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On the Hausdorff dimension distortions of quasi-symmetric homeomorphisms

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Abstract. In this paper, we first prove that for a Fuchsian group G of divergence type and non-lattice, if h is a quasi-symmetric homeomorphism of the real axis $\mathbb{R}$ corresponding to a quasi-conformal compact deformation of G, then h is not strongly singular for divergence groups. This generalizes a result of Bishop and Steger (1993). Furthermore, we show that Bishop and Steger's result does not hold for the covering groups of all d-dimensional 'Jungle Gyms' (d is any positive integer) which generalizes Gönye's results (2007) where the author discussed the case of 1-dimensional 'Jungle Gym'.

Hausdorffin ulottuvuuden vääristymät kvasisymmetrisissä homeomorfismeissa

Tiivistelmä. Tässä työssä todistamme ensinnäkin, että jos h on hajaantumistyyppisen, ei-hilamaisen Fuchsin ryhmän G kvasikonformista kompaktia muodonmuutosta vastaava reaaliakselin $\mathbb{R}$ kvasisymmetrinen homeomorfismi, niin h ei ole vahvasti singulaarinen hajaantumisryhmien suhteen. Tämä yleistää Bishopin ja Stegerin tulosta (1993). Lisäksi todistamme, että Bishopin ja Stegerin tulos ei päde kaikkien d-ulotteisten "kiipeilytelineiden" peiteryhmille (d on mikä tahansa positiivinen kokonaisluku). Tämä yleistää Gönyen tuloksia (2007), joissa kirjoittaja tarkasteli 1-ulotteisen "kiipeilytelineen" tapausta.

1. Introduction

Let G be a non-elementary torsion free discrete Möbius transformations group acting on $\bar{\mathbb{R}}^n = \mathbb{R}^n \cup \infty$ or $S^n = \partial\mathbb{B}^n$; the action of G can extend to the $(n+1)$-dimension hyperbolic upper half plane $\mathbb{H}^{n+1} = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \colon x_{n+1} > 0\}$ or the hyperbolic unit ball $\mathbb{B}^{n+1}$. A discrete group G is called a Kleinian group if $n = 2$ and a Fuchsian group if $n = 1$. In this paper, we mainly focus our attention on Fuchsian groups.

Let $\Lambda(G)$ be the accumulation set of any orbit. A Fuchsian group G is said to be of the first kind if the limit set $\Lambda(G)$ is the entire circle. Otherwise, it is of the second kind. A point $x \in \bar{\mathbb{R}}$ is a conical limit point or radial limit point of G if there is a sequence $(g_i)_{i\geq 1}$ of elements $g_i \in G$ such that for any $z \in \mathbb{H}$, there exists a constant C and a hyperbolic line L with endpoint x such that the hyperbolic distance between $g_i(z)$ and L are bounded by C. Denote by $\Lambda_c(G)$ the set of all the conical limit points and $\Lambda_e(G)$ the set of all the escaping limit points. Let $S = \mathbb{H}/G$ be the corresponding surface of G. The points in $\Lambda_c(G)$ are just corresponding to the geodesics in S which return to some compact set infinitely often and the points in $\Lambda_e(G)$ are corresponding to the geodesics in S which eventually leave every compact subset of S. An important subset of the escaping limit set $\Lambda_e(G)$ is the linear escaping

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limit set which contains all the linear escaping limit points. Parameterize geodesics γ by hyperbolic arclength by $\gamma(t)$ and for any $0 < \delta < 1$, let $\Lambda_\delta(G)$ correspond to the set of geodesics γ such that

$$\mathrm{dist}_S(\gamma(t), \gamma(0)) \geq \delta t$$

for t sufficiently large and let $\Lambda_L(G) = \bigcup_{0<\delta<1} \Lambda_\delta(G)$ denote the linear escaping limit set.

For g in G, we denote by $\mathcal{D}_z(g)$ the closed hyperbolic half plane containing z, bounded by the perpendicular bisector of the segment $[z, g(z)]_h$. The Dirichlet fundamental domain $\mathcal{F}_z(G)$ of G centered at z is the intersection of all the sets $\mathcal{D}_z(g)$ with g in $G\backslash\{\mathrm{id}\}$. For simplicity, in this paper we use the notation $\mathcal{F}$ for the Dirichlet fundamental domain $\mathcal{F}_z(G)$ of G centered at $z = 0$. A Fuchsian group Γ is called a lattice if the area of its one Dirichlet fundamental domain is finite. Moreover, a lattice is said to be uniform if each of its Dirichlet fundamental domain is compact, for more details, see [12].

A Fuchsian group G is said to be of divergence type if $\Sigma_{g\in G}(1 - |g(0)|) = \infty$. Otherwise, we say it is of convergence type. All the second kind groups are of convergence type but the converse is not true.

We call F a quasi-conformal deformation of G if it is a quasi-conformal homeomorphism of the upper half plane $\mathbb{H}$ such that

$$G' = \{g' \colon g' = F \circ g \circ F^{-1} \text{ for every } g \in G\}$$

is also a Fuchsian group and a compact quasi-conformal deformation of G if it is just a lifted mapping of a quasi-conformal mapping f defined on the surface $\mathbb{H}/G$ whose Beltrami coefficient is supported on a compact subset of $\mathbb{H}/G$. Such an F will extend unique to a homeomorphism of the real axis $\bar{\mathbb{R}}$, denoted by h. The homeomorphism h is a quasi-symmetric mapping of $\bar{\mathbb{R}}$.

The quasi-symmetric mappings can be very singular in the measure theoretic sense. It is known that the quasi-conformal mappings preserve the null-sets. However, the quasi-symmetric mappings may be very singular, which will not preserve null-sets, see [5].

In [21], Tukia showed that, for the unit interval $I = [0, 1]$, there are a quasi-symmetric self mapping of I and a set $E \subset I$ such that the Hausdorff dimensions of both $I \setminus E$ and $f(E)$ are less than 1. In [11], Bishop and Steger got the following result: for a lattice group G (i.e. G is finitely generated of first kind), there is a set $E \subset \mathbb{R}$ such that the Hausdorff dimensions of both E and $h(\mathbb{R} \setminus E)$ are less than 1, where h is a quasi-symmetric conjugating homeomorphism of the real axis $\mathbb{R}$. Bishop and Steger's result implies that any conjugation h of the real axis $\mathbb{R}$ of a lattice group G must either be Möbius or strongly singular, i.e., h maps a set of Hausdorff dimension < 1 to the complement of a set of Hausdorff dimension < 1.

Concerning the negative results we first give the definition of the d-dimensional 'Jungle Gym'. Let S_0 be a compact surface of genus d and G_0 its covering group. Let N_0 be a normal subgroup of G_0 such that G_0/N_0 is isomorphic to $\mathbb{Z}^d$. The surface $S^* = \mathbb{H}/N_0$ is the so called infinite d-dimensional 'Jungle Gym', that is, $S^* = \mathbb{H}/N_0$ can be quasi-isometrically embedded into $\mathbb{R}^d$ as a surface S which is invariant under translations t_j, $1 \leq j \leq d$, in d orthogonal directions. Moreover S_0 is conformal equivalent to $S/\langle t_1, \cdots, t_d\rangle$.

In [14], Gönye showed that Tukia–Bishop–Steger's results do not hold for the covering groups of 1-dimensional 'Jungle Gyms'. Gonye constructed a conjugating map f between covering groups of two 1-dimensional 'Jungle Gyms' with the Beltrami coefficient being compactly supported, for which

$$\max(\dim(E), \dim f(\mathbb{R} \setminus E)) = 1$$

for all $E \subset R$, where and in the following of the paper, $\dim(\cdot)$ always denotes the Hausdorff dimension of the set.

In this paper, we continue to investigate the range of validity of Tukia–Bishop–Steger's results. The divergence Fuchsian groups have Mostow's rigidity property, so if h is any quasi-symmetric homeomorphism which conjugates a divergence Fuchsian group to another one, then h is either Möbius or singular (see, e.g., Agard [1], Tukia [20] or Bishop [6]), i.e. h is continuous but the derivation of h vanishes almost everywhere in the real axis $\mathbb{R}$. For the quasi-symmetric homeomorphisms corresponding to a compact deformation of a divergence Fuchsian group, we have

Theorem 1.1. *Let G be a Fuchsian group of divergence type and non-lattice. If h is a homeomorphism of the real axis $\mathbb{R}$ corresponding to a compact deformation of G, then for any $E \subset R$, we have*

$$\max(\dim(E), \dim h(\mathbb{R} \setminus E)) = 1.$$

In other words, for the covering groups of the divergence surfaces of infinite area, a non-Möbius conjugating homeomorphism of the real axis $\mathbb{R}$ must be singular in the sense that it maps a set of zero Lebesgue measure to the complement of zero measure, but can't be strongly singular.

Combine with Bishop and Steger's result [11], we have

Theorem 1.2. *Let G be a Fuchsian group and h a quasi-symmetric homeomorphism of the real axis $\mathbb{R}$ corresponding to a compact deformation of G. Then there exists a subset $E \subset \mathbb{R}$, such that*

$$\max(\dim(E), \dim h(\mathbb{R} \setminus E)) < 1 \tag{1.1}$$

if and only if G is a lattice.

Concerning the 'Jungle Gyms', note that the area of a 'Jungle Gym' is infinite and by Theorem 1.2, we can easily generalize Gönye's result to d-dimensional 'Jungle Gyms', where d is any positive integer number.

Corollary 1.3. *For any positive integer number d, suppose G is a covering group of a d-dimensional 'Jungle Gym' and h a homeomorphism of the real axis $\mathbb{R}$ corresponding to a compact deformation of G, for any $E \subset \mathbb{R}$, we have*

$$\max(\dim(E), \dim h(\mathbb{R} \setminus E)) = 1. \tag{1.2}$$

Remark. By [2] we know that when $d = 1$ or 2, the covering groups of 'd-dimensional 'Jungle Gyms' are of divergence type and when $d \geq 3$, the covering group of d-dimensional 'Jungle Gyms' are of convergence type.

The remainder of the paper is organized as follows: In Section 2, we recall some definitions. In Section 3, we give some results about the differentiability of quasi-conformal mappings at escaping limit points and give some applications. In Section 4, we prove Theorem 1.1 and in Section 5, we prove Theorem 1.2.

2. Preliminaries

Before giving the proofs of the above results, we first recall some definitions.

2.1. Quasi-conformal mapping. Let $\mathbb{H}$ be the upper half-plane in the complex plane $\mathbb{C}$. We denote by $M(\mathbb{H})$ the unit ball of the space $L^\infty(\mathbb{H})$ of all essentially bounded Lebesgue measurable functions in $\mathbb{H}$. For a given $\mu \in M(\mathbb{H})$, there exists a unique quasi-conformal self-mapping f^μ of $\mathbb{H}$ fixing 0, 1 and ∞, and satisfying the following equation

$$\frac{\partial}{\partial \bar{z}} f^\mu(z) = \mu(z) \frac{\partial}{\partial z} f^\mu(z), \quad \text{a.e. } z \in \mathbb{H}.$$

We call μ the Beltrami coefficient of f^μ. It is well known that f^μ can be extended continuously to the real axis $\mathbb{R}$ such that f^μ restricted to $\mathbb{R}$ is a quasi-symmetric homeomorphism.

Similarly, there exists a unique quasi-conformal homeomorphism f_μ of the complex plane $\mathbb{C}$ which is holomorphic in the lower half plane, fixing 0, 1 and ∞ and satisfying

$$\frac{\partial}{\partial \bar{z}} f_\mu(z) = \mu(z) \frac{\partial}{\partial z} f_\mu(z), \quad \text{a.e. } z \in \mathbb{H}.$$

2.2. Poincaré exponent. The critical exponent (or Poincaré exponent) of a Fuchsian group G is defined as

$$\delta(G) = \inf \left\{ t\colon \sum_{g \in G} \exp(-t\rho(0, g(0))) < \infty \right\} \tag{2.1}$$

$$= \inf \left\{ t\colon \sum_{g \in G} (1 - |g(0)|)^t < +\infty \right\}, \tag{2.2}$$

where ρ denotes the hyperbolic metric. It has been proven in[8] that for any non-elementary group G, $\delta(G)$ equals to $dim(\Lambda_c(G))$, the Hausdorff dimension of the conical limit set.

2.3. Hausdorff dimension. Let E be a subset of the complex plane $\mathbb{C}$. Suppose φ is an nonnegative increasing homeomorphism of $[0, \infty)$. For φ and $0 < \delta \leq \infty$, we define

$$\mathcal{H}^\varphi_\delta(E) = \inf \left\{ \sum_{i=1}^{\infty} \varphi(|B_i|)\colon E \subset \bigcup_{i=1}^{\infty} B_i, \ |B_i| \leq \delta \right\},$$

where $B_i \subset \mathbb{C}$ is a set and $|B_i|$ denotes its diameter, the infimum is taken over all open coverings of E. Then the Hausdorff measure of E to be

$$\mathcal{H}^\varphi(E) = \lim_{\delta \to 0} \mathcal{H}^\varphi_\delta(E) = \sup_{\delta > 0} \mathcal{H}^\varphi_\delta(E)$$

and the Hausdorff content of E is $\mathcal{H}^\varphi_\infty(E)$.

If $\varphi(t) = t^\alpha$, $\alpha \in [0, 2]$, we denote $\mathcal{H}^\varphi(E)$ by $\mathcal{H}^\alpha(E)$. Then one defines the α-dimensional Hausdorff measure of E to be

$$\mathcal{H}^\alpha(E) = \lim_{\delta \to 0} \mathcal{H}^\alpha_\delta(E) = \sup_{\delta > 0} \mathcal{H}^\alpha_\delta(E).$$

One defines the Hausdorff dimension of E to be

$$\dim E = \inf\{\alpha\colon \mathcal{H}^\alpha(E) = 0\}.$$

3. Differentiability of quasi-conformal mappings at escaping limit points revisited

It is well known that the quasi-conformal mappings of a domian Ω are differentiable almost everywhere in Ω. In this paper we need the following criterion for pointwise conformality, see ([4, 15, 18, 19, 22] or [16, Theorem 6.1]) (I would like to thank Professor Liu Jinsong pointing reference [18] out to me).

Lemma 3.1. *Let Ω and Ω' be two domains in the complex plane $\mathbb{C}$ and $0 \in \Omega$, and let f be a quasi-conformal mapping from Ω to Ω' with Beltrami coefficient $\mu(z)$, where $|\mu(z)| \leq k < 1$ almost everywhere in Ω. If $\mu(z)$ satisfies*

$$\frac{1}{2\pi} \iint_{|z|<r} \frac{|\mu(z)|}{|z|^2}\, dx\, dy < \infty \tag{3.1}$$

for some $r > 0$, then f is conformal at $z = 0$.

Let G be non-lattice Fuchsian group of divergence type. Let f be a quasi-conformal mapping on the surface $S = \mathbb{H}/G$ whose Beltrami coefficient μ is supported on a compact subset of S. Thus we can choose a point $z_0 \in S$ and a sufficiently large r_0 such that the support set of μ is contained in the disk $B(z_0, r_0)$. Let $S_{r_0} = S \setminus B(z_0, r_0)$ and let Ω_{r_0} be the lifting of S_{r_0} to the upper plane $\mathbb{H}$. By the definition of escaping limit points we know that an escaping geodesic eventually stays inside in the region Ω_{r_0} and far from the support of μ. We can lift f to the upper half plane $\mathbb{H}$ and get a quasi-conformal homeomorphism $F^\mu(z)\colon \mathbb{H} \to \mathbb{H}$. Induced by the Beltrami coefficient of F^μ we can get a quasi-conformal homeomorphism of the complex plane $\mathbb{C}$ such that the Beltrami coefficient of F_μ is almost everywhere equal to the one of F^μ on the upper half plane $\mathbb{H}$ and vanishes almost every on the lower half plane $\mathbb{L}$.

In the set of finitely generated Kleinian groups, the differentiability of conjugating maps at linear escaping limit points (where McMullen called them "deep points" of the limit set) has been noted by McMullen [17]. Bishop and Jones [9] proved that one only needs a logarithm escaping to infinity to get differentiability. For the quasi-conformal mapping F_μ, we have

Theorem 3.2. *Let G be a Fuchsian group of divergence type and non-lattice, and let f be a quasi-conformal mapping on the surface $S = \mathbb{H}/G$ so that the Beltrami coefficient μ of f is compactly supported on S. Let F^μ be the lifted mapping of f to the upper half plane $\mathbb{H}$ extended to the real axis $\mathbb{R}$ with 0, 1 and ∞ fixed. If F_μ is the quasi-conformal homeomorphism of the complex $\mathbb{C}$ whose Beltrami coefficient is equal to the one of F^μ almost everywhere on the upper half plane $\mathbb{H}$ and vanishes on the lower half plane $\mathbb{L}$, then F_μ is conformal at the linear escaping limit points $x \in \Lambda_L(G)$.*

Proof. Under the assumptions in the theorem, we can choose a point $p_0 \in S$ and a sufficiently large R_0 such that the support set of f is contained in the disk $B(p_0, R_0)$. Let $S_{R_0} = S \setminus B(p_0, R_0)$ and let Ω_{R_0} be the lifting of S_{R_0} to the upper half plane $\mathbb{H}$. By the definition of escaping limit points, we know that an escaping geodesic eventually stays inside the region Ω_{R_0} and far from the support of μ.

Since the Möbius transformations which keep the upper half plane invariant do not change the hyperbolic geometry properties(such as hyperbolic area of subset of $\mathbb{H}$ and hyperbolic distance between two points) of the upper half plane, with the conjugation of such Möbius transformations, we can suppose $x = 0$ and the initial point of the geodesic ray is i, denote the geodesic by $\gamma(t)$, where t is the arc-length parametrization with $\gamma(0) = i$ and $\lim_{t\to\infty}\gamma(t) = 0$. By the definition of the linear escaping geodesics, there is a region such that none of the lifted pre-images of $B(p_0, R_0)$ will hit the escaping geodesics eventually. Hence there is a sufficiently large t_0 ($t_0 > 1$) and a $\delta \in (0, 1)$, for $t > t_0$, $\mathrm{dist}(\gamma(t), \mathbb{H} \setminus \Omega_{R_0}) > \delta t > R_0$, where $\mathrm{dist}(\cdot,\cdot)$ denotes the hyperbolic distance between two points.

Let $r_0 = e^{-t_0}$ and μ_F be the Beltrami coefficient of F_μ. In the following, we will show that the integral

$$\frac{1}{2\pi}\iint_{|z|<r_0} \frac{|\mu_F(z)|}{|z|^2}\,dx\,dy \tag{3.2}$$

is finite.

Since the Beltrami coefficient μ_F vanishes in the regions Ω_{R_0} and the lower half plane $\mathbb{L}$, we need to show that the integral (3.2) is finite in a neighborhood of 0 outside the regions Ω_{R_0} and $\mathbb{L}$. We will use polar coordinate to estimate the integral (3.2). We have

$$\frac{1}{2\pi}\iint_{|z|<r_0} \frac{|\mu_F(z)|}{|z|^2}\,dx\,dy \le \frac{1}{2\pi}\int_0^{r_0} dr \int_0^{\theta_1(r)} \frac{1}{r}\,d\theta + \frac{1}{2\pi}\int_0^{r_0} dr \int_0^{\theta_2(r)} \frac{1}{r}\,d\theta, \tag{3.3}$$

where $\theta_1(r)$ and $\theta_2(r)$ are the arguments of the points which are the intersction of the hyperbolic circle $\mathrm{dist}(ir, z) = -\delta \ln r$ with the Euclidean circle $|z| = r$, where $r < 1$. Since the region is symmetric with respect to $\gamma(t)$, we only need to show that the integral

$$\int_0^{r_0} dr \int_0^{\theta_1(r)} \frac{1}{r}\,d\theta \tag{3.4}$$

is finite. By some easy calculations or see [3, p. 131], we know that the hyperbolic circle $\mathrm{dist}(ir, z) = -\delta \ln r$ is just the Euclidean circle

$$\left| z - ir\frac{r^\delta + r^{-\delta}}{2} \right| = r\left(\frac{r^{-\delta} - r^\delta}{2}\right). \tag{3.5}$$

Let $Q = x + iy$ be the intersection points of the hyperbolic circle $\mathrm{dist}(ir, z) = -\delta \ln r$ with the Euclidean circle $|z| = r$ in the first quadrant. Combine (3.5), we have the imaginary part of Q satisfies the equation

$$y = \frac{2r}{r^\delta + r^{-\delta}}.$$

Therefore

$$\sin\theta_1(r) = \frac{y}{r} = \frac{2}{r^\delta + r^{-\delta}} \le r^\delta.$$

Since for $\theta \in (0, \frac{\pi}{2})$, $\frac{2}{\pi}\theta \le \sin\theta$, we have

$$\theta_1(r) \le \frac{\pi}{2} r^\delta. \tag{3.6}$$

Hence the integral

$$\int_0^{r_0} dr \int_0^{\theta_1(r)} \frac{1}{r}\,d\theta \le \int_0^{r_0} \frac{\pi}{2} r^{\delta-1}\,dr$$

is finite. Furthermore, we have

$$\frac{1}{2\pi}\iint_{|z|<r_0} \frac{|\mu_F(z)|}{|z|^2}\,dx\,dy < \infty. \tag{3.7}$$

By Lemma 3.1, we know that F_μ is conformal at 0. □

For the quasi-conformal homeomorphism F^μ of the upper half plane $\mathbb{H}$, we have the following results which is similar to Gönye's result [14, Theorem 1.1] where Gönye discussed the differentiability of the conjugating quasi-symmetric homeomorphisms of the covering groups of 1-dimensional 'Jungle Gyms'.

Theorem 3.3. *Under the assumptions in Theorem 3.2, let $h := F^\mu|\mathbb{R}$ be the quasi-symmetric homeomorphism of F^μ extended and restricted to the real axis $\mathbb{R}$. Then h is differentiable at the linear escaping limit points $x \in \Lambda_L(G)$ with $h'(x) \neq 0$.*

Proof. Let

$$F(z) = \begin{cases} F^\mu(z), & z \in \mathbb{H}; \\ \overline{F^\mu(\bar{z})}, & z \in \mathbb{C} \setminus \mathbb{H}. \end{cases}$$

By the symmetry of the Beltrami coefficient of F relative to the real axis $\mathbb{R}$ and (3.6), we have that the quasi-symmetric homeomorphism h are differentiable at the linear escaping points $x \in \Lambda_L(G)$. □

For a compact quasi-conformal deformation, in [8], Bishop and Jones use the properties of Schwarzian derivative of F_μ to estimate the Hausdorff dimension of the escaping limit set $\Lambda_e(G)$ and $F_\mu(\Lambda_e(G))$, they showed

Theorem 3.4. [8] *Under the assumptions in Theorem 3.2,* $\dim \Lambda_e(G)$ *is equal to* $\dim F_\mu(\Lambda_e(G))$.

As an application of Theorem 3.2, we show

Theorem 3.5. *Under the assumptions in Theorem 3.2,* $\dim \Lambda_L(G)$ *is equal to* $\dim F_\mu(\Lambda_L(G))$.

Proof. By Theorem 3.2 we know

$$\Lambda_L(G) = \{x\colon F'_\mu(x) \text{ exists and non-zero, } x \in \Lambda_L(G)\}.$$

Define the set

$$\Lambda_n = \{x\colon \tfrac{1}{n} \leq |F'_\mu(x)| \leq n,\ x \in \Lambda_L(G)\},$$

it is easy to see $\Lambda_L(G) = \bigcup_{n=1}^\infty \Lambda_n$ and $\Lambda_n \subset \Lambda_{n+1}$. Hence $\Lambda_L(G) = \lim_{n\to\infty} \Lambda_n$.

For $x \in \Lambda_n$, we can choose a δ_x such that, for $|z - x| < \delta_x$,

$$\frac{1}{2n} \leq \frac{|F_\mu(z) - F_\mu(x)|}{|z - x|} \leq 2n.$$

This means that for each $x \in \Lambda_n$, there exists a constant δ_x, such that for all neighborhood B_x of x with $|B_x| < \delta_x$,

$$\frac{1}{2n}|B_x| \leq F_\mu(|B_x|) \leq 2n|B_x|. \tag{3.8}$$

Note that for a fixed number n, the choice of constant δ_x depends on the points x. To get rid of the dependence on x, define the set

$$\Lambda_{n,k} = \left\{x\colon x \in \Lambda_n, \forall |B_x| \text{ with } |B_x| < \frac{1}{k},\ \frac{1}{2n}|B_x| \leq F_\mu(|B_x|) \leq 2n|B_x|\right\}. \tag{3.9}$$

It is easy to see that $\Lambda_{n,k} \subset \Lambda_{n,k+1}$ and $\Lambda_n = \lim_{k\to\infty} \Lambda_{n,k}$.

In the following, we will show that for $\alpha \in (0, 2)$, the Hausdorff measures of $\mathcal{H}^\alpha(F_\mu(\Lambda_{n,k}))$ and $\mathcal{H}^\alpha(\Lambda_{n,k})$ satisfy

$$\left(\frac{1}{2n}\right)^\alpha \mathcal{H}^\alpha(\Lambda_{n,k}) \leq \mathcal{H}^\alpha(F_\mu(\Lambda_{n,k})) \leq (2n)^\alpha \mathcal{H}^\alpha(\Lambda_{n,k}). \tag{3.10}$$

We first show that the second inequality of (3.10) holds.

For fixed n and k, suppose $\{B_i\}$ is a cover of $\Lambda_{n,k}$ with $|B_i| < \frac{1}{2nj}$, where $j \geq k$. Then by the definition of $\Lambda_{n,k}$, we know that the sequence $(F_\mu(B_i))_{i\geq 1}$ is a cover of $F_\mu(\Lambda_{n,k})$ with

$$|F_\mu(B_i)| < \frac{1}{j}.$$

For any $\alpha \in (0, 2)$, we have

$$\mathcal{H}^\alpha_{1/j}(F_\mu(\Lambda_{n,k})) \leq \sum_{i=1}^{\infty} |F_\mu(B_i)|^\alpha \leq \sum_{i=1}^{\infty} (2n|B_i|)^\alpha. \tag{3.11}$$

Take the infimum of the right of (3.10), we obtain

$$\mathcal{H}^\alpha_{1/j}(F_\mu(\Lambda_{n,k})) \leq (2n)^\alpha \mathcal{H}^\alpha_{1/2nj}(\Lambda_{n,k}).$$

Let j tend to infinity, the α-dimensional Hausdorff measures of $F_\mu(\Lambda_{n,k})$ and $\Lambda_{n,k}$ satisfy

$$\mathcal{H}^\alpha(F_\mu(\Lambda_{n,k})) \leq (2n)^\alpha \mathcal{H}^\alpha(\Lambda_{n,k}). \tag{3.12}$$

On the other hand, take a covering $\{V_i\}$ of the set $F_\mu(\Lambda_{n,k})$ with

$$|V_i| < \frac{1}{2nj},$$

where $j > 2k$. Without loss of generality we may suppose $V_i \cap F_\mu(\Lambda_{n,k}) \neq \emptyset$. By the definition of the set $\Lambda_{n,k}$, we can choose a point $x \in F_\mu^{-1}(V_i) \cap \Lambda_{n,k}$ and a neighborhood B_x of it with $\frac{1}{k} > |B_x| > \frac{2}{j}$, for which

$$\frac{1}{2n}|B_x| \leq F_\mu(|B_x|) \leq 2n|B_x|. \tag{3.13}$$

Hence $|F_\mu^{-1}(V_i)| < |B_x|$. So we can suppose $F_\mu^{-1}(V_i) \subset B_x$ and

$$\mathcal{H}^\alpha(\Lambda_{n,k}) \leq \sum |F_\mu^{-1}(V_i)|^\alpha \leq \sum (2n|V_i|)^\alpha. \tag{3.14}$$

Note that $F_\mu^{-1}(V_i)$ is a covering of $\Lambda_{n,k}$ and take the infimum of the covering $\{V_i\}$ we have

$$\mathcal{H}^\alpha(\Lambda_{n,k}) \leq (2n)^\alpha \mathcal{H}^\alpha_{1/2nj}(F_\mu(\Lambda_{n,k})).$$

Let j tend to infinity, we can get that the first inequality of (3.9) also holds. By the definition of Hausdorff dimension, the inequalities (3.10) show that, for fixed n, k, the Hausdorff dimension of $\Lambda_{n,k}$ is the same as its image under the map F_μ. Since the dimension is preserve for every n and k, hence we have

$$\dim F_\mu(\Lambda_L) = \dim(\Lambda_L).$$

This completes the proof of this theorem. □

Note that in [7], the Hausdorff dimension of the linear escaping limit points of a Fuchsian group G is always equal to the dimenson of the geodesics that escape to infinity at linear speed, by Theorem 3.5, this can give a new proof of Bishop and Jones' result of Theorem 3.4.

For a non-lattice divergence type Fuchsian group G, Fernandez and Melian proved

Theorem 3.6. [13] *Let G be a non-lattice divergence type Fuchsian group. Then $\Lambda_e(G)$ has zero Lebesgue measure, but its Hausdorff dimension is* 1.

For the convenience of the reader to better understand the distribution of the escaping limit set $\Lambda_e(G)$, appendix 1 reproduces a sketch proof of this theorem.

As a subset of the escaping limit set $\Lambda_e(G)$, the linear escaping limit set $\Lambda_L(G)$ satisfies:

Theorem 3.7. *Let G be a non-lattice divergence type Fuchsian group. Then $\Lambda_L(G)$ has zero 1-dimensional Hausdorff measure, but its Hausdorff dimension is 1.*

Proof. Let G be a non-lattice divergence Fuchsian group and f be a quasi-conformal mapping of the surface $S = \mathbb{H}/G$ whose Beltrami coefficient is compactly supported on S. As the statements of Theorem 3.2, let F_μ be a quasi-conformal mapping of the complex plane $\mathbb{C}$ which has the same Beltrami coefficient with the lifted mapping F^μ of f to the upper half plane $\mathbb{H}$ and is conformal on the lower half plane $\mathbb{L}$. By [10, Theorem 1.3], we know that the 1-dimensional Hausdorff measure of $F_\mu(\Lambda_L(G))$ is zero. Hence, as the notations in the proof of Theorem 3.2, for fixed numbers n and k, the Hausdorff measure of the subset $F_\mu(\Lambda_{n,k})$ is zero. By (3.10) in the proof of Theorem 3.2, we know, for fixed n and k, the 1-dimensional Hausdorff measure of $\Lambda_{n,k}$ is also zero. Furthermore the 1-dimensional Hausdorff measure of $\Lambda_L(G)$ is zero.

By Theorem 3.6, the Hausdorff dimension of $\Lambda_e(G)$ is 1. By [7], we know that the Hausdorff dimension of the linear escaping limit set $\Lambda_L(G)$ is always equal to the dimension of the escaping limit set $\Lambda_e(G)$. □

4. Proof of Theorem 1.1

To prove this theorem, we need the following lemma which is essentially due to Gönye, see [14, p. 29].

Lemma 4.1. *Let F be a quasi-symmetric homeomorphism of the real axis $\mathbb{R}$ and A be a subset of $\mathbb{R}$ with Hausdorff dimension equal to 1. If for any $x \in A$, $F'(x)$ exists and is non-zero, then the Hausdorff dimension of $F(A)$ is also 1.*

Now we give the proof of Theorem 1.1.

Proof. Let G be a Fuchsian group of divergence type and not a lattice. Let f be a quasi-conformal mapping on the surface $\mathbb{H}/G$. The lifting mapping F^μ of f to the upper half plane $\mathbb{H}$ can extend to the real axis $\mathbb{R}$ naturally. We denote by $h = F^\mu|\mathbb{R}$. The mapping h is a quasi-symmetric homeomorphism of $\mathbb{R}$. By Theorem 3.3, the quasi-symmetric homeomorphism h is differentiable at x in $\Lambda_L(G)$ with $|h'(x)| \neq 0$. By Theorem 3.7, we know that the Hausdorff dimension of the escaping limit set $\Lambda_L(G)$ is 1. By Lemma 4.1, we know, for any $E \subset \mathbb{R}$,

$$\max(\dim(E \cap \Lambda_L(G)), \dim(\Lambda_L(G) \setminus E)) = 1.$$

Hence

$$\max(\dim(E), \dim f(\mathbb{R} \setminus E)) = 1$$

and the theorem holds. □

5. Proof of Theorem 1.2

Proof. The necessity of the equivalence is from [11, Theorem 4].

For the sufficient condition, by Theorem 1.1, we only need to show the case when G is a Fuchsian group of the convergence type.

If G is a Fuchsian group of the second kind, the boundary of any one Dirichlet fundamental domain $\mathcal{F}$ contains at least an arc (denoted by α^*) in $\mathbb{R}$. It is easy to see that the homeomorphism is smooth on α^*. Hence the sufficient condition holds. If G is a Fuchsian group of the convergence type and of the first kind, we need to show that the Hausdorff dimension of the escaping limit set $\Lambda_e(G)$ is 1, actually it has positive 1-dimensional Hausdorff measure.

Suppose γ be a closed geodesic on the surface $S = \mathbb{H}/G$. Consider the lifting of the closed geodesic γ in the upper half plane $\mathbb{H}$. It consists of a nested set Σ of hyperbolic lines: the one intersecting the Dirichlet fundamental domain $\mathcal{F}$ cuts it into two parts and we may assume that the point i belongs to a part that has infinite (hyperbolic) area. The hyperbolic lines in Σ of the first generation define a two-by-two disjoint family (I_j) of intervals of the real axis $\mathbb{R}$. Suppose $\bigcup_{i=1} I_j$ is equal to $\mathbb{R}$ except a zero Lebesgue measure set, then almost every geodesic issued from i would visit γ infinitely often, contradicting of [13, Theorem 1]. Thus the set of geodesics from i that never visit γ has positive measure. It follows that the escaping limit set of S has positive Lebesgue measure.

Hence if F^μ corresponding to a compact quasi-conformal deformation of G, we always have, for any $E \subset \mathbb{R}$, $\max(\dim(E), \dim f(\mathbb{R} \setminus E)) = 1$. □

6. Appendix 1: Proof of Theorem 3.6

Since G is non-lattice, the area of the surface S and the generators of G are both infinity. The method we used here is from [13]. We first recall the definition of geodesic domain. A domain $D \subset S$ is called a geodesic domain if its relative boundary consists of finitely many non-intersecting closed simple geodesics and its area is finite. Fix a point $P_0 \in S$, by [13, Theorem 4.1], we know that there exists a family $\{D_i\}_{i=0}^{\infty}$ of pairwise disjoint (except the boundary) geodesic domains in S satisfying that the boundary of D_i and D_{i+1} have at least a simple closed geodesic in common and $\lim_{i\to\infty} \operatorname{dist}(P_0, D_i) = \infty$, where $\operatorname{dist}(\cdot,\cdot)$ denotes the hyperbolic distance of the surface S.

Let $\{D_i\}_{i=0}^{+\infty}$ be the family of geodesic domains of S constructed as above. For any i, let S_i be the Riemann surface obtained from D_i by gluing a funnel along each one of the simple closed geodesics of its boundary. For each i, we choose a simple closed geodesic γ_i from the common boundary $D_i \cap D_{i+1}$ and a point $P_i \in \gamma_i$. By [13, Theorem 4.1], we have $\delta_i \to 1$ when i tends to infinity, where δ_i is the Poincare exponent of the surface S_i.

For $\theta \in (0, \frac{1}{2}\pi)$, by ([13], Theorem 5.1), we can choose a collection $\mathcal{B}_i$ of geodesics in S_i with initial and final endpoint P_i such that

$$L_i \le \operatorname{length}(\gamma) \le L_i + C(P_i), \quad \gamma \in \mathcal{B}_i,$$

where L_i is a constant such that $L_i \to \infty$ as $i \to \infty$, $C(P_i)$ is a constant depending only on the length of the geodesic γ_i, and $\sigma_i < \delta(S_i)$, $\sigma_i \to 1$ as $i \to \infty$. The number of geodesic arcs in B_i is at least $e^{L_i\sigma_i}$, and both the absolute value of the angles between γ and the closed geodesic γ_i are less than or equal to θ.

Note that for each i, D_i is the convex core of S_i, implying that every geodesic arc $\gamma \in \mathcal{B}_i$ is contained in the convex core D_i. Furthermore, for each i, we may choose a geodesic arcs γ_i^* with initial point P_i and final endpoint P_{i+1} such that

$$L_i \le \operatorname{length}(\gamma_i^*) \le L_i + C(P_{i+1}),$$

and both the absolute value of the angles between γ_i, γ_i^*, and γ_i^*, γ_{i+1} are less than or equal to θ.

In order to show the distribution of geodesics on S, we are going to construct a tree $\mathfrak{T}$ consisting of oriented geodesic arcs in the unit disk Δ.

Let us first lift γ_0^* to the unit disk starting at 0 (without loss of generality we may suppose that 0 projects onto P_0). From the endpoint of the lifted γ_0^* (which project onto P_1), lift the family $\mathfrak{B}_1$; from each of the end points of these liftings (which still project onto P_1), lift again $\mathfrak{B}_1$. Keep lifting $\mathfrak{B}_1$ in this way a total of M_1 times.

Next, from each one of the endpoints obtained in the process above, we lift γ_1^*, and from each one of the endpoints of the liftings of γ_1^* (which project onto P_2), we lift the collection $\mathfrak{B}_2$ sucessively M_2 times as above. Continuously this process indefinitely we obtain a tree $\mathfrak{T}$.

It is easy to see that $\mathfrak{T}$ contains uncountably many branches. The tips of the branches of $\mathfrak{T}$ are contained in the escaping limit set $\Lambda_e(G)$ of the covering group of S. For suitably choosing the sequence $\{M_i\}$ of repetitions, the dimension of the rims of tree $\mathfrak{T}$ is 1. By the construction of the tree $\mathfrak{T}$, we see that the tree $\mathfrak{T}$ is a unilaterally connected graph. Hence the geodesic corresponding to any branch of $\mathfrak{T}$ does not tend to the funnel with boundary γ. Hence the dimension of the escaping limit set $\Lambda_e(G)$ of the covering group G is 1.

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