

# Existence and multiplicity of normalized solutions for a class of fractional Schrödinger–Poisson equations

ZHIPENG YANG, FUKUN ZHAO\* and SHUNNENG ZHAO

**Abstract.** We consider the fractional Schrödinger–Poisson equation

$$\begin{cases} (-\Delta)^s u - \lambda u + \phi u = |u|^{p-2}u, & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & x \in \mathbb{R}^3, \end{cases}$$

where  $s, t \in (0, 1)$  satisfies  $2s + 2t > 3$ ,  $p \in (\frac{4s+6}{3}, 2_s^*)$  and  $\lambda \in \mathbb{R}$  is an undetermined parameter. We deal with the case where the associated functional is not bounded below on the  $L^2$ -unit sphere and show the existence of infinitely many solutions  $(u, \lambda)$  with  $u$  having prescribed  $L^2$ -norm.

## Murtoasteisten Schrödingerin–Poissonin yhtälöiden luokan normitettujen ratkaisujen olemassaolo ja monikäsitteisyys

**Tiivistelmä.** Tarkastelemme murtoasteista Schrödingerin–Poissonin yhtälöä

$$\begin{cases} (-\Delta)^s u - \lambda u + \phi u = |u|^{p-2}u, & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & x \in \mathbb{R}^3, \end{cases}$$

missä luvut  $s, t \in (0, 1)$  toteuttavat ehdon  $2s + 2t > 3$ ,  $p \in (\frac{4s+6}{3}, 2_s^*)$  ja  $\lambda \in \mathbb{R}$  on määrittämätön parametri. Käsittelemme tapausta, jossa vastaava funktionaali ei ole alhaalta rajattu avaruuden  $L^2$  yksikköpallonkuorella, ja osoitamme, että em. yhtälöllä on äärettömästi ratkaisuja  $(u, \lambda)$ , jossa funktiolla  $u$  on annettu  $L^2$ -normi.

## 1. Introduction and the main results

In this paper, we study the following stationary fractional Schrödinger–Poisson equation

$$(1.1) \quad \begin{cases} (-\Delta)^s u - \lambda u + \phi u = |u|^{p-2}u, & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & x \in \mathbb{R}^3, \end{cases}$$

where  $s, t \in (0, 1)$  satisfies  $2s + 2t > 3$ ,  $p \in (\frac{4s+6}{3}, 2_s^*)$  and  $\lambda \in \mathbb{R}$ ,  $2_s^* = \frac{6}{3-2s}$  is the fractional critical exponent. Part of the interest is the fact that solutions  $(u(x), \phi(x))$  of (1.1) are related to standing wave solutions  $(e^{-i\lambda t}u(x), \phi(x))$  of the time-dependent system

$$(1.2) \quad \begin{cases} i \frac{\partial \Psi}{\partial t} = (-\Delta)^s \Psi + \phi \Psi - \tilde{f}(x, |\Psi|) \Psi & \text{in } \mathbb{R}^3 \times \mathbb{R}, \\ (-\Delta)^t \phi = |\Psi|^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where  $i$  is the imaginary unit and  $\tilde{f}(x, |u|)u = f(x, u)$ .

The first equation in (1.2) was introduced by Laskin (see [21, 22]) and comes from an expansion of the Feynman path integral from Brownian-like to Lévy-like quantum mechanical paths. This class of fractional Schrödinger equation with a

---

<https://doi.org/10.54330/afm.119450>

2020 Mathematics Subject Classification: Primary 35Q40, 35J50, 58E05.

Key words: Variational method, fractional Schrödinger–Poisson, normalized solutions.

\*Corresponding author.

© 2022 The Finnish Mathematical Society

repulsive nonlocal Coulombic potential is obtained by approximation of the Hartree–Fock equation describing a quantum mechanical system of many particles; see, for instance, [14, 23, 24, 38]. It also appeared in several areas such as optimization, finance, phase transitions, stratified materials, crystal dislocation, flame propagation, conservation laws, materials science and water waves (see [9]).

A first line of studying (1.1) is to consider  $\lambda \in \mathbb{R}$  as a fixed parameter and then to search for critical points of the functional

$$(1.3) \quad I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx - \frac{\lambda}{2} \int_{\mathbb{R}^3} |u|^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|^{3-2t}} dx dy - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx,$$

which is well defined and  $C^1$  in  $H_r^s(\mathbb{R}^3)$ . In that direction, to the best of our knowledge, there are only few papers considered the existence and multiplicity of solutions to the fractional Schrödinger–Poisson system (1.1). In [40], the authors studied the existence of radial solutions by using the constrained minimization methods for system (1.1) with  $\lambda = 0$  and Berestycki–Lions type conditions [7]. In [33, 34], Teng considers the fractional Schrödinger–Poisson system (1.1) with subcritical and critical nonlinearity respectively. By the monotone trick, concentration-compactness principle and a global compactness Lemma he establishes the existence of ground state solutions. For other existence results we refer to [13, 19, 25, 26, 27, 29, 35, 36, 37, 39] and the references therein.

In present paper, motivated by the fact that physicists are often interested in “normalized solutions”, we look for solutions in  $H_r^s(\mathbb{R}^3)$  having a prescribed  $L^2$ -norm. More precisely, for given  $c > 0$ , we look to

$$(u_c, \lambda_c) \in H_r^s(\mathbb{R}^3) \times \mathbb{R} \text{ is solution of (1.1) with } |u_c|_2^2 = c.$$

In this case, a solution  $u_c \in H_r^s(\mathbb{R}^3)$  of (1.1) can be obtained as a constrained critical point of the functional

$$J(u) = \frac{1}{2} |(-\Delta)^{\frac{s}{2}} u|_2^2 + \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|^{3-2t}} dx dy - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx.$$

on the constraint

$$S_r(c) := \{u \in H_r^s(\mathbb{R}^3) : |u|_2^2 = c, c > 0\}.$$

The parameter  $\lambda_c \in \mathbb{R}$ , in this situation, can not be fixed any more and it appears as a Lagrange parameter.

Set

$$(1.4) \quad e(c) := \inf_{u \in S_r(c)} J(u).$$

It is standard that minimizers of  $e(c)$  are exactly critical points of  $J(u)$  restricted to  $S_r(c)$ , and thus solutions of (1.1). By the  $L^2$ -preserving scaling and the fractional Gagliardo–Nirenberg inequality with best constant [8]: Let  $p \in (2, 2_s^*)$ , then

$$|u|_p^p \leq \frac{p}{2|Q|_p^{p-2}} |(-\Delta)^{\frac{s}{2}} u|_2^{\frac{3(p-2)}{2s}} |u|_2^{p - \frac{3(p-2)}{2s}},$$

with equality for  $u = Q$ , where  $Q$  is, up to translations, the unique positive ground state solution of

$$\frac{3(p-2)}{4s} (-\Delta)^s u + \left(1 + \frac{p-2}{4} \left(2 - \frac{3}{s}\right)\right) u - |u|^{p-2} u = 0, \quad x \in \mathbb{R}^3.$$

Therefore, we can prove that  $p = \frac{4s+6}{3}$  is the  $L^2$ -critical exponent for (1.4), i.e. for all  $c > 0$ ,  $e(c) > -\infty$  if  $2 < p < \frac{4s+6}{3}$  and  $e(c) = -\infty$  if  $\frac{4s+6}{3} < p < 2_s^*$ .

The above normalized problem associated to (1.4) with  $s = 1$ , has been studied in the literature [3, 5, 6, 16, 18, 20, 30]. In the cited references, the existence and no-existence of normalized solutions are established, depending strongly on the value  $p \in (2, 2^*)$  and of the parameter  $c > 0$ . Precisely, it is proved that a solution which minimized globally  $J$  on  $S_r(c)$ , exists when  $p \in (2, 3)$  and  $c > 0$  small enough. When  $p \in (3, \frac{10}{3})$ , there exists a  $c_0 > 0$  such that such a solution exists if and only if  $c \geq c_0$ . When  $p \in (\frac{10}{3}, 2^*)$ , it is not possible to find a solution as a global minimizer of  $J$  on  $S_r(c)$  since the associated functional is not bounded below on the  $L^2$ -unit sphere. However, it is proved in [4] that for  $c > 0$  sufficiently small, there exists a critical point which minimizes the energy among all solutions on  $S_r(c)$  and infinitely many normalized solutions in [28].

For the nonlocal problem, that  $s \in (0, 1)$ , up to our knowledge, in the existing literature, results in this direction do not exist yet. Our contribution in this paper is that there exist normalized solutions of (1.1) for  $p \in (\frac{4s+6}{3}, 2_s^*)$ . The solutions are obtained as critical points of the functional  $J$  on a suitable submanifold of the constraint set  $S_r(c)$ . We state our main results as follows.

**Theorem 1.1.** *Assume that  $s, t \in (0, 1)$  satisfies  $2s + 2t > 3$  and  $p \in (\frac{4s+6}{3}, 2_s^*)$ . There exists a  $c_0 > 0$  such that for any  $c \in (0, c_0)$ , Eq. (1.1) admits an unbounded sequence of distinct pairs of radial solutions  $(\pm u_n, \lambda_n)$  with  $|u_n|_2^2 = c$  and  $\lambda_n < 0$  for each  $n \in \mathbb{N}$ .*

We give the main idea in the proof of our main results. To prove Theorem 1.1, because  $e(c) = -\infty$  for all  $c > 0$ , the genus of the sublevel sets  $J^c = \{u \in S_r(c) \mid J(u) \leq c\}$  is always infinite, so classical arguments based on the Kranoselski genus, see [32], do not apply. Secondly, it can be easily checked that the functional  $J$ , restricted to  $S_r(c)$ , does not satisfy the Palais–Smale condition, even working on the subspace  $H_r^s(\mathbb{R}^3)$  of radially symmetric functions where one has the advantage of the compact embedding of  $H_r^s(\mathbb{R}^3)$  into  $L^q(\mathbb{R}^3)$  for  $q \in (2, 2_s^*)$ . To overcome these difficulties we are inspired by a recent work [2] and [28]. The authors present a new type of linking geometry for the functional  $J$  on  $S_r(c)$  and set up a min-max scheme where the cohomological index for spaces with an action on the group  $G := \{-1, 1\}$  is used. Following [17], for each fixed  $n \in \mathbb{N}$ , we can construct a special Palais–Smale sequence. That construction leads easily to get the boundedness and further non-vanishing of the Palais–Smale sequence.

## 2. Variational settings and preliminary results

Throughout this paper, we denote  $|\cdot|_q$  the usual norm of the space  $L^q(\mathbb{R}^3)$ ,  $1 \leq q < \infty$ ,  $B_r(x)$  denotes the open ball with center at  $x$  and radius  $r$ ,  $C$  or  $C_i$  ( $i = 1, 2, \dots$ ) denote some positive constants may change from line to line.  $\rightharpoonup$  and  $\rightarrow$  mean the weak and strong convergence.

**2.1. The functional space setting.** Firstly, fractional Sobolev spaces are the convenient setting for our problem, so we will give some sketches of the fractional order Sobolev spaces and the complete introduction can be found in [11]. We recall that, for any  $s \in (0, 1)$ , the fractional Sobolev space  $H^s(\mathbb{R}^3) = W^{s,2}(\mathbb{R}^3)$  is defined

as follows:

$$H^s(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\xi|^{2s} |\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2) d\xi < \infty\},$$

whose norm is defined as

$$\|u\|^2 = \int_{\mathbb{R}^3} (|\xi|^{2s} |\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2) d\xi,$$

where  $\mathcal{F}$  denotes the Fourier transform. We also define the homogeneous fractional Sobolev space  $\mathcal{D}^{s,2}(\mathbb{R}^3)$  as the completion of  $\mathcal{C}_0^\infty(\mathbb{R}^3)$  with respect to the norm

$$[u]_{H^s(\mathbb{R}^3)} := \left( \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} dx dy \right)^{\frac{1}{2}}.$$

Obviously,  $H^s(\mathbb{R}^3)$  is a Hilbert space. A function is radial if and only if it is invariant under all rotations leaving the origin fixed. Let  $H_r^s(\mathbb{R}^3)$  denote the subset of  $H^s(\mathbb{R}^3)$  containing only the radial function, and equipped topology with  $H^s(\mathbb{R}^3)$ , which is also a Hilbert space.

The fractional Laplace,  $(-\Delta)^s u$ , of a smooth function  $u: \mathbb{R}^3 \rightarrow \mathbb{R}$ , is defined by

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s} \mathcal{F}(u)(\xi), \quad \xi \in \mathbb{R}^3.$$

Moreover,  $(-\Delta)^s u$  can be equivalently represented [11] as

$$(-\Delta)^s u(x) = -\frac{1}{2} C(s) \int_{\mathbb{R}^3} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{3+2s}} dy, \quad \forall x \in \mathbb{R}^3,$$

where

$$C(s) = \left( \int_{\mathbb{R}^3} \frac{(1 - \cos \xi_1)}{|\xi|^{3+2s}} d\xi \right)^{-1}, \quad \xi = (\xi_1, \xi_2, \xi_3).$$

Also, by the Plancherel formular in Fourier analysis, we have

$$[u]_{H^s(\mathbb{R}^3)}^2 = \frac{2}{C(s)} |(-\Delta)^{\frac{s}{2}} u|_2^2.$$

As a consequence, the norms on  $H^s(\mathbb{R}^3)$  defined below are equivalent:

$$\begin{aligned} u &\longmapsto \left( \int_{\mathbb{R}^3} |u|^2 dx + \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} dx dy \right)^{\frac{1}{2}}; \\ u &\longmapsto \left( \int_{\mathbb{R}^3} (|\xi|^{2s} |\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2) d\xi \right)^{\frac{1}{2}}; \\ u &\longmapsto \left( \int_{\mathbb{R}^3} |u|^2 dx + |(-\Delta)^{\frac{s}{2}} u|_2^2 \right)^{\frac{1}{2}}. \end{aligned}$$

For the reader's convenience, we review some useful results for  $H^s(\mathbb{R}^3)$  and  $H_r^s(\mathbb{R}^3)$ , which will be used later.

**Lemma 2.1.** [11] *Let  $0 < s < 1$ , then there exists a constant  $C = C(s) > 0$ , such that*

$$|u|_{2_s^*}^2 \leq C [u]_{H^s(\mathbb{R}^3)}^2$$

for every  $u \in H^s(\mathbb{R}^3)$ . Moreover, the embedding  $H^s(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$  is continuous for any  $q \in [2, 2_s^*]$  and is locally compact whenever  $q \in [2, 2_s^*)$ .

**Lemma 2.2.** [31] *If  $\{u_n\}$  is bounded in  $H^s(\mathbb{R}^3)$  and for some  $R > 0$  we have*

$$\limsup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^3} \int_{B_R(y)} |u_n|^2 dx = 0,$$

then  $u_n \rightarrow 0$  in  $L^q(\mathbb{R}^3)$  for any  $2 < q < 2_s^*$ .

**Lemma 2.3.** [15] *Let  $2 < q < 2_s^*$ , then every bounded sequence  $\{u_n\} \subset H_r^s(\mathbb{R}^3)$  has a convergent subsequence in  $L^q(\mathbb{R}^3)$ .*

**2.2. Some preliminary lemmas.** We first establish some useful preliminary results. Let  $\{V_n\} \subset H_r^s(\mathbb{R}^3)$  be a strictly increasing sequence of finite-dimensional linear subspaces in  $H_r^s(\mathbb{R}^3)$ , such that  $\bigcup_n V_n$  is dense in  $H_r^s(\mathbb{R}^3)$ . We denote by  $V_n^\perp$  the orthogonal space of  $\{V_n\}$  in  $H_r^s(\mathbb{R}^3)$ .

**Lemma 2.4.** *Assume that  $p \in (2, 2_s^*)$ . Then there holds*

$$\mu_n := \inf_{u \in V_{n-1}^\perp} \frac{\int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u|^2 + |u|^2) dx}{\left(\int_{\mathbb{R}^3} |u|^p dx\right)^{\frac{2}{p}}} = \inf_{u \in V_{n-1}^\perp} \frac{\|u\|^2}{|u|_p^2} \rightarrow \infty, \quad \text{as } n \rightarrow \infty.$$

*Proof.* Arguing by contradiction, suppose there exists a sequence  $\{u_n\} \subset H_r^s(\mathbb{R}^3)$  such that  $u_n \in V_{n-1}^\perp$ ,  $|u_n|_p = 1$ , and  $\|u_n\| \rightarrow c < \infty$ . Then there exists  $u \in H_r^s(\mathbb{R}^3)$  with  $u_n \rightharpoonup u$  in  $H_r^s(\mathbb{R}^3)$  and  $u_n \rightarrow u$  in  $L^p(\mathbb{R}^3)$  up to a subsequence. Let  $v \in H_r^s(\mathbb{R}^3)$  and  $\{v_n\} \subset H_r^s(\mathbb{R}^3)$  such that  $v_n \in V_{n-1}$  and  $v_n \rightarrow v$  in  $H_r^s(\mathbb{R}^3)$ . We have, in  $H_r^s(\mathbb{R}^3)$ ,

$$|\langle u_n, v \rangle| \leq |\langle u_n, v - v_n \rangle| + |\langle u_n, v_n \rangle| \leq \|u_n\| \|v - v_n\| \rightarrow 0,$$

so that  $u_n \rightarrow 0 = u$ , while  $|u|_p = 1$ , a contradiction. □

Now for  $c > 0$  fixed and for each  $n \in \mathbb{N}$ , we define

$$\rho_n := \frac{\mu_n^{\frac{p}{p-2}}}{K^{\frac{2}{p-2}}} \quad \text{with} \quad K = \max_{|(-\Delta)^{\frac{s}{2}} u|_2 > 0} \frac{(|(-\Delta)^{\frac{s}{2}} u|_2^2 + c)^{\frac{p}{2}}}{|(-\Delta)^{\frac{s}{2}} u|_2^p + c^{\frac{p}{2}}},$$

and

$$(2.1) \quad B_n := \{u \in V_{n-1}^\perp \cap S_r(c) : |(-\Delta)^{\frac{s}{2}} u|_2^2 = \rho_n\}.$$

We also define

$$(2.2) \quad b_n := \inf_{u \in B_n} J(u).$$

Then we have

**Lemma 2.5.** *For any  $p \in (2, 2_s^*)$ ,  $b_n \rightarrow +\infty$  as  $n \rightarrow \infty$ . In particular, we can assume without restriction that  $b_n \geq 1$  for all  $n \in \mathbb{N}$ .*

*Proof.* For any  $u \in B_n$ , we have that

$$\begin{aligned} J(u) &= \frac{1}{2} |(-\Delta)^{\frac{s}{2}} u|_2^2 + \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|^{3-2t}} dx dy - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx \\ &\geq \frac{1}{2} |(-\Delta)^{\frac{s}{2}} u|_2^2 - \frac{1}{2} \frac{(|(-\Delta)^{\frac{s}{2}} u|_2^2 + c)^{\frac{p}{2}}}{p \mu_n^{\frac{p}{p-2}}} \\ &\geq \frac{1}{2} |(-\Delta)^{\frac{s}{2}} u|_2^2 - \frac{K}{2} \frac{(|(-\Delta)^{\frac{s}{2}} u|_2^p + c^{\frac{p}{2}})}{p \mu_n^{\frac{p}{p-2}}} \\ &= \left(\frac{1}{2} - \frac{1}{p}\right) \rho_n - \frac{1}{2} c^{\frac{p}{2}}. \end{aligned}$$

From this estimate and Lemma 2.4, it follow since  $p > 2$ , that  $b_n \rightarrow +\infty$  as  $n \rightarrow \infty$ . Next, considering the sequence  $\{V_n\} \subset H_r^s(\mathbb{R}^3)$  only from an  $n_0 \in \mathbb{N}$  such that  $b_n \geq 1$  for any  $n \geq n_0$  it concludes the proof of the Lemma.  $\square$

Next we start to set up our min-max scheme. First we introduce the map

$$(2.3) \quad m: H_r^s(\mathbb{R}^3) \times \mathbb{R} \rightarrow H_r^s(\mathbb{R}^3), \quad m(u, \theta) = u * \theta,$$

be the action of group  $\mathbb{R}$  on  $H_r^s(\mathbb{R}^3)$  defined by

$$(2.4) \quad m(u, \theta)(x) = (u * \theta)(x) = e^{\frac{3\theta}{2}} u(e^\theta x).$$

Observe that for any given  $u \in S_r(c)$ , we have  $m(u, \theta) \in S_r(c)$  for all  $\theta \in \mathbb{R}$ .

**Lemma 2.6.** *Assume that  $u \in S_r(c)$  be arbitrary but fixed. Let  $A(u) := \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx$ , then we have*

- (i)  $A(m(u, \theta)) \rightarrow 0$  and  $J(m(u, \theta)) \rightarrow 0$  as  $\theta \rightarrow -\infty$ .
- (ii)  $A(m(u, \theta)) \rightarrow +\infty$  and  $J(m(u, \theta)) \rightarrow -\infty$  as  $\theta \rightarrow +\infty$ .

*Proof.* A straightforward calculation shows that

$$A(m(u, \theta)) = e^{2s\theta} |(-\Delta)^{\frac{s}{2}} u|_2^2 \rightarrow 0 \quad \text{as } \theta \rightarrow -\infty,$$

and

$$A(m(u, \theta)) = e^{2s\theta} |(-\Delta)^{\frac{s}{2}} u|_2^2 \rightarrow +\infty \quad \text{as } \theta \rightarrow +\infty.$$

Next, we get for  $\theta < 0$ ,

$$\begin{aligned} |J(m(u, \theta))| &= \left| \frac{1}{2} |(-\Delta)^{\frac{s}{2}} m(u, \theta)|_2^2 + \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|m(u, \theta)(x)|^2 |m(u, \theta)(y)|^2}{|x - y|^{3-2t}} dx dy \right. \\ &\quad \left. - \frac{1}{p} \int_{\mathbb{R}^3} |m(u, \theta)|^p dx \right| \\ &\leq \frac{e^{2s\theta}}{2} |(-\Delta)^{\frac{s}{2}} u|_2^2 + \frac{e^{(3-2t)\theta}}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x - y|^{3-2t}} dx dy \\ &\quad + \frac{e^{\frac{3\theta(p-2)}{2}}}{p} \int_{\mathbb{R}^3} |u|^p dx. \end{aligned}$$

Thus  $J(m(u, \theta)) \rightarrow 0$  as  $\theta \rightarrow -\infty$  and point (i) holds. Moreover, we have for  $\theta > 0$ ,

$$\begin{aligned} J(m(u, \theta)) &= \frac{1}{2} |(-\Delta)^{\frac{s}{2}} m(u, \theta)|_2^2 + \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|m(u, \theta)(x)|^2 |m(u, \theta)(y)|^2}{|x - y|^{3-2t}} dx dy \\ &\quad - \frac{1}{p} \int_{\mathbb{R}^3} |m(u, \theta)|^p dx \\ &= \frac{e^{2s\theta}}{2} |(-\Delta)^{\frac{s}{2}} u|_2^2 + \frac{e^{(3-2t)\theta}}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x - y|^{3-2t}} dx dy \\ &\quad - \frac{e^{\frac{3\theta(p-2)}{2}}}{p} \int_{\mathbb{R}^3} |u|^p dx. \end{aligned}$$

Taking into account that  $2s + 2t > 3$  and  $p \in (\frac{4s+6}{3}, 2_s^*)$  it shows that  $J(m(u, \theta)) \rightarrow -\infty$  as  $\theta \rightarrow +\infty$ .  $\square$

Due to Lemma 2.6, using the fact that  $V_n$  is finite dimensional, we deduce that, for each  $n \in \mathbb{N}$ , there exists a  $\theta_n > 0$ , such that

$$\bar{\gamma}_n: [0, 1] \times (S_r(c) \cap V_n) \rightarrow S_r(c), \quad \bar{\gamma}_n(t, u) = m(u, (2t - 1)\theta_n)$$

satisfies

$$(2.5) \quad A(\bar{\gamma}_n(0, u)) < \rho_n, \quad A(\bar{\gamma}_n(1, u)) > \rho_n,$$

and

$$(2.6) \quad J(\bar{\gamma}_n(0, u)) < b_n, \quad J(\bar{\gamma}_n(1, u)) < b_n.$$

Now we define

$$\Gamma_n := \left\{ \gamma: [0, 1] \times (S_r(c) \cap V_n) \rightarrow S_r(c) \mid \gamma \text{ is continuous, odd in } u \right. \\ \left. \text{and such that } \forall u: \gamma(0, u) = \bar{\gamma}_n(0, u), \gamma(1, u) = \bar{\gamma}_n(1, u) \right\}.$$

Clearly we have  $\bar{\gamma}_n \in \Gamma_n$ . Now we give the follow linking property, due to [2].

**Lemma 2.7.** *For every  $\gamma \in \Gamma_n$ , there exists  $(t, u) \in [0, 1] \times (S_r(c) \cap V_n)$  such that  $\gamma(t, u) \in B_n$ .*

*Proof.* In order to prove this lemma, we first recall some properties of the cohomological index for spaces with an action of the group  $G = \{1, -1\}$ . This index goes back to [10] and has been used in a variational setting in [12]. It associates to a  $G$ -space  $X$  an element  $i(X) \in \mathbb{N}_0 \cup \{\infty\}$ . We shall need the following properties[1, 32]:

- (I<sub>1</sub>) If  $G$  acts on  $\mathbb{S}^{n-1}$  via multiplication, then  $i(\mathbb{S}^{n-1}) = n$ .
- (I<sub>2</sub>) If there exists an equivariant map  $X \rightarrow Y$ , then  $i(X) \leq i(Y)$ .
- (I<sub>3</sub>) Let  $X = X_0 \cup X_1$  be metrisable and  $X_0, X_1 \subset X$  be closed  $G$ -invariant subspaces. Let  $Y$  be a  $G$ -space, and consider a continuous map  $\phi: [0, 1] \times Y \rightarrow X$  such that each  $\phi_t = \phi(t, \cdot): Y \rightarrow X$  is equivariant. If  $\phi_0(Y) \subset X_0$  and  $\phi_1(Y) \subset X_1$ , then

$$i(\text{Im}(\phi) \cap X_0 \cap X_1) \geq i(Y).$$

Now, let  $P_{n-1}: H_r^s(\mathbb{R}^3) \rightarrow V_{n-1}$  be the orthogonal projection, and set

$$h_n: S_r(c) \rightarrow V_{n-1} \times \mathbb{R}^+, \quad u \mapsto (P_{n-1}u, |(-\Delta)^{\frac{s}{2}}u|_2^2).$$

Then clearly  $B_n = h_n^{-1}(0, \rho_n)$ . We fix  $\gamma \in \Gamma_n$  and consider the map

$$\phi = h_n \circ \gamma: [0, 1] \times (S_r(c) \cap V_n) \rightarrow V_{n-1} \times \mathbb{R}^+ := X.$$

Since

$$\phi_0(S_r(c) \cap V_n) \subset V_{n-1} \times (0, \rho_n] := X_0$$

and

$$\phi_1(S_r(c) \cap V_n) \subset V_{n-1} \times [\rho_n, \infty) := X_1,$$

it follows from (I<sub>1</sub>) to (I<sub>3</sub>) that

$$i(\text{Im}(\phi) \cap X_0 \cap X_1) \geq i(S_r(c) \cap V_n) = \dim V_n.$$

If there would not exist  $(t, u) \in [0, 1] \times (S_r(c) \cap V_n)$  with  $\gamma(t, u) \in B_n$ , then

$$\text{Im}(\phi) \cap X_0 \cap X_1 \subset (V_{n-1} \setminus \{0\}) \times \{\rho_n\}.$$

Therefore (I<sub>1</sub>), (I<sub>2</sub>) imply that

$$i(\text{Im}(\phi) \cap X_0 \cap X_1) \leq i((V_{n-1} \setminus \{0\}) \times \{\rho_n\}) = \dim V_{n-1},$$

contradicting  $\dim V_{n-1} < \dim V_n$ . □

**Remark 2.1.** Note that by Lemma 2.7 we have that for each  $n \in \mathbb{N}$ ,

$$c_n := \inf_{\gamma \in \Gamma_n} \max_{t \in [0,1], u \in S_r(c) \cap V_n} J(\gamma(t, u)) \geq b_n \rightarrow \infty.$$

Then we have that for any  $\gamma \in \Gamma_n$ ,

$$c_n \geq b_n > \max \left\{ \max_{u \in S_r(c) \cap V_n} J(\gamma(0, u)), \max_{u \in S_r(c) \cap V_n} J(\gamma(1, u)) \right\}.$$

### 3. Proofs of the main results

In this section, we shall prove that the sequence  $\{c_n\}$  is indeed a sequence of critical values for  $J$  restricted to  $S_r(c)$ . To this purpose, we first show that there exists a bounded Palais–Smale sequence at each level  $c_n$ . From now on, we fix an arbitrary  $n \in \mathbb{N}$ .

**Lemma 3.1.** *There exists a Palais–Smale sequence  $\{u_k\} \subset S_r(c)$  for  $J$  at the level  $c_n$  satisfying*

$$(3.1) \quad \begin{aligned} Q(u_k) &= s|(-\Delta)^{\frac{s}{2}} u_k|_2^2 + \frac{3-2t}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_k(x)|^2 |u_k(y)|^2}{|x-y|^{3-2t}} dx dy \\ &\quad - \frac{3(p-2)}{2p} \int_{\mathbb{R}^3} |u_k|^p dx \rightarrow 0. \end{aligned}$$

In particular,  $\{u_k\} \subset S_r(c)$  is bounded.

*Proof.* In order to find such a Palais–Smale sequence, we apply the approach developed by Jeanjean [17], which already applied in [2] and [28]. First, we introduce the auxiliary functional

$$\tilde{J}: S_r(c) \times \mathbb{R} \rightarrow \mathbb{R}, \quad (u, \theta) \mapsto J(m(u, \theta)),$$

where  $m(u, \theta)$  is given in (2.4), and we define the set

$$\begin{aligned} \tilde{\Gamma}_n &:= \{ \tilde{\gamma}: [0, 1] \times (S_r(c) \cap V_n) \rightarrow S_r(c) \times \mathbb{R} \mid \tilde{\gamma} \text{ is continuous, odd in } u \\ &\quad \text{and such that } m \circ \tilde{\gamma} \in \Gamma_n \}. \end{aligned}$$

Clearly, for any  $\gamma \in \Gamma_n$ ,  $\tilde{\gamma} := (\gamma, 0) \in \tilde{\Gamma}_n$ .

Observe that defining

$$\tilde{c}_n := \inf_{\tilde{\gamma} \in \tilde{\Gamma}_n} \max_{t \in [0,1], u \in S_r(c) \cap V_n} \tilde{J}(\tilde{\gamma}(t, u)),$$

we have that  $\tilde{c}_n = c_n$ . Indeed, by the definitions of  $\tilde{c}_n$  and  $c_n$ , this identity follows immediately from the fact that the maps

$$\varphi: \Gamma_n \rightarrow \tilde{\Gamma}_n, \quad \gamma \mapsto \varphi(\gamma): (\gamma, 0),$$

and

$$\psi: \tilde{\Gamma}_n \rightarrow \Gamma_n, \quad \tilde{\gamma} \mapsto \psi(\tilde{\gamma}): m \circ \tilde{\gamma},$$

satisfy

$$\tilde{J}(\varphi(\gamma)) = J(\gamma) \quad \text{and} \quad J(m \circ \tilde{\gamma}) = \tilde{J}(\tilde{\gamma}).$$

Now from the definition of  $c_n$ , we know that for each  $k \in \mathbb{N}$ , there exists an  $\gamma_k \in \Gamma_n$  such that

$$\max_{t \in [0,1], u \in S_r(c) \cap V_n} J(\gamma_k(t, u_k)) \leq c_n + \frac{1}{k}.$$



Since  $\tilde{c}_n = c_n$ ,  $\tilde{\gamma}_k = (\gamma_k, 0) \in \tilde{\Gamma}_n$  satisfies

$$\max_{t \in [0,1], u \in S_r(c) \cap V_n} \tilde{J}(\tilde{\gamma}_k(t, u)) \leq \tilde{c}_n + \frac{1}{k}.$$

We can apply the Ekeland’s variational principle to obtain a sequence  $\{(u_k, \theta_k)\} \subset S_r(c) \times \mathbb{R}$  such that:

- (i)  $\tilde{J}(u_k, \theta_k) \in [c_n - \frac{1}{k}, c_n + \frac{1}{k}]$ ;
- (ii)  $\min_{t \in [0,1], u \in S_r(c) \cap V_n} \|(u_k, \theta_k) - (\gamma_k(t, u), 0)\|_E \leq \frac{1}{\sqrt{k}}$ ;
- (iii)  $\|\tilde{J}'|_{S_r(c) \cap \mathbb{R}}(u_k, \theta_k)\|_{E^*} \leq \frac{2}{\sqrt{k}}$ , i.e.  $\|\langle \tilde{J}'(u_k, \theta_k), z \rangle_{E^* \times E}\| \leq \frac{2}{\sqrt{k}}\|z\|$ , holds for all  $z \in \tilde{T}_{(u_k, \theta_k)} := \{(z_1, z_2) \in E, \langle u_k, z_1 \rangle_{L^2} = 0\}$ .

Here we denote by  $E$  the set  $H_r^s(\mathbb{R}^3) \times \mathbb{R}$  equipped with  $\|\cdot\|_E^2 = \|\cdot\|_{H_r^s}^2 + |\cdot|_{\mathbb{R}}^2$ , and by  $E^*$  its dual space. For each  $k \in \mathbb{N}$ , let  $v_k = m(u_k, \theta_k)$ . We shall prove that  $v_k \in S_r(c)$  is the sequence we need.

Indeed, first, since  $J(v_k) = J(m(u_k, \theta_k)) = \tilde{J}(u_k, \theta_k)$ , from (i) we have that  $J(v_k) \xrightarrow{k} c_n$ . Secondly, note that

$$\begin{aligned} Q(v_k) &= s|(-\Delta)^{\frac{s}{2}}v_k|_2^2 + \frac{3-2t}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|v_k(x)|^2|v_k(y)|^2}{|x-y|^{3-2t}} dx dy \\ &\quad - \frac{3(p-2)}{2p} \int_{\mathbb{R}^3} |v_k|^p dx = \langle \tilde{J}'(u_k, \theta_k), (0, 1) \rangle_{E^* \times E}, \end{aligned}$$

and  $(0, 1) \in \tilde{T}_{(u_k, \theta_k)}$ . Thus (iii) yields  $Q(v_k) \xrightarrow{k} 0$ . Finally, to verify that  $J'|_{S_r(c)}(v_k) \xrightarrow{k} 0$ , it suffices to prove for  $k \in \mathbb{N}$  sufficiently large, that

$$(3.2) \quad |\langle J'(v_k), w \rangle_{(H_r^s)^* \times H_r^s}| \leq \frac{4}{\sqrt{k}}\|w\|, \quad \text{for all } w \in T_{v_k},$$

where  $T_{v_k} := \{w \in H_r^s(\mathbb{R}^3), \langle v_k, w \rangle_{L^2} = 0\}$ . To this end, we note that, for  $w \in T_{v_k}$ , setting  $\tilde{w} = m(w, -\theta_k)$ , we have

$$\begin{aligned} &\langle J'(v_k), w \rangle_{(H_r^s)^* \times H_r^s} \\ &= \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}}v_k(-\Delta)^{\frac{s}{2}}w dx + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|v_k(x)|^2v_k(y)w(y)}{|x-y|^{3-2t}} dx dy - \int_{\mathbb{R}^3} |v_k|^{p-2}v_kw dx \\ &= e^{2s\theta_k} \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}}u_k(-\Delta)^{\frac{s}{2}}\tilde{w} dx + e^{(3-2t)\theta_k} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_k(x)|^2u_k(y)\tilde{w}(y)}{|x-y|^{3-2t}} dx dy \\ &\quad - e^{\frac{3(p-2)}{2}\theta_k} \int_{\mathbb{R}^3} |u_k|^{p-2}u_k\tilde{w} dx = \langle \tilde{J}'(u_k, \theta_k), (\tilde{w}, 0) \rangle_{E^* \times E}. \end{aligned}$$

If  $(\tilde{w}, 0) \in \tilde{T}_{(u_k, \theta_k)}$  and  $\|(\tilde{w}, 0)\|_E^2 \leq 4\|w\|^2$  when  $k \in \mathbb{N}$  is sufficiently large, then (iii) implies (3.2). To verify these conditions, observe that  $(\tilde{w}, 0) \in \tilde{T}_{(u_k, \theta_k)} \Leftrightarrow w \in T_{v_k}$ . Also from (ii) it follows that

$$|\theta_k| = |\theta_k - 0| \leq \min_{t \in [0,1], u \in S_r(c) \cap V_n} \|(u_k, \theta_k) - (\gamma_k(t, u), 0)\|_E \leq \frac{1}{\sqrt{k}},$$

by which we deduce that

$$\|(\tilde{w}, 0)\|_E^2 = \|\tilde{W}\|_{H_r^s}^2 \leq 4\|w\|^2,$$

holds for  $k \in \mathbb{N}$  large enough. At this point, (3.2) has been verified. To end the proof of the lemma it remains to show that  $\{v_k\} \subset S_r(c)$  is bounded. Notes that for any

$u \in H_r^s(\mathbb{R}^3)$ , there holds that

$$(3.3) \quad \begin{aligned} J(u) - \frac{2}{3(p-2)}Q(u) &= \frac{3p - (6 + 4s)}{6(p-2)}|(-\Delta)^s u|_2^2 \\ &+ \frac{3(p-2) - 2(3-2t)}{12(p-2)} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|^{3-2t}} dx dy. \end{aligned}$$

Thus we have

$$(3.4) \quad \begin{aligned} c_n + o_k(1) &= \frac{3p - (6 + 4s)}{6(p-2)}|(-\Delta)^s v_k|_2^2 \\ &+ \frac{3(p-2) - 2(3-2t)}{12(p-2)} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|v_k(x)|^2 |v_k(y)|^2}{|x-y|^{3-2t}} dx dy. \end{aligned}$$

Since  $p \in (\frac{4s+6}{3}, 2_s^*)$  and  $2s+2t > 3$  it follows immediately from (3.4) that  $\{v_k\} \subset S_r(c)$  is bounded in  $H_r^s(\mathbb{R}^3)$ .  $\square$

**Lemma 3.2.** *If  $u_0$  is a critical point of  $J(u)$  on  $S_r(c)$ , then  $Q(u_0) = 0$ .*

*Proof.* First, we denote

$$(3.5) \quad \begin{aligned} F_\lambda(u) &:= \langle I'_\lambda(u), u \rangle = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx - \lambda \int_{\mathbb{R}^3} |u|^2 dx \\ &+ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|^{3-2t}} dx dy - \int_{\mathbb{R}^3} |u|^p dx. \end{aligned}$$

$$(3.6) \quad \begin{aligned} P_\lambda(u) &= \frac{3-2s}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx - \frac{3}{2} \lambda \int_{\mathbb{R}^3} u^2 dx \\ &+ \frac{3+2t}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|^{3-2t}} dx dy - \frac{3}{p} \int_{\mathbb{R}^3} |u|^p dx. \end{aligned}$$

Here,  $\lambda \in \mathbb{R}$  is a parameter and  $I_\lambda$  is the energy functional corresponding to the equation (1.1), that is

$$(3.7) \quad \begin{aligned} I_\lambda(u) &= \frac{1}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx - \frac{\lambda}{2} \int_{\mathbb{R}^3} |u|^2 dx \\ &+ \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|^{3-2t}} dx dy - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx. \end{aligned}$$

Clearly,  $I_\lambda(u) = J(u) - \frac{\lambda}{2} \int_{\mathbb{R}^3} |u|^2 dx$  and simple calculations imply that

$$(3.8) \quad \frac{3}{2}F_\lambda(u) - P_\lambda(u) = Q(u).$$

Now, from [34], we know that  $P_\lambda(u) = 0$  is a Pohožaev identity for the fractional Schrödinger–Poisson equation (1.1). In particular, any critical point  $u$  of  $I_\lambda(u)$  satisfies  $P_\lambda(u) = 0$ .

On the other hand, since  $u_0$  is a critical point of  $J(u)$  restricted to  $S_r(c)$ , there exists a Lagrange multiplier  $\lambda_0 \in \mathbb{R}$ , such that

$$J'(u_0) = \lambda_0 u_0.$$

Thus, for any  $\phi \in H_r^s(\mathbb{R}^3)$ ,

$$(3.9) \quad \langle I'_{\lambda_0}(u_0), \phi \rangle = \langle J'(u_0) - \lambda_0 u_0, \phi \rangle = 0,$$

which shows that  $u_0$  is also a critical point of  $I_{\lambda_0}(u)$ . Hence,

$$P_{\lambda_0}(u_0) = 0, \quad F_{\lambda_0}(u_0) = \langle I'_{\lambda_0}(u_0), u_0 \rangle = 0,$$

and  $Q(u_0) = 0$  follows from (3.8). □

**Lemma 3.3.** *Let  $\{u_k\} \subset S_r(c)$  be the Palais–Smale sequence obtained in Lemma 3.1. Then there exist  $\lambda_n \in \mathbb{R}$  and  $u_n \in H_r^s(\mathbb{R}^3)$ , such that, up to a subsequence,*

- (i)  $u_k \rightharpoonup u_n \neq 0$ , in  $H_r^s(\mathbb{R}^3)$ ;
- (ii)  $(-\Delta)^s u_k - \lambda_n u_k + (|x|^{2t-3} * |u_k|^2)u_k - |u_k|^{p-2}u_k \rightarrow 0$ , in  $H_r^{-s}(\mathbb{R}^3)$ ;
- (iii)  $(-\Delta)^s u_n - \lambda_n u_n + (|x|^{2t-3} * |u_n|^2)u_n - |u_n|^{p-2}u_n = 0$ , in  $H_r^{-s}(\mathbb{R}^3)$ .

Moreover, if  $\lambda_n < 0$ , then we have

$$u_k \rightarrow u_n, \quad \text{in } H_r^s(\mathbb{R}^3), \text{ as } k \rightarrow \infty.$$

In particular,  $|u_n|_2^2 = c$ ,  $J(u_n) = c_n$  and  $J'(u_n) - \lambda_n u_n = 0$  in  $H_r^{-s}(\mathbb{R}^3)$ .

*Proof.* Since  $\{u_k\} \subset S_r(c)$  is bounded, up to a subsequence, there exists a  $u_n \in H_r^s(\mathbb{R}^3)$ , such that

$$\begin{aligned} u_k &\xrightarrow{k} u_n, \quad \text{in } H_r^s(\mathbb{R}^3), \\ u_k &\xrightarrow{k} u_n, \quad \text{in } L^p(\mathbb{R}^3). \end{aligned}$$

Next, we have  $u_n \neq 0$ . Indeed suppose by contradiction that  $u_n = 0$ . Then by the strong convergence in  $L^p(\mathbb{R}^3)$  it follows that  $\int_{\mathbb{R}^3} |u_k|^p dx \rightarrow 0$ . Taking into account that  $Q(u_k) \rightarrow 0$  it then implies that  $J(u_k) \rightarrow 0$  and this contradicts the fact that  $c_n \geq b_n > 0$ . Thus point (i) holds.

Since  $\{u_k\} \subset S_r(c)$  is bounded, we know that:

$$J'|_{S_r(c)}(v_k) \rightarrow 0 \iff J'(v_k) - \langle J'(v_k), v_k \rangle v_k \rightarrow 0 \quad \text{in } H_r^{-s}(\mathbb{R}^3).$$

Thus, for any  $w \in H_r^s(\mathbb{R}^3)$ ,

$$\begin{aligned} \langle J'(v_k) - \langle J'(v_k), v_k \rangle v_k, w \rangle &= \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} v_k (-\Delta)^{\frac{s}{2}} w dx \\ &\quad + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_k(x)|^2 v_k(y) w(y)}{|x-y|^{3-2t}} dx dy \\ &\quad - \lambda_n \int_{\mathbb{R}^3} v_k w dx - \int_{\mathbb{R}^3} |u_k|^{p-2} v_k w dx, \end{aligned}$$

with

$$(3.10) \quad \lambda_n = \frac{1}{|v_k|_2} \left\{ |(-\Delta)^{\frac{s}{2}} v_k|_2^2 + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_k(x)|^2 |v_k(y)|^2}{|x-y|^{3-2t}} dx dy - |v_k|_p^p \right\}.$$

Thus, we obtain (ii) with  $\{\lambda_n\} \subset \mathbb{R}$  defined by (3.10). Moreover, we refer to [33, Lemma 2.4] for a proof of (iii).

Finally, using point (ii)–(iii) and the convergence  $u_k \xrightarrow{k} u_n$  in  $L^p(\mathbb{R}^3)$ , it follows that

$$\|u_k\|^2 - \lambda_n |u_k|_2^2 + (|x|^{2t-3} * |u_k|^2) |u_k|^2 \xrightarrow{k} \|u_n\|^2 - \lambda_n |u_n|_2^2 + (|x|^{2s-3} * |u_n|^2) |u_n|^2.$$

If  $\lambda_n < 0$ , then we conclude from the weak convergence of  $u_k \xrightarrow{k} u_n$  in  $H_r^s(\mathbb{R}^3)$  and [33, Lemma 2.3], that  $u_k \xrightarrow{k} u_n$  in  $H_r^s(\mathbb{R}^3)$ . And in particular,  $|u_n|_2^2 = c$ ,  $J(u_n) = c_n$  and  $J'(u_n) - \lambda_n u_n = 0$  in  $H_r^{-s}(\mathbb{R}^3)$ . □

*Proof of Theorem 1.1.* Similar the proof in [4, Lemma 4.2], we can prove that if  $(u, \lambda) \in S_r(c) \times \mathbb{R}$  solves (1.1), then necessarily  $\lambda < 0$  provided  $c > 0$  is sufficiently small. Thus by Lemma 3.1 and Lemma 3.3, when  $c > 0$  is small enough, for each  $n \in \mathbb{N}$ , we obtain a couple solution  $(u_n, \lambda_n) \in H_r^s(\mathbb{R}^3) \times \mathbb{R}^-$  solving (1.1) with  $|u_n|_2^2 = c$  and  $J(u_n) = c_n$ . Note from Lemma 2.5 and Remark 2.1 that  $c_n \rightarrow \infty$  as  $n \rightarrow \infty$

and then we deduce that the sequence of solutions  $\{(u_n, \lambda_n)\}$  is unbounded. At this point, the proof of the theorem is completed.  $\square$

*Acknowledgement.* We would like to thank the anonymous referee for his/her careful readings of our manuscript and the useful comments made for its improvement. This work is partially supported by National Natural Science Foundation of China (No. 11771385).

## References

- [1] BARTSCH, T.: Topological methods for variational problems with symmetries. - Lecture Notes in Math. 1560, Springer-Verlag, Berlin, 1993.
- [2] BARTSCH, T., and DE VALERIOLO, S.: Normalized solutions of nonlinear Schrödinger equations. - Arch. Math. (Basel) 100:1, 2013, 75–83.
- [3] BARTSCH, T., and N. SOAVE: A natural constraint approach to normalized solutions of nonlinear Schrödinger equations and systems. - J. Funct. Anal. 272:12, 2017, 4998–5037.
- [4] BELLAZZINI, J., L. JEANJEAN, and T. LUO: Existence and instability of standing waves with prescribed norm for a class of Schrödinger–Poisson equations. - Proc. Lond. Math. Soc. 107:2, 2013, 303–339.
- [5] BELLAZZINI, J., and G. SICILIANO: Scaling properties of functionals and existence of constrained minimizers. - J. Funct. Anal. 261:9, 2011, 2486–2507.
- [6] BELLAZZINI, J., and G. SICILIANO: Stable standing waves for a class of nonlinear Schrödinger–Poisson equations. - Z. Angew. Math. Phys. 62:2, 2011, 267–280.
- [7] BERESTYCKI, H., and P.-L. LIONS: Nonlinear scalar field equations. I. Existence of a ground state. - Arch. Rational Mech. Anal. 82:4, 1983, 313–345.
- [8] BOULENGER, T., D. HIMMELSBACH, and E. LENZMANN: Blowup for fractional NLS. - J. Funct. Anal. 271:9, 2016, 2569–2603.
- [9] BUCUR, C., and E. VALDINOCI: Nonlocal diffusion and applications. - Lect. Notes Unione Mat. Ital. 20, Unione Matematica Italiana, Bologna, Springer, 2016.
- [10] CONNER, P. E., and E. E. FLOYD: Fixed point free involutions and equivariant maps. II. - Trans. Amer. Math. Soc. 105, 1962, 222–228.
- [11] DI NEZZA, E., and G. PALATUCCI, and E. VALDINOCI: Hitchhiker’s guide to the fractional Sobolev spaces. - Bull. Sci. Math. 136:5, 2012, 521–573.
- [12] FADELL, E. R., and P. H. RABINOWITZ: Bifurcation for odd potential operators and an alternative topological index. - J. Funct. Anal. 26:1, 1977, 48–67.
- [13] FENG, X.: Nontrivial solution for Schrödinger–Poisson equations involving a fractional nonlocal operator via perturbation methods. - Z. Angew. Math. Phys. 67:3, 2016, Art. 74, 10.
- [14] FRÖHLICH, J., and E. LENZMANN: Dynamical collapse of white dwarfs in Hartree and Hartree–Fock theory. - Comm. Math. Phys. 274:3, 2007, 737–750.
- [15] GOU, T., and H. SUN: Solutions of nonlinear Schrödinger equation with fractional Laplacian without the Ambrosetti–Rabinowitz condition. - Appl. Math. Comput. 257, 2015, 409–416.
- [16] HUANG, Y., Z. LIU, and Y. WU: Existence of prescribed  $L^2$ -norm solutions for a class of Schrödinger–Poisson equation. - Abstr. Appl. Anal. 2013, Art. ID 398164, 11.
- [17] JEANJEAN, L.: Existence of solutions with prescribed norm for semilinear elliptic equations. - Nonlinear Anal. 28:10, 1997, 1633–1659.
- [18] JEANJEAN, L., and T. LUO: Sharp nonexistence results of prescribed  $L^2$ -norm solutions for some class of Schrödinger–Poisson and quasi-linear equations. - Z. Angew. Math. Phys. 64:4, 2013, 937–954.
- [19] JIN, T., and Z. YANG: The fractional Schrödinger–Poisson systems with infinitely many solutions. - J. Korean Math. Soc. 57:2, 2020, 489–506.

- [20] KIKUCHI, H.: Existence and stability of standing waves for Schrödinger–Poisson–Slater equation. - *Adv. Nonlinear Stud.* 7:3, 2007, 403–437.
- [21] LASKIN, N.: Fractional quantum mechanics and Lévy path integrals. - *Phys. Lett. A* 268:4-6, 2000, 298–305.
- [22] LASKIN, N.: Fractional Schrödinger equation. - *Phys. Rev. E* (3) 66:5, 2002, 056108.
- [23] LENZMANN, E.: Well-posedness for semi-relativistic Hartree equations of critical type. - *Math. Phys. Anal. Geom.* 10:1, 2007, 43–64.
- [24] LIEB, E. H., and B. SIMON: *The Hartree–Fock theory for Coulomb systems.* - Springer, Berlin Heidelberg, 2005.
- [25] LIU, W.: Infinitely many positive solutions for the fractional Schrödinger–Poisson system. - *Pacific J. Math.* 287:2, 2017, 439–464.
- [26] LIU, Z., and J. ZHANG: Multiplicity and concentration of positive solutions for the fractional Schrödinger–Poisson systems with critical growth. - *ESAIM Control Optim. Calc. Var.* 23:4, 2017, 1515–1542.
- [27] LUO, H., and X. TANG: Ground state and multiple solutions for the fractional Schrödinger–Poisson system with critical Sobolev exponent. - *Nonlinear Anal. Real World Appl.* 42, 2018, 24–52.
- [28] LUO, T.: Multiplicity of normalized solutions for a class of nonlinear Schrödinger–Poisson–Slater equations. - *J. Math. Anal. Appl.* 416:1, 2014, 195–204.
- [29] MURCIA, E., and G. SICILIANO: Positive semiclassical states for a fractional Schrödinger–Poisson system. - *Differential Integral Equations* 30, 2017, 231–258.
- [30] SÁNCHEZ, Ó., and J. SOLER: Long-time dynamics of the Schrödinger–Poisson–Slater system. - *J. Statist. Phys.* 114:1-2, 2004, 179–204.
- [31] SECCHI, S.: Ground state solutions for nonlinear fractional Schrödinger equations in  $\mathbb{R}^N$ . - *J. Math. Phys.* 54:3, 2013, 031501, 17.
- [32] STRUWE, M.: *Variational methods: Applications to nonlinear partial differential equations and Hamiltonian systems.* Fourth edition. - *Ergeb. Math. Grenzgeb.* (3) 34, Springer-Verlag, Berlin, 2008.
- [33] TENG, K.: Existence of ground state solutions for the nonlinear fractional Schrödinger–Poisson system with critical Sobolev exponent. - *J. Differential Equations* 261:6, 2016, 3061–3106.
- [34] TENG, K.: Ground state solutions for the non-linear fractional Schrödinger–Poisson system. - *Appl. Anal.* 98:11, 2019, 1959–1996.
- [35] YANG, Z., Y. YU, and F. ZHAO: The concentration behavior of ground state solutions for a critical fractional Schrödinger–Poisson system. - *Math. Nachr.* 292:8, 2019, 1837–1868.
- [36] YANG, Z., Y. YU, and F. ZHAO: Concentration behavior of ground state solutions for a fractional Schrödinger–Poisson system involving critical exponent. - *Commun. Contemp. Math.* 21:6, 2019, 1850027, 46.
- [37] YANG, Z., W. ZHANG, and F. ZHAO: Existence and concentration results for fractional Schrödinger–Poisson system via penalization method. - *Electron. J. Differential Equations* 2021, Paper No. 14, 31.
- [38] YANG, Z., and F. ZHAO: Multiplicity and concentration behaviour of solutions for a fractional Choquard equation with critical growth. - *Adv. Nonlinear Anal.* 10:1, 2021, 732–774.
- [39] YU, Y., and F. ZHAO, and L. ZHAO: The concentration behavior of ground state solutions for a fractional Schrödinger–Poisson system. - *Calc. Var. Partial Differential Equations* 56:4, 2017, Art. 116, 25.
- [40] ZHANG, J., J. M. DO Ó., and M. SQUASSINA: Fractional Schrödinger–Poisson systems with a general subcritical or critical nonlinearity. - *Adv. Nonlinear Stud.* 16, 2016, 15–30.

Received 19 May 2021 • Accepted 7 March 2022 • Published online 18 May 2022

Zhipeng Yang

Yunnan Normal University  
Department of Mathematics  
Kunming 650500, P. R. China  
yangzhipeng326@163.com

Fukun Zhao

Yunnan Normal University  
Department of Mathematics  
Kunming 650500, P. R. China  
fukunzhao@163.com

Shunneng Zhao

Yunnan Normal University  
Department of Mathematics  
Kunming 650500, P. R. China  
Zhejiang Normal University  
Department of Mathematics  
Jinhua 321004, P. R. China  
snzhao@zjnu.edu.cn