The open n-dimensional ball of radius r centered at the point x is denoted by $B^n(x, r)$. For two points $x, y \in \mathbb{R}^n$, $[x, y]$ means the segment starting from x to y.

Definition 2.1. A domain $\Omega \subset \mathbb{R}^n$ is said to be star-shaped, if there exists a point $x_o \in \Omega$ such that for every $x \in \Omega$, the segment $[x, x_o]$ between x and x_o is contained in Ω. The point x_o is called the star center of Ω.

For a star-shaped domain, the choice of the star center may not be unique. For example, for a convex domain, every point inside the domain is a star center. From now on, whenever we mention a star-shaped domain Ω, it means we have already fixed a star center $x_o \in \Omega$. Let $\Omega \subset \mathbb{R}^n$ be a bounded star-shaped domain. For every $0 < \lambda < \infty$, we define

$$\lambda\Omega := \left\{ x \in \mathbb{R}^n \colon \frac{x - x_o}{\lambda} + x_o \in \Omega \right\}.$$

We write

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} := \left\{ z := (t, x) \in \mathbb{R} \times \mathbb{R}^{n-1} \right\}.$$

Let $\Omega \subset \mathbb{R}^{n-1}$ be a bounded star-shaped domain. We consider a left continuous and increasing function $\psi \colon (0,1] \to (0,\infty)$, extend the definition of ψ to the interval $(0,2)$ by setting

$$\psi(t) = \psi(1), \quad \text{for every} \ \ t \in (1,2)$$

and write

$$\Omega_\psi = \left\{ (t,x) \in (0,2) \times \mathbb{R}^{n-1} \colon \frac{x - x_o}{\psi(t)} + x_o \in \Omega \right\}.$$

The space of locally integrable functions is denoted by $L^1_{\mathrm{loc}}(U)$. For every measurable set $Q \subset U$ with $0 < |Q| < \infty$, and every non-negative measurable or integrable function f on Q we define the integral average of f over Q by

$$\fint_Q f(w)\, dw := \frac{1}{|Q|} \int_Q f(w)\, dw.$$

Let us give the definitions of Sobolev space $W^{1,p}(U)$ and Hajłasz-Sobolev space $M^{1,p}(U)$.

Definition 2.2. We define the first order Sobolev space $W^{1,p}(U)$, $1 \le p \le \infty$, as the set

$$\{ u \in L^p(U) \colon \nabla u \in L^p(U; \mathbb{R}^n) \}.$$

Here $\nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n} \right)$ is the weak (or distributional) gradient of a locally integrable function u.

We equip $W^{1,p}(U)$ with the norm

$$\|u\|_{W^{1,p}(U)} = \|u\|_{L^p(U)} + \|\nabla u\|_{L^p(U)}$$

for $1 \le p \le \infty$, where $\|\cdot\|_{L^p(U)}$ denotes the usual L^p-norm for $p \in [1,\infty]$. The following lemma from [11, page 13] tells us that on a star-shaped domain, Sobolev functions can be approximated by global smooth functions.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a star-shaped domain. Then $C^\infty(\overline{\Omega}) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$ for $1 \le p \le \infty$.*

For $u \in L^p(\Omega)$, we denote by $\mathcal{D}_p(u)$ the class of functions $0 \le g \in L^p(\Omega)$ for which there exists $E \subset U$ with $|E| = 0$, so that

$$|u(z_1) - u(z_2)| \le |z_1 - z_2| \left(g(z_1) + g(z_2) \right), \quad \text{for} \ \ z_1, z_2 \in U \setminus E.$$

Definition 2.4. We define the Hajłasz–Sobolev space $M^{1,p}(U)$, $1 \le p \le \infty$, as the set

$$\{ u \in L^p(U) \colon \mathcal{D}_p(u) \ne \emptyset \}.$$

We equip $M^{1,p}(U)$ with the norm

$$\|u\|_{M^{1,p}(U)} = \|u\|_{L^p(U)} + \inf_{g \in \mathcal{D}_p(u)} \|g\|_{L^p(U)},$$

for $1 \le p \le \infty$. For $1 < p \le \infty$, we write $W^{1,p}(U) =_C M^{1,p}(U)$ if we have $W^{1,p}(U) = M^{1,p}(U)$ with

$$\frac{1}{C}\|\nabla u\|_{L^p(U)} \le \inf_{g\in\mathcal{D}_p(u)} \|g\|_{L^p(U)} \le C\|\nabla u\|_{L^p(U)}$$

for a positive constant $C > 1$ independent of $u \in W^{1,p}(U)$. The following lemma tells us that the equivalence of the Sobolev space and the Hajłasz–Sobolev space on bounded star-shaped domain is invariant under linear stretching.

Lemma 2.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded star-shaped domain with $W^{1,p}(\Omega) =_C M^{1,p}(\Omega)$ for some $1 < p \le \infty$. Then for every $0 < \lambda < \infty$, we have $W^{1,p}(\lambda\Omega) =_C M^{1,p}(\lambda\Omega)$ with a same constant C.*

Proof. Without loss of generality, we may assume $0 \in \Omega$ is a star center. Fix $0 < \lambda < \infty$. Let $u \in W^{1,p}(\lambda\Omega)$ be arbitrary. We define a function u_λ on Ω by setting

$$u_\lambda(z) := u(\lambda z)$$

for every $z \in \Omega$. Then, by the change of variables formula, we have $u_\lambda \in W^{1,p}(\Omega)$ with

$$\|\nabla u_\lambda\|_{L^p(\Omega)} = \lambda^{1-\frac{n}{p}}\|\nabla u\|_{L^p(\lambda\Omega)}. \tag{2.6}$$

Since $W^{1,p}(\Omega) =_C M^{1,p}(\Omega)$, there exists a function $g_{u_\lambda} \in \mathcal{D}_p(u_\lambda)$ with

$$\frac{1}{C}\|\nabla u_\lambda\|_{L^p(\Omega)} \le \|g_{u_\lambda}\|_{L^p(\Omega)} \le C\|\nabla u_\lambda\|_{L^p(\Omega)}. \tag{2.7}$$

Then we define a function g_u on $\lambda\Omega$ by setting

$$g_u(z) := \frac{1}{\lambda} g_{u_\lambda}\left(\frac{z}{\lambda}\right)$$

for every $z \in \lambda\Omega$. Then for almost every $z_1, z_2 \in \lambda\Omega$, we have

$$\begin{aligned}|u(z_1) - u(z_2)| &= \left|u_\lambda\left(\frac{z_1}{\lambda}\right) - u_\lambda\left(\frac{z_2}{\lambda}\right)\right| \\ &\le |z_1 - z_2|\left(\frac{1}{\lambda}g_{u_\lambda}\left(\frac{z_1}{\lambda}\right) + \frac{1}{\lambda}g_{u_\lambda}\left(\frac{z_2}{\lambda}\right)\right) \\ &\le |z_1 - z_2|(g_u(z_1) + g_u(z_2)).\end{aligned}$$

The change of variables formula implies

$$\|g_u\|_{L^p(\lambda\Omega)} = \lambda^{\frac{n}{p}-1}\|g_{u_\lambda}\|_{L^p(\Omega)}. \tag{2.8}$$

Hence $g_u \in \mathcal{D}_p(u)$. By combining inequalities (2.6), (2.7) and (2.8), we obtain the desired inequality

$$\frac{1}{C}\|\nabla u\|_{L^p(\lambda\Omega)} \le \|g_u\|_{L^p(\lambda\Omega)} \le C\|\nabla u\|_{L^p(\lambda\Omega)}.$$ □

3. Maximal functions

We will define a maximal function $M^\tau[f]$. That will vary only the first component t. For every $x \in \psi(1)\Omega \subset \mathbb{R}^{n-1}$ set

$$S_x := \{t \in \mathbb{R} \colon (t, x) \in \Omega_\psi\}.$$

Let $f \colon \Omega_\psi \to \mathbb{R}$ be measurable and let $(t, x) \in \Omega_\psi$. We define the one-dimensional maximal function in the direction of the first variable by setting

$$M^\tau[f](t, x) := \sup_{[a,b]\ni t} \fint_{[a,b]\cap S_x} |f(s, x)|\, ds\,. \tag{3.1}$$

The supremum is taken over all intervals $[a, b]$ containing t.

The next lemmas tell us that M^τ enjoys the usual L^p-boundedness property. See [3, Lemma 3.1] for a proof.

Lemma 3.2. *Let $1 < p < \infty$. Then for every $f \in L^p(\Omega_\psi)$, $M^\tau[f]$ is measurable and we have*

$$\int_{\Omega_\psi} |M^\tau[f](z)|^p \, dz \leq C \int_{\Omega_\psi} |f(z)|^p \, dz, \tag{3.3}$$

where the constant C is independent of f.

4. Proof of Theorem 1.2

Let $\Omega \subset \mathbb{R}^{n-1}$ be a star-shaped bounded domain with $W^{1,p}(\Omega) = M^{1,p}(\Omega)$ for some $1 < p < \infty$. Then $\Omega_\psi \subset \mathbb{R}^n$ is also a star-shaped domain. By Lemma 2.3, $C^\infty(\overline{\Omega_\psi}) \cap W^{1,p}(\Omega_\psi)$ is dense in $W^{1,p}(\Omega)$.

Lemma 4.1. *Let $u \in C^\infty(\overline{\Omega_\psi}) \cap W^{1,p}(\Omega_\psi)$ be arbitrary. Fix $0 < t < 2$, define the restriction of u to $\{t\} \times \psi(t)\Omega$ by setting*

$$u_t(x) = u(t, x) \quad \text{on every } x \in \psi(1)\Omega. \tag{4.2}$$

Then $u_t \in W^{1,p}(\psi(t)\Omega)$ for every $0 < t < 2$. And there exists a nonnegative function $g_u \in L^p(\Omega_\psi)$ such that for every $t \in (0, 2)$ and every $x, y \in \psi(t)\Omega$ we have

$$|u_t(x) - u_t(y)| \leq |x - y|(g_u(t, x) + g_u(t, y))$$

and

$$\int_{\Omega_\psi} g_u^p(z) \, dz \leq C \int_{\Omega_\psi} |\nabla u(z)|^p \, dz$$

with a constant C independent of u.

Proof. If

$$\int_{\Omega_\psi} |\nabla u(z)|^p \, dz = 0,$$

then $u \equiv c$ on Ω_ψ for some constant $c \in \mathbb{R}$. In this case, we simply define $g_u \equiv 0$ on Ω_ψ. Then we have

$$|u(z_1) - u(z_2)| \leq |z_1 - z_2|(g_u(z_1) + g_u(z_2))$$

for every $z_1, z_2 \in \Omega_\psi$ and

$$\|g_u\|_{L^p(\Omega_\psi)} = \|\nabla u\|_{L^p(\Omega_\psi)}.$$

Let us consider the case that

$$T_u := \int_{\Omega_\psi} |\nabla u(z)|^p \, dz > 0.$$

Denote the gradient with respect to the x-variable by ∇^χ. By Lemma 2.5, for every $t \in (0, 2)$, there exists a nonnegative function $g_t \in L^p(\psi(t)\Omega)$ with

$$|u(x_1) - u(x_2)| \leq |x_1 - x_2|(g_t(x_1) + g_t(x_2)) \tag{4.3}$$

for almost every x_1, $x_2 \in \psi(t)\Omega$ and

$$\|g_t\|_{L^p(\psi(t)\Omega)} \leq C\|\nabla^\chi u_t\|_{L^p(\psi(t)\Omega)} \tag{4.4}$$

for a constant C independent of t. Simply resetting g_t to be ∞ on a measure zero set, we can assume inequality (4.3) holds for every $x_1, x_2 \in \psi(t)\Omega$. We define

$$\hat{g}_t(x) := 2g_t(x) + (T_u)^{\frac{1}{p}}$$

for every $x \in \psi(t)\Omega$. Since $u \in C^\infty(\overline{\Omega_\psi}) \cap W^{1,p}(\Omega_\psi)$, u' is uniformly continuous on $\psi(t)\Omega$, there exists a small enough $0 < \delta < 1$ such that for every $x, y \in \psi(t)\Omega$ with $0 < |x-y| < \delta$, we have

$$|u_t(x) - u_t(y)| < |x-y|(g_t(x) + (T_u)^{\frac{1}{p}}) \le |x-y|(\hat{g}_t(x) + \hat{g}_t(y)). \tag{4.5}$$

Since $\psi(s)\Omega \subset \psi(t)\Omega$ for every $0 < s < t$, there exists a small enough $0 < \epsilon_t^1 < t$ such that for every $s \in (t - \epsilon_t^1, t]$ and every $x, y \in \psi(s)\Omega$ with $|x-y| < \delta$, we have

$$|u_s(x) - u_s(y)| < |x-y|(\hat{g}_t(x) + \hat{g}_t(y)). \tag{4.6}$$

Due to $u \in C^\infty(\overline{\Omega_\psi}) \cap W^{1,p}(\Omega_\psi)$ again, there exists a small enough $0 < \epsilon_t^2 < t$ such that for every $s \in (t - \epsilon_t^2, t]$ and every $x, y \in \psi(s)\Omega$ with $|x-y| \ge \delta$, we have

$$|u_s(x) - u_s(y)| \le |x-y|(\hat{g}_t(x) + \hat{g}_t(y)). \tag{4.7}$$

Hence, we can find a sufficiently small $0 < \epsilon^t \le \min\{\epsilon_t^1, \epsilon_t^2\}$ such that for every $s \in (t - \epsilon_t, t]$ and every $x, y \in \psi(s)\Omega$, we have

$$|u_s(x) - u_s(y)| \le |x-y|(\hat{g}_t(x) + \hat{g}_t(y)), \tag{4.8}$$

and for every $s \in (t - \epsilon_t, t]$, we have

$$\|g_t\|_{L^p(\psi(s)\Omega)} \le C\|\nabla^\chi u_s\|_{L^p(\psi(s)\Omega)} \tag{4.9}$$

with a constant C independent of s and t. By the simply geometry of the line segment $(0, 2]$, there exists an at most countable class $\{(t_i - \epsilon_{t_i}, t_i]\}_{i \in I \subset \mathbb{N}}$ such that

$$(0, 2] \subset \bigcup_{i \in I} (t_i - \epsilon_{t_i}, t_i]$$

and

$$\sum_{i \in I} \chi_{(t_i - \epsilon_{t_i}, t_i]}(t) \le 2$$

for every $t \in (0, 2]$. Simply extend $\hat{g}_t$ to $\mathbb{R}^{n-1}$ by setting it to be 0 outside $\psi(t)\Omega$ and define a function g_u on Ω_ψ by setting

$$g_u(t, x) := \sum_{i \in I} \hat{g}_{t_i}(x) \chi_{(t_i - \epsilon_{t_i}, t_i]}(t) \tag{4.10}$$

for every $z = (t, x) \in \Omega_\psi$. By (4.8) and (4.10), for every $t \in (0, 2]$ and every $x, y \in \psi(t)\Omega$, we have

$$|u_t(x) - u_t(y)| \le |x-y|(g_u(t, x) - g_u(t, y)). \tag{4.11}$$

By the argument above, we obtain $g_u \in L^p(\Omega_\psi)$ with

$$\begin{aligned}
\int_{\Omega_\psi} g_u^p(z)\,dz &\le C \sum_{i \in I} \int_{t_i - \epsilon_{t_i}}^{t_i} \int_{\psi(t)\Omega} \hat{g}_{t_i}^p(x)\,dx\,dt \\
&\le C \sum_{i \in I} \int_{t_i - \epsilon_{t_i}}^{t_i} \int_{\psi(t)\Omega} (g_{t_i}^p(x) + T_u)\,dx\,dt \\
&\le C \int_0^2 \int_{\psi(t)\Omega} |\nabla^\chi u_t(x)|^p\,dx\,dt + CT_u \\
&\le C \int_{\Omega_\psi} |\nabla u(z)|^p\,dz.
\end{aligned}$$

The third inequality above comes from the bounded overlaps of the intervals $\{(t_i - \epsilon_{t_i}, t_i]\}_{i \in I \subset \mathbb{N}}$. □

First, we introduce some results which will be used in the proof of that $W^{1,p}(\Omega) = M^{1,p}(\Omega)$ implies $W^{1,p}(\Omega_\psi) = M^{1,p}(\Omega_\psi)$. By Hajłasz [5], there is a bounded inclusion $\iota\colon M^{1,p}(\Omega_\psi) \hookrightarrow W^{1,p}(\Omega_\psi)$. To show that ι is an isomorphism, it suffices to show that its inverse ι^{-1} is both densely defined and bounded on $W^{1,p}(\Omega_\psi)$. Since $C^\infty(\overline{\Omega_\psi}) \cap W^{1,p}(\Omega_\psi)$ is dense in $W^{1,p}(\Omega_\psi)$, it suffices to show that $C^\infty(\overline{\Omega_\psi}) \cap W^{1,p}(\Omega_\psi) \subset M^{1,p}(\Omega_\psi)$ and that for each $u \in C^\infty(\overline{\Omega_\psi}) \cap W^{1,p}(\Omega_\psi)$ we have

$$||u||_{M^{1,p}(\Omega_\psi)} \le C||u||_{W^{1,p}(\Omega_\psi)},$$

for a positive constant independent of u. The proof of the next lemma is obtained by following the proof of Lemma 4.1 in [3] and replacing instances $M^\chi[|\nabla u|]$ with g_u and repeat the argument.

Lemma 4.12. *Let $\Omega \subset \mathbb{R}^{n-1}$ be a bounded star-shaped domain with $W^{1,p}(\Omega) = M^{1,p}(\Omega)$ for $1 < p < \infty$ and $\psi\colon (0,1] \to (0,\infty)$ be a left continuous and increasing function. Define an outward cuspidal domain Ω_ψ as in (1.1). Let $z_1 = (t_1, x_1), z_2 := (t_2, x_2) \in \Omega_\psi$ be two points with $t_1 < t_2$. Suppose that $u \in W^{1,p}(\Omega_\psi) \cap C^1(\Omega_\psi)$. Then we have*

$$\begin{aligned} |u(z_1) - u(z_2)| \le C|z_1 - z_2|\big(&M^\tau[|\nabla u|](z_1) + M^\tau[g_u](z_1) \\ &+ M^\tau[|\nabla u|](z_2) + M^\tau[g_u](z_2)\big), \end{aligned} \tag{4.13}$$

where g_u comes from Lemma 4.1.

Let us prove the main result in this note.

Proof of Theorem 1.2. By [6, Theorem 7], if U is a bounded domain, $W^{1,\infty}(U) = M^{1,\infty}(U)$ if and only if U is quasiconvex. Recall that a domain U is quasiconvex if there exists a constant $C \ge 1$ such that, for every pair of points $x, y \in U$, there is a rectifiable curve $\gamma \subset U$ joining x to y so that $\mathrm{len}(\gamma) \le C|x - y|$ for a constant C independent of x, y. For every left continuous and increasing function $\psi\colon (0,1] \to (0,\infty)$, Ω_ψ is quasiconvex if and only if Ω is quasiconvex. Hence, we have $W^{1,\infty}(\Omega_\psi) = M^{1,\infty}(\Omega_\psi)$ if and only if $W^{1,\infty}(\Omega) = M^{1,\infty}(\Omega)$.

Fix $1 < p < \infty$. First, we show $M^{1,p}(\Omega) = W^{1,p}(\Omega)$ implies $M^{1,p}(\Omega_\psi) = W^{1,p}(\Omega_\psi)$. By [5], we know that $M^{1,p}(\Omega_\psi)$ can be boundedly embedded into $W^{1,p}(\Omega_\psi)$. To show $W^{1,p}(\Omega_\psi) = M^{1,p}(\Omega_\psi)$ it suffices to show that the dense subspace $C^\infty(\overline{\Omega_\psi}) \cap W^{1,p}(\Omega_\psi)$ of $W^{1,p}(\Omega_\psi)$ is contained in $M^{1,p}(\Omega_\psi)$ with $M^{1,p}$-norm is controlled by $W^{1,p}$-norm from above uniformly. Let $u \in C^\infty(\overline{\Omega_\psi}) \cap W^{1,p}(\Omega_\psi)$ be arbitrary. Set

$$\hat{g}(t,x) = M^\tau[|\nabla u|](t,x) + g_u(t,x) + M^\tau[g_u](t,x). \tag{4.14}$$

Here g_u is defined as in (4.10).

By (4.3) and Lemma 4.12, for every $z_1, z_2 \in \Omega_\psi$, we get the estimate

$$|u(z_1) - u(z_2)| \le C|z_1 - z_2|\left(\hat{g}(z_1) + \hat{g}(z_2)\right).$$

Define $g := C\hat{g} \in \mathcal{D}_p(u)$ for a suitable constant $C > 1$. The triangle inequality gives

$$\int_{\Omega_\psi} |g(z)|^p\,dz \le C\left(\int_{\Omega_\psi} M^\tau[|\nabla u|](z)^p\,dz + \int_{\Omega_\psi} g_u(z)^p\,dz + \int_{\Omega_\psi} M^\tau[g_u](z)^p\,dz\right).$$

The inequality (4.4) and the fact that $|\nabla^\chi u(z)| \le |\nabla u(z)|$ almost everywhere leads to the estimate

$$\int_{\Omega_\psi} g_u(z)^p\,dz \le C\int_{\Omega_\psi} |\nabla u(z)|^p dz. \tag{4.15}$$

Lemma 3.2 leads to the estimates

$$\int_{\Omega_\psi} |M^\tau[|\nabla u|](z)|^p\, dz \le C \int_{\Omega_\psi} |\nabla u(z)|^p\, dz$$

and

$$\int_{\Omega_\psi} |M^\tau[g_u](z)|^p\, dz \le C \int_{\Omega_\psi} g_u(z)^p\, dz \le C \int_{\Omega_\psi} |\nabla u(z)|^p\, dz,$$

which imply that $g \in \mathcal{D}_p(u)$ and that

$$\|u\|_{M^{1,p}(\Omega_\psi)} \le C\|u\|_{W^{1,p}(\Omega_\psi)}.$$

That is, $C^\infty(\overline{\Omega_\psi}) \cap W^{1,p}(\Omega_\psi)$ can be boundedly embedded into $M^{1,p}(\Omega_\psi)$. Hence, we proved that $W^{1,p}(\Omega) = M^{1,p}(\Omega)$ implies $W^{1,p}(\Omega_\psi) = M^{1,p}(\Omega_\psi)$.

Next, we prove $W^{1,p}(\Omega_\psi) = M^{1,p}(\Omega_\psi)$ implies $W^{1,p}(\Omega) = M^{1,p}(\Omega)$. Since Ω is star-shaped, by a similar argument as above, it suffices to show the dense subspace $C^\infty(\overline{\Omega}) \cap W^{1,p}(\Omega)$ can be boundedly embedded into $M^{1,p}(\Omega)$. Let $u \in C^\infty(\overline{\Omega}) \cap W^{1,p}(\Omega)$ be arbitrary. If $u \equiv c$ for some constant $c \in \mathbb{R}$, then $u \in M^{1,p}(\Omega)$ with

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{M^{1,p}(\Omega)}.$$

Hence, we assume u is not a constant function. Now, suppose that

$$\|\nabla u\|_{L^p(\Omega)} > 0.$$

As before, we assume $0 \in \Omega$ is a star center. We define a function $\tilde{u}$ on $\psi(1)\Omega$ by setting

$$\tilde{u}(x) = u\left(\frac{x}{\psi(1)}\right) \quad \text{for every } x \in \psi(1)\Omega.$$

The change of variables formula implies

$$\|\tilde{u}\|_{L^p(\psi(1)\Omega)} = \psi(1)^{\frac{-n}{p}}\|u\|_{L^p(\Omega)} \quad \text{and} \quad \|\nabla^\chi \tilde{u}\|_{L^p(\psi(1)\Omega)} = \psi(1)^{1-\frac{n}{p}}\|\nabla^\chi u\|_{L^p(\Omega)}. \tag{4.16}$$

Hence, $\tilde{u} \in C^1(\psi(1)\Omega) \cap W^{1,p}(\psi(1)\Omega)$. Simply by the geometry, we can write

$$\Omega_\psi := \bigcup_{x \in \psi(1)\Omega} S_x.$$

We define a function $\hat{u}$ on Ω_ψ by setting

$$\hat{u}(t,x) := \tilde{u}(x) \quad \text{for every } (t,x) \in \Omega_\psi.$$

Since $\psi(t)\Omega \subset \psi(1)\Omega$ for every $t \in (0,2)$, we have $\hat{u} \in C^1(\Omega_\psi)$ with

$$\|\hat{u}\|_{L^p(\Omega_\psi)} \le 2\|\tilde{u}\|_{L^p(\psi(1)\Omega)} \quad \text{and} \quad \|\nabla \hat{u}\|_{L^p(\Omega_\psi)} \le 2\|\nabla^\chi \tilde{u}\|_{L^p(\psi(1)\Omega)}.$$

Hence, $\hat{u} \in C^1(\Omega_\psi) \cap W^{1,p}(\Omega_\psi)$. Since $W^{1,p}(\Omega_\psi) = M^{1,p}(\Omega_\psi)$, there exists $g \in \mathcal{D}_p(\hat{u})$ with

$$\|g\|_{L^p(\Omega_\psi)} \le C\|\nabla \hat{u}\|_{L^p(\Omega_\psi)}$$

and

$$|\hat{u}(z_1) - \hat{u}(z_2)| \le |z_1 - z_2|(g(z_1) + g(z_2))$$

for almost every $z_1, z_2 \in \Omega_\psi$. We assume last inequality holds at every $z_1, z_2 \in \Omega_\psi$ by simply setting $g = \infty$ on a measure-zero set. Set g_t to be the restriction of g to $\{t\} \times \psi(t)\Omega$ and define

$$A := \inf_{t \in (1,2)} \|g_t\|_{L^p(\psi(t)\Omega)}.$$

Since u is not a constant function, we have

$$0 < A \le C\|\nabla \hat{u}\|_{L^p(\Omega_\psi)} \le C\|\nabla^\chi \tilde{u}\|_{L^p(\psi(1)\Omega)}.$$

There exists $\hat{t} \in (1,2)$ with

$$A \leq \|g_{\hat{t}}\|_{L^p(\psi(\hat{t})\Omega)} \leq 2A.$$

Then for every $x_1, x_2 \in \psi(1)\Omega$, we have

$$|\tilde{u}(x_1) - \tilde{u}(x_2)| = |\hat{u}(\hat{t}, x_1) - \hat{u}(\hat{t}, x_2)| \leq |x_1 - x_2|(g_{\hat{t}}(x_1) + g_{\hat{t}}(x_2)).$$

Hence, we have $g_{\hat{t}} \in \mathcal{D}_p(\tilde{u})$ with

$$\|g_{\hat{t}}\|_{L^p(\psi(1)\Omega)} \leq C\|\nabla^\chi \tilde{u}\|_{L^p(\psi(1)\Omega)}.$$

Define a function g on Ω by setting

$$g(x) := \frac{1}{\psi(1)} g_{\hat{t}}(\psi(1)x) \quad \text{for every } x \in \Omega.$$

Then, we have

$$|u(x_1) - u(x_2)| \leq |x_1 - x_2|(g(x_1) + g(x_2))$$

for every $x_1, x_2 \in \Omega$, and

$$\|g\|_{L^p(\Omega)} = \psi(1)^{\frac{n}{p}-1}\|g_{\hat{t}}\|_{L^p(\psi(1)\Omega)}. \tag{4.17}$$

Hence, we obtain $g \in \mathcal{D}_p(u)$ with

$$\|g\|_{L^p(\Omega)} \leq C\|\nabla^\chi u\|_{L^p(\Omega)}$$

for a constant C independent of u. Hence, we have $C^\infty(\overline{\Omega}) \cap W^{1,p}(\Omega) \subset M^{1,p}(\Omega)$ with

$$\|u\|_{M^{1,p}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$$

as desired. □

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