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Existence and multiplicity of normalized solutions for a class of fractional Schrödinger–Poisson equations

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Abstract. We consider the fractional Schrödinger–Poisson equation

$$\begin{cases}(-\Delta)^s u-\lambda u+\phi u=|u|^{p-2}u, & x\in\mathbb{R}^3,\\ (-\Delta)^t\phi=u^2, & x\in\mathbb{R}^3,\end{cases}$$

where $s,t\in(0,1)$ satisfy $2s+2t>3$, $p\in(\frac{4s+6}{3},2_s^*)$ and $\lambda\in\mathbb{R}$ is an undetermined parameter. We deal with the case where the associated functional is not bounded below on the L^2-unit sphere and show the existence of infinitely many solutions (u,λ) with u having prescribed L^2-norm.

Murtoasteisten Schrödingerin–Poissonin yhtälöiden luokan normitettujen ratkaisujen olemassaolo ja monikäsitteisyys

Tiivistelmä. Tarkastelemme murtoasteista Schrödingerin–Poissonin yhtälöä

$$\begin{cases}(-\Delta)^s u-\lambda u+\phi u=|u|^{p-2}u, & x\in\mathbb{R}^3,\\ (-\Delta)^t\phi=u^2, & x\in\mathbb{R}^3,\end{cases}$$

missä luvut $s,t\in(0,1)$ toteuttavat ehdon $2s+2t>3$, $p\in(\frac{4s+6}{3},2_s^*)$ ja $\lambda\in\mathbb{R}$ on määrittämätön parametri. Käsittelemme tapausta, jossa vastaava funktionaali ei ole alhaalta rajattu avaruuden L^2 yksikköpallonkuorella, ja osoitamme, että em. yhtälöllä on äärettömästi ratkaisuja (u,λ), jossa funktiolla u on annettu L^2-normi.

1. Introduction and the main results

In this paper, we study the following stationary fractional Schrödinger–Poisson equation

$$\begin{cases}(-\Delta)^s u-\lambda u+\phi u=|u|^{p-2}u, & x\in\mathbb{R}^3,\\ (-\Delta)^t\phi=u^2, & x\in\mathbb{R}^3,\end{cases} \tag{1.1}$$

where $s,t\in(0,1)$ satisfies $2s+2t>3$, $p\in(\frac{4s+6}{3},2_s^*)$ and $\lambda\in\mathbb{R}$, $2_s^*=\frac{6}{3-2s}$ is the fractional critical exponent. Part of the interest is the fact that solutions $(u(x),\phi(x))$ of (1.1) are related to standing wave solutions $(e^{-i\lambda t}u(x),\phi(x))$ of the time-dependent system

$$\begin{cases}i\frac{\partial\Psi}{\partial t}=(-\Delta)^s\Psi+\phi\Psi-\widetilde{f}(x,|\Psi|)\Psi & \text{in } \mathbb{R}^3\times\mathbb{R},\\ (-\Delta)^t\phi=|\Psi|^2 & \text{in } \mathbb{R}^3,\end{cases} \tag{1.2}$$

where i is the imaginary unit and $\widetilde{f}(x,|u|)u=f(x,u)$.

The first equation in (1.2) was introduced by Laskin (see [21, 22]) and comes from an expansion of the Feynman path integral from Brownian-like to Lévy-like quantum mechanical paths. This class of fractional Schrödinger equations with a

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repulsive nonlocal Coulombic potential is obtained by approximation of the Hartree–Fock equation describing a quantum mechanical system of many particles; see, for instance, [14, 23, 24, 38]. It also appeared in several areas such as optimization, finance, phase transitions, stratified materials, crystal dislocation, flame propagation, conservation laws, materials science and water waves (see [9]).

A first line of studying (1.1) is to consider $\lambda \in \mathbb{R}$ as a fixed parameter and then to search for critical points of the functional

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2\,dx - \frac{\lambda}{2}\int_{\mathbb{R}^3}|u|^2\,dx \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|^{3-2t}}\,dx\,dy - \frac{1}{p}\int_{\mathbb{R}^3}|u|^p\,dx, \end{aligned} \tag{1.3}$$

which is well defined and C^1 in $H_r^s(\mathbb{R}^3)$. In that direction, to the best of our knowledge, there are only few papers considering the existence and multiplicity of solutions to the fractional Schrödinger–Poisson system (1.1). In [40], the authors studied the existence of radial solutions by using the constrained minimization methods for system (1.1) with $\lambda = 0$ and Berestycki–Lions type conditions [7]. In [33, 34], Teng considers the fractional Schrödinger–Poisson system (1.1) with subcritical and critical nonlinearity respectively. By the monotone trick, concentration-compactness principe and a global compactness lemma he establishes the existence of ground state solutions. For other existence results we refer to [13, 19, 25, 26, 27, 29, 35, 36, 37, 39] and the references therein.

In present paper, motivated by the fact that physicists are often interested in "normalized solutions", we look for solutions in $H_r^s(\mathbb{R}^3)$ having a prescribed L^2-norm. More precisely, for given $c > 0$, we look at

$$(u_c, \lambda_c) \in H_r^s(\mathbb{R}^3) \times \mathbb{R} \ \text{ is solution of (1.1) with } |u_c|_2^2 = c.$$

In this case, a solution $u_c \in H_r^s(\mathbb{R}^3)$ of (1.1) can be obtained as a constrained critical point of the functional

$$J(u) = \frac{1}{2}|(-\Delta)^{\frac{s}{2}}u|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|^{3-2t}}\,dx\,dy - \frac{1}{p}\int_{\mathbb{R}^3}|u|^p\,dx.$$

on the constraint

$$S_r(c) := \{u \in H_r^s(\mathbb{R}^3) \colon |u|_2^2 = c,\ c > 0\}.$$

The parameter $\lambda_c \in \mathbb{R}$, in this situation, can not be fixed any more and it appears as a Lagrange parameter.

Set

$$e(c) := \inf_{u \in S_r(c)} J(u). \tag{1.4}$$

It is standard that minimizers of $e(c)$ are exactly critical points of $J(u)$ restricted to $S_r(c)$, and thus solutions of (1.1). By the L^2-preserving scaling and the fractional Gagliardo–Nirenberg inequality with best constant [8]: Let $p \in (2, 2_s^*)$, then

$$|u|_p^p \le \frac{p}{2|Q|_p^{p-2}}|(-\Delta)^{\frac{s}{2}}u|_2^{\frac{3(p-2)}{2s}}|u|_2^{p-\frac{3(p-2)}{2s}},$$

with equality for $u = Q$, where Q is, up to translations, the unique positive ground state solution of

$$\frac{3(p-2)}{4s}(-\Delta)^s u + \left(1 + \frac{p-2}{4}\left(2 - \frac{3}{s}\right)\right)u - |u|^{p-2}u = 0, \quad x \in \mathbb{R}^3.$$

Therefore, we can prove that $p = \frac{4s+6}{3}$ is the L^2-critical exponent for (1.4), i.e. for all $c > 0$, $e(c) > -\infty$ if $2 < p < \frac{4s+6}{3}$ and $e(c) = -\infty$ if $\frac{4s+6}{3} < p < 2_s^*$.

The above normalized problem associated to (1.4) with $s = 1$, has been studied in the literature[3, 5, 6, 16, 18, 20, 30]. In the cited references, the existence and non-existence of normalized solutions are established, depending strongly on the value $p \in (2, 2^*)$ and of the parameter $c > 0$. Precisely, it is proved that a solution which minimized globally J on $S_r(c)$, exists when $p \in (2, 3)$ and $c > 0$ small enough. When $p \in (3, \frac{10}{3})$, there exists a $c_0 > 0$ such that such a solution exists if and only if $c \geq c_0$. When $p \in (\frac{10}{3}, 2^*)$, it is not possible to find a solution as a global minimizer of J on $S_r(c)$ since the associated functional is not bounded below on the L^2-unit sphere. However, it is proved in [4] that for $c > 0$ sufficiently small, there exists a critical point which minimizes the energy among all solutions on $S_r(c)$ and infinitely many normalized solutions in [28].

For the nonlocal problem, that $s \in (0, 1)$, up to our knowledge, in the existing literature, results in this direction do not exist yet. Our contribution in this paper is that there exist normalized solutions of (1.1) for $p \in (\frac{4s+6}{3}, 2_s^*)$. The solutions are obtained as critical points of the functional J on a suitable submanifold of the constraint set $S_r(c)$. We state our main results as follows.

Theorem 1.1. *Assume that $s, t \in (0, 1)$ satisfies $2s + 2t > 3$ and $p \in (\frac{4s+6}{3}, 2_s^*)$. There exists a $c_0 > 0$ such that for any $c \in (0, c_0)$, Eq.* (1.1) *admits an unbounded sequence of distinct pairs of radial solutions $(\pm u_n, \lambda_n)$ with $|u_n|_2^2 = c$ and $\lambda_n < 0$ for each $n \in \mathbb{N}$.*

We give the main idea in the proof of our main results. To prove Theorem 1.1, because $e(c) = -\infty$ for all $c > 0$, the genus of the sublevel sets $J^c = \{u \in S_r(c) \mid J(u) \leq c\}$ is always infinite, so classical arguments based on the Kranoselski genus, see [32], do not apply. Secondly, it can be easily checked that the functional J, restricted to $S_r(c)$, does not satisfy the Palais–Smale condition, even working on the subspace $H_r^s(\mathbb{R}^3)$ of radially symmetric functions where one has the advantage of the compact embedding of $H_r^s(\mathbb{R}^3)$ into $L^q(\mathbb{R}^3)$ for $q \in (2, 2_s^*)$. To overcome these difficulties we are inspired by a recent work [2] and [28]. The authors present a new type of linking geometry for the functional J on $S_r(c)$ and set up a min-max scheme where the cohomological index for spaces with an action on the group $G := \{-1, 1\}$ is used. Following [17], for each fixed $n \in \mathbb{N}$, we can construct a special Palais–Smale sequence. That construction leads easily to get the bounededness and further non-vanishing of the Palais–Smale sequence.

2. Variational settings and preliminary results

Throughout this paper, we denote $|\cdot|_q$ the usual norm of the space $L^q(\mathbb{R}^3)$, $1 \leq q < \infty$, $B_r(x)$ denotes the open ball with center at x and radius r, C or C_i $(i = 1, 2, \cdots)$ denote some positive constants may change from line to line. $\rightharpoonup$ and $\to$ mean the weak and strong convergence.

2.1. The functional space setting. Firstly, fractional Sobolev spaces are the convenient setting for our problem, so we will give some sketches of the fractional order Sobolev spaces and the complete introduction can be found in [11]. We recall that, for any $s \in (0, 1)$, the fractional Sobolev space $H^s(\mathbb{R}^3) = W^{s,2}(\mathbb{R}^3)$ is defined as follows:

$$H^s(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3)\colon \int_{\mathbb{R}^3} \left(|\xi|^{2s}|\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2\right) d\xi < \infty\},$$

whose norm is defined as

$$\|u\|^2 = \int_{\mathbb{R}^3} \left(|\xi|^{2s}|\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2\right) d\xi,$$

where $\mathcal{F}$ denotes the Fourier transform. We also define the homogeneous fractional Sobolev space $\mathcal{D}^{s,2}(\mathbb{R}^3)$ as the completion of $\mathcal{C}_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$[u]_{H^s(\mathbb{R}^3)} := \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,dx\,dy\right)^{\frac{1}{2}}.$$

Obviously, $H^s(\mathbb{R}^3)$ is a Hilbert space. A function is radial if and only if it is invariant under all rotations leaving the origin fixed. Let $H_r^s(\mathbb{R}^3)$ denote the subset of $H^s(\mathbb{R}^3)$ containing only the radial function, and equipped topology with $H^s(\mathbb{R}^3)$, which is also a Hilbert space.

The fractional Laplace, $(-\Delta)^s u$, of a smooth function $u\colon \mathbb{R}^3 \to \mathbb{R}$, is defined by

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s}\mathcal{F}(u)(\xi), \quad \xi \in \mathbb{R}^3.$$

Moreover, $(-\Delta)^s u$ can be equivalently represented [11] as

$$(-\Delta)^s u(x) = -\frac{1}{2}C(s)\int_{\mathbb{R}^3} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{3+2s}}\,dy, \quad \forall x \in \mathbb{R}^3,$$

where

$$C(s) = \left(\int_{\mathbb{R}^3} \frac{(1-cos\xi_1)}{|\xi|^{3+2s}}\,d\xi\right)^{-1}, \quad \xi = (\xi_1,\xi_2,\xi_3).$$

Also, by the Plancherel formular in Fourier analysis, we have

$$[u]^2_{H^s(\mathbb{R}^3)} = \frac{2}{C(s)}|(-\Delta)^{\frac{s}{2}}u|_2^2.$$

As a consequence, the norms on $H^s(\mathbb{R}^3)$ defined below are equivalent:

$$u \longmapsto \left(\int_{\mathbb{R}^3} |u|^2\,dx + \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,dx\,dy\right)^{\frac{1}{2}};$$

$$u \longmapsto \left(\int_{\mathbb{R}^3} (|\xi|^{2s}|\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2)\,d\xi\right)^{\frac{1}{2}};$$

$$u \longmapsto \left(\int_{\mathbb{R}^3} |u|^2 dx + |(-\Delta)^{\frac{s}{2}}u|_2^2\right)^{\frac{1}{2}}.$$

For the reader's convenience, we review some useful results for $H^s(\mathbb{R}^3)$ and $H_r^s(\mathbb{R}^3)$, which will be used later.

Lemma 2.1. [11] *Let $0 < s < 1$, then there exists a constant $C = C(s) > 0$, such that*

$$|u|^2_{2_s^*} \le C[u]^2_{H^s(\mathbb{R}^3)}$$

for every $u \in H^s(\mathbb{R}^3)$. Moreover, the embedding $H^s(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ is continuous for any $q \in [2, 2_s^*]$ and is locally compact whenever $q \in [2, 2_s^*)$.

Lemma 2.2. [31] *If $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$ and for some $R>0$ we have*

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_R(y)}|u_n|^2dx=0,$$

then $u_n\to 0$ in $L^q(\mathbb{R}^3)$ for any $2<q<2_s^*$.

Lemma 2.3. [15] *Let $2<q<2_s^*$, then every bounded sequence $\{u_n\}\subset H_r^s(\mathbb{R}^3)$ has a convergent subsequence in $L^q(\mathbb{R}^3)$.*

2.2. Some preliminary lemmas. We first establish some useful preliminary results. Let $\{V_n\}\subset H_r^s(\mathbb{R}^3)$ be a strictly increasing sequence of finite-dimensional linear subspaces in $H_r^s(\mathbb{R}^3)$, such that $\bigcup_n V_n$ is dense in $H_r^s(\mathbb{R}^3)$. We denote by $V_n^\perp$ the orthogonal space of $\{V_n\}$ in $H_r^s(\mathbb{R}^3)$.

Lemma 2.4. *Assume that $p\in(2,2_s^*)$. Then there holds*

$$\mu_n:=\inf_{u\in V_{n-1}^\perp}\frac{\int_{\mathbb{R}^3}(|(-\Delta)^{\frac{s}{2}}u|^2+|u|^2)\,dx}{\left(\int_{\mathbb{R}^3}|u|^p\,dx\right)^{\frac{2}{p}}}=\inf_{u\in V_{n-1}^\perp}\frac{\|u\|^2}{|u|_p^2}\to\infty,\quad \text{as } n\to\infty.$$

Proof. Arguing by contradiction, suppose there exists a sequence $\{u_n\}\subset H_r^s(\mathbb{R}^3)$ such that $u_n\in V_{n-1}^\perp$, $|u_n|_p=1$, and $\|u_n\|\to c<\infty$. Then there exists $u\in H_r^s(\mathbb{R}^3)$ with $u_n\rightharpoonup u$ in $H_r^s(\mathbb{R}^3)$ and $u_n\to u$ in $L^p(\mathbb{R}^3)$ up to a subsequence. Let $v\in H_r^s(\mathbb{R}^3)$ and $\{v_n\}\subset H_r^s(\mathbb{R}^3)$ such that $v_n\in V_{n-1}$ and $v_n\to v$ in $H_r^s(\mathbb{R}^3)$. We have, in $H_r^s(\mathbb{R}^3)$,

$$|\langle u_n,v\rangle|\le|\langle u_n,v-v_n\rangle|+|\langle u_n,v_n\rangle|\le\|u_n\|\|v-v_n\|\to 0,$$

so that $u_n\rightharpoonup 0=u$, while $|u|_p=1$, a contradiction. □

Now for $c>0$ fixed and for each $n\in\mathbb{N}$, we define

$$\rho_n:=\frac{\mu_n^{\frac{p}{p-2}}}{K^{\frac{2}{p-2}}}\quad\text{with}\quad K=\max_{|(-\Delta)^{\frac{s}{2}}|_2>0}\frac{(|(-\Delta)^{\frac{s}{2}}u|_2^2+c)^{\frac{p}{2}}}{|(-\Delta)^{\frac{s}{2}}u|_2^p+c^{\frac{p}{2}}},$$

and

$$B_n:=\{u\in V_{n-1}^\perp\cap S_r(c)\colon |(-\Delta)^{\frac{s}{2}}u|_2^2=\rho_n\}. \tag{2.1}$$

We also define

$$b_n:=\inf_{u\in B_n}J(u). \tag{2.2}$$

Then we have

Lemma 2.5. *For any $p\in(2,2_s^*)$, $b_n\to+\infty$ as $n\to\infty$. In particular, we can assume without restriction that $b_n\ge 1$ for all $n\in\mathbb{N}$.*

Proof. For any $u\in B_n$, we have that

$$\begin{aligned}J(u)&=\frac12|(-\Delta)^{\frac{s}{2}}u|_2^2+\frac14\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|^{3-2t}}\,dx\,dy-\frac1p\int_{\mathbb{R}^3}|u|^p\,dx\\&\ge\frac12|(-\Delta)^{\frac{s}{2}}u|_2^2-\frac{1}{p\mu_n^{\frac{2}{p}}}(|(-\Delta)^{\frac{s}{2}}u|_2^2+c)^{\frac{p}{2}}\\&\ge\frac12|(-\Delta)^{\frac{s}{2}}u|_2^2-\frac{K}{p\mu_n^{\frac{2}{p}}}(|(-\Delta)^{\frac{s}{2}}u|_2^p+c^{\frac{p}{2}})\\&=\left(\frac12-\frac1p\right)\rho_n-\frac{1}{p\mu_n^{\frac{2}{p}}}c^{\frac{p}{2}}.\end{aligned}$$

From this estimate and Lemma 2.4, it follow since $p > 2$, that $b_n \to +\infty$ as $n \to \infty$. Next, considering the sequence $\{V_n\} \subset H_r^s(\mathbb{R}^3)$ only from an $n_0 \in \mathbb{N}$ such that $b_n \geq 1$ for any $n \geq n_0$ it concludes the proof of the Lemma. □

Next we start to set up our min-max scheme. First we introduce the map

$$m\colon H_r^s(\mathbb{R}^3) \times \mathbb{R} \to H_r^s(\mathbb{R}^3), \quad m(u,\theta) = u * \theta, \tag{2.3}$$

be the action of group $\mathbb{R}$ on $H_r^s(\mathbb{R}^3)$ defined by

$$m(u,\theta)(x) = (u * \theta)(x) = e^{\frac{3\theta}{2}} u(e^{\theta} x). \tag{2.4}$$

Observe that for any given $u \in S_r(c)$, we have $m(u,\theta) \in S_r(c)$ for all $\theta \in \mathbb{R}$.

Lemma 2.6. *Assume that $u \in S_r(c)$ be arbitrary but fixed. Let $A(u) := \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2\, dx$, then we have*

(i) *$A(m(u,\theta)) \to 0$ and $J(m(u,\theta)) \to 0$ as $\theta \to -\infty$.*
(ii) *$A(m(u,\theta)) \to +\infty$ and $J(m(u,\theta)) \to -\infty$ as $\theta \to +\infty$.*

Proof. A straightforward calculation shows that

$$A(m(u,\theta)) = e^{2s\theta} |(-\Delta)^{\frac{s}{2}} u|_2^2 \to 0 \ \text{ as } \theta \to -\infty,$$

and

$$A(m(u,\theta)) = e^{2s\theta} |(-\Delta)^{\frac{s}{2}} u|_2^2 \to +\infty \ \text{ as } \theta \to +\infty.$$

Next, we get for $\theta < 0$,

$$\begin{aligned} |J(m(u,\theta))| &= \left|\frac{1}{2} |(-\Delta)^{\frac{s}{2}} m(u,\theta)\right|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|m(u,\theta)(x)|^2 |m(u,\theta)(y)|^2}{|x-y|^{3-2t}}\, dx\, dy \\ &\quad - \frac{1}{p}\int_{\mathbb{R}^3} |m(u,\theta)|^p\, dx\big| \\ &\leq \frac{e^{2s\theta}}{2} |(-\Delta)^{\frac{s}{2}} u|_2^2 + \frac{e^{(3-2t)\theta}}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|^{3-2t}}\, dx\, dy \\ &\quad + \frac{e^{\frac{3\theta(p-2)}{2}}}{p}\int_{\mathbb{R}^3} |u|^p\, dx. \end{aligned}$$

Thus $J(m(u,\theta)) \to 0$ as $\theta \to -\infty$ and point (i) holds. Moreover, we have for $\theta > 0$,

$$\begin{aligned} J(m(u,\theta)) &= \frac{1}{2} |(-\Delta)^{\frac{s}{2}} m(u,\theta)|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|m(u,\theta)(x)|^2 |m(u,\theta)(y)|^2}{|x-y|^{3-2t}}\, dx\, dy \\ &\quad - \frac{1}{p}\int_{\mathbb{R}^3} |m(u,\theta)|^p dx \\ &= \frac{e^{2s\theta}}{2} |(-\Delta)^{\frac{s}{2}} u|_2^2 + \frac{e^{(3-2t)\theta}}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|^{3-2t}}\, dx\, dy \\ &\quad - \frac{e^{\frac{3\theta(p-2)}{2}}}{p}\int_{\mathbb{R}^3} |u|^p\, dx. \end{aligned}$$

Taking into account that $2s + 2t > 3$ and $p \in (\frac{4s+6}{3}, 2_s^*)$ it shows that $J(m(u,\theta)) \to -\infty$ as $\theta \to +\infty$. □

Due to Lemma 2.6, using the fact that V_n is finite dimensional, we deduce that, for each $n \in \mathbb{N}$, there exists a $\theta_n > 0$, such that

$$\bar{\gamma}_n\colon [0,1] \times (S_r(c) \cap V_n) \to S_r(c), \quad \bar{\gamma}_n(t,u) = m(u, (2t-1)\theta_n)$$

satisfies

$$A(\bar{\gamma}_n(0,u)) < \rho_n, \quad A(\bar{\gamma}_n(1,u)) > \rho_n, \tag{2.5}$$

and

$$J(\bar{\gamma}_n(0,u)) < b_n, \quad J(\bar{\gamma}_n(1,u)) < b_n. \tag{2.6}$$

Now we define

$$\begin{aligned}\Gamma_n := \big\{\gamma\colon [0,1]\times(S_r(c)\cap V_n) \to S_r(c) \mid \gamma \text{ is continuous, odd in } u \\ \text{and such that } \forall\, u\colon \gamma(0,u) = \bar{\gamma}_n(0,u),\ \gamma(1,u) = \bar{\gamma}_n(1,u)\big\}.\end{aligned}$$

Clearly we have $\bar{\gamma}_n \in \Gamma_n$. Now we give the follow linking property, due to [2].

Lemma 2.7. *For every $\gamma \in \Gamma_n$, there exists $(t,u) \in [0,1]\times(S_r(c)\cap V_n)$ such that $\gamma(t,u) \in B_n$.*

Proof. In order to prove this lemma, we first recall some properties of the cohomological index for spaces with an action of the group $G = \{1,-1\}$. This index goes back to [10] and has been used in a variational setting in [12]. It associates to a G-space X an element $i(X) \in \mathbb{N}_0 \cup \{\infty\}$. We shall need the following properties[1, 32]:

(I_1) If G acts on $\mathbb{S}^{n-1}$ via multiplication, then $i(\mathbb{S}^{n-1}) = n$.
(I_2) If there exists an equivariant map $X \to Y$, then $i(X) \le i(Y)$.
(I_3) Let $X = X_0 \cup X_1$ be metrisable and $X_0, X_1 \subset X$ be closed G-invariant subspaces. Let Y be a G-space, and consider a continuous map $\phi\colon [0,1]\times Y \to X$ such that each $\phi_t = \phi(t,\cdot)\colon Y \to X$ is equivariant. If $\phi_0(Y) \subset X_0$ and $\phi_1(Y) \subset X_1$, then

$$i(\mathrm{Im}(\phi)\cap X_0 \cap X_1) \ge i(Y).$$

Now, let $P_{n-1}\colon H^s_r(\mathbb{R}^3) \to V_{n-1}$ be the orthogonal projection, and set

$$h_n\colon S_r(c) \to V_{n-1}\times\mathbb{R}^+, \quad u \mapsto (P_{n-1}u, |(-\Delta)^{\frac{s}{2}}u|_2^2).$$

Then clearly $B_n = h_n^{-1}(0,\rho_n)$. We fix $\gamma \in \Gamma_n$ and consider the map

$$\phi = h_n \circ \gamma\colon [0,1]\times(S_r(c)\cap V_n) \to V_{n-1}\times\mathbb{R}^+ := X.$$

Since

$$\phi_0(S_r(c)\cap V_n) \subset V_{n-1}\times(0,\rho_n] := X_0$$

and

$$\phi_1(S_r(c)\cap V_n) \subset V_{n-1}\times[\rho_n,\infty) := X_1,$$

it follows from (I_1) to (I_3) that

$$i(\mathrm{Im}(\phi)\cap X_0\cap X_1) \ge i(S_r(c)\cap V_n) = \dim V_n.$$

If there would not exist $(t,u) \in [0,1]\times(S_r(c)\cap V_n)$ with $\gamma(t,u) \in B_n$, then

$$\mathrm{Im}(\phi)\cap X_0\cap X_1 \subset (V_{n-1}\backslash\{0\})\times\{\rho_n\}.$$

Therefore (I_1), (I_2) imply that

$$i(\mathrm{Im}(\phi)\cap X_0\cap X_1) \le i((V_{n-1}\backslash\{0\})\times\{\rho_n\}) = \dim V_{n-1},$$

contradicting $\dim V_{n-1} < \dim V_n$. □

Remark 2.1. Note that by Lemma 2.7 we have that for each $n \in \mathbb{N}$,

$$c_n := \inf_{\gamma\in\Gamma_n} \max_{t\in[0,1],u\in S_r(c)\cap V_n} J(\gamma(t,u)) \geq b_n \to \infty.$$

Then we have that for any $\gamma \in \Gamma_n$,

$$c_n \geq b_n > \max\Big\{ \max_{u\in S_r(c)\cap V_n} J(\gamma(0,u)), \max_{u\in S_r(c)\cap V_n} J(\gamma(1,u))\Big\}.$$

3. Proofs of the main results

In this section, we shall prove that the sequence $\{c_n\}$ is indeed a sequence of critical values for J restricted to $S_r(c)$. To this purpose, we first show that there exists a bounded Palais–Smale sequence at each level c_n. From now on, we fix an arbitrary $n \in \mathbb{N}$.

Lemma 3.1. *There exists a Palais–Smale sequence $\{u_k\} \subset S_r(c)$ for J at the level c_n satisfying*

$$\begin{aligned} Q(u_k) = s|(-\Delta)^{\frac{s}{2}}u_k|_2^2 + \frac{3-2t}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_k(x)|^2|u_k(y)|^2}{|x-y|^{3-2t}}\,dx\,dy \\ - \frac{3(p-2)}{2p}\int_{\mathbb{R}^3}|u_k|^p\,dx \to 0. \end{aligned} \tag{3.1}$$

In particular, $\{u_k\} \subset S_r(c)$ is bounded.

Proof. In order to find such a Palais–Smale sequence, we apply the approach developed by Jeanjean [17], which already applied in [2] and [28]. First, we introduce the auxiliary functional

$$\tilde{J}\colon S_r(c)\times\mathbb{R} \to \mathbb{R}, \quad (u,\theta) \mapsto J(m(u,\theta)),$$

where $m(u,\theta)$ is given in (2.4), and we define the set

$$\begin{aligned} \tilde{\Gamma}_n := \big\{\tilde{\gamma}\colon [0,1]\times(S_r(c)\cap V_n) \to S_r(c)\times\mathbb{R} \;|\tilde{\gamma} \text{ is continuous, odd in } u \\ \text{and such that } m\circ\tilde{\gamma}\in\Gamma_n\big\}. \end{aligned}$$

Clearly, for any $\gamma \in \Gamma_n$, $\tilde{\gamma} := (\gamma, 0) \in \tilde{\Gamma}_n$.

Observe that defining

$$\tilde{c}_n := \inf_{\tilde{\gamma}\in\tilde{\Gamma}_n} \max_{t\in[0,1],u\in S_r(c)\cap V_n} \tilde{J}(\tilde{\gamma}(t,u)),$$

we have that $\tilde{c}_n = c_n$. Indeed, by the definitions of $\tilde{c}_n$ and c_n, this identity follows immediately from the fact that the maps

$$\varphi\colon \Gamma_n \to \tilde{\Gamma}_n, \quad \gamma \mapsto \varphi(\gamma)\colon (\gamma, 0),$$

and

$$\psi\colon \tilde{\Gamma}_n \to \Gamma_n, \quad \tilde{\gamma} \mapsto \psi(\tilde{\gamma})\colon m\circ\tilde{\gamma},$$

satisfy

$$\tilde{J}(\varphi(\gamma)) = J(\gamma) \quad \text{and} \quad J(m\circ\tilde{\gamma}) = \tilde{J}(\tilde{\gamma}).$$

Now from the definition of c_n, we know that for each $k \in \mathbb{N}$, there exists an $\gamma_k \in \Gamma_n$ such that

$$\max_{t\in[0,1],u\in S_r(c)\cap V_n} J(\gamma_k(t,u_k)) \leq c_n + \frac{1}{k}.$$

Since $\tilde{c}_n = c_n$, $\tilde{\gamma}_k = (\gamma_k, 0) \in \tilde{\Gamma}_n$ satisfies

$$\max_{t\in[0,1],u\in S_r(c)\cap V_n} \tilde{J}(\tilde{\gamma}_k(t,u)) \le \tilde{c}_n + \frac{1}{k}.$$

We can apply the Ekeland's variational principle to obtain a sequence $\{(u_k, \theta_k)\} \subset S_r(c) \times \mathbb{R}$ such that:

(i) $\tilde{J}(u_k, \theta_k) \in [c_n - \frac{1}{k}, c_n + \frac{1}{k}]$;

(ii) $\min_{t\in[0,1],u\in S_r(c)\cap V_n} \|(u_k, \theta_k) - (\gamma_k(t,u), 0)\|_E \le \frac{1}{\sqrt{k}}$;

(iii) $\|\tilde{J}'|_{S_r(c)\cap\mathbb{R}}(u_k, \theta_k)\|_{E^*} \le \frac{2}{\sqrt{k}}$, i.e. $\|\langle \tilde{J}'(u_k, \theta_k), z\rangle_{E^*\times E}| \le \frac{2}{\sqrt{k}}\|z\|$, holds for all $z \in \tilde{T}_{(u_k,\theta_k)} := \{(z_1, z_2) \in E, \langle u_k, z_1\rangle_{L^2} = 0\}$.

Here we denote by E the set $H^s_r(\mathbb{R}^3) \times \mathbb{R}$ equipped with $\|\cdot\|^2_E = \|\cdot\|^2_{H^s_r} + |\cdot|^2_{\mathbb{R}}$, and by E^* its dual space. For each $k \in \mathbb{N}$, let $v_k = m(u_k, \theta_k)$. We shall prove that $v_k \in S_r(c)$ is the sequence we need.

Indeed, first, since $J(v_k) = J(m(u_k, \theta_k)) = \tilde{J}(u_k, \theta_k)$, from (i) we have that $J(v_k) \overset{k}{\to} c_n$. Secondly, note that

$$Q(v_k) = s|(-\Delta)^{\frac{s}{2}} v_k|^2_2 + \frac{3-2t}{4} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v_k(x)|^2|v_k(y)|^2}{|x-y|^{3-2t}}\, dx\, dy - \frac{3(p-2)}{2p}\int_{\mathbb{R}^3} |v_k|^p\, dx = \langle \tilde{J}'(u_k, \theta_k), (0,1)\rangle_{E^*\times E},$$

and $(0,1) \in \tilde{T}_{(u_k,\theta_k)}$. Thus (iii) yields $Q(v_k) \overset{k}{\to} 0$. Finally, to verify that $J'|_{S_r(c)}(v_k) \overset{k}{\to} 0$, it suffices to prove for $k \in \mathbb{N}$ sufficiently large, that

$$|\langle J'(v_k), w\rangle_{(H^s_r)^*\times H^s_r}| \le \frac{4}{\sqrt{k}}\|w\|, \quad \text{for all } w \in T_{v_k}, \tag{3.2}$$

where $T_{v_k} := \{w \in H^s_r(\mathbb{R}^3), \langle v_k, w\rangle_{L^2} = 0\}$. To this end, we note that, for $w \in T_{v_k}$, setting $\tilde{w} = m(w, -\theta_k)$, we have

$$\begin{aligned}
&\langle J'(v_k), w\rangle_{(H^s_r)^*\times H^s_r}\\
&= \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} v_k (-\Delta)^{\frac{s}{2}} w\, dx + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v_k(x)|^2 v_k(y) w(y)}{|x-y|^{3-2t}}\, dx\, dy - \int_{\mathbb{R}^3} |v_k|^{p-2} v_k w\, dx\\
&= e^{2s\theta_k}\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_k (-\Delta)^{\frac{s}{2}} \tilde{w}\, dx + e^{(3-2t)\theta_k}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_k(x)|^2 u_k(y)\tilde{w}(y)}{|x-y|^{3-2t}}\, dx\, dy\\
&\quad - e^{\frac{3(p-2)}{2}\theta_k}\int_{\mathbb{R}^3} |u_k|^{p-2} u_k \tilde{w}\, dx = \langle \tilde{J}'(u_k, \theta_k), (\tilde{w}, 0)\rangle_{E^*\times E}.
\end{aligned}$$

If $(\tilde{w}, 0) \in \tilde{T}_{(u_k,\theta_k)}$ and $\|(\tilde{w}, 0)\|^2_E \le 4\|w\|^2$ when $k \in \mathbb{N}$ is sufficiently large, then (iii) implies (3.2). To verify these conditions, observe that $(\tilde{w}, 0) \in \tilde{T}_{(u_k,\theta_k)} \Leftrightarrow w \in T_{v_k}$. Also from (ii) it follows that

$$|\theta_k| = |\theta_k - 0| \le \min_{t\in[0,1],u\in S_r(c)\cap V_n} \|(u_k, \theta_k) - (\gamma_k(t,u), 0)\|_E \le \frac{1}{\sqrt{k}},$$

by which we deduce that

$$\|(\tilde{w}, 0)\|^2_E = \|\tilde{W}\|^2_{H^s_r} \le 4\|w\|^2,$$

holds for $k \in \mathbb{N}$ large enough. At this point, (3.2) has been verified. To end the proof of the lemma it remains to show that $\{v_k\} \subset S_r(c)$ is bounded. Notes that for any

$u \in H_r^s(\mathbb{R}^3)$, there holds that

$$J(u) - \frac{2}{3(p-2)}Q(u) = \frac{3p-(6+4s)}{6(p-2)}|(-\Delta)^s u|_2^2 + \frac{3(p-2)-2(3-2t)}{12(p-2)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|^{3-2t}}\,dx\,dy. \tag{3.3}$$

Thus we have

$$c_n + o_k(1) = \frac{3p-(6+4s)}{6(p-2)}|(-\Delta)^s v_k|_2^2 + \frac{3(p-2)-2(3-2t)}{12(p-2)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|v_k(x)|^2|v_k(y)|^2}{|x-y|^{3-2t}}\,dx\,dy. \tag{3.4}$$

Since $p \in (\frac{4s+6}{3}, 2_s^*)$ and $2s+2t > 3$ it follows immediately from (3.4) that $\{v_k\} \subset S_r(c)$ is bounded in $H_r^s(\mathbb{R}^3)$. □

Lemma 3.2. *If u_0 is a critical point of $J(u)$ on $S_r(c)$, then $Q(u_0) = 0$.*

Proof. First, we denote

$$F_\lambda(u) := \langle I'_\lambda(u), u\rangle = \int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,dx - \lambda\int_{\mathbb{R}^3}|u|^2\,dx + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|^{3-2t}}\,dx\,dy - \int_{\mathbb{R}^3}|u|^p\,dx. \tag{3.5}$$

$$P_\lambda(u) = \frac{3-2s}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,dx - \frac{3}{2}\lambda\int_{\mathbb{R}^3}u^2\,dx + \frac{3+2t}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|^{3-2t}}\,dx\,dy - \frac{3}{p}\int_{\mathbb{R}^3}|u|^p\,dx. \tag{3.6}$$

Here, $\lambda \in \mathbb{R}$ is a parameter and I_λ is the energy functional corresponding to the equation (1.1), that is

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,dx - \frac{\lambda}{2}\int_{\mathbb{R}^3}|u|^2\,dx + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|^{3-2t}}\,dx\,dy - \frac{1}{p}\int_{\mathbb{R}^3}|u|^p\,dx. \tag{3.7}$$

Clearly, $I_\lambda(u) = J(u) - \frac{\lambda}{2}\int_{\mathbb{R}^3}|u|^2\,dx$ and simple calculations imply that

$$\frac{3}{2}F_\lambda(u) - P_\lambda(u) = Q(u). \tag{3.8}$$

Now, from [34], we know that $P_\lambda(u) = 0$ is a Pohožaev identity for the fractional Schrödinger–Poisson equation (1.1). In particular, any critical point u of $I_\lambda(u)$ satisfies $P_\lambda(u) = 0$.

On the other hand, since u_0 is a critical point of $J(u)$ restricted to $S_r(c)$, there exists a Lagrange multiplier $\lambda_0 \in \mathbb{R}$, such that

$$J'(u_0) = \lambda_0 u_0.$$

Thus, for any $\phi \in H_r^s(\mathbb{R}^3)$,

$$\langle I'_{\lambda_0}(u_0), \phi\rangle = \langle J'(u_0) - \lambda_0 u_0, \phi\rangle = 0, \tag{3.9}$$

which shows that u_0 is also a critical point of $I_{\lambda_0}(u)$. Hence,

$$P_{\lambda_0}(u_0) = 0,\ F_{\lambda_0}(u_0) = \langle I'_{\lambda_0}(u_0), u_0\rangle = 0,$$

and $Q(u_0) = 0$ follows from (3.8). □

Lemma 3.3. *Let $\{u_k\} \subset S_r(c)$ be the Palais–Smale sequence obtained in Lemma 3.1. Then there exist $\lambda_n \in \mathbb{R}$ and $u_n \in H_r^s(\mathbb{R}^3)$, such that, up to a subsequence,*

(i) $u_k \rightharpoonup u_n \neq 0$, *in* $H_r^s(\mathbb{R}^3)$;

(ii) $(-\Delta)^s u_k - \lambda_n u_k + (|x|^{2t-3} * |u_k|^2)u_k - |u_k|^{p-2}u_k \to 0$, *in* $H_r^{-s}(\mathbb{R}^3)$;

(iii) $(-\Delta)^s u_n - \lambda_n u_n + (|x|^{2t-3} * |u_n|^2)u_n - |u_n|^{p-2}u_n = 0$, *in* $H_r^{-s}(\mathbb{R}^3)$.

Moreover, if $\lambda_n < 0$, then we have

$$u_k \to u_n, \quad \text{in } H_r^s(\mathbb{R}^3), \text{ as } k \to \infty.$$

In particular, $|u_n|_2^2 = c$, $J(u_n) = c_n$ and $J'(u_n) - \lambda_n u_n = 0$ in $H_r^{-s}(\mathbb{R}^3)$.

Proof. Since $\{u_k\} \subset S_r(c)$ is bounded, up to a subsequence, there exists a $u_n \in H_r^s(\mathbb{R}^3)$, such that

$$u_k \overset{k}{\rightharpoonup} u_n, \quad \text{in } H_r^s(\mathbb{R}^3),$$
$$u_k \overset{k}{\to} u_n, \quad \text{in } L^p(\mathbb{R}^3).$$

Next, we have $u_n \neq 0$. Indeed suppose by contradiction that $u_n = 0$. Then by the strong convergence in $L^p(\mathbb{R}^3)$ it follows that $\int_{\mathbb{R}^3} |u_k|^p\,dx \to 0$. Taking into account that $Q(u_k) \to 0$ it then implies that $J(u_k) \to 0$ and this contradicts the fact that $c_n \geq b_n > 0$. Thus point (i) holds.

Since $\{u_k\} \subset S_r(c)$ is bounded, we know that:

$$J'|_{S_r(c)}(v_k) \to 0 \iff J'(v_k) - \langle J'(v_k), v_k\rangle v_k \to 0 \quad \text{in } H_r^{-s}(\mathbb{R}^3).$$

Thus, for any $w \in H_r^s(\mathbb{R}^3)$,

$$\begin{aligned}\langle J'(v_k) - \langle J'(v_k), v_k\rangle v_k, w\rangle &= \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} v_k (-\Delta)^{\frac{s}{2}} w\,dx \\ &\quad + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_k(x)|^2 v_k(y) w(y)}{|x-y|^{3-2t}}\,dx\,dy \\ &\quad - \lambda_n \int_{\mathbb{R}^3} v_k w\,dx - \int_{\mathbb{R}^3} |u_k|^{p-2} v_k w\,dx,\end{aligned}$$

with

$$\lambda_n = \frac{1}{|v_k|_2}\left\{|(-\Delta)^{\frac{s}{2}} v_k|_2^2 + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_k(x)|^2 |v_k(y)|^2}{|x-y|^{3-2t}}\,dx\,dy - |v_k|_p^p\right\}. \tag{3.10}$$

Thus, we obtain (ii) with $\{\lambda_n\} \subset \mathbb{R}$ defined by (3.10). Moreover, we refer to [33, Lemma 2.4] for a proof of (*iii*).

Finally, using point (ii)–(iii) and the convergence $u_k \overset{k}{\to} u_n$ in $L^p(\mathbb{R}^3)$, it follows that

$$\|u_k\|^2 - \lambda_n |u_k|_2^2 + (|x|^{2t-3} * |u_k|^2)|u_k|^2 \overset{k}{\to} \|u_n\|^2 - \lambda_n |u_n|_2^2 + (|x|^{2s-3} * |u_n|^2)|u_n|^2.$$

If $\lambda_n < 0$, then we conclude from the weak convergence of $u_k \overset{k}{\rightharpoonup} u_n$ in $H_r^s(\mathbb{R}^3)$ and [33, Lemma 2.3], that $u_k \overset{k}{\to} u_n$ in $H_r^s(\mathbb{R}^3)$. And in particular, $|u_n|_2^2 = c$, $J(u_n) = c_n$ and $J'(u_n) - \lambda_n u_n = 0$ in $H_r^{-s}(\mathbb{R}^3)$. □

Proof of Theorem 1.1. Similar the proof in [4, Lemma 4.2], we can prove that if $(u, \lambda) \in S_r(c) \times \mathbb{R}$ solves (1.1), then necessarily $\lambda < 0$ provided $c > 0$ is sufficiently small. Thus by Lemma 3.1 and Lemma 3.3, when $c > 0$ is small enough, for each $n \in \mathbb{N}$, we obtain a couple solution $(u_n, \lambda_n) \in H_r^s(\mathbb{R}^3) \times \mathbb{R}^-$ solving (1.1) with $|u_n|_2^2 = c$ and $J(u_n) = c_n$. Note from Lemma 2.5 and Remark 2.1 that $c_n \to \infty$ as $n \to \infty$

and then we deduce that the sequence of solutions $\{(u_n, \lambda_n)\}$ is unbounded. At this point, the proof of the theorem is completed. □

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