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Complex flows, escape to infinity and a question of Rubel

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Abstract. Let f be a transcendental entire function. It was shown in a previous paper (2017) that the holomorphic flow $\dot{z} = f(z)$ always has infinitely many trajectories tending to infinity in finite time. It will be proved here that such trajectories are in a certain sense rare, although an example will be given to show that there can be uncountably many. In contrast, for the classical antiholomorphic flow $\dot{z} = \bar{f}(z)$, such trajectories need not exist at all, although they must if f belongs to the Eremenko–Lyubich class $\mathcal{B}$. It is also shown that for transcendental entire f in $\mathcal{B}$ there exists a path tending to infinity on which f and all its derivatives tend to infinity, thus affirming a conjecture of Rubel for this class.

Kompleksiset virtaukset, pako äärettömyyteen ja Rubelin kysymys

Tiivistelmä. Olkoon f transkendenttinen kokonainen funktio. Aiemmassa tutkimuksessa (2017) osoitettiin, että holomorfisella virtauksella $\dot{z} = f(z)$ on aina äärettömän montaa rataa, jotka lähestyvät äärettömyyttä äärellisessä ajassa. Tässä työssä osoitetaan, että tällaiset radat ovat tietyssä mielessä harvinaisia, vaikka toisaalta näytetään esimerkillä, että näitä ratoja voi olla ylinumeroituva määrä. Tälle vastakkainen ilmiö on se, että klassisella antiholomorfisella virtauksella $\dot{z} = \bar{f}(z)$ tällaisia ratoja ei tarvitse olla lainkaan, paitsi siinä tapauksessa, että f kuuluu Eremenkon–Lyubichin luokkaan $\mathcal{B}$. Lisäksi osoitetaan, että transkendenttisella kokonaisella funktiolla $f \in \mathcal{B}$ on olemassa äärettömyyttä lähestyvä polku, jota pitkin sekä f että sen kaikki derivaatat lähestyvät ääretöntä, mikä vahvistaa Rubelin otaksuman tälle funktioluokalle.

1. Introduction

The starting point of this note is the flow

$$\dot{z} = f(z), \tag{1}$$

in which f or its conjugate $\bar{f}$ is an entire function. A trajectory for (1) is a path $z(t)$ in the plane with $z'(t) = f(z(t)) \in \mathbb{C}$ for t in some maximal interval $(\alpha, \beta) \subseteq \mathbb{R}$. It was shown in [15, Theorem 5] that if f is a polynomial in z of degree $n \geq 2$ then there exist $n-1$ disjoint trajectories for (1) which tend to infinity in finite increasing time, that is, which satisfy $\beta \in \mathbb{R}$ and $\lim_{t\to\beta-} z(t) = \infty$. The following theorem for holomorphic flows with transcendental entire f was proved in [12, Theorem 1.1].

Theorem 1.1. [12] *Let the function f be transcendental entire: then (1) has infinitely many pairwise disjoint trajectories which tend to infinity in finite increasing time.*

For meromorphic functions in general, such trajectories need not exist at all [12], but a result was also proved in [12] for the case where f is transcendental and meromorphic in the plane and the inverse function f^{-1} has a logarithmic singularity

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over ∞: this means that there exist $M > 0$ and a simply connected component U of the set $\{z \in \mathbb{C}\colon |f(z)| > M\}$ such that U contains no poles of f and $\log f$ maps U conformally onto the half-plane $H = \{v \in \mathbb{C}\colon \operatorname{Re} v > \log M\}$ [3, 16]. In this case [12, Theorem 1.2], (1) has infinitely many pairwise disjoint trajectories tending to infinity in finite increasing time from within a neighbourhood $\{z \in U\colon |f(z)| > M' \geq M\}$ of the singularity.

On the other hand, for entire f in (1), it seems that trajectories which tend to infinity in finite increasing time are somewhat exceptional. For the simple example $\dot{z} = -\exp(-z)$, it is easy to check that all trajectories satisfy $\exp(z(t)) = \exp(z(0)) - t$ and so tend to infinity as t increases, but take infinite time to do so unless $\exp(z(0))$ is real and positive.

It will be shown that for transcendental entire f there is, in a certain sense, zero probability of landing on a trajectory of (1) which tends to infinity in finite increasing time. To state the theorem, let f be transcendental entire and let

$$z_0 \in \mathbb{C}, \quad f(z_0) \neq 0, \quad F(z) = \int_{z_0}^{z} \frac{du}{f(u)}. \tag{2}$$

Then $F(z)$ is defined near z_0 and is real and increasing as z follows the trajectory $\zeta_{z_0}(t)$ of (1) starting at z_0. Let δ be small and positive and take the pre-image $L_\delta(z_0)$ of the real interval $(-\delta, \delta)$ under the function $-iF(z)$; then $L_\delta(z_0)$ is perpendicular to $\zeta_{z_0}(t)$ at z_0. The proof of the following result is adapted from that of the Gross star theorem [16, p. 292].

Theorem 1.2. *Let f be a transcendental entire function and let z_0 and F be as in (2). For small positive δ let Y_δ be the set of $y \in (-\delta, \delta)$ such that the trajectory of (1) starting at $F^{-1}(iy)$ tends to infinity in finite increasing time. Then Y_δ has Lebesgue measure* 0.

Theorem 1.2 seems unlikely to be best possible, but a construction from [18] (see §3) gives rise to a transcendental entire f for which (1) has uncountably many trajectories tending to infinity in finite increasing time.

It seems natural to ask similar questions in respect of the antiholomorphic flow

$$\dot{z} = \frac{dz}{dt} = \bar{g}(z), \tag{3}$$

where g is a non-constant entire function. Equation (3) appears widely in textbooks as a model for incompressible irrotational plane fluid flow [19, pp. 85–86], and is linked to (1) insofar as if $f = 1/g$ then (3) has the same trajectories as (1), since $\bar{g} = f/|f|^2$, although zeros of one of f and g are of course poles of the other and in general the speeds of travel differ. The trajectories of (3) are determined by choosing G with $G'(z) = g(z)$ and writing

$$v = G(z), \quad \dot{v} = g(z)\dot{z} = |g(z)|^2 \geq 0, \tag{4}$$

which leads to the classical fact that trajectories for (3) are level curves of $\operatorname{Im} G(z)$ on which $\operatorname{Re} G(z)$ increases with t. By the maximum principle, $\operatorname{Im} G(z)$ cannot be constant on a closed curve. Thus, apart from the countably many which tend to a zero of $G' = g$, all non-constant trajectories for (3) go to infinity, but how long they take to do so is less evident.

If a non-constant trajectory Γ of (3) passes from z_1 to z_2 along an arc avoiding zeros of g, then (4) implies that $\operatorname{Im} v = \beta$ is constant on Γ and $X = \operatorname{Re} v$ increases from $X_1 = \operatorname{Re} G(z_1)$ to $X_2 = \operatorname{Re} G(z_2)$, the transit time being given by

$$(5) \qquad \int_{X_1+i\beta}^{X_2+i\beta} \frac{1}{|g(z)|^2}\, dv = \int_{X_1+i\beta}^{X_2+i\beta} \left|\frac{dz}{dv}\right|^2 dv = \int_{X_1}^{X_2} \left|\frac{dz}{dX}\right|^2 dX.$$

Suppose that $G' = g$ is a polynomial of degree $n \geq 1$ in (3), (4) and (5). If $S \in \mathbb{R}$ and R is sufficiently large and positive then each pre-image under $v = G(z)$ of the half-line $v = r + iS, r \geq R$, gives a trajectory of (3) which tends to infinity, on which (4) delivers

$$\frac{dt}{dv} = \frac{1}{|g(z)|^2} \sim \frac{c_1}{|z|^{2n}} \sim \frac{c_2}{|v|^{2n/(n+1)}},$$

with c_1, c_2 positive constants. Hence (5) implies that the transit time to infinity is finite for $n \geq 2$ and infinite for $n = 1$. Thus, if g is a non-linear polynomial, (3) always has uncountably many trajectories tending to infinity in finite increasing time, but it turns out that this need not be the case for transcendental entire g.

Theorem 1.3. *There exists a transcendental entire function g such that (3) has no trajectories tending to infinity in finite increasing time.*

Theorem 1.3 also marks a sharp contrast with Theorem 1.1, and its proof rests on the following immediate consequence of a result of Barth, Brannan and Hayman [1, Theorem 2].

Theorem 1.4. [1] *There exists a transcendental entire function G such that any unbounded connected plane set contains a sequence (w_n) tending to infinity on which $U = \operatorname{Re} G$ satisfies $(-1)^n U(w_n) \leq |w_n|^{1/2}$.*

To establish Theorem 1.4, take the plane harmonic function v constructed in [1, Theorem 2], using $\psi(r)$ as given by [1, p. 364]. With $U = v$, and V a harmonic conjugate of U, elementary considerations show that the resulting entire function $G = U + iV$ cannot be a polynomial.

On the other hand, in the presence of a logarithmic singularity of the inverse function over infinity, trajectories of (3) tending to infinity in finite increasing time exist in abundance.

Theorem 1.5. *Let g and G be transcendental meromorphic functions in the plane such that $G' = g$ and either G^{-1} or g^{-1} has a logarithmic singularity over ∞. Then in each neighbourhood of the singularity the flow (3) has a family of pairwise disjoint trajectories $\gamma_Y, Y \in \mathbb{R}$, each of which tends to infinity in finite increasing time.*

Theorem 1.5 applies in particular if g or its antiderivative G is a transcendental entire function and belongs to the Eremenko–Lyubich class $\mathcal{B}$, which plays a salient role in complex dynamics [2, 6, 17] and is defined by the property that $F \in \mathcal{B}$ if the finite critical and asymptotic values of F form a bounded set, from which it follows that if $F \in \mathcal{B}$ is transcendental entire then F^{-1} automatically has a logarithmic singularity over ∞. A specific function to which Theorem 1.5 may be applied is $g(z) = e^{-z} + 1$; here g is in $\mathcal{B}$, but its antiderivative G is not, and this example also gives uncountably many trajectories of (3) taking infinite time to reach infinity through the right half-plane.

Theorem 1.5 is quite straightforward to prove when the inverse of G has a logarithmic singularity over infinity, but the method turns out to have a bearing on the following question of Rubel [7, pp. 595–596]: if f is a transcendental entire function, must there exist a path tending to infinity on which f and its derivative f' both have asymptotic value ∞? This problem was motivated by the classical theorem of Iversen [16], which states that ∞ is an asymptotic value of every non-constant entire function. For transcendental entire f of finite order, a strongly affirmative answer to Rubel's question was provided by the following result [11, Theorem 1.5].

Theorem 1.6. [11] *Let the function f be transcendental and meromorphic in the plane, of finite order of growth, and with finitely many poles. Then there exists a path γ tending to infinity such that, for each non-negative integer m and each positive real number c,*

$$\lim_{z\to\infty, z\in\gamma} \frac{\log |f^{(m)}(z)|}{\log |z|} = +\infty \quad \textit{and} \quad \int_\gamma |f^{(m)}(z)|^{-c}|dz| < +\infty. \tag{6}$$

For functions of infinite order, Rubel's question appears to be difficult, although a path satisfying (6) for $m = 0$ is known to exist for any transcendental entire function f [14]. However, a direct analogue of Theorem 1.6 goes through relatively straightforwardly for transcendental entire functions f in the Eremenko–Lyubich class $\mathcal{B}$.

Theorem 1.7. *Let f be a transcendental meromorphic function in the plane such that f^{-1} has a logarithmic singularity over ∞, and let $D \in \mathbb{R}$. Then there exists a path γ tending to infinity in a neighbourhood of the singularity, such that $f(z) - iD$ is real, positive and increasing on γ and (6) holds for each integer $m \geq 0$ and real $c > 0$.*

This paper is organised as follows: Theorem 1.2 is proved in §2, followed by an example in §3 and the proof of Theorem 1.3 in §4. It is then convenient to give the proof of Theorem 1.7 in §5, prior to that of Theorem 1.5 in §6.

2. Proof of Theorem 1.2

Let f, F, z_0 and δ be as in the statement of Theorem 1.2. For $y \in (-\delta, \delta)$ let $g(y) = F^{-1}(iy)$ and let $T(y)$ be the supremum of $s > 0$ such that the trajectory $\zeta_{g(y)}(t)$ of (1) with $\zeta_{g(y)}(0) = g(y)$ is defined and injective for $0 \leq t < s$. If the trajectory $\zeta_{g(y)}(t)$ is periodic with minimal period S_y then $T(y) = S_y$ and $\zeta_{g(y')}(t)$ has the same period for y' close to y [4]. Furthermore, if $\zeta_{g(y)}(t)$ tends to infinity in finite time then $T(y) < +\infty$, while if $T(y)$ is finite but $\zeta_{g(y)}(t)$ is not periodic then $\lim_{t\uparrow T(y)} \zeta_{g(y)}(t) = \infty$ [12, Lemma 2.1]. Set

$$A = \{iy + t \colon y \in (-\delta, \delta),\ 0 < t < T(y)\}, \quad B = \{\zeta_{g(y)}(t) \colon y \in (-\delta, \delta),\ 0 < t < T(y)\}.$$

Then $G(iy + t) = \zeta_{g(y)}(t)$ is a bijection from A to B.

For $u = \zeta_{g(y)}(t)$, where $y \in (-\delta, \delta)$ and $0 < t < T(y)$, let σ_u be the subarc of $L_\delta(z_0)$ from z_0 to $g(y)$ followed by the sub-trajectory of (1) from $g(y)$ to u, and define F by (2) on a simply connected neighbourhood D_u of σ_u. Then F maps σ_u bijectively to the line segment $[0, iy]$ followed by the line segment $[iy, iy + t]$, and taking a sub-domain if necessary makes it possible to assume that F is univalent on D_u, with inverse function defined on a neighbourhood of $[iy, iy + t]$.

Let y' and t' be real and close to y and t respectively. Then the image under F^{-1} of the line segment $[iy', iy' + t']$ is an injective sub-trajectory of (1) joining $g(y') \in L_\delta(z_0)$ to $F^{-1}(iy' + t') = \zeta_{g(y')}(t') = G(iy' + t')$, and so $T(y') \geq t'$. Thus $y \to T(y)$ is lower semi-continuous and A is a domain, while $G \colon A \to B$ is analytic. Moreover, A is simply connected, because its complement in $\mathbb{C} \cup \{\infty\}$ is connected, and so is B. Furthermore, F extends to be analytic on B, by (2) and the fact that $f \neq 0$ on B, and $F \circ G$ is the identity on A because $F(G(t)) = t$ for small positive t.

For $N \in (0, +\infty)$, let M_N be the set of all y in $(-\delta, \delta)$ such that $\zeta_{g(y)}(t)$ tends to infinity and $T(y) < N$. To prove Theorem 1.2, it suffices to show that each such M_N has measure 0, and the subsequent steps will be adapted from the proof of the Gross star theorem [16, p.292] and its extensions due to Kaplan [10]. Let $\Lambda_N \subseteq B$ be the image of $\Omega_N = \{w \in A\colon \operatorname{Re} w < N\}$ under G, let r be large and positive and denote the circle $|z| = r$ by $S(0, r)$. Then $S(0, r) \cap \Lambda_N$ is a union of countably many open arcs Σ_r.

If $y \in M_N$ then $T(y) < N$ and as $t \to T(y)$ the image $z = G(iy + t)$ tends to infinity in Λ_N and so crosses $S(0, r)$, and hence there exists ζ in some Σ_r with $\operatorname{Im} F(\zeta) = y$, since $F\colon B \to A$ is the inverse of G. Thus the measure μ_N of M_N is at most the total length $s(r)$ of the arcs $F(\Sigma_r)$. It follows from the Cauchy–Schwarz inequality that, as $t \to +\infty$,

$$\begin{aligned}\mu_N^2 \leq s(t)^2 &= \left(\int_{te^{i\phi} \in \Lambda_N} |F'(te^{i\phi})| t \, d\phi \right)^2 \\ &\leq \left(\int_{te^{i\phi} \in \Lambda_N} |F'(te^{i\phi})|^2 t \, d\phi \right) \left(\int_{te^{i\phi} \in \Lambda_N} t \, d\phi \right) \leq 2\pi t \left(\int_{te^{i\phi} \in \Lambda_N} |F'(te^{i\phi})|^2 t \, d\phi \right).\end{aligned}$$

Thus $\mu_N = 0$, since dividing by $2\pi t$ and integrating from r to r^2 yields, as $r \to +\infty$,

$$\begin{aligned}\frac{\mu_N^2 \log r}{2\pi} &\leq \int_r^{r^2} \int_{te^{i\phi} \in \Lambda_N} |F'(te^{i\phi})|^2 \, t \, d\phi \, dt \leq \int_{\Lambda_N} |F'(te^{i\phi})|^2 \, t \, d\phi \, dt \\ &= \operatorname{area}(\Omega_N) \leq 2\delta N.\end{aligned}$$

□

3. An example

Suppose that G is a locally univalent meromorphic function in the plane, whose set of asymptotic values is an uncountable subset E of the unit circle $\mathbb{T}$. Suppose further that there exists a simply connected plane domain D, mapped univalently onto the unit disc Δ by G, such that the branch ϕ of G^{-1} mapping Δ to D has no analytic extension to a neighbourhood of any $\beta \in E$.

Let $F = S(G)$, where S is a Möbius transformation mapping Δ onto $\{w \in \mathbb{C}\colon \operatorname{Re} w < 0\}$, and for $\beta \in E$ let $\alpha = S(\beta)$ and let L be the half-open line segment $[\alpha - 1, \alpha)$. Then $M = S^{-1}(L)$ is a line segment or circular arc in Δ which meets $\mathbb{T}$ orthogonally at β. Moreover, $\phi(M)$ is a level curve of $\operatorname{Im} F$ in D, which cannot tend to a simple β-point of G in $\mathbb{C}$ because this would imply that ϕ extends to a neighbourhood of β. Hence $\phi(M)$ is a path tending to infinity in D, on which $\operatorname{Im} F(z)$ is constant and $F(z)$ tends to α.

Since G and F are locally univalent, $f = 1/F'$ is entire. As $t \to 0-$ write, on $\phi(M)$,

$$F(z) = \alpha + t, \quad \frac{dt}{dz} = F'(z) = \frac{1}{f(z)}, \quad \frac{dz}{dt} = f(z),$$

so that $\phi(M)$ is a trajectory of (1) which tends to infinity in finite increasing time, and there exists one of these for every β in the uncountable set E.

A suitable G is furnished by a construction of Volkovyskii [5, 18], in which $\mathbb{T} \setminus E$ is a union of disjoint open circular arcs $I_k = (a_k, b_k)$, oriented counter-clockwise. For each k, take the multi-sheeted Riemann surface onto which $(a_k - b_k e^z)/(1 - e^z)$ maps the plane, cut it along a curve which projects to I_k, and glue to Δ that half which lies to the right as I_k is followed counter-clockwise. This forms a simply connected Riemann surface R with no algebraic branch points. By [18, Theorem 17, p. 71] (see also [5, p. 6]), the I_k can be chosen so that R is parabolic and is thereby the image surface of a locally univalent meromorphic function G in the plane. □

4. Proof of Theorem 1.3

Following the notation of the introduction, suppose that $v = G(z)$ is a transcendental entire function with derivative g in (3), (4) and (5).

Proposition 4.1. *Let Γ be a level curve tending to infinity on which $Y = \operatorname{Im} G(z) = \beta \in \mathbb{R}$ and $X = \operatorname{Re} G(z)$ increases, with $X \geq \alpha \in \mathbb{R}$, and assume that Γ meets no zero of g. Suppose that (z_n) is a sequence tending to infinity on Γ such that $v_n = G(z_n) = X_n + i\beta$ satisfies $v_n = o(|z_n|)^2$. Then the trajectory of (3) which follows Γ takes infinite time in tending to infinity.*

Here it is not assumed that $X \to +\infty$ as $z \to \infty$ on Γ.

Proof of Proposition 4.1. It may be assumed that Γ starts at z^* and $G(z^*) = \alpha + i\beta$. Denote positive constants, independent of n, by C_j. Then the Cauchy–Schwarz inequality gives, as n and z_n tend to infinity,

$$\begin{aligned} |z_n|^2 &\leq \left(C_1 + \int_\alpha^{X_n} \left| \frac{dz}{dX} \right| dX \right)^2 \leq 2 \left(\int_\alpha^{X_n} \left| \frac{dz}{dX} \right| dX \right)^2 \\ &\leq 2 \left(\int_\alpha^{X_n} dX \right) \left(\int_\alpha^{X_n} \left| \frac{dz}{dX} \right|^2 dX \right) \leq 2\left(|v_n| + C_2\right) \left(\int_\alpha^{X_n} \left| \frac{dz}{dX} \right|^2 dX \right) \\ &\leq o\left(|z_n|^2\right) \left(\int_\alpha^{X_n} \left| \frac{dz}{dX} \right|^2 dX \right). \end{aligned}$$

Thus (5) shows that the transit time from z^* to z_n tends to infinity with n. □

The assumption in Proposition 4.1 that Γ meets no zero of g represents no real restriction since if $\hat{z}$ is a zero of g of multiplicity $m \geq 1$, then the trajectory of (3) starting at $\hat{z}$ is constant. Indeed, if z tends to $\hat{z}$ as $X = \operatorname{Re} G(z) \to \hat{X}$ then, with c_j denoting non-zero constants,

$$\begin{aligned} X - \hat{X} &= G(z) - G(\hat{z}) \sim c_1 (z - \hat{z})^{m+1}, \\ \left| \frac{dz}{dX} \right|^2 &= \frac{1}{|g(z)|^2} \sim \frac{c_2}{|X - \hat{X}|^{2m/(m+1)}} \geq \frac{c_2}{|X - \hat{X}|}. \end{aligned}$$

Thus formula (5) shows that $\hat{z}$ cannot be reached in finite (increasing or decreasing) time. □

Proof of Theorem 1.3. Let G be the entire function given by Theorem 1.4, and set $g = G'$. As already noted, no trajectory of (3) can pass through a zero of g, and it takes infinite time for a trajectory to approach a zero of g. Furthermore, if Γ is a level curve, starting at z^* say, on which $\operatorname{Im} G(z)$ is constant and $U(z) = \operatorname{Re} G(z)$ increases, and on which g has no zeros, then there exists a sequence $z_n = w_{2n}$ which tends to infinity on Γ and satisfies

$$U(z^*) \leq U(z_n) \leq |z_n|^{1/2}, \quad |G(z_n)| \leq |U(z_n)| + O(1) \leq |z_n|^{1/2} + O(1).$$

Hence Γ satisfies the hypotheses of Proposition 4.1. It now follows that (3) has no trajectories tending to infinity in finite increasing time. Since time can be reversed for these flows by setting $s = -t$ and $dz/ds = -\bar{g}(z)$, the same example has no trajectories tending to infinity in finite decreasing time either. $\square$

5. Proof of Theorem 1.7

Let f be as in the hypotheses. Then there exist $M > 0$ and a component U of $\{z \in \mathbb{C}\colon |f(z)| > M\}$ such that $v = \log f(z)$ is a conformal bijection from U to the half-plane H given by $\mathrm{Re}\, v > N = \log M$; it may be assumed that $0 \notin U$. Let $\phi : H \to U$ be the inverse function. If $u \in H$ then ϕ and $\log \phi$ are univalent on the disc $|w - u| < \mathrm{Re}\, u - N$ and so Bieberbach's theorem and Koebe's quarter theorem [9, Chapter 1] imply that

$$\left|\frac{\phi''(u)}{\phi'(u)}\right| \le \frac{4}{\mathrm{Re}\, u - N}, \quad \left|\frac{\phi'(u)}{\phi(u)}\right| \le \frac{4\pi}{\mathrm{Re}\, u - N}. \tag{7}$$

Lemma 5.1. *Let v_0 be large and positive and for $0 \le k \in \mathbb{Z}$ write*

$$V_k = \left\{v_0 + te^{i\theta}\colon t \ge 0,\ -\frac{\pi}{2^{k+2}} \le \theta \le \frac{\pi}{2^{k+2}}\right\}, \quad G_k(v) = \frac{f^{(k)}(z)}{f(z)}, \quad z = \phi(v). \tag{8}$$

Then there exist positive constants d and c_k such that

$$|\log \phi(v)| + |\log \phi'(v)| \le d \log(\mathrm{Re}\, v) \tag{9}$$

as $v \to \infty$ in V_1 and $|\log |G_k(v)|| \le c_k \log(\mathrm{Re}\, v)$ as $v \to \infty$ in V_k.

Proof. For $v \in V_1$, parametrise the straight line segment from v_0 to v with respect to $s = \mathrm{Re}\, u$. Then (9) follows from (7) and the simple estimate $|du| \le \sqrt{2}ds$. Next, the assertion for G_k is trivially true for $k = 0$, so assume that it holds for some $k \ge 0$ and write

$$\begin{aligned} G_{k+1}(v) &= \frac{f^{(k+1)}(z)}{f(z)} = \frac{f^{(k)}(z)}{f(z)} \cdot \frac{f'(z)}{f(z)} + \frac{d}{dz}\left(\frac{f^{(k)}(z)}{f(z)}\right) \\ &= G_k(v)G_1(v) + \frac{G_k'(v)}{\phi'(v)} = \frac{G_k(v)}{\phi'(v)}\left(1 + \frac{G_k'(v)}{G_k(v)}\right). \end{aligned}$$

Thus it suffices to show that $G_k'(v)/G_k(v) \to 0$ as as $v \to \infty$ in V_{k+1}. By (8) there exists a small positive d_1 such that if $v \in V_{k+1}$ is large then the circle $|u - v| = r_v = d_1 \mathrm{Re}\, v$ lies in V_k, and the differentiated Poisson–Jensen formula [8, p. 22] delivers

$$\frac{G_k'(v)}{G_k(v)} = \frac{1}{\pi}\int_0^{2\pi} \frac{\log |G_k(v + r_v e^{i\theta})|}{r_v e^{i\theta}}\, d\theta = O\left(\frac{\log(\mathrm{Re}\, v)}{\mathrm{Re}\, v}\right) \to 0$$

as $v \to \infty$ in V_{k+1}. This proves the lemma. $\square$

To establish Theorem 1.7, take any $D \in \mathbb{R}$. Then there exist $v_1 \in [1, +\infty)$ and a path

$$\Gamma \subseteq \{v \in \mathbb{C}\colon \mathrm{Re}\, v > N,\ |\mathrm{Im}\, v| < \pi/4\} \subseteq H$$

which is mapped by e^v to the half-line $\{t + iD\colon t \ge v_1\}$. Thus $f(z) - iD = e^v - iD$ is real and positive for z on $\gamma = \phi(\Gamma)$, and $\Gamma \setminus V_k$ is bounded for each $k \ge 0$. Now write, on Γ,

$$e^v = t + iD, \quad \frac{dv}{dt} = \frac{1}{t + iD}, \quad s = \mathrm{Re}\, v = \frac{1}{2}\ln(t^2 + D^2).$$

Hence, for any non-negative integers k, m, Lemma 5.1 gives, as $v \to \infty$ on Γ,

$$\left|\frac{f^{(k)}(z)}{z^m}\right| = \left|\frac{f(z)G_k(v)}{z^m}\right| = \left|\frac{e^v G_k(v)}{\phi(v)^m}\right| \ge \frac{e^s}{s^{c_k + md}} \ge e^{s/2} \to \infty.$$

It then follows that, for $c > 0$,

$$\int_\gamma |f^{(k)}(z)|^{-c}\,|dz| \leq O(1) + \int_\Gamma e^{-cs/2}|\phi'(v)|\,|dv| \leq O(1) + \int_\Gamma e^{-cs/4}\,|dv| = O(1) + \int_{v_1}^{+\infty} \frac{1}{(t^2+D^2)^{1/2+c/8}}\,dt < +\infty. \qquad \square$$

6. Proof of Theorem 1.5

Suppose first that the inverse function of the antiderivative G of g has a logarithmic singularity over infinity, and take $D \in \mathbb{R}$. Then Theorem 1.7 may be applied with $f = G$ and $m = c = 1$, giving a level curve $\gamma = \gamma_D$, lying in a neighbourhood of the singularity, on which $\operatorname{Im} G(z) = D$ and $\operatorname{Re} G(z)$ increases. This curve is a trajectory for (3), traversed in time

$$\int_\gamma \frac{1}{\bar{g}(z)}\,dz \leq \int_\gamma |G'(z)|^{-1}\,|dz| < +\infty,$$

which completes the proof in this case.

For the proof of the following lemma the reader is referred to the statement and proof of [13, Lemma 3.1].

Lemma 6.1. [13] *Let the function $\phi\colon H \to \mathbb{C}\setminus\{0\}$ be analytic and univalent, where $H = \{v \in \mathbb{C}\colon \operatorname{Re} v > 0\}$, and for $v, v_1 \in H$ define $Z(v) = Z(v, v_1)$ by*

$$(10) \qquad Z(v,v_1) = \int_{v_1}^{v} e^{u/2}\phi'(u)\,du = 2e^{v/2}\phi'(v) - 2e^{v_1/2}\phi'(v_1) - 2\int_{v_1}^{v} e^{u/2}\phi''(u)\,du.$$

Let ε be a small positive real number. Then there exists a large positive real number N_0, depending on ε but not on ϕ, with the following property.

Let $v_0 \in H$ be such that $S_0 = \operatorname{Re} v_0 \geq N_0$, and define v_1, v_2, v_3, K_2 and K_3 by

$$v_j = \frac{2^j S_0}{128} + iT_0, \quad T_0 = \operatorname{Im} v_0, \quad K_j = \left\{ v_j + re^{i\theta}\colon r \geq 0,\ -\frac{\pi}{2^j} \leq \theta \leq \frac{\pi}{2^j} \right\}.$$

Then the following two conclusions both hold:

(i) *$Z = Z(v, v_1)$ satisfies, for $v \in K_2$,*

$$(11) \qquad Z(v,v_1) = \int_{v_1}^{v} e^{u/2}\phi'(u)\,du = 2e^{v/2}\phi'(v)(1+\delta(v)), \quad |\delta(v)| < \varepsilon.$$

(ii) *$\psi = \psi(v, v_1) = \log Z(v, v_1)$ is univalent on a domain H_1, with $v_0 \in H_1 \subseteq K_3$, and $\psi(H_1)$ contains the strip*

$$(12) \qquad \left\{ \psi(v_0) + \sigma + i\tau\colon \sigma \geq \log\frac{1}{8},\ -2\pi \leq \tau \leq 2\pi \right\}. \qquad \square$$

Assume henceforth that g is as in the hypotheses of Theorem 1.5 and the inverse function of g has a logarithmic singularity over infinity. This time there exist $M > 0$ and a component C of $\{z \in \mathbb{C}\colon |g(z)| > M\}$ such that $\zeta = \log g(z)$ is a conformal mapping of C onto the half-plane given by $\operatorname{Re}\zeta > \log M$. Since (3) may be re-scaled via $z = Mw$ and $g(z) = Mh(w)$, it may be assumed that $M = 1$ and $0 \notin C$. In order to apply Lemma 6.1, let $\phi\colon H \to C$ be the inverse function $z = \phi(v)$ of the mapping from C onto H given by

$$v = 2\zeta = 2\log g(z), \quad g(z) = e^{v/2}.$$

As in the proof of Theorem 1.7, (7) holds for $u \in H$, with $N = 0$.

By (12) there exists $X_0 > 0$ such that $Z(v, v_1)$ maps a domain $H_2 \subseteq H_1 \subseteq K_3 \subseteq H$ univalently onto a half-plane $\operatorname{Re} Z > X_0$. Hence, for any $Y_0 \in \mathbb{R}$, there exists a path Γ which tends to infinity in H_2 and is mapped by $Z(v, v_1)$ onto the half-line $L_0 = \{X + iY_0,\ X \geq X_0 + 1\}$. Consider the flow in H_2 given by

$$\phi'(v)\dot{v} = \overline{e^{v/2}}; \tag{13}$$

by (11) this transforms under $Z = Z(v, v_1)$ to

$$\dot{Z} = \frac{dZ}{dv}\,\dot{v} = e^{v/2}\phi'(v)\dot{v} = |e^v|. \tag{14}$$

Combining (7) and (11) shows that $|e^v| \geq |Z(v)|^{3/2}$ for large v on Γ. Hence there exists a trajectory of (14) which starts at $X_0 + 1 + iY_0$ and tends to infinity along L_0 in time

$$T_0 \leq \int_{X_0+1}^{\infty} \left|\frac{dt}{dX}\right| dX \leq O(1) + \int_{X_0+1}^{\infty} (X^2 + Y_0^2)^{-3/4}\, dX < +\infty.$$

This gives a trajectory of (13) tending to infinity along Γ and taking finite time to do so, and hence a trajectory γ of (3) in C, tending to infinity in finite increasing time. Since $Y_0 \in \mathbb{R}$ may be chosen at will, this proves Theorem 1.5. □

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