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On the full regularity of the free boundary for minima of Alt–Caffarelli functionals in Orlicz spaces

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Abstract. In this paper, we discuss two issues about the full regularity of the free boundary for overdetermined Bernoulli-type problems in Orlicz spaces. First, we show that in dimension $n = 2$ there are no singular points on the free boundary $F(u) := \partial\{u > 0\} \cap \Omega$ of minimizers of the Alt–Caffarelli functional

$$J_G(u) := \int_\Omega \left(G(|\nabla u|) + \lambda\chi_{\{u>0\}}\right) dx$$

for suitable N-functions G. Next, we prove as a consequence of our main results that there exist a critical dimension $5 \le n_0 \le 7$ and a universal constant $\varepsilon_0 \in (0, 1)$ such that if $G(t)$ is "ε_0-close" of t^2, then for $2 \le n < n_0$, $F(u)$ is a real analytic hypersurface.

Altin–Caffarellin funktionaalin minimoijien vapaan reunan täyssäännöllisyys Orliczin avaruuksissa

Tiivistelmä. Tässä työssä tarkastelemme kahta ylimääritettyjen Bernoullin-tyyppisten ongelmien vapaan reunan täyssäännöllisyyttä koskevaa kysymystä Orliczin avaruuksissa. Ensinnäkin osoitamme, että Altin–Caffarellin funktionaalin

$$J_G(u) := \int_\Omega \left(G(|\nabla u|) + \lambda\chi_{\{u>0\}}\right) dx$$

minimoijien vapaalla reunalla $F(u) := \partial\{u > 0\} \cap \Omega$ ei sopivilla N-funktioilla G ole lainkaan singulaarisia pisteitä ulottuvuudessa $n = 2$. Päätuloksemme seurauksena todistamme, että on olemassa sellainen kriittinen ulottuvuus $5 \le n_0 \le 7$ ja yleinen vakio $\varepsilon_0 \in (0, 1)$, että $F(u)$ on reaalianalyyttinen hypertaso kaikissa ulottuvuuksissa $2 \le n < n_0$, jos $G(t)$ on "ε_0-lähellä" funktiota t^2.

1. Introduction

In [13], the authors consider the following optimization problem

$$\min_{v\in K_\varphi} J_G(u), \tag{1.1}$$

where

$$J_G(u) := \int_\Omega \left(G(|\nabla u|) + \lambda\chi_{\{u>0\}}\right) dx,$$

$\Omega \subseteq \mathbb{R}^n$ ($n \ge 2$) is a smooth and bounded domain, G is a suitable N-function, λ is a positive constant, $0 \le \varphi \in W^{1,G}(\Omega) \cap L^\infty(\partial\Omega)$ and $K_\varphi := \{v \in W^{1,G}(\Omega)\colon v - \varphi \in W_0^{1,G}(\Omega)\}$. In this work was proved that if G satisfies the Lieberman's conditions (see [12]), i.e,

- Primitive Condition:

$$G'(t) = g(t), \quad \text{where } g \in C^0([0, +\infty)) \cap C^1((0, +\infty)); \tag{PC}$$

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- Quotient Condition: for $0 < \delta \leq g_0$ fixed constants,

$$0 < \delta \leq Q_g(t) := \frac{tg'(t)}{g(t)} \leq g_0, \quad \forall\, t > 0, \tag{QC}$$

then any minimizer u of (1.1) is a bounded and nonnegative function, locally Lipschitz continuous and satisfies (in some weak sense) the following one-phase Free Boundary Problem (FBP)

$$\begin{cases} \Delta_g u = 0 & \text{in } \{u > 0\} \cap \Omega, \\ |\nabla u| = \lambda^* & \text{on } F(u) := \partial\{u > 0\} \cap \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where

$$\Delta_g u := \operatorname{div}\left(\frac{g(|\nabla u|)}{|\nabla u|}\nabla u\right), \quad g = G' \quad \text{and} \quad H(\lambda^*) := \lambda^* g(\lambda^*) - G(\lambda^*) = \lambda.$$

Besides, it was shown that the free boundary $F(u)$ is locally of finite Hausdorff $\mathcal{H}^{n-1}$ measure, the measure-theoretic reduced boundary $F(u)_{red} := \partial_{red}\{u > 0\} \cap \Omega$ is a union of $C^{1,\alpha}$ hypersurfaces and the set of singular points of $F(u)$ satisfies $\mathcal{H}^{n-1}(F(u) \setminus F(u)_{red}) = 0$ (see Theorem 1.3 of [13]). The original purpose of this paper was to extend the free boundary theory for minimizers developed by Alt–Caffarelli in the pioneering work [2] (the Laplacian operator) and the paper [6] (the p-Laplacian scenario).

Among the differences observed between the results presented in [2] and the papers [6, 13], we can highlight that in two dimensions, if $G(t) = t^2$, the free boundary has no singular points (see Theorem 8.3 of [2]). Actually, it is well known that if $G(t) = t^2$, then the free boundary for any minimizer of (1.1) is also a full regular surface for $n = 3$ and 4 (see for instance [5, 9]) and if $n = 7$ there exists a singular cone u_0 as an absolute minimizer (see [8]). By results of Weiss [16], we also know that there exists a critical dimension n_0 such that if u is a minimizer of (1.1), for $G(t) = t^2$, the set $F(u)$ is full regular since $n < n_0$. In the case where $n = n_0$, the singular set $F(u) \setminus F(u)_{red}$ consists of at most isolated points and has Hausdorff dimension at most $n - n_0$, if $n > n_0$. Particularly, we have $5 \leq n_0 \leq 7$. Such information reveals an interesting and curious parallel between FBPs and the theory of minimal surfaces. For this reason, the problem of the full regularity of the free boundary of the Bernoulli-type problem (1.2) has attracted the attention of specialists from both areas of mathematics.

The issue about the full regularity of the free boundary for minimizers in the p-Laplace case was investigated in [7]. In this work, the authors provided a partial answer to this question. Precisely, they proved that in two dimensions, if u is an absolute minimizer of (1.1), where $G(t) = t^p$, then there exists a universal constant $0 < \varepsilon_0 << 1$ such that if p is in the range $2 - \varepsilon_0 < p < \infty$ necessarily $F(u)$ is fully regular. The fundamental step to prove this result was to show that any absolute minimizer is sufficiently close to a *halfspace solution*[1] near any free boundary point. Very recently, a new geometric approach has been considered in [10] to study the regularity of the free boundary in dimension two for more general weak solutions of (1.2). These techniques work well in the presence of the finite Morse index solutions when $G(t) = t^p$ with $p \geq 2$, and there exists the curvature measure of $F(u)$, and it

[1] A *halfspace solution* has the form $u(x) = \langle x - x_0, e\rangle_+$ for some free boundary point x_0 and unit vector e.

is nonnegative. For the general case, uniform density property on the set $\{u = 0\}$ around points of the free boundary needs to be imposed, and the set $F(u)$ needs to have finitely connectivity.

The critical dimension results for the p-Laplacian operators has been considered in [14]. Again the partial result was obtained. In fact, it was shown that if $1 < p < \infty$ belongs to in a small (universal) interval around 2, then the free boundary of a minimizer in dimension $2 \leq n < n_0$ is an analytic hypersurface.

In this paper, we use a similar method as in [2, 7, 14] to study the full regularity of the free boundary for minimizers of (1.1) under Liberman's conditions (PC) and (QC). From now on, the minimization problem (1.1) and the Lieberman's conditions are associated. In this point, an important fact that should be noted in the papers [7, 14]. In most results, arguments of compactness are crucial, especially in the ranges $p \in (1, 2)$ and $p \in (2 - \epsilon, 2 + \epsilon)$. In the specific cases, the authors rely on indirect arguments and compactness results. Unfortunately, classes of N-functions that meet Lieberman's conditions are weak to provide compactness (see examples in [3, 4]). As pointed in [4], the main reason for this failure is the absence of a uniform modulus of continuity for the quotient Q_g. To address this lack of compactness in more general cases, we assume that Q_g satisfies the following Dini type control

$$\text{(DTC)} \qquad \int_0^{L-l} \frac{\omega_g^{l,L}(t)}{t}\, dt \leq C(\delta, g_0)\xi_1\left(\frac{L}{l}\right) \cdot \xi_2\left(\frac{L-l}{l}\right),$$

for a modulus of continuity $\omega_g^{l,L}$ of Q_g and some functions $\xi_i \colon (0,\infty) \to (0,\infty)$ (for more details see Definition 2.1).

Inspired by works [2, 7, 14], we provide the following improvement of the mentioned results above.

Theorem 1.1. *Let $n = 2$ and u be an absolute minima of (1.1). If $\delta \geq 1$, then $F(u)$ is fully regular. For $\delta \in (0, 1)$ we have two possibilities:*

i) *There exists a universal constant $\varrho \in (0, 1)$ such that if*

$$1 - \varrho < \delta \leq g_0 < 1 + \varrho,$$

then $F(u)$ is fully regular;

ii) *If additionally Q_g satisfies a Dini type control (DTC), then $F(u)$ is a smooth surface provided*

$$1 - \mu < \delta \leq g_0 < \infty,$$

for some universal (small) constant $\mu > 0$.

Indeed, we will provide a result that has a more general statement than Theorem 1.1. The above theorem will be established in terms of classes of minimizers (see Theorem 6.1).

To state our next theorem we establish the following definition.

Definition 1.1. Fixed the constants $2 \leq n \in \mathbb{N}$, $\lambda > 0$ and $\eta \in (0, 1]$, we define $\mathcal{G}(n, \lambda, \eta)$ be the set of pairs (δ, g_0) with $0 < \delta \leq g_0 < \infty$ such that any minimizer of (1.1) with $\eta \leq G(1) \leq \eta^{-1}$ on any open subset $\Omega \subset \mathbb{R}^n$ has no singular free boundary points.

If we consider only N-functions G such that Q_g satisfies (DTC) we replace $\mathcal{G}(n, \lambda, \eta)$ by $\mathcal{G}_{\xi_1,\xi_2}(n, \lambda, \eta)$. Clearly, $\mathcal{G}_{\xi_1,\xi_2}(n, \lambda, \eta) \subseteq \mathcal{G}(n, \lambda, \eta)$, for any functions ξ_1, ξ_2 satisfying the conditions of Definition 2.1. We know that $(1, 1) \notin \mathcal{G}(7, \lambda, \eta)$ and by Corollary 1.3 of [14], there exists a universal constant $\varepsilon_0 > 0$ such that if

$1-\varepsilon_0 < p < 1+\varepsilon_0$, then

$$(p,p) \in \mathcal{G}(n,\lambda,\eta), \quad 2 \le n < n_0.$$

Theorem 1.2. *Consider the following constants* $2 \le n \in \mathbb{N}$, $0 < \delta^* \le g_0^* < \infty$, $\lambda > 0$ *and* $\eta \in (0,1]$. *Then,*

i) *If* $\delta^* = g_0^*$ *and* $(\delta^*, \delta^*) \in \mathcal{G}(n,\lambda,\eta)$, *there exists a constant* $\varrho_0 = \varrho_0(n,\delta^*,\eta,\lambda) > 0$ *such that* $\forall$ (δ, g_0) *with*

$$\delta^* - \varrho_0 < \delta \le g_0 < \delta^* + \varrho_0,$$

we have $(\delta, g_0) \in \mathcal{G}(n,\lambda,\eta)$.

ii) *If* $(\delta^*, g_0^*) \in \mathcal{G}_{\xi_1,\xi_2}(n,\lambda,\eta)$, *there exists a universal constant* $\mu_0 = \mu_0(n,\delta^*,g_0^*,\eta,\lambda,\xi_1,\xi_2) > 0$ *such that* $\forall$ (δ, g_0) *with*

$$\delta^* - \mu_0 < \delta \le g_0 < g_0^* + \mu_0,$$

holds $(\delta, g_0) \in \mathcal{G}_{\xi_1,\xi_2}(n,\lambda,\eta)$.

The following corollary is an immediate consequence of Theorem 1.2.

Corollary 1.1. *Let* u *be a minimizer of (1.1) in dimension* $2 \le n < n_0$. *There exists a universal constant* $\varepsilon_0 = \varepsilon_0(n, G(1), \lambda) \in (0,1)$ *such that if*

$$1-\varepsilon_0 < \delta \le g_0 < 1+\varepsilon_0,$$

then the free boundary $F(u)$ *is an analytic hypersurface.*

We observe that if $G(t) = t^p$ for some $1 < p < \infty$, then Q_g satisfies trivially (DTC), in particular, Theorem 1.1 and Theorem 1.2 extend the results obtained in [7, 14] for more general singular/degenerate elliptic equations.

2. Background results and main definitions

In this section, we present some background results that will be used throughout the paper. In this point we remember that G is a N-function if $G(t) = \int_0^t g(s)ds$, where $g\colon [0,\infty) \to \mathbb{R}$ is a positive nondecreasing function such that $g(0) = 0$, $\lim_{t\to\infty} g(t) = \infty$ and g is right continuous, that is, if $t \ge 0$ then $\lim_{s\to t+} g(s) = g(t)$. Basics properties and results in Orlicz Spaces theory can be found in [1]. Here we also present part of the theory of Orlicz–Sobolev spaces and the regularity theory of singular/degenerate elliptic equations of the type $\Delta_g u = B(x,u,\nabla u)$. Some proofs can be found in [12, 13]. Here we use freely the definitions, results, and properties of the N-functions obtained in Section 2 of [13].

Initially, we observe that the conditions (PC) and (QC) imply the properties below.

Lemma 2.1. *Let* G *a N-function satisfying the conditions (PC) and (QC). Then, for all* $t, s > 0$*:*

$(g-1)$ $\min\{s^\delta, s^{g_0}\} g(t) \le g(st) \le \max\{s^\delta, s^{g_0}\} g(t)$;

$(g-2)$ $\dfrac{tg(t)}{1+g_0} \le G(t) \le tg(t)$;

$(G-1)$ G *is convex and* $C^2(0,\infty)$;

$(G-2)$ $\dfrac{1}{1+g_0}\min\{s^{1+\delta}, s^{1+g_0}\} G(t) \le G(st) \le (1+g_0)\max\{s^{1+\delta}, s^{1+g_0}\} G(t)$;

$(G-3)$ $G(a+b) \le 2^{g_0}(1+g_0)(G(a)+G(b)), \forall\ a, b > 0$.

Proof. See Lemma 1.1 of [12], Lemma 2.1 and Remark 2.2 of [13]. □

The next lemma is an important tool to prove the Theorem 1.1.

Lemma 2.2. *Let G a N-function satisfying the conditions (PC) and (QC). Then, there exists a constant $C_0 = C_0(\delta, g_0) > 0$ such that*

$$\left|\frac{g(|\xi|)}{|\xi|}\xi - \frac{g(|\eta|)}{|\eta|}\eta\right| \leq C_0 \cdot \frac{g(|\xi| + |\eta|)}{|\xi| + |\eta|}|\xi - \eta|, \quad \forall\, \xi, \eta \in \mathbb{R}^n. \tag{2.3}$$

In particular, if $\delta \in (0, 1)$, then

$$\left|\frac{g(|\xi|)}{|\xi|}\xi - \frac{g(|\eta|)}{|\eta|}\eta\right| \leq C_0 \cdot g(1) \cdot \max\left\{1, (|\xi| + |\eta|)^{g_0 - \delta}\right\}|\xi - \eta|^{\delta}. \tag{2.4}$$

Proof. For any $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$ define

$$F_g(\xi, \eta) := \left|\frac{g(|\xi|)}{|\xi|}\xi - \frac{g(|\eta|)}{|\eta|}\eta\right| \quad \text{and} \quad H_g(\xi, \eta) := \frac{g(|\xi| + |\eta|)}{|\xi| + |\eta|}|\xi - \eta|.$$

In this case, it is enough to show that

$$\Psi_g(\xi, \eta) := \frac{F_g(\xi, \eta)}{H_g(\xi, \eta)} \leq C, \quad \forall\, \xi \neq \eta.$$

Since for $\xi = 0$ we have $\Psi(0, \eta) = 1$, we can assume that $\xi \neq 0$. Still, because Ψ is invariant by orthogonal transformations we can also assume that $\xi = |\xi| e_1$ where $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^n$. Now, we notice that

$$\Psi_g(|\xi| e_1, \eta) = \Psi_{\widetilde{g}}\left(e_1, \frac{\eta}{|\xi|}\right), \tag{2.5}$$

for $\widetilde{g}(t) = g(|\xi| t)$. In particular, if

$$\widetilde{G}(t) = \int_0^t \widetilde{g}(s)\, ds$$

we know that $\widetilde{G}$ also satisfies the conditions (PC) and (QC). By (2.5), it is sufficient to prove that $\Psi_{\widetilde{g}}(e_1, \eta) \leq C$. We divide the proof in two cases:

Case 1: Assume that $|e_1 - \eta| < \frac{1}{2}$. By mean value theorem, (QC) and $(g-1)$ we get

$$\begin{aligned} F_{\widetilde{g}}(e_1, \eta) &\leq \widetilde{g}(1)|e_1 - \eta| + \left|\widetilde{g}(1) - \frac{\widetilde{g}(|\eta|)}{|\eta|}\right| \cdot |\eta| \\ &\leq \widetilde{g}(1)|e_1 - \eta| + \widetilde{g}(1) C_1 |e_1 - \eta| \cdot |\eta| \leq \widetilde{g}(1) C_2 |e_1 - \eta|. \end{aligned}$$

Thus,

$$\Psi_{\widetilde{g}}(e_1, \eta) \leq \frac{\widetilde{g}(1) C_2 |e_1 - \eta|}{\frac{2\widetilde{g}(1)}{5}|e_1 - \eta|} \leq C_3. \tag{2.6}$$

Case 2: Suppose that $|e_1 - \eta| \geq \frac{1}{2}$. Since $\widetilde{g}$ is increasing

$$\Psi_{\widetilde{g}}(e_1, \eta) \leq \frac{(1 + |\eta|) \cdot \left(\widetilde{g}(1) + \widetilde{g}(|\eta|)\right)}{|e_1 - \eta| \cdot \widetilde{g}(1 + |\eta|)} \leq 2\frac{(1 + |\eta|)}{|e_1 - \eta|} \leq C_4. \tag{2.7}$$

Combination of the estimates (2.6) and (2.7) assures (2.3). Finally, note that if $\delta \in (0,1)$, using (2.3),

$$\begin{aligned}\left|\frac{g(|\xi|)}{|\xi|}\xi - \frac{g(|\eta|)}{|\eta|}\eta\right| &\leq C \cdot g(1) \cdot \max\left\{(|\xi|+|\eta|)^{\delta-1}, (|\xi|+|\eta|)^{g_0-1}\right\}|\xi-\eta| \\ &\leq C \cdot g(1) \cdot \max\left\{1, (|\xi|+|\eta|)^{g_0-\delta}\right\}|\xi-\eta|^{\delta}.\end{aligned}$$

This proves (2.4). □

Now, we remember the definition of modulus of continuity. A modulus of continuity is a nondecreasing continuous function $\omega\colon [0,\infty) \to [0,\infty)$ where $\omega(0) = 0$ and $\omega(t) > 0, \forall t > 0$. Now, for $G \in \mathcal{G}(\delta, g_0)$, Q_g as in (QC) and for any $0 < l < L < \infty$, we define

$$\omega_g^{l,L}(t) := \sup\left\{|Q_g(x) - Q_g(y)|\colon l \leq x, y \leq L \text{ and } |x-y| \leq t\right\}. \tag{2.8}$$

Definition 2.1. (A Dini modulus of continuity for Q_g) Let G a N-function satisfying (PC) and (QC). We say that Q_g satisfies a Dini type control if, for the nondecreasing functions $\xi_1, \xi_2\colon (0,\infty) \to [0,\infty)$ with $\lim_{t\to 0^+} \xi_2(t) = 0$ and any $0 < l < L < \infty$, the following estimate holds

$$\int_0^{L-l} \frac{\omega_g^{l,L}(t)}{t}\,dt \leq C_1^*(\delta, g_0) \cdot \xi_1\left(\frac{L}{l}\right) \cdot \xi_2\left(\frac{L-l}{l}\right). \tag{DTC}$$

In the sequence we present the main results about the regularity theory of weak solutions to

$$\Delta_g u = 0 \quad \text{in } B_1.$$

Theorem 2.1. *Let G a N-function satisfying the conditions (PC) and (QC). Suppose that*

$$\Delta_g u(x) = 0 \quad x \in B_1,$$

in the distributional sense. Then,

i) (Harnack inequality) *There exists a constant $C_1 = C_1(n, \delta, g_0) > 1$ such that*

$$\sup_{B_{1/2}} u \leq C \inf_{B_{1/2}} u.$$

ii) (Regularity) *There exist constants $\alpha \in (0,1)$ and $C_2 > 0$ depending only on n, δ and g_0 such that*

$$||u||_{C^{1,\alpha}(B_{1/2})} \leq C_2 \cdot ||u||_{L^\infty(B_1)}.$$

Proof. See Theorem 1.7 and Lemma 5.1 in [12] or Theorem 3.1 in [3]. □

3. Regularity and nondegeneracy of minimizers

We start this section with a definition.

Definition 3.1. Let the constants $0 < \delta \leq g_0 < +\infty$, $\eta \in (0,1]$, $\lambda > 0$ and $x_0 \in \Omega$ for some open set $\Omega \subset \mathbb{R}^n$. We say that $u \in S(\delta, g_0, \eta, \lambda, x_0, \Omega)$ if:

(1) There exists a N-function G satisfying the conditions (PC) and (QC) with $\eta \leq G(1) \leq \eta^{-1}$;
(2) $u \geq 0$ in Ω and $u \in W^{1,G}(\Omega) \cap L^\infty(\Omega)$;
(3) u is an absolute minimizer for (1.1) in Ω;
(4) $x_0 \in F(u)$.

If $g = G'$ and Q_g satisfies (DTC), then we say that $u \in S_{\xi_1,\xi_2}(\delta, g_0, \eta, \lambda, x_0, \Omega)$. Observe still that

$$S_{\xi_1,\xi_2}(\delta, g_0, \eta, \lambda, x_0, \Omega) \subseteq S(\delta, g_0, \eta, \lambda, x_0, \Omega).$$

For $\Omega = B_r(x_0)$ we use $S(\delta, g_0, \eta, \lambda, B_r(x_0))$ instead $S(\delta, g_0, \eta, \lambda, x_0, \Omega)$. Finally, if $u \in S(\delta, g_0, \eta, \lambda, x_0, \Omega)$ for every open set $\Omega \subset \mathbb{R}^n$, then u is called a global minimizer. We denote the class of global minimizers by $S(\delta, g_0, \eta, \lambda, x_0, \mathbb{R}^n)$.

Remark 3.1. The class $S(\delta, g_0, \eta, \lambda, B_r(x_0))$ enjoys the following scaling and translating property:

$$u \in S(\delta, g_0, \eta, \lambda, B_r(x_0)) \implies u_{\rho,x_0} \in S(\delta, g_0, \eta, \lambda, B_{\frac{r}{\rho}}(0)),$$

where

$$u_{\rho,x_0}(x) = \frac{u(x_0 + \rho x)}{\rho}, \quad \rho > 0.$$

More precisely, if u is an absolute minimizer of (1.1) in $B_1(0)$, then u_{ρ,x_0} is an absolute minimizer of (1.1) in $B_{\frac{1}{\rho}}(x_0)$. The same facts are true for the class $S_{\xi_1,\xi_2}(\delta, g_0, \eta, \lambda, B_r(x_0))$.

The next Theorem is a combination of the important results in [13].

Theorem 3.1. *Let $u \in S(\delta, g_0, \eta, \lambda, B_1(0))$. Then:*

i) (Uniform Lipschitz regularity) *There exists a universal constant $C = C(n, \delta, g_0, \eta, \lambda) > 0$ such that*

$$||\nabla u||_{L^\infty(B_{1/2}(0))} \leq C.$$

ii) (Nondegeneracy) *There exists a positive and universal constant $c = c(n, \delta, g_0, \eta, \lambda)$ such that*

$$\left(\fint_{B_r(0)} u^2 \, dx \right)^{\frac{1}{2}} \geq c \cdot r, \qquad 0 < r \leq \frac{1}{2}.$$

iii) (Density property) *There exists a universal constant $c_0 = c_0(n, \delta, g_0, \eta, \lambda) \in (0, 1)$ such that*

$$c_0 \leq \frac{|\{u = 0\} \cap B_r(0)|}{|B_r(0)|} \leq 1 - c_0, \qquad 0 < r < \frac{1}{2}.$$

Proof. Following the same strategy of [13], we observe that the proof of i) depends in a crucial way of a variant on the Theorem 4.1 and Lemma 4.3 of [13]. Such results, in turn, depend on the estimates in Theorem 2.1. However, we need these results in the context of classes. Thanks to Theorem 6.1 of [3], we obtain direct proof to i) via Theorem 2.1. For the proof of ii), we indicate Lemma 5.1 of [13]. Finally, the proof of iii) follows similarly as in Theorem 5.1 of [13] (see still Theorem 7.1 and Theorem 7.2 of [3]). □

Now, we present a gradient Hölder estimate for minimizers. Before, we consider a useful remark.

Remark 3.2. (Non divergence structure of the g-Laplace operator) Denote $\mathcal{A}_g(x, \nabla u) = \frac{g(|\nabla u|)}{|\nabla u|} \nabla u$. In this case, we have the g-Laplace operator

$$\Delta_g u := \operatorname{div}(\mathcal{A}_g(x, \nabla u)).$$

Thus,

$$\mathcal{A}_g(x,\xi) := \begin{cases} H_g(|\xi|)\cdot\xi & \text{for } \xi \neq 0; \\ \\ 0 & \text{for } \xi = 0, \end{cases}$$

where $H_g(t) = \frac{g(t)}{t}$ for $t > 0$. We observe that for any $w \in C^2(\Omega)$ such that $\nabla w \neq 0$ the non divergence structure of the g-Laplace operator is

$$\Delta_g w(x) = H_g(|\nabla w(x)|)\; Tr(A_w(x)\cdot D^2 w(x)) \quad \forall x \in \Omega,$$

where

$$A_w(x) := \left(\left[\frac{g'(|\nabla w(x)|)}{g(|\nabla w(x)|)} |\nabla w(x)| - 1 \right] \cdot \left(\frac{\nabla w(x)}{|\nabla w(x)|} \otimes \frac{\nabla w(x)}{|\nabla w(x)|} \right) + I_n \right), \quad \forall x \in \Omega.$$

In particular A_w is $(\lambda_\delta, \Lambda_{g_0})$-elliptic where $\lambda_\delta := \min\{1,\delta\}$ and $\Lambda_{g_0} := \max\{1, g_0\}$.

Theorem 3.2. (Gradient Hölder estimate) *Let $u \in S(\delta, g_0, \eta, \lambda, B_1(0))$. Then, there exist positive constants $C = C(n,\delta,g_0,\eta,\lambda)$ and $\alpha = \alpha(n,\delta,g_0,\eta,\lambda) < 1$ such that*

$$\sup_{B_r(0)} |\nabla u| \leq \lambda^* + C r^\alpha, \quad 0 < r \leq \frac{1}{4}.$$

Proof. Let G the N-function satisfying the conditions of the Definition 3.1, $\lambda^* > 0$ such that $H(\lambda^*) = \lambda$ and $\varepsilon \in (0,1)$. We consider the function

$$v_\varepsilon(x) = \left[|\nabla u|^2 - (\lambda^*)^2 - \varepsilon\right]^+.$$

By Lemma 7.4 of [13] it vanishes in a neighbourhood of the free boundary. Since $v_\varepsilon > 0$ implies $|\nabla u| > \lambda^* + \varepsilon$ the closure of $\{v_\varepsilon > 0\}$ is contained in $\{|\nabla u| > \lambda^* + \varepsilon/2\}$. By Remark 3.2, u satisfies

$$Tr(A_u(x)\cdot D^2 u(x)) = 0, \quad \forall\, x \in \{|\nabla u| > \lambda^* + \varepsilon/2\}.$$

The proof now follows similarly as in Theorem 7.1 of [6]. □

A simple consequence of above theorem is the gradient bounds for global minimizers.

Corollary 3.1. *Suppose $u \in S(\delta, g_0, \eta, \lambda, 0, \mathbb{R}^n)$. Then,*

$$|\nabla u| \leq \lambda^* \quad \text{in } \{u > 0\}.$$

4. Flatness implies regularity

In this short section, we recall substantial results from [13] related to the regularity of the free boundary. As in the previous section, we state the result based on the classes of Definition 3.1. The proof can be obtained by small changes from the corresponding results in [13]. First, we establish a definition of the flatness class.

Definition 4.1. Let $0 \leq \sigma_+, \sigma_- \leq 1$ and $\tau > 0$. We say that u is of the flatness class $F(\sigma_+, \sigma_-; \tau)$ in the ball $B_r(0)$ if:

i) $u \in S(\delta, g_0, \eta, \lambda, B_r(0))$;
ii) $u(x) = 0$, for $x_n \geq \sigma_+ r$;
iii) $u(x) \geq -(x_n + \sigma_- r)$, for $x_n \leq -\sigma_- r$;
iv) $|\nabla u| \leq 1 + \tau$, in $B_r(0)$.

If $u \in S_{\xi_1,\xi_2}(\delta, g_0, \eta, \lambda, B_r(0))$, then we say that u is of the flatness class $F_{\xi_1,\xi_2}(\sigma_+, \sigma_-; \tau)$ in the ball $B_r(0)$. We observe that more generally, changing the direction e_n by a unit vector ν and the origin by x_0 in the definition above, we obtain definition of the flatness classes $F(\sigma_+, \sigma_-; \tau)$ and $F_{\xi_1,\xi_2}(\sigma_+, \sigma_-; \tau)$ in the ball $B_r(x_0)$ in the direction ν.

Lemma 4.1. (Improvement of flatness) *Given* $u \in S(\delta, g_0, \eta, \lambda, B_1(0))$ *and* $\theta \in (0,1)$, *there exist positive constants* $\sigma_\theta = \sigma_\theta(n,\theta)$, $c_\theta = c_\theta(n,\theta)$ *and* $C = C(n, \delta, g_0)$ *such that*

$$u \in F(\sigma, 1; \tau) \quad \text{in } B\rho(0) \text{ in direction } \nu$$

with $\sigma \le \sigma_\theta$ *and* $\tau \le \sigma_\theta \sigma^2$, *then*

$$u \in F(\theta\sigma, \theta\sigma; \theta^2\tau) \quad \text{in } B_r(0) \text{ in direction } \widetilde{\nu}$$

for some $c_\theta \rho < r \le \frac{1}{4}\rho$ *and* $|\widetilde{\nu} - \nu| \le C\sigma$.

Proof. See Lemma 9.5 of [13]. □

Theorem 4.1. (Flatness implies regularity) *Let* $u \in S(\delta, g_0, \eta, \lambda, B_1(0))$. *There exist positive constants* $\alpha, \beta, \sigma_0, \tau_0$ *depending on* n, δ, g_0, η *and* λ *such that if* $u \in F(\sigma, 1; \infty)$ *in* $B_r(0)$ *where* $\sigma \le \sigma_0$ *and* $r \le \tau_0 \sigma_0^{\beta/2}$, *then* $F(u) \cap B_{r/4}(0)$ *is a* $C^{1,\alpha}$ *surface.*

Proof. See Theorem 9.3 of [13]. □

Similar theorems holds if u is of the flatness class $F_{\xi_1,\xi_2}(\sigma, 1; \infty)$ in $B_1(0)$.

5. Blowups and halfspace solutions

In the current section, we present convergence lemmas for absolute minimizers. These results are easily applicable to blow-up sequences of minimizers. Let u a minimum of (1.1) in $B_1(0)$ and $\rho_k \to 0^+$. We can define a blow-up sequence

$$u_k(x) = \frac{u(\rho_k x)}{\rho_k}, \quad x \in B_{1/\rho_k}(0).$$

By previous results, we can assume that, up to a subsequence, u_k is converging in $C^{0,\alpha}_{loc}(\mathbb{R}^n)$ to a function u_∞. The function u_∞ will be called the blow-up limit. This definition can be used in classes as in Definition 3.1.

Lemma 5.1. *Suppose* u_k *be an absolute minimizer of (1.1) in* $B_{R_k}(0)$ *for some* $R_k \to \infty$. *Then, there exists a Lipschitz continuous function* u_∞ *in* $\mathbb{R}^n$ *such that, up to subsequence, for any* $\alpha \in (0,1)$,

$$\begin{aligned} u_k &\longrightarrow u_\infty \quad \text{in } C^{0,\alpha}_{\text{loc}}(\mathbb{R}^n) \\ \nabla u_k &\longrightarrow \nabla u_\infty \quad \text{a.e. in } \mathbb{R}^n. \end{aligned}$$

Moreover, u_∞ *is a global minimizer, i.e.,* u *is an absolute minimizer for (1.1) in* $B_R(0)$ *for any* $R > 0$.

The proof of Lemma 5.1 follows as in the lemma below put in the context of classes.

Lemma 5.2. *Suppose* $u_k \in S_{\xi_1,\xi_2}(\delta_k, g_{0,k}, \eta, \lambda, B_{R_k}(0))$ *where* $R_k \to \infty$. *Assume still that there exist* $0 < \delta \le g_0 < \infty$ *such that* $\delta \le \delta_k \le g_{0,k} \le g_0$. *Then, there* *exists a Lipschitz continuous function u_∞ in $\mathbb{R}^n$ such that, up to subsequence, for any $\alpha \in (0,1)$,*

$$u_k \longrightarrow u_\infty \quad \text{in } C^{0,\alpha}_{\text{loc}}(\mathbb{R}^n)$$
$$\nabla u_k \longrightarrow \nabla u_\infty \quad \text{a.e. in } \mathbb{R}^n.$$

Moreover, $u_\infty \in S(\delta, g_0, \eta, \lambda, B_R(0))$, for any $R > 0$.

Proof. The convergences mentioned above follows similarly as in Lemma 4.1 of [7] by using Theorem 2.1 and Theorem 3.1. We proof that $u_\infty \in S(\delta, g_0, \eta, \lambda, B_R(0))$, for any $R > 0$. In fact, firstly we know that there exists $k_0 \in \mathbb{N}$ such that $R_k \geq R$ for any $k \geq k_0$. In this point, let $\psi \in C_0^\infty(B_R(0))$ where $0 \leq \psi \leq 1$. Then, for any v_∞ with $u_\infty - v_\infty \in C_0^\infty(B_R)$ we define

$$v_k = v_\infty + (1-\psi)(u_k - u_\infty).$$

We note that $v_k = u_k$ on ∂B_R. Now, since $u_k \in S_{\xi_1,\xi_2}(\delta_k, g_{0,k}, \eta, \lambda, B_{R_k}(0))$ there exists G_k a N-function satisfying the conditions of Definition 3.1 related to u_k. In particular,

$$\int_{B_R(0)} \left(G_k(|\nabla u_k|) + \lambda \chi_{\{u_k>0\}}\right) dx \leq \int_{B_R(0)} \left(G_k(|\nabla v_k|) + \lambda \chi_{\{v_k>0\}}\right) dx. \tag{5.9}$$

By Theorem 6.1 of [3] there exists a N-function G_∞ satisfying the conditions (PC) and (QC) such that G_k converges to G_∞ in C^2 topology on compact subsets of $(0,\infty)$ and in the C^1 topology on compact subsets of $[0,\infty)$. By i) of Theorem 3.1, definition of v_k, and since $\nabla u_k \to \nabla u_\infty$ a.e., we conclude by dominated convergence theorem that

$$\int_{B_R(0)} G_k(|\nabla u_k|)\, dx \longrightarrow \int_{B_R(0)} G_\infty(|\nabla u_\infty|)$$

and

$$\int_{B_R(0)} G_k(|\nabla v_k|)\, dx \longrightarrow \int_{B_R(0)} G_\infty(|\nabla v_\infty|).$$

Still, as

$$\chi_{\{v_k>0\}} \leq \chi_{\{v_\infty>0\}} + \chi_{\{\psi<1\}}$$

we obtain by (5.9),

$$\int_{B_R(0)} \left(G_\infty(|\nabla u_\infty|) + \lambda\chi_{\{u_\infty>0\}}\right) dx$$
$$\leq \int_{B_R(0)} \left(G_\infty(|\nabla v_\infty|) + \lambda\chi_{\{v_\infty>0\}}\right) dx + \lambda \int_{B_R(0)} \chi_{\{\psi<1\}}\, dx.$$

Now, the proof of lemma follows by choosing a sequence of functions ψ such that $|\{\psi < 1\}| \to 0$ and by ii) of Theorem 3.1. □

By the proof of lemma above we obtain the following corollary.

Corollary 5.1. *Suppose $u_k \in S(\delta_k, g_{0,k}, \eta, \lambda, B_{R_k}(0))$ where $R_k \to \infty$. Assume still that there exist $0 < \delta \leq g_0 < \infty$ such that*

$$\delta \leq \delta_k \leq g_{0,k} \leq g_0 \quad \text{and} \quad g_k - \delta_k = o(k).$$

Then, there exists a Lipschitz continuous function u_∞ *in* $\mathbb{R}^n$ *such that, up to subsequence, for any* $\alpha \in (0,1)$,

$$u_k \longrightarrow u_\infty \quad \text{in } C^{0,\alpha}_{\text{loc}}(\mathbb{R}^n)$$
$$\nabla u_k \longrightarrow \nabla u_\infty \quad \text{a.e. in } \mathbb{R}^n.$$

Moreover, for some $\delta \le p \le g_0$, $u_\infty \in S(p,p,\eta,\lambda,B_R(0))$, *for any* $R > 0$.

Lemma 5.3. (Characterization of halfspace solutions) *Let* u_∞ *be as in Lemma 5.1, Lemma 5.2 or Corollary 5.1. Suppose that*

$$|\nabla u_\infty| = \lambda^* \quad \text{a.e. in } \{u_\infty > 0\}.$$

Then, there exists a unit vector e *such that*

$$u_\infty(x) = \lambda^* \langle x, e\rangle^+$$

for any $x \in \mathbb{R}^n$.

Proof. The proof follows the same guide lines of Lemma 4.2 of [7] by using the non-divergence structure of g-Laplacian operators (Remark 3.2) and iii) of Theorem 3.1. □

By Remark 3.1, the three lemmas above and the Corollary 5.1 we can state the conditions that guarantee that the blowup limit is a halfspace solution. For a proof, we indicate Lemma 4.3 of [7].

Lemma 5.4. *Let* u_k *be as in Lemma 5.1 or Corollary 5.1 and suppose that for some sequence* $\varepsilon_k \to 0$ *we have*

(i) $|\nabla u_k| \le \lambda^* + \varepsilon_k$ *in* B_{R_k};
(ii) *for all* $0 < r < R_k$

$$\frac{1}{r^2}\int_{B_r\cap\{u_k>0\}} \big[H(\lambda^*) - H(|\nabla u_k|)\big]\,dx \le \varepsilon_k.$$

Then, there exists a unit vector e *such that over a subsequence*

$$u_k \longrightarrow \lambda^*\langle x, e\rangle^+ \quad \text{in } C^{0,\alpha}_{\text{loc}}(\mathbb{R}^n).$$

A similar lemma holds in the context of classes. Precisely,

Lemma 5.5. *Let* u_k *is as in Lemma 5.2 or Corollary 5.1 and suppose that for some sequence* $\varepsilon_k \to 0$ *we have*

(i) $|\nabla u_k| \le \lambda_k^* + \varepsilon_k$ *in* B_{R_k};
(ii) *for all* $0 < r < R_k$

$$\frac{1}{r^2}\int_{B_r\cap\{u_k>0\}} \big[H_k(\lambda_k^*) - H_k(|\nabla u_k|)\big]\,dx \le \varepsilon_k,$$

where

$$H_k(\lambda_k^*) = \lambda_k^* g_k(\lambda_k^*) - G_k(\lambda_k^*) = \lambda$$

with $g_k = G_k'$ *and* G_k *is the N-function associated with* u_k *according to Definition 3.1.*

Then, there exists a unit vector e *and a positive constant* λ^* *such that, up to a subsequence,* $\lambda_k^* \to \lambda^*$ *and*

$$u_k \longrightarrow \lambda^*\langle x, e\rangle^+ \quad \text{in } C^{0,\alpha}_{\text{loc}}(\mathbb{R}^n).$$

6. Full regularity of the free boundary

The proof of the main result will follow by three lemmas. First, we prove that any absolute minimizer is sufficiently close to a halfspace solution in a small neighbourhood of the origin.

Lemma 6.1. *Let $n = 2$, $u \in S(\delta, g_0, \eta, \lambda, B_1(0))$ and $0 < \delta_0 < \delta \le g_0 < g_0^* < \infty$. Then, if $\delta \ge 1$,*

$$\limsup_{r\searrow 0} \frac{1}{r^2}\int_{B_r\cap\{u>0\}} \big[H(\lambda^*) - H(|\nabla u|)\big]\,dx \le 0. \tag{6.10}$$

If $\delta \in (0,1)$ there exist a positive constants C^* depending only on δ_0, g_0^*, η and λ, and $\gamma = \gamma(\delta)$ such that

$$\limsup_{r\searrow 0} \frac{1}{r^2}\int_{B_r\cap\{u>0\}} \big[H(\lambda^*) - H(|\nabla u|)\big]\,dx \le C^*\cdot\gamma. \tag{6.11}$$

In particular, $\gamma \to 0$ when $\delta \nearrow 1$. Moreover, the both inequalities are uniform in the sense that for every $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon,\delta) > 0$ such that, for any $0 < r < r_0$

$$\frac{1}{r^2}\int_{B_r\cap\{u>0\}} \big[H(\lambda^*) - H(|\nabla u|)\big]\,dx \le C^*\cdot\gamma + \varepsilon,$$

where we assume $\gamma = 0$ in the case $\delta \ge 1$.

Proof. Let G the N-function associated to u by Definition 3.1. Still, consider $\psi \in C_0^\infty(B_{\frac{1}{2}}(0))$, $\psi \ge 0$ and $\varepsilon > 0$. Clearly, $u_\varepsilon = \max\{u - \varepsilon\psi, 0\}$ is an admissible function. In particular,

$$J_G(u) \le J_G(u_\varepsilon).$$

Thus, by above inequality, convexity of G and $\Delta_g u = 0$ in $\{u > 0\}$ we have

$$\begin{aligned}
&\int_{\{0<u\le\varepsilon\psi\}} \lambda\,dx \le -\int_{\{0<u\le\varepsilon\psi\}} G(|\nabla u|)\,dx + \int_{\{u>\varepsilon\psi\}} [G(|\nabla u_\varepsilon|) - G(|\nabla u|)]\,dx\\
&\le -\int_{\{0<u\le\varepsilon\psi\}} G(|\nabla u|)\,dx - \int_{\{u>\varepsilon\psi\}} \frac{g(|\nabla u_\varepsilon|)}{|\nabla u_\varepsilon|}\nabla u_\varepsilon(\nabla u_\varepsilon - \nabla u)\,dx\\
&\le -\int_{\{0<u\le\varepsilon\psi\}} G(|\nabla u|)\,dx - \int_{\{u>\varepsilon\psi\}} \frac{g(|\nabla u_\varepsilon|)}{|\nabla u_\varepsilon|}\nabla u_\varepsilon\cdot\nabla(\varepsilon\psi)\,dx\\
&\quad + \int_{B_1(0)} \frac{g(|\nabla u|)}{|\nabla u|}\nabla u\cdot\nabla(\min\{\varepsilon\psi, u\})\,dx\\
&\le \int_{\{0<u\le\varepsilon\psi\}} H(|\nabla u|)\,dx + \int_{\{u>\varepsilon\psi\}} \left[\frac{g(|\nabla u|)}{|\nabla u|}\nabla u - \frac{g(|\nabla u_\varepsilon|)}{|\nabla u_\varepsilon|}\nabla u_\varepsilon\right]\cdot\nabla(\varepsilon\psi)\,dx
\end{aligned}$$

Since $\lambda = H(\lambda^*)$ we conclude that

$$\begin{aligned}
&\int_{\{0<u\le\varepsilon\psi\}} [H(\lambda^*) - H(|\nabla u|)]\,dx\\
&\le \int_{\{u>\varepsilon\psi\}} \left[\frac{g(|\nabla u|)}{|\nabla u|}\nabla u - \frac{g(|\nabla u_\varepsilon|)}{|\nabla u_\varepsilon|}\nabla u_\varepsilon\right]\cdot\nabla(\varepsilon\psi)\,dx.
\end{aligned} \tag{6.12}$$

We proceed with the proof by dividing it into two cases.

Case 1: $\delta \geq 1$. By i) of Theorem 3.1 we know that there exists $C = C(n, \delta_0, g_0^*, \eta, \lambda) > 0$ such that

$$u \leq C \cdot r, \quad 0 < r < 1/2.$$

Thus, if we choose $\varepsilon = C \cdot r$ and

$$\psi(x) := \begin{cases} 1 & \text{in } B_r(0), \\ \dfrac{\log\left(\frac{R}{|x|}\right)}{\log\left(\frac{R}{r}\right)} & \text{in } B_R(0) \setminus B_r(0), \\ 0 & \text{in } B_1(0) \setminus \overline{B}_R(0), \end{cases} \tag{6.13}$$

there exists $C_* = C_*(n, \delta_0, g_0^*, \eta, \lambda) \geq C$ where

$$\max\Big\{|\nabla u_\varepsilon|, |\nabla(\varepsilon\psi)|\Big\} \leq C_* \quad \text{in } \ B_1(0).$$

Combining (2.3) of Lemma 2.2, (6.12), $(g-1)$ and $(g-2)$ of Lemma 2.1 we get

$$\begin{aligned} &\int_{\{0<u\leq\varepsilon\psi\}} [H(\lambda^*) - H(|\nabla u|)]\, dx \leq C_0 \cdot \int_{\{u>\varepsilon\psi\}} \frac{g\left(|\nabla u| + |\nabla u_\varepsilon|\right)}{|\nabla u| + |\nabla u_\varepsilon|} |\nabla(\varepsilon\psi)|^2\, dx \\ &\leq C_0 \cdot \max\left\{(C + C_*)^{\delta - 1}, (C + C_*)^{g_0 - 1}\right\} \cdot g(1) \int_{\{u>\varepsilon\psi\}} |\nabla(\varepsilon\psi)|^2\, dx \\ &\leq C_1 \int_{\{u>\varepsilon\psi\}} |\nabla(\varepsilon\psi)|^2\, dx, \end{aligned}$$

where $C_0 = C_0(\delta_0, g_0^*)$ and $C_1 = C_1(n, \delta_0, g_0^*, \eta, \lambda)$ are positive constants.

Observing that $B_r \subset \{0 < u \leq \varepsilon\psi\}$ and by direct computation of the integral of ψ in the ring $B_R(0) \setminus B_r(0)$ we conclude that

$$\limsup_{r\to 0} \frac{1}{r^2} \int_{\{0<u\leq\varepsilon\psi\}\cap B_r} [H(\lambda^*) - H(|\nabla u|)]\, dx \leq C_2 \cdot \limsup_{r\to 0} \frac{1}{\log\left(\frac{R}{r}\right)} = 0,$$

para some $C_2 > 0$ with the same dependence of C_1. This proves the Case 1 and (6.10).

Case 2: $\delta \in (0, 1)$. By using estimates (6.12) and (2.4) of the Lemma 2.2, we obtain similarly as in the previous case

$$\int_{\{0<u\leq\varepsilon\psi\}} [H(\lambda^*) - H(|\nabla u|)]\, dx \leq C_3 r^{1+\delta} \cdot \int_{\{u>\varepsilon\psi\}} |\nabla\psi|^{1+\delta}\, dx, \tag{6.14}$$

for some $C_3 = C_3(n, \delta_0, g_0^*, \eta, \lambda) > 0$. Now, putting $\psi(x) = \varphi(x/r)$, where

$$\varphi(x) := \begin{cases} 1 & \text{in } B_1(0), \\ \dfrac{|x|^{\frac{\delta-1}{\delta}} - \rho^{\frac{\delta-1}{\delta}}}{1 - \rho^{\frac{\delta-1}{\delta}}} & \text{in } B_\rho(0) \setminus B_1(0), \\ 0 & \text{elsewhere}, \end{cases} \tag{6.15}$$

we remark that for $0 < r < \left(\dfrac{2^{\frac{1}{\delta}} - 1}{2^{\frac{2-\delta}{\delta}}}\right)^{\frac{\delta}{1-\delta}}$,

$$r^{1+\delta} \cdot \int_{B_R(0)\setminus B_r(0)} |\nabla\psi|^{1+\delta}\, dx \leq 4\pi r^2 \left(\frac{1-\delta}{\delta}\right)^{\delta}. \tag{6.16}$$

Taking $\gamma = \left(\frac{1-\delta}{\delta}\right)^{\delta}$, observing that $B_r(0) \subset \{0 < u \leq \varepsilon\psi\}$ and combining (6.14) and (6.16), we obtain that

$$\limsup_{r\to 0} \int_{\{u>0\}\cap B_r(0)} [H(\lambda^*) - H(|\nabla u|)]\, dx \leq C^* \cdot \gamma,$$

where $C^* = 4\pi C_3$. The proof is now complete. □

Lemma 6.2. *Let $n = 2$ and u an absolute minimizer of (1.1). Assume still that $\delta \geq 1$. Then, for any $\sigma > 0$ there exists $\rho = \rho(\sigma, \delta, g_0, G(1), \lambda) > 0$ such that u is of the flatness class $F(\sigma, 1; \infty)$ in $B_\rho(0)$ in some direction ν.*

Proof. Assume that the conclusion of the lemma is not true. Then there exists $\sigma > 0$ and sequences $\rho_k \to 0$ and u_k absolute minimizers of (1.1) in $B_1(0)$ such that u_k is not of class $F(\sigma, 1; \infty)$ in $B_{\rho_k}(0)$ in any direction ν. Now, consider the following rescaling

$$v_k(x) = \frac{u_k(\rho_k x)}{\rho_k}, \quad x \in B_{\frac{1}{\rho_k}}(0).$$

By Remark 3.1, Lemma 6.1, Theorem 3.2 and Lemma 5.4, up to subsequence, $v_k \to \lambda^*\langle x, e\rangle_+$ uniformly on every compact subset of $\mathbb{R}^2$. By iii) of Theorem 3.1, we obtain that for sufficiently large k, v_k must vanish on $B_1(0) \cap \{\lambda^*\langle x, e\rangle \leq -\sigma\}$. But, this implies that v_k is of the flatness class $F(\sigma, 1; \infty)$ in $B_1(0)$. In particular, u_k is of the flatness class $F(\sigma, 1; \infty)$ in $B_{\rho_k}(0)$, contrary to our assumption. The proof of lemma is complete. □

Lemma 6.3. *Let $n = 2$ and $u \in S(\delta, g_0, \eta, \lambda, B_1(0))$ where $\delta \in (0, 1)$. For any $\sigma > 0$ there exists a small and positive constant $\varrho_0 = \varrho_0(\sigma, \eta, \lambda) \in (0, 1)$ such that if*

$$1 - \varrho_0 < \delta \leq g_0 < 1 + \varrho_0,$$

then u will be of the flatness class $F(\sigma, 1; \infty)$ in $B_\rho(0)$ in some direction ν and for some radius $\rho = \rho(\sigma, \eta, \lambda) > 0$. For $u \in S_{\xi_1,\xi_2}(\delta, g_0, \eta, \lambda, B_1(0))$ there exists $\mu = \mu(\sigma, g_0, \eta, \lambda, \xi_1, \xi_2) \in (0, 1)$ such that if

$$1 - \mu < \delta \leq g_0 < \infty,$$

then u will be of the flatness class $F_{\xi_1,\xi_2}(\sigma, 1; \infty)$ in $B_\rho(0)$ in some direction ν and for some radius $\rho = \rho(\sigma, g_0, \eta, \lambda, \xi_1, \xi_2) > 0$.

Proof. We prove the case where $u \in S(\delta, g_0, \eta, \lambda, B_1(0))$ the other case follows by a similar argument. Since we expect $1 - \delta$ and $g_0 - \delta$ close enough to zero, we can choose without loss of generality that $\delta_0 = 1/8$ and $g_0^* = 2$ to use the Lemma 6.1. Thus, for $1/8 < \delta \leq g_0 < 2$ there exists $r_\delta = r_\delta(\delta) > 0$ such that for any $u \in S(\delta, g_0, \eta, \lambda, B_1(0))$ and $0 < r < r_\delta$ we have

$$\frac{1}{r^2} \int_{B_r\cap\{u>0\}} \left[H(\lambda^*) - H(|\nabla u|)\right] dx \leq (C^* + 1) \cdot \gamma_\delta,$$

for some constant $C^* > 0$ depending on η and λ. Still, $\gamma_\delta \to 0$ as $\delta \nearrow 1$. Besides, by Theorem 3.2, we can assume that there exists $\varepsilon_\delta = \varepsilon_\delta(g_0, \eta, \lambda, \delta)$ such that

$$|\nabla u| \leq \lambda^* + \varepsilon_\delta \quad \text{in } B_r(0),$$

and $\varepsilon_\delta \to 0$ as $\delta \nearrow 1$. We claim that one can take $\rho_\delta = r_\delta^2$ in the assertion of the lemma. Assuming the contrary, there exist $\sigma > 0$, sequences $g_{0,k}$ and δ_k such that

$$\delta_k \nearrow 1 \quad \text{and} \quad g_{0,k} - \delta_k = o(k),$$

and $u_k \in S(\delta_k, g_{0,k}, \eta, \lambda, B_1(0))$ such that u_k does not belong to the flatness class $F(\sigma, 1; \infty)$ in $B_{\rho_{\delta_k}}(0)$ in any direction ν. Similarly as in Lemma 5.3 of [7] we define

$$v_k(x) = \frac{u_k(\rho_{\delta_k} x)}{\rho_{\delta_k}}, \quad \text{in } B_{1/\sqrt{\rho_{\delta_k}}}(0).$$

Now, since we can assume that $\rho_{\delta_k} \to 0$ as $\delta_k \nearrow 1$ follows by Lemma 5.5 there exists a unit vector e and a positive constant λ^* such that over a subsequence, $v_k \to \lambda^* \langle x, e\rangle_+$ uniformly on every compact subset of $\mathbb{R}^2$. By iii) of Theorem 3.1, for k sufficiently large,

$$v_k = 0 \quad \text{on } \{\lambda^* \langle x, e\rangle \leq -\sigma\}.$$

We conclude that v_k is of the flatness class $F(\sigma, 1; \infty)$ in $B_1(0)$. Thus, u_k is of the flatness class $F(\sigma, 1; \infty)$ in $B_{\rho_k}(0)$, contrary to our assumption. The proof of the second part of the lemma is analogous to the first part. □

The proof of Theorem 1.1 is an immediate consequence of theorem below.

Theorem 6.1. *Let $n = 2$, constants $\eta \in (0, 1]$, $\lambda > 0$ and $0 < \delta \leq g_0 < \infty$. Consider still functions ξ_1, ξ_2 as in Definition 2.1. If $\delta \geq 1$, the free boundary of any $u \in S(\delta, g_0, \eta, \lambda, B_1(0))$ is real analytic. If $\delta \in (0, 1)$ two possibilities can occur:*

i) *There exists a universal constant $\varrho = \varrho(\eta, \lambda) \in (0, 1)$ such that if*

$$1 - \varrho < \delta \leq g_0 < 1 + \varrho,$$

then the free boundary of any $u \in S(\delta, g_0, \eta, \lambda, B_1(0))$ is real analytic;

ii) *There exists a small constant $\mu = \mu(g_0, \eta, \lambda, \xi_1, \xi_2) \in (0, 1)$ such that if*

$$1 - \mu < \delta \leq g_0 < \infty,$$

then the free boundary of any $u \in S_{\xi_1,\xi_2}(\delta, g_0, \eta, \lambda, B_1(0))$ is an analytic hypersurface.

Proof. Suppose initially $\delta \geq 1$ and consider $u \in S(\delta, g_0, \eta, \lambda, B_1(0))$. There exists a N-function G associated to u by Definition 3.1 satisfying conditions (PC) and (QC) and $\eta \leq G(1) \leq \eta^{-1}$ such that u is an absolute minimizer of (1.1). By Lemma 6.2, for any $\sigma > 0$, there exists $\rho = \rho(\sigma, \delta, g_0, G(1), \lambda) > 0$ such that u is of the flatness class $F(\sigma, 1; \infty)$ in some direction ν. Thus, choosing $0 < \sigma \leq \sigma_0$ and $0 < \rho \leq \tau_0 \sigma_0^{\beta/2}$ as in Theorem 4.1 we conclude that $F(u) \cap B_{\rho/4}(0)$ is a $C^{1,\alpha}$ surface. The analyticity of $F(u)$ follows by [11]. The proof of the case $\delta \in (0, 1)$ follows the same guide lines of the previous case by Lemma 6.3 and Theorem 4.1. □

7. The critical dimension result

In the last section we establish the proof of Theorem 1.2 that will follow by two lemmas. First, we state a Bernstein type result similarly as in the minimal surface theory [15]. The proof is analogous to Theorem 3.1 of [14] by using Theorem 3.1, Lemma 4.1 and Corollary 3.1.

Lemma 7.1. (Bernstein Lemma) *Let $u \in S(\delta, g_0, \eta, \lambda, 0, \mathbb{R}^n)$. Suppose that $(\delta, g_0) \in \mathcal{G}(n, \lambda, \eta)$. Then, there exists a direction e such that*

$$u(x) = \lambda^* \langle x, e\rangle_+, \quad \forall\, x \in \mathbb{R}^n.$$

The next lemma is the core of Theorem 1.2, and the statement and the proof keep the same spirit of the Lemma 6.3. In fact, the proof follows by a contradiction argument similar to Lemma 6.3, where the use of Lemma 5.4 is replaced by the use of Lemma 7.1.

Lemma 7.2. *Consider the following constants* $0 < \delta^* \le g_0^* < \infty$, $\lambda > 0$, $\eta \in (0,1]$ *and* $2 \le n \in \mathbb{N}$, *and some functions* ξ_1, ξ_2 *as in Definition 2.1. Then, for any* $\sigma > 0$,

i) *If* $\delta^* = g_0^*$ *and* $(\delta^*, \delta^*) \in \mathcal{G}(n, \lambda, \eta)$, *there exist small and positive constants*

$$\varrho_0 = \varrho_0(n, \sigma, \delta^*, \eta, \lambda) \quad \text{and} \quad r_0 = r_0(n, \sigma, \delta^*, \eta, \lambda)$$

such that for every $u \in S(\delta, g_0, \eta, \lambda, B_1(0))$ *we have*

$$u \in F(\sigma, 1; \infty) \quad \text{in } B_r(0) \text{ in some direction } \nu,$$

provided

$$\delta^* - \varrho_0 < \delta \le g_0 < \delta^* + \varrho_0 \quad \text{and} \quad 0 < r \le r_0.$$

ii) *If* $(\delta^*, g_0^*) \in \mathcal{G}_{\xi_1,\xi_2}(n, \lambda, \eta)$, *there exist positive constants*

$$\mu_0 = \mu_0(n, \sigma, \delta^*, g_0^*, \eta, \lambda, \xi_1, \xi_2) \quad \text{and} \quad r_0 = r_0(n, \sigma, \delta^*, g_0^*, \eta, \lambda, \xi_1, \xi_2)$$

such that for every $u \in S_{\xi_1,\xi_2}(\delta, g_0, \eta, \lambda, B_1(0))$ *we have*

$$u \in F(\sigma, 1 : \infty) \quad \text{in } B_r(0) \text{ in some direction } \nu,$$

provided

$$\delta^* - \mu_0 < \delta \le g_0 < g_0^* + \mu_0 \quad \text{and} \quad 0 < r \le r_0.$$

Proof of Theorem 1.2. We prove i). The proof of ii) is analogous. For $(\delta^*, \delta^*) \in \mathcal{G}(n, \lambda, \eta)$ we take β, σ_0 and τ_0 as in Theorem 4.1 and choose $0 < \sigma_1 < \sigma_0$. Now, let

$$\varrho_0 = \varrho_0(n, \sigma_1, \delta^*, \eta, \lambda) \quad \text{and} \quad r_0 = r_0(n, \sigma_1, \delta^*, \eta, \lambda)$$

as in previous lemma. Define

$$r_1 = \min\left\{r_0, \tau_0 \sigma_0^{\beta/2}\right\}.$$

Again by Lemma 7.2, for every $u \in S(\delta, g_0, \eta, \lambda, B_1(0))$ we have

$$u \in F(\sigma, 1; \infty) \quad \text{in } B_r(0) \text{ in some direction } \nu,$$

provided

$$\delta^* - \varrho_0 < \delta \le g_0 < \delta^* + \varrho_0 \quad \text{and} \quad 0 < r \le r_1.$$

Then, by Theorem 4.1, there exists a universal $\alpha \in (0,1)$ such that $F(u) \cap B_{\frac{r_1}{4}}(0)$ is a $C^{1,\alpha}$ surface. By results of [11] and scaling properties we conclude that $(\delta, g_0) \in \mathcal{G}(n, \lambda, \eta)$. □

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