On existence of Becker extension

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Abstract. A well-known theorem by Becker states that if a normalized univalent function f in the unit disk $\mathbb{D}$ can be embedded as the initial element into a Loewner chain $(f_t)_{t\geqslant 0}$ such that the Herglotz function p in the Loewner–Kufarev PDE

$$\partial f_t(z)/\partial f = z f_t'(z) p(z, t), \quad z \in \mathbb{D}, \text{ a.e. } t \geqslant 0,$$

satisfies $\big|(p(z,t)-1)/(p(z,t)+1)\big| \leqslant k < 1$, then f admits a k-q.c. ($=$ "k-quasiconformal") extension $F\colon \mathbb{C} \to \mathbb{C}$. The converse is not true. However, a simple argument shows that if f has a q-q.c. extension with $q \in (0, 1/6)$, then Becker's condition holds with $k := 6q$. In this paper we address the following problem: find the largest $k_* \in (0, 1]$ with the property that for any $q \in (0, k_*)$ there exists $k_0(q) \in (0, 1)$ such that every normalized univalent function $f\colon \mathbb{D} \to \mathbb{C}$ with a q-q.c. extension to $\mathbb{C}$ satisfies Becker's condition with $k := k_0(q)$. We prove that $k_* \geqslant 1/3$.

Beckerin jatkeen olemassaolosta

Tiivistelmä. Beckerin tunnettu lause kertoo, että jos yksikkökiekon $\mathbb{D}$ normitettu injektiivinen funktio f voidaan upottaa Loewnerin ketjun $(f_t)_{t\geqslant 0}$ aloitusalkioksi sillä tavalla, että Loewnerin–Kufarevin osittaisdifferentiaaliyhtälössä

$$\partial f_t(z)/\partial f = z f_t'(z) p(z, t), \quad z \in \mathbb{D}, \text{ m.k. } t \geqslant 0,$$

esiintyvä Herglotzin funktio p toteuttaa epäyhtälön $\big|(p(z,t)-1)/(p(z,t)+1)\big| \leqslant k < 1$, niin tällöin funktiolla f on k-kvasikonforminen jatke $F\colon \mathbb{C} \to \mathbb{C}$. Käänteinen väite ei pidä paikkaansa, mutta yksinkertainen päättely osoittaa, että jos funktiolla f on q-kvasikonforminen jatke jollakin $q \in (0, 1/6)$, niin Beckerin ehto pätee arvolla $k := 6q$. Tässä työssä tarkastelemme seuraavaa kysymystä: mikä on suurin sellainen $k_* \in (0, 1]$, että jokaista $q \in (0, k_*)$ kohti on sellainen $k_0(q) \in (0, 1)$, että jokainen normitettu injektiivinen funktio $f\colon \mathbb{D} \to \mathbb{C}$, jolla on q-kvasikonforminen jatke tasoon $\mathbb{C}$, toteuttaa Beckerin ehdon arvolla $k := k_0(q)$? Osoitamme, että $k_* \geqslant 1/3$.

1. Introduction

Univalent functions admitting a quasiconformal extension is one of the classical topics in Geometric Function Theory closely related to Teichmüller Theory, see e.g. [6, 22, 38]. In 1972, Becker [4, 5] found a witty construction of q.c.-extensions for holomorphic functions based on Loewner's parametric method. Although Becker's extensions are quasiconformal mappings of quite special nature [16, Theorem 2], his result is interesting from several points of view. Precise definitions and a brief discussion on this matter can be found in Sect. 2.1.

Taking advantage of Becker's result, the author and I. Hotta [15] answered recently a question concerning the sharp bound of the third coefficient raised in 1977 by Kühnau and Niske [25]. In the same paper [15] the following problem was stated.

Problem. Find the largest $k_* \in (0, 1]$ such that there exists a function $k_0 \colon (0, k_*) \to (0, 1)$ with the following property: if $q \in (0, k_*)$, then any univalent function $f \colon \mathbb{D} \to \mathbb{C}$ admitting a q-q.c. extension to $\mathbb{C}$ has also a $k_0(q)$-q.c. Becker extension.

A simple observation concerning conditions of q.c.-extendibility in terms of the pre-Schwarzian shows that $k_* \geqslant 1/6$, see [15, Sect. 5]. At the same time, for $q \in (0, 1)$ close to 1, it is not known whether every univalent function f admitting a q-q.c. extension to $\mathbb{C}$ has also a k-q.c. Becker extension with some $k \in (0, 1)$, even if k is allowed to depend on f. In fact, it was shown [15, Proposition 5.2] that two natural ways to construct a Loewner chain starting from such an f fail to produce Becker q.c.-extensions in general. The main result of this paper is as follows.

Theorem 1. *In the above notation, $k_* \geqslant 1/3$; i.e. for any $q \in (0, 1/3)$ there exists $k_0 \in (0, 1)$ depending only on q such that every $f \in \mathcal{S}_q$ admits a k_0-q.c. Becker extension.*

An optimistic conjecture is that $k_* = 1$. At the same time, it would not be much surprising if k_* turned out to be equal to $1/3$.

It is worth pointing out that the quasiconformal extensions F of holomorphic functions in $\mathbb{D}$ considered in the above theorem satisfy $F(\infty) = \infty$. This condition is not merely a normalization. In fact, it turns out to be rather essential, see e.g. Remark 2.1. An analogue of the above theorem for quasiconformal extensions of functions $g \in \Sigma$ follows easily from known results, but its does not imply Theorem 1 (which was unfortunately overlooked in [16, Sect. 5]). More precisely, on the one hand, by a result of Kühnau [23], every univalent function $f \colon \mathbb{D} \to \mathbb{C}$ admitting a k-q.c. extension to $\mathbb{C}$ satisfies

$$\|S_f\| := \sup_{z \in \mathbb{D}} |S_f(z)|(1 - |z|^2)^2 \leqslant 6k, \quad \text{where} \quad S_f(z) := \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

On the other hand, a sort of partial converse of this result was found by Ahlfors and Weill [3]: if $\|S_f\| \leqslant 2q$, then f admits a q-q.c. extension to $\overline{\mathbb{C}}$. The Ahlfors–Weill extension is given by an explicit formula, and moreover, if $f''(0) = 0$ then it is also a Becker extension; but for $f''(0) \neq 0$, it does not fix ∞ and hence cannot be a Becker extension. In this paper we combine a variety of techniques from the theory of conformal mappings to show that the Ahlfors–Weill extension can be modified in a suitable way in order to obtain a Becker extension.

The paper is organized as follows. The proof of Theorem 1 is given in Sect. 4.6. It is preceded by a block of auxiliary statements in Sect. 4.2–4.5. To facilitate the reader, we explain the main idea in a more detail and give a short overview of the main steps of the proof in Sect. 4.1.

Necessary preliminaries on the Parametric Method and Becker's extensions are given the next section. The Loewner chain we construct in the proof of Theorem 1 is defined as the family of conformal mappings onto domains bounded by images of concentric circles under a certain diffeomorphic mapping. In Sect. 3 we combine results from [36] and [37] to show that such families are differentiable w.r.t. the parameter and satisfy the Loewner–Kufarev PDE.

Unfortunately, the methods employed in this paper allows us only to establish existence of a Becker extension in a quite implicit form. Similarly, it is hardly possible to trace the dependence of k_0 on q in Theorem 1. Therefore, it would be interesting to find some more explicit modifications of the Ahlfors–Weill construction that produce

Becker extensions. One modification of this kind was discovered in [15, Corollary 6.7], but it leads to only a slight improvement of the trivial estimate $k_* \geqslant 1/6$.

2. Preliminaries

2.1. Loewner chains and Becker's extensions. An important role in the study of univalent functions is played by the Parametric Method, going to back Loewner [29].

Let $p \colon \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ be a (classical) *Herglotz function*, i.e.

HF1: for each $z \in \mathbb{D}$, $p(z, \cdot)$ is locally integrable on $[0, +\infty)$;
HF2: for a.e. $t \geqslant 0$, $p(\cdot, t)$ is holomorphic in $\mathbb{D}$ and satisfies $\mathtt{Re}\, p(\cdot, t) \geqslant 0$;
HF3: $\int_0^{+\infty} \mathtt{Re}\, p(0, t)\, \mathrm{d}t = +\infty$.

Usually instead of HF3, one assumes the normalization $\mathtt{Re}\, p(0, t) = 1$ for a.e. $t \geqslant 0$. Following Becker [5], [6, Sect. 5.1], as a matter of convenience we prefer to work with a less restrictive condition HF3. With suitable slight modifications, most of the classical results remain valid, because the normalization can be achieved by a change of variables. In particular, it is known, see e.g. [31, Theorem 6.3 on p. 160], that the initial value problem for the Loewner–Kufarev ODE

$$(2.1) \qquad \mathrm{d}w/\mathrm{d}t = -w\, p(w, t), \quad t \geqslant 0, \quad w(z, 0) = z \in \mathbb{D},$$

has a unique solution $w = w(z, t)$ and that the limit

$$(2.2) \qquad f(z) := \lim_{t \to +\infty} \frac{w(z, t)}{w'(0, t)}, \quad z \in \mathbb{D},$$

exists locally uniformly in $\mathbb{D}$ and belongs to the class $\mathcal{S}$ of all univalent functions $f \colon \mathbb{D} \to \mathbb{C}$ normalized by $f(0) = 0$, $f'(0) = 1$.

Moreover, a much deeper result, see [17] and [31, Theorem 6.1 on p. 160], states that for any $f \in \mathcal{S}$ there exists a Herglotz function p, not necessarily unique, such that f is given by (2.2). The proof of this fact is based on the possibility to embed f as the initial element into a suitable Loewner chain (f_t).

Definition 1. A family $(f_t)_{t \geqslant 0}$ of holomorphic functions is called a (radial) *Loewner chain* in $\mathbb{D}$ if it satisfies the following conditions:

LC1: for each $t \geqslant 0$, $f_t \colon \mathbb{D} \to \mathbb{C}$ is univalent, with $f_t(0) = 0$;
LC2: for any $s, t \geqslant 0$ with $s < t$, $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$;
LC3: the function $t \mapsto f'_t(0)$ is locally absolutely continuous in $[0, +\infty)$, $f'_0(0) = 1$, and

$$\lim_{t \to +\infty} |f'_t(0)| \to +\infty.$$

Every Herglotz function p generates a Loewner chain (f_t) by

$$(2.3) \qquad f_s(z) := \lim_{t \to +\infty} \frac{w(z; s, t)}{w'(0; 0, t)}, \quad z \in \mathbb{D}, \ \ s \geqslant 0,$$

where for any $s \geqslant 0$, $t \geqslant s$, and $z \in \mathbb{D}$, $w(t) = w(z; s, t)$ stands for the unique solution to

$$\mathrm{d}w/\mathrm{d}t = -w\, p(w, t), \quad t \geqslant s, \quad w(z; s, s) = z \in \mathbb{D}.$$

The Loewner chain (f_t) defined in this way satisfies the Loewner–Kufarev PDE:

$$\partial f_t(z)/\partial t = z f'_t(z) p(z, t), \quad z \in \mathbb{D}, \ \ t \geqslant 0.$$

Furthermore, the correspondence between the Herglotz functions p and Loenwer chains defined by (2.3) is a bijection, i.e. for any Loewner chain (f_t) there exists a unique Herglotz function p such that equality (2.3) holds, see e.g. [30, §6.1].

Thus, the class $\mathcal{S}$ can be seen as the image of the set of all Herglotz functions in $\mathbb{D}$ under the map defined via the Loewner–Kufarev ODE (2.1) and formula (2.2). It is natural to ask which properties of a Herglotz function p ensure that the corresponding function $f = f_0$ belongs to a given subclass of $\mathcal{S}$. One important result in this direction was discovered in 1972 by Becker [4]. To state Becker's theorem, we need the following definition.

Definition 2. Let with $k \in [0,1)$. We say that a univalent (holomorphic or meromorphic) function f defined in a domain $D \subset \overline{\mathbb{C}}$ *admits a k-q.c. extension to* $\overline{\mathbb{C}}$ if there exists a k-q.c. mapping $F \colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $f = F|_D$. Moreover, a holomorphic function $f \colon D \to \mathbb{C}$, $D \subset \mathbb{C}$, is said to *admits a k-q.c. extension to* $\mathbb{C}$ if there exists a k-q.c. mapping $F \colon \mathbb{C} \to \mathbb{C}$ such that $f = F|_D$.

For the theory of quasiconformal mappings in the plane, we refer to e.g. [2] or [28].

Definition 3. Denote by $\mathcal{S}_k$ the class of all $f \in \mathcal{S}$ admitting k-q.c. extensions to $\mathbb{C}$.

Remark 2.1. Due to the fact that all isolated singularities of quasiconformal mappings are removable, the q.c.-extendibility of a function f to $\mathbb{C}$ is in fact a bit stronger condition than the q.c.-extendibility to $\overline{\mathbb{C}}$: in addition to existence of a k-q.c. extension $F \colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, it is required that $F(\infty) = \infty$. As a result, certain properties of the class $\mathcal{S}_k$ are different from those of the wider class formed all $f \in \mathcal{S}$ having k-q.c. extensions to $\overline{\mathbb{C}}$. In particular, $\mathcal{S}_k$ admits Lehto's Majorant Principle [27], while the latter class does not. Moreover, if $f \in \mathcal{S}_k$, then $|f(z)| < M(k) \leqslant 4^{K-1}$, $K := (1 + k)/(1 - k)$, for all $z \in \mathbb{D}$, see [24, 18, 11], but this is clearly not the case in general for functions $f \in \mathcal{S}$ admitting k-q.c. extensions to $\overline{\mathbb{C}}$.

Theorem A. (Becker [4, 5]) *Let $k \in [0,1)$ and let (f_t) be a radial Loewner chain whose Herglotz function p satisfies*

$$(2.4) \qquad p(\mathbb{D}, t) \subset U(k) := \left\{ w \in \mathbb{C} \colon \left| \frac{w-1}{w+1} \right| \leqslant k \right\} \quad \text{for a.e. } t \geqslant 0.$$

Then for every $t \geqslant 0$, the function f_t admits a k-q.c. extension to $\mathbb{C}$. In particular, such an extension for f_0 is given by

$$(2.5) \qquad F(\rho e^{i\theta}) := \begin{cases} f_0(\rho e^{i\theta}), & \text{if } 0 \leqslant \rho < 1, \\ f_{\log \rho}(e^{i\theta}), & \text{if } \rho \geqslant 1. \end{cases}$$

Definition 4. The extension F of $f = f_0$ given in the above theorem is called a *Becker extension* of $f \in \mathcal{S}$.

Remark 2.2. Theorem A has a sort of converse [16, Theorem 2 (II)]: if (f_t) is a Loewner chain such that for any $t \geqslant 0$, f_t extends continuously to $\partial \mathbb{D}$ and if $F \colon \mathbb{C} \to \mathbb{C}$ defined by (2.5) is a k-q.c. mapping, then the Herglotz function p of the Loewner chain (f_t) satisfies (2.4).

For a fixed $k \in (0, 1)$, the above theorem gives a sufficient condition for a holomorphic function f in $\mathbb{D}$ to admit a k-q.c. extension to $\mathbb{C}$: if f can be embedded as the initial element into a Loewner chain (f_t) such that $p(z) := \frac{\partial}{\partial t} f_t(z) / \big(z f'(z) \big)$

satisfies (2.4), then $f \in \mathcal{S}_k$. Unfortunately, this condition is far from being necessary, see e.g. [16, Theorem 3].

At the same time, most of other known sufficient conditions can be obtained as corollaries of Theorem A, see e.g. [6, Sect. 5.3–5.4], [19, 20], and [32]. Moreover, Becker's condition (2.4) remains to be sufficient for q.c.-extendibility in other variants of the Parametric Method, see e.g. [14], [16, Theorem 1], and [21]. Finally, there are many examples, see e.g. [15, Proposition 4.2], in which Becker's extension is the best possible in the sense that it has the smallest sharp upper bound $\operatorname{ess\,sup}_{|z|>1} |\mu_F(z)|$ for the modulus of the Beltrami coefficient μ_F among all q.c.-extensions $F\colon \mathbb{C} \to \mathbb{C}$ of a given $f \in \mathcal{S}$.

Therefore, the study of Becker's extensions represents a considerable interest. One of the questions to investigate is whether any $f \in \mathcal{S}$ with a q.c.-extension to $\mathbb{C}$ admits also a Becker q.c.-extension. From the main result of this paper, Theorem 1, it follows that the answer is positive at least for all q-q.c. extendible functions with $q \in (0, 1/3)$. The main difficulty resides however in proving existence of k-q.c. Becker extensions with some $k = k_0(q) \in (0, 1)$ depending only on q but not on the function f.

2.2. Loewner chains in Δ. In some cases, it is more convenient to work with univalent functions in $\Delta := \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ normalized by $g(\infty) = \infty$. Most of the classical Loewner Theory extends easily to this case. In particular, we say that $(g_t)_{t \geqslant 0}$ is a (radial) *Loewner chain in Δ* if:

   (i) for any $t \geqslant 0$, g_t is a univalent meromorphic function in Δ with $g_t(\infty) = \infty$;
   (ii) for any $s \geqslant 0$ and any $t \geqslant s$, $g_s(\Delta) \subset g_t(\Delta)$;
   (iii) the function $t \mapsto g_t'(\infty) \in \mathbb{C}$ is locally absolutely continuous in $[0, +\infty)$, with $g_0'(\infty) = 1$ and

$$\lim_{t \to +\infty} g_t'(\infty) = 0,$$

     where as usual by $g_t'(\infty)$ we mean the coefficient of z in the Laurent expansion of $g_t(z)$ in $\Delta \setminus \{\infty\}$.

Moreover, a function $p\colon \Delta \times [0, +\infty)$ is said to be a *Herglotz function in Δ*, if $(\zeta, t) \mapsto p(1/\zeta, t)$ is a Herglotz function in $\mathbb{D}$.

If (g_t) is a Loewner chain in Δ, then

$$\bigcap_{t \geqslant 0} \mathbb{C} \setminus g_t(\Delta)$$

is non-empty. In fact, this set contains exactly one point, which we denote by w_0. The functions $f_t(\zeta) := \big(g_t(1/\zeta) - w_0\big)^{-1}$, $\zeta \in \mathbb{D}$, form a Loewner chain in $\mathbb{D}$. It follows that there exists a null-set $N \subset [0, +\infty)$ such that the limit

$$\frac{\partial g_t(z)}{\partial t} := \lim_{s \to t} \frac{g_s(z) - g_t(z)}{s - t}$$

exists locally uniformly in $\Delta \setminus \{\infty\}$. Moreover, using the Loewner–Kufarev PDE for (f_t) we easily see that (g_t) satisfies

$$(2.6) \qquad \frac{\partial g_t(z)}{\partial t} = -z g_t'(z) p(z, t), \quad \text{a.e. } t \geqslant 0, \ z \in \Delta \setminus \{\infty\},$$

for a suitable Herglotz function p in Δ, uniquely defined for all $t \in [0, +\infty) \setminus N$.

Conversely, if p is a Herglotz function in Δ, then there exists a Loewner chain (g_t) in Δ, satisfying (2.6). The only difference from the case of $\mathbb{D}$, which one has to keep

in mind, is that to each Herglotz function there corresponds a one-parameter family of Loewner chains (g_t) that differ from each other by an additive constant.

3. Subordination chains defined by diffeomorphisms

In this section we establish an auxiliary assertion concerning the Herglotz function of a Loewner chain, assuming that the boundaries $\partial f_t(\mathbb{D})$ and their evolution in t is sufficiently regular. Since this result might have some independent interest, we state and prove it separately from the main discussion of this paper.

Let $\Psi\colon \{e^{t+i\tau}\colon t \in (a,b),\ \tau \in [0,2\pi]\} \to \mathbb{C}$ be a homeomorphic map such that for any $s,t \in (a,b)$, $s < t$, the curve $\Gamma_s := \Psi(\{z\colon |z| = e^s\})$ is contained in the Jordan domain Ω_t bounded by Γ_t. Replacing Ψ with $z \mapsto \Psi(\bar{z})$ if necessary, we may suppose that for each $t \in (a,b)$ the parametrization $[0,2\pi] \ni \tau \mapsto \Psi(e^{t+i\tau})$ induces on Γ_t the positive (i.e. counterclockwise) orientation. Finally, using translations we may assure that $0 \in \Omega_t$ for all $t \in (a,b)$.

For each $t \in (a,b)$, let f_t be the conformal mapping of $\mathbb{D}$ onto Ω_t and normalized by $f_t(0) = 0$, $f_t'(0) > 0$. Then the family $(f_t)_{t\in(a,b)}$ is a subordination chain. We combine results from [37] and [36] to show that if Ψ is regular enough, then (f_t) is differentiable for all $t \in (a,b)$ and satisfies the Loewner–Kufarev equation with a Herglotz function having a continuous extension to $\partial\mathbb{D}$.

Proposition 3.1. *If Ψ is C^2-diffeomorphic (i.e. it is of class C^2 and its Jacobian determinant does not vanish), then for any $t \in (a,b)$ the limit*

$$(3.1) \qquad \frac{\partial f_t(z)}{\partial t} = \lim_{s \to t} \frac{f_s(z) - f_t(z)}{s - t}$$

exists locally uniformly in $\mathbb{D}$. Moreover, (f_t) satisfies the Loewner–Kufarev PDE

$$(3.2) \qquad \frac{\partial f_t(z)}{\partial t} = z f_t'(z) p(z,t) \quad \text{for all } t \in (a,b),\ z \in \mathbb{D},$$

where $p(\cdot,t)$ is a holomorphic function in $\mathbb{D}$ with positive real part and continuous extension to $\partial\mathbb{D}$ which is uniquely determined by $\mathrm{Im}\, p(0,t) = 0$ and

$$(3.3) \qquad \mathrm{Re}\, p(e^{i\theta},t) = -\frac{1}{|f_t'(e^{i\theta})|}\, \mathrm{Im}\left(\overline{\frac{\partial \Psi(e^{t+i\tau})}{\partial \tau}}\, \frac{\partial \Psi(e^{t+i\tau})}{\partial t}\right) \left|\frac{\partial \Psi(e^{t+i\tau})}{\partial \tau}\right|^{-1}$$

for all $\theta \in [0,2\pi]$ with $\tau = \tau(\theta)$ satisfying $f_t(e^{i\theta}) = \Psi(e^{t+i\tau(\theta)})$.

Remark 3.1. An argument used in the proof of Theorem 1 shows that in conditions of the above proposition a stronger assertion holds. Namely, the functions $(z,t) \mapsto \log|f_t'(z)|$ and p are bounded on compact subsets of $\overline{\mathbb{D}} \times (a,b)$. It follows that $t \mapsto f_t \in \mathsf{Hol}(\mathbb{D},\mathbb{C})$ is locally absolutely continuous on (a,b) and hence (f_t) is a *solution* to the Loewner–Kufarev equation (3.2); for a precise definition, see e.g. [7, Definition 2.1].

Proof of Proposition 3.1. Since Γ_t's are of class C^2, the functions f_t extend C^1-smoothly to $\partial\mathbb{D}$ and f_t' does not vanish in $\overline{\mathbb{D}}$, see e.g. [33, Theorem 3.5 on p. 48].

Fix some $t \in (a,b)$. Following [37] denote by n_u and $\mathcal{L}_u$, $u \in \Gamma_t$, the outward normal unit vector and the normal line to the curve Γ_t at the point u, respectively. Furthermore, for $s \in (a,b)$, $s \neq t$, we denote by $\Omega_{s,t}$ the doubly connected domain bounded by Γ_s and Γ_t and let $I(s,\tau)$ be the unique connected component of $\mathcal{L}_u \cap \Omega_{s,t}$ for which u is one of the end-points. Denote by $w = w(s,u)$ the other end-point. If $|t-s|$ is small enough, then $w(s,u)$ lies on Γ_s and moreover, the map $u \mapsto w(s,u)$

is a bijection between Γ_t and Γ_s. Write $\Delta n_u(s) := \mathrm{sgn}(s - t)|u - w(s, u)|$. It is not difficult to see that $w(s, u)$ is differentiable in s. Therefore,

$$\nu(u) := \left. \frac{\mathrm{d}\Delta n_u(s)}{\mathrm{d}s} \right|_{s=t}$$

is well-defined for all $u \in \Gamma_t$.

Denote by g_s the Green function of Ω_s. By [37, Theorem 1], the function $t \mapsto g_s(z, 0)$ is differentiable at $s = t$ locally uniformly w.r.t. $z \in \Omega_t$ and

$$(3.4) \qquad h_t(z) := \left. \frac{\mathrm{d}}{\mathrm{d}s} \right|_{s=t} g_s(z, 0) = \frac{1}{2\pi} \int_{\Gamma_t} \frac{\partial g_t(u, z)}{\partial n} \frac{\partial g_t(u, 0)}{\partial n} \nu(u)\, |\mathrm{d}u|,$$

where $\partial/\partial n$ stands for the derivative along the outward normal direction to Γ_t and $|\mathrm{d}u|$ is the length element of Γ_t.

The r.h.s. of (3.4) is a Poisson integral in the domain Ω_t. Taking into account regularity of the boundary, we conclude that $h_t(z)$ is harmonic in Ω_t and continuous on its closure, with

$$(3.5) \qquad h_t(u) = \frac{\partial g(u, 0)}{\partial n} \nu(u) \quad \text{for all } u \in \Gamma_t.$$

Notice that $\nu(u)$ coincides with the projection of $\partial\Psi/\partial t$ onto n_u. More precisely,

$$(3.6) \qquad \nu(\Psi(e^{t+i\tau})) = -\,\mathtt{Im}\left(\overline{\frac{\partial\Psi(e^{t+i\tau})}{\partial\tau}} \frac{\partial\Psi(e^{t+i\tau})}{\partial t} \right) \left| \frac{\partial\Psi(e^{t+i\tau})}{\partial\tau} \right|^{-1}$$

for all $\tau \in [0, 2\pi]$. Indeed, fix some $\tau \in [0, 2\pi]$ and let τ_s be a solution to $\Psi(e^{s+i\tau_s}) = w(s, u)$ with $u := \Psi(e^{t+i\tau})$, which is clearly unique modulo 2π. Then $\tau_t = \tau$ and $s \mapsto \tau_s$ is differentiable. Denote $v := (\mathrm{d}\tau_s/\mathrm{d}s)|_{s=t}$. By the Chain Rule,

$$\nu(\Psi(e^{t+i\tau})) = \overline{n_u} \left. \frac{\partial w(s, u)}{\partial s} \right|_{s=t} = \mathtt{Re}\left(\overline{n_u} \left. \frac{\partial w(s, u)}{\partial s} \right|_{s=t} \right)$$

$$= \mathtt{Re}\left(\overline{n_u} \left(\frac{\partial\Psi(e^{t+i\tau})}{\partial t} + v\frac{\partial\Psi(e^{t+i\tau})}{\partial\tau} \right) \right) = \mathtt{Re}\left(\overline{n_u} \frac{\partial\Psi(e^{t+i\tau})}{\partial t} \right),$$

where the last equality holds because $\partial\Psi/\partial\tau$ is orthogonal to n_u. To obtain (3.6) it remains to substitute $n_u = -i\partial\Psi/\partial\tau |\partial\Psi/\partial\tau|^{-1}$.

It is shown in [36, Proof of Theorem 6] that differentiability of the Green function g_t w.r.t. the parameter t implies existence of the locally uniform limit (3.1) and that in such a case (f_t) satisfies the Loewner–Kufarev equation with $p(\cdot, t) = H_t \circ f_t$, where H_t is the holomorphic function in Ω_t satisfying $\mathtt{Im}\, H_t(0) = 0$ and $\mathtt{Re}\, H_t = h_t$.

Note that $\partial g(u, 0)/\partial n = |(f_t^{-1})'(u)|$. According to the Kellogg–Warschawski Theorem, see e.g. [33, Theorem 3.6 on p. 49], f_t' is Hölder continuous on $\partial\mathbb{D}$. Using formulas (3.5) and (3.6), we conclude that $h_t \circ f_t$ is also Hölder continuous on $\partial\mathbb{D}$. By Privalov's Theorem on conjugate functions, it follows that $p(\cdot, t)$ extend continuously to the boundary. Combining now (3.5) and (3.6), we obtain (3.3). $\qquad\square$

4. Proof of the main result

As usual we will denote by Σ the class of all univalent functions g in $\Delta := \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ with the expansion of the form $g(z) = z + b_0 + b_1/z + b_2/z^2 + \ldots$ and let $\Sigma(k)$, $k \in (0, 1)$, stands for the class of all $g \in \Sigma$ admitting k-q.c. extentions to $\overline{\mathbb{C}}$.

4.1. Notation and the main construction. In this section we explain the ideas used in the proof of Theorem 1. Fix $q \in (0, 1/3)$ and $f \in \mathcal{S}_q$, $f(\zeta) = \zeta + a_2(f)\zeta^2 + \dots$ Then $g_0(z) := 1/f(1/z)$ belongs to $\Sigma(q)$. Moreover,

$$(4.1) \qquad (1 - |\zeta|^2)^2 |S_f(\zeta)| \leqslant 6q \quad \text{for all } \zeta \in \mathbb{D},$$

where S_f stands for the Schwarzian derivative of f, see [23, Satz 3*] or [26, Example 9 on p. 134]. It follows that g_0 has a k-q.c. extension $G \colon \overline{\mathbb{C}} \setminus \{0\} \to \overline{\mathbb{C}}$, with $k := 3q$, given explicitly by

$$(4.2) \qquad G(z) := \begin{cases} g_0(z), & \text{if } \zeta \in \Delta, \\ g_t(z/|z|), \ t := -\log|z|, & \text{if } 0 < |z| \leqslant 1, \end{cases}$$

where

$$(4.3) \qquad 1/g_t(w) := f(e^{-t}/w) + \frac{(1 - e^{-2t})f'(e^{-t}/w)}{e^{-t}w - \frac{1}{2}(1 - e^{-2t})P_f(e^{-t}/w)}, \quad w \in \Delta, \ t \geqslant 0,$$

and $P_f(\zeta) := f''(\zeta)/f'(\zeta)$, $\zeta \in \mathbb{D}$, is the so-called pre-Schwarzian of f.

The above well-known extension is originally due to Ahlfors and Weill [3], see also [1]. The relation to Loewner chains was discovered by Becker, see [4, Sect. 4] and [6, Sect. 5].

Remark 4.1. Note that every isolated singularity of a q.c.-map is removable; hence in fact, G is a k-q.c. automorphism of $\overline{\mathbb{C}}$. Moreover, expressing f via g_0 and substituting $e^t w = 1/\bar{z}$, $e^{-2t} = z\bar{z}$ in the above formulas, we can rewrite (4.2) for $z \in \Delta$ as follows:

$$(4.4) \qquad G(z) = g_0(1/\bar{z}) - \frac{(1 - z\bar{z})g_0'(1/\bar{z})}{\bar{z} + \frac{1}{2}(1 - z\bar{z})P_{g_0}(1/\bar{z})} = b_0 + z + 3b_1\bar{z} + 6b_2\bar{z}^2 + \dots,$$

where b_j's are the Laurent coefficients of g_0, i.e. $g_0(w) = w + b_0 + b_1/w + b_2/w^2 + \dots$ In particular, it follows that G is real-analytic in $\mathbb{D}$, including the point 0.

The extension (4.2) can be obtained with the help of Becker's construction. However, $G(0) = -a_2(f)$. Therefore, if $a_2(f) \neq 0$, then it does provide a Becker extension for $\hat{f}(z) := f(z)/(1 + a_2(f)f(z))$, $z \in \mathbb{D}$, with the Loewner chain $\hat{f}_t(z) := 1/(g_t(z) + a_2(f))$, but not for the function f itself.

In this paper we show that the family (g_t) can be modified in such a way that it defines a k_0-q.c. extension of g, with some $k_0 \in (0, 1)$, having a fixed point at the origin. This would yield the desired Becker extension of f.

The idea is as follows. Denote by D_t the Jordan domain bounded by $\Gamma_t := g_t(\partial\mathbb{D})$. It is known that $|f(z)| < 4^{Q-1}$, $Q := (1+q)/(1-q)$, for all $z \in \mathbb{D}$ and any $f \in \mathcal{S}_q$, see e.g. [11][1]. It follows that $0 \in D_0$. Hence, there exists also $t_1 > 0$ such that $0 \in D_{t_1}$. Furthermore, we fix some $t_0 \in (0, t_1)$. Note also that $-a_2(f) \in D_t$ for all $t \geqslant 0$. Let L be a diffeomorphism of D_{t_1} onto itself that sends $-a_2(f)$ to 0. To have control on its properties, we choose L of the form $L = F \circ T \circ F^{-1}$, where $F(z) := G(e^{-t_0}z)$ and T is a suitable Moebius transformation mapping $e^{t_0 - t_1}\mathbb{D}$ onto itself. Denote by $\tilde{g}_t$, $t \geqslant t_1$, the conformal mapping of Δ onto the unbounded component of $\mathbb{C} \setminus L(\Gamma_t)$ normalized by $\tilde{g}_t(\infty) = \infty$, $\tilde{g}_t'(\infty) > 0$. For $t \in [0, t_1]$ we set $\tilde{g}_t := g_t$. Then using Proposition 3.1 and Remark 3.1, it is not difficult to show that the functions

$$f_t(z) := \frac{1}{\tilde{g}_t(1/z)}, \quad t \geqslant 0,$$

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[1]This estimate is not sharp. See [24, 18] for sharp estimates.

form a Loewner chain in $\mathbb{D}$ starting from $f_0 = f$.

The reader might ask why we do need to fix $t_0 < t_1$. As it will be clear from the proof, it is crucial to have certain control over the behaviour of L in a domain slightly larger than D_{t_1}. However, for a moment we may simply assume that $t_0 = t_1$.

Denote by p the Herglotz function of (f_t). It is easy to check that for $t \in [0, t_1)$,

$$\frac{1 - p(z, t)}{1 + p(z, t)} = \frac{1}{2} z^2 (1 - e^{-2t})^2 S_f(e^{-t} z), \quad z \in \mathbb{D}.$$

Taking into account (4.1), we see that p satisfies Becker's condition (2.4) with $k := 3q$ for all $t \in [0, t_1)$.

Moreover, there exists another suitable value of $k \in (0, 1)$ such that condition (2.4) holds also for all $t > t_1$. Indeed, the curves $L(\Gamma_t)$ are images of the concentric circles in $e^{t_0 - t_1} \mathbb{D}$ w.r.t. the map $\Psi := L \circ F = F \circ T$, which is a real-analytic diffeomorphism of $e^{t_0 - t_1} \overline{\mathbb{D}}$ onto $\overline{D_{t_1}}$. Therefore, $\mathtt{Re}\, p(z, t)$ is positive and real-analytic for all $(z, t) \in \overline{\mathbb{D}} \times (t_1, +\infty)$, and converges to a holomorphic function with positive real part as $t \to +\infty$ or $t \to t_1$. As a result, the values of p on $\mathbb{D} \times (t_1, +\infty)$ are contained in some compact set $X \subset \mathbb{H} := \{w : \mathtt{Re}\, w > 0\}$. This means that (2.4) holds if $k \in (0, 1)$ is chosen sufficiently close to 1.

The main difficulty is to show that one can choose k depending only on $q \in (0, 1/3)$, but not on $f \in \mathcal{S}_q$. Although this is very plausible to be indeed the case, a rigourous proof requires considerable work. In particular, we need to estimate certain quantities describing the Riemann map on $\partial \mathbb{D}$ via quantities measuring the regularity of the boundary. There are many studies on the boundary behaviour of conformal mappings addressing such problems. However, most of known results contain constants depending on the Riemann map itself, while in our situation the constant may depend only on q. This makes impossible to apply standard results directly.

One of the main ingredients of our proof is the following proposition. By a careful choice of the parameters t_0 and t_1, given at the beginning of Sect. 4.3, we ensure that T^{-1} has no pole in $\mathbb{D}$ and hence $\Psi^{-1} = T^{-1} \circ F^{-1}$ extends to a real-analytic function in D_{t_0}.

Proposition 4.1. *There exists a constant $a = a(q) > 0$ depending only on q such that the function $\varphi(w) := |\Psi^{-1}(w)|^{-a(q)}$ is subharmonic in D_{t_0}.*

For each $t \geqslant t_1$, $w \mapsto \varphi(w) - e^{a(q)(t - t_0)}$ is a defining function for the domain $\overline{\mathbb{C}} \setminus \overline{L(D_t)}$. The latter means that it is defined in a neigbourhood of its boundary, vanishes on the boundary itself, takes negative values in the domain and positive ones in its exterior. The fact that it is subharmonic helps us to derive a lower estimate for the derivative of the conformal mapping $\tilde{g}_t \colon \Delta \to \overline{\mathbb{C}} \setminus \overline{L(D_t)}$, $t > t_1$, on the boundary. This is Step 1 in the proof of Theorem 1, which we give in Sect. 4.6.

Furthermore, a somewhat similar argument, borrowed from [34], is used in Step 2 to give an upper estimate for $|\tilde{g}_t'|$.

In Step 3, we apply Proposition 3.1 to show that $(\tilde{g}_t)_{t > t_1}$ satisfies the Loewner–Kufarev PDE in Δ, with the Herglotz function p continuous on $\overline{\Delta}$ for each fixed $t > t_1$. Formula (3.3) allows us to find upper and lower bounds for $\mathtt{Re}\, p$.

In Step 4, we estimate the modulus of continuity of $\mathtt{Re}\, p$ on $\partial \Delta$. Using the Hilbert transform on $\partial \Delta$ we conclude that the values of p lie in some compact set $X(q) \subset \mathbb{H} := \{w : \mathtt{Re}\, w > 0\}$ depending only on q, which is equivalent to the conclusion of the theorem.

4.2. Estimates for the partial derivatives. Keeping the notation introduced above, we establish a few estimates, which will be used in the proofs of Proposition 4.1 and Theorem 1.

Lemma 4.2. *For any $t \in [0, t_*)$, where $t_* := -\log(3q)$, we have $0 \in D_t$.*

Proof. Let τ be the smallest $t > 0$ for which $0 \notin D_t$. Then $g_\tau(w) = 0$ for some $w \in \partial\Delta$. Using formula (4.3), we see that

$$(4.5) \qquad e^{-\tau}w = \frac{1}{2}(1 - e^{-2\tau})P_f(e^{-\tau}/w).$$

The estimate $|P_\varphi(z)| \leqslant 6/(1 - |z|^2)$, $z \in \mathbb{D}$, holds for all $\varphi \in \mathcal{S}$, see e.g. [8, Theorem 2.4 on p. 32]. Since $f \in \mathcal{S}_q$, thanks to Lehto's Majorant Principle [27] we have $|P_f(z)| \leqslant 6q/(1 - |z|^2)$ for all $z \in \mathbb{D}$. In combination with (4.5) this yields the desired conclusion that $\tau \geqslant t_*$. $\qquad\square$

Let us now choose $t_0 := t_*/2 = -\frac{1}{2}\log(3q)$. Denote by ∂ and $\bar{\partial}$ the formal partial derivatives w.r.t. z and $\bar{z}$, respectively:

$$\partial := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial x}\right), \quad \bar{\partial} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial x}\right).$$

Lemma 4.3. *Let $F(z) := G(e^{-t_0}z)$. For any $z \in \overline{\mathbb{D}}$, we have*

$$(4.6) \qquad e^{-t_0}\frac{(1 - k)^q}{(1 + k^2)^2} \leqslant |\partial F(z)| \leqslant \frac{e^{-t_0}}{(1 - k^2)^2(1 - k)^q}, \quad k := 3q.$$

Moreover, the directional derivatives $\mathsf{D}_\alpha F(z) := \frac{\mathrm{d}}{\mathrm{d}s}F(z + se^{i\alpha})|_{s=0}$ and the Jacobian determinant J_f of F satisfy for all $z \in \overline{\mathbb{D}}$ and all $\alpha \in \mathbb{R}$ the following inequalities:

$$e^{-t_0}\frac{(1 - k)^{1+q}}{(1 + k^2)^2} \leqslant \mathsf{D}_* F(z) := \min_{\alpha \in \mathbb{R}}|\mathsf{D}_\alpha F(z)|$$

$$(4.7) \qquad\qquad \leqslant \mathsf{D}^* F(z) := \max_{\alpha \in \mathbb{R}}|\mathsf{D}_\alpha F(z)| \leqslant \frac{e^{-t_0}}{(1 - k^2)(1 - k)^{1+q}},$$

$$(4.8) \qquad \frac{k(1 - k^2)(1 - k)^{2q}}{(1 + k^2)^4} \leqslant |J_F(z)| \leqslant \frac{k}{(1 - k^2)^4(1 - k)^{2q}}.$$

Proof. Since F is real-analytic in $\overline{\mathbb{D}}$, see Remark 2.1, it is sufficient to establish the estimates for $z \in \mathbb{D} \setminus \{0\}$. Recall that $g_0(z) = 1/f(1/z)$ belongs to $\Sigma(q)$. Therefore,

$$(4.9) \qquad\qquad \left|\log g_0'(w)\right| \leqslant q\log\frac{|w|^2}{|w|^2 - 1} \quad \text{and}$$

$$(4.10) \qquad\qquad \left|wP_{g_0}(w)\right| \leqslant 6q/(|w|^2 - 1) \quad \text{for all } w \in \Delta.$$

Inequality (4.9) is due to Kühnau [23, Satz 4]. Inequality (4.10) can be obtained with the help of Lehto's Majorant Principle [27] from the simple estimate $\left|wP_g(w)\right| \leqslant 6/(|w|^2 - 1)$ valid in the whole class Σ, which in turn follows from a more precise result due to Goluzion, see [12] or [13, Theorem 4 in §IV.3].

From (4.4) we obtain

$$(4.11) \qquad \partial G(z) = \frac{g_0'(1/\bar{z})}{\left(1 - \frac{1}{2}(z - 1/\bar{z})P_{g_0}(1/\bar{z})\right)^2} \quad \text{for all } z \in \mathbb{D} \setminus \{0\}.$$

Recall that $\partial F(z) = e^{-t_0}\partial G(e^{-t_0}z)$. So we replace z in (4.11) by $e^{-t_0}z$ and apply (4.9) with $w := e^{t_0}/\bar{z}$. Taking into account that $|w| \geqslant e^{t_0} = 1/\sqrt{k}$, we see that the absolute

value of the numerator in (4.11) is contained between $(1-k)^q$ and $(1-k)^{-q}$. Similarly, inequality (4.10) implies that the absolute value of the denominator in (4.11) is between $(1-k^2)^2$ and $(1+k^2)^2$. This proves (4.6).

Since F is a smooth k-q.c. mapping of $\mathbb{D}$, $|\bar{\partial}F(z)| \leqslant k|\partial F(z)|$ for all $z \in \mathbb{D}$. Therefore, to obtain (4.7) and (4.8) it remains to notice that

$$|\partial F| - |\bar{\partial}F| \leqslant |\mathsf{D}_\alpha F| \leqslant |\partial F| + |\bar{\partial}F| \quad \text{and} \quad J_F = |\partial F|^2 - |\bar{\partial}F|^2.$$

The proof is complete. $\qquad\qquad\square$

Lemma 4.4. *There exists a constant $M = M(q)$ depending only on $q \in (0, 1/3)$ such that*

$$(4.12) \qquad \max\left\{|\partial^2 F(z)|, |\bar{\partial}^2 F(z)|, |\partial\bar{\partial}F(z)|\right\} \leqslant M(q)|\partial F(z)| \quad \text{for all } z \in \overline{\mathbb{D}},$$

where the map F is defined in Lemma 4.3. In particular,

$$(4.13) \qquad\qquad |\partial J_F| = |\bar{\partial}J_F| \leqslant 4M(q)|\partial F|^2 \quad \text{for all } z \in \overline{\mathbb{D}}.$$

Proof. As in the proof of Lemma 4.3, it is sufficient to establish the estimates for $z \in \mathbb{D} \setminus \{0\}$. Denote $w := e^{t_0}/\bar{z}$. Note that by (4.6), ∂F does not vanish in $\overline{\mathbb{D}}$. Hence, using equality (4.11) we obtain

$$\frac{|\partial^2 F(z)|}{|\partial F(z)|} = |\partial \log \partial F(z)| = e^{-t_0}|\partial \log \partial G(1/\bar{w})| = \frac{e^{-t_0}|P_{g_0}(w)|}{|1 + \frac{1}{2}(w - 1/\bar{w})P_{g_0}(w)|}.$$

Taking into account that $|w| \geqslant e^{t_0} = 1/\sqrt{k}$, one can use (4.10) to see that the denominator in the last expression is greater or equal to $1 - k^2$, while the numerator $e^{-t_0}|P_{g_0}(w)| \leqslant 2e^{-t_0}k|w|^{-1}(|w|^2 - 1)^{-1} \leqslant 2k^3/(1-k)$. This leads to

$$(4.14) \qquad\qquad \frac{|\partial^2 F(z)|}{|\partial F(z)|} \leqslant \frac{2k^3}{(1+k)(1-k)^2}.$$

Similarly,

$$(4.15) \qquad \frac{|\bar{\partial}\partial F(z)|}{|\partial F(z)|} = e^{-t_0}|\bar{\partial}\log\partial G(1/\bar{w})| = e^{-t_0}|w|\frac{(|w|^2 - 1)|S_{g_0}(w)|}{|1 + \frac{1}{2}(w - 1/\bar{w})P_{g_0}(w)|}.$$

Taking into account that $S_{g_0}(w) = w^{-4}S_f(1/w)$, by (4.1) we have

$$(4.16) \qquad |S_{g_0}(w)| \leqslant 6q/(|w|^2 - 1)^2 = 2k/(|w|^2 - 1)^2 \quad \text{for all } w \in \Delta.$$

As above, the denominator in (4.15) is separated from zero by $1 - k^2$, while the numerator can be estimated with the help of (4.16). In this way we obtain

$$(4.17) \qquad \frac{|\partial\bar{\partial}F(z)|}{|\partial F(z)|} = \frac{|\bar{\partial}\partial F(z)|}{|\partial F(z)|} \leqslant \frac{2k^2}{(1+k)(1-k)^2}.$$

The estimate of $|\bar{\partial}^2 F(z)|/|\partial F(z)|$ is a bit more tricky. Using (4.4), we find that

$$(4.18) \qquad\qquad \bar{\partial}G(1/\bar{w}) = -\frac{\frac{1}{2}(w - 1/\bar{w})^2 w^2 S_{g_0}(w)g_0'(w)}{\left(1 + \frac{1}{2}(w - 1/\bar{w})P_{g_0}(w)\right)^2}.$$

If $S_{g_0} \equiv 0$ in Δ, then G is a Moebius transformation and hence $\bar{\partial}^2 F(z) \equiv 0$ in $\mathbb{D}$. Therefore, we may suppose that S_{g_0} does not vanish identically. From (4.18) it follows

by a simple calculation that for all $w \in \Delta$ with $S_{g_0}(w) \neq 0$,

$$(4.19) \qquad Q(w) := \bar{\partial} \log \bar{\partial} G(1/\bar{w}) + w^2 \frac{\mathrm{d}}{\mathrm{d}w} \log \left(w^4 S_{g_0}(w) \right)$$

$$= -\frac{2w}{|w|^2 - 1} + w^2 \frac{(w - 1/\bar{w}) S_{g_0}(w)}{1 + \frac{1}{2}(w - 1/\bar{w}) P_{g_0}(w)}.$$

Using the same technique as above, it is not difficult to see that

$$(4.20) \qquad |e^{-t_0} Q(w)| \leqslant \frac{2k}{1 - k} + \frac{2k^2}{(1 + k)(1 - k)^2} \leqslant \frac{2k}{(1 - k)^2}.$$

Furthermore,

$$(4.21) \quad w^2 \frac{\mathrm{d}}{\mathrm{d}w} \log \left(w^4 S_{g_0}(w) \right) = w^2 \frac{\mathrm{d}}{\mathrm{d}w} \log S_f(1/w) = -\frac{S_f'(1/w)}{S_f(1/w)} = -\frac{S_f'(1/w)}{w^4 S_{g_0}(w)}.$$

Recall that $|w| \geqslant e^{t_0} = 1/\sqrt{k}$ and apply the Cauchy estimate for the derivative of S_f in the disk D of radius $(1 - \sqrt{k})/2$ centered at the point $1/w$. By (4.1), for all $\zeta \in \partial D$ we have $|S_f(\zeta)| \leqslant 2k/(1 - |\zeta|^2)^2 \leqslant 2k/\left(1 - \frac{1}{4}(1 + \sqrt{k})^2\right)^2$. Hence,

$$(4.22) \qquad |S_f'(1/w)| \leqslant \frac{4k}{\left(1 - \sqrt{k}\right)\left(1 - \frac{1}{4}(1 + \sqrt{k})^2\right)^2} \leqslant \frac{32k}{(1 - k)^3}.$$

From (4.11) and (4.18) we get immediately that

$$\frac{\bar{\partial} F(z)}{\partial F(z)} = -\frac{1}{2}(w - 1/\bar{w})^2 w^2 S_{g_0}(w).$$

Combining this equality with (4.16), (4.21) and (4.22), we see that

$$\frac{|\bar{\partial} F(z)|}{|\partial F(z)|} \leqslant k \quad \text{and} \quad \left| w^2 \frac{\mathrm{d}}{\mathrm{d}w} \log \left(w^4 S_{g_0}(w) \right) \right| \cdot \frac{|\bar{\partial} F(z)|}{|\partial F(z)|} \leqslant \frac{16k}{1 - k}.$$

Taking into account (4.20), it follows that

$$\frac{|\bar{\partial}^2 F(z)|}{|\partial F(z)|} = e^{-t_0} \left| \bar{\partial} \log \bar{\partial} G(1/\bar{w}) \right| \cdot \frac{|\bar{\partial} F(z)|}{|\partial F(z)|}$$

$$\leqslant |e^{-t_0} Q(w)| \cdot \frac{|\bar{\partial} F(z)|}{|\partial F(z)|} + e^{-t_0} \left| w^2 \frac{\mathrm{d}}{\mathrm{d}w} \log \left(w^4 S_{g_0}(w) \right) \right| \cdot \frac{|\bar{\partial} F(z)|}{|\partial F(z)|}$$

$$(4.23) \qquad \leqslant \frac{2k^2}{(1 - k)^2} + \frac{16k^{3/2}}{1 - k} =: M(q).$$

The last inequality is obtained for all $z \in \mathbb{D}$, $z \neq 0$, such that $S_{g_0}(e^{t_0}/\bar{z}) \neq 0$. Since S_{g_0} is holomorphic, its zeros are isolated and hence (4.23) holds everywhere in $\overline{\mathbb{D}}$.

Inequalities (4.14), (4.17), and (4.23) imply (4.12), which in turn implies (4.13) since

$$|\bar{\partial} F| \leqslant k |\partial F| \quad \text{and} \quad \bar{\partial} J_F = \overline{\partial J_F} = \partial \bar{\partial} F \, \overline{\partial F} + \overline{\partial^2 F} \, \partial F - \bar{\partial}^2 F \, \overline{\partial F} - \overline{\partial \bar{\partial} F} \, \bar{\partial} F,$$

where the equality $\bar{\partial} J_F = \overline{\partial J_F}$ holds because J_F is real-valued. $\square$

4.3. Proof of Proposition 4.1. Referring to the construction explained in Sect. 4.1, we start by making an appropriate choice of $t_1 > t_0$ and of the Moebius transformation $T \colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$. Let us recall that by Lemma 4.2, $0 \in D_t$ for all $t \in$

$(0, t_*)$, where $t_* = -\log(3q)$. Recall also that we fixed $t_0 := t_*/2$. Therefore, for $z_0 := e^{t_0}G^{-1}(0)$ we have

$$(4.24) \qquad\qquad\qquad |z_0| \leqslant e^{-t_0} \leqslant \sqrt{3q}.$$

A central element in our construction is the diffeomorphism $L \colon D_{t_1} \to D_{t_1}$ sending the point $-a_2(f)$ to 0 and satisfying $L \circ F = F \circ T =: \Psi$, where F is defined as in Lemma 4.3, i.e. $F(z) := G(e^{-t_0}z)$, $z \in \mathbb{D}$. The Moebius transformation T must therefore map the disk $r\mathbb{D}$, $r := e^{t_0-t_1}$, onto itself, with $T(0) = z_0$. It is easy to check that

$$(4.25) \qquad\qquad T(z) := (1 + |z_0|^2)\frac{z + z_0}{1 + |z_0|^2 + 2\bar{z}_0 z}, \quad z \in \mathbb{D},$$

satisfies the above two requirements with $r := \sqrt{(1 + |z_0|^2)/2}$, which corresponds to

$$(4.26) \qquad\qquad\qquad t_1 = t_0 + \tfrac{1}{2}\log(2/(1 + |z_0|^2)).$$

Note also that T^{-1} has no pole in $\mathbb{D}$.

Denote $\eta(z) := \log(T^{-1}(z))$, $z \in \mathbb{D} \setminus \{z_0\}$. This function itself is multivalued, but $\mathtt{Re}\,\eta$ and η' are single-valued in $\mathbb{D} \setminus \{z_0\}$. Calculate the Laplacian of $\varphi(w) = \exp(-a\,\mathtt{Re}\,\eta(H(w)))$, where $H := F^{-1}$ maps D_{t_0} onto $\mathbb{D}$. Denote $u(w) := \mathtt{Re}\,\eta(H(w))$. Then

$$\Delta\varphi(w) = a\varphi(w)\big(a|\nabla u(w)|^2 - \Delta u(w)\big), \quad w \in D_{t_0} \setminus \{-a_2(f)\}.$$

Therefore, in order to prove that there exists $a > 0$ depending only on $q \in (0, 1/3)$ such that φ is subharmonic in D_{t_0}, we have to show that $\Delta u(w)/|\nabla u(w)|^2$ has an upper bound depending only on q.

Note that $u(w)$ is real-valued. Regarding the vector $\nabla u(w)$ as a complex number, we have $\nabla u = 2\bar{\partial} u = \overline{(\eta' \circ H)\partial H} + (\eta' \circ H)\bar{\partial} H$. By Becker's result [6, Sect. 5.2], G is a k-q.c. map with $k := 3q$. (This can be seen also directly by calculating $|\bar{\partial}G|/|\partial G|$.) Therefore, H is also k-q.c. and hence

$$(4.27) \qquad \begin{aligned} |\nabla u(w)| &\geqslant \left|\,|\overline{\eta'(H(w))\partial H(w)}| - |\eta'(H(w))\bar{\partial} H(w)|\,\right| \\ &\geqslant (1 - k)|\eta'(H(w))\partial H(w)| \end{aligned}$$

for all $w \in D_{t_0} \setminus \{-a_2(f)\}$. Moreover,

$$\Delta u(w) = 4\partial\bar{\partial} u(w) = \mathtt{Re}\left(\eta'(H(w))\Delta H(w) + 4\eta''(H(w))\partial H(w)\bar{\partial} H(w)\right).$$

Therefore,

$$\frac{|\Delta u(w)|}{|\nabla u(w)|^2} \leqslant \frac{1}{(1 - k)^2}\left(\left|\frac{\Delta H(w)}{\eta'(H(w))\partial H(w)^2}\right| + 4\left|\frac{\eta''(H(w))\bar{\partial} H(w)}{\eta'(H(w))^2\partial H(w)}\right|\right).$$

The second term in the r.h.s. is easy to estimate. Indeed, $|\bar{\partial}H| \leqslant k|\partial H|$ because H is a k-q.c. map. Moreover,

$$\frac{\eta''(z)}{\eta'(z)^2} = \frac{4\bar{z}_0 z - (1 + 3|z_0|^2)}{1 - |z_0|^2}.$$

Hence $|\eta''(z)|/|\eta'(z)|^2 \leqslant 8/(1 - |z_0|^2) \leqslant 8/(1 - k)$ for all $z \in \mathbb{D}$.

To estimate the first term, we notice that

$$\eta'(z) = \frac{1 - |z_0|^2}{(z - z_0)(1 + |z_0|^2 - 2\bar{z}_0 z)}.$$

It follows that

$$|\eta'(z)| \geqslant \frac{1 - |z_0|}{(1 + |z_0|)^2} \geqslant \frac{1 - \sqrt{k}}{(1 + \sqrt{k})^2} \quad \text{for all } z \in \mathbb{D} \setminus \{z_0\}.$$

Furthermore, $|\partial H(w)|^2 \geqslant |\partial H(w)|^2 - |\bar{\partial} H(w)|^2 = J_H(w) = 1/J_F(H(w))$, which is greater or equal to $k^{-1}(1 - k^2)^4(1 - k)^{2q}$ by Lemma 4.3. Therefore, it remains to show that $|\Delta H(w)|$ has an upper bound in D_{t_0} depending only on q.

Denote $U(z) := -\bar{\partial} F(z)/J_F(z)$. Then

$$(4.28) \qquad \partial U = -\frac{\partial \bar{\partial} F}{J_F^2} + 2\bar{\partial} F \frac{\partial J_F}{J_F^3}, \quad \bar{\partial} U = -\frac{\bar{\partial}^2 F}{J_F^2} + 2\bar{\partial} F \frac{\bar{\partial} J_F}{J_F^3}.$$

Taking into account that $J_F = |\partial F|^2 - |\bar{\partial} F|^2 \geqslant (1 - k^2)|\partial F|^2$, with the help of Lemma 4.4 we see that for all $z \in \mathbb{D}$,

$$(4.29) \qquad \max\left\{|\partial U|, |\bar{\partial} U|\right\} \leqslant M_1(q)\,|\partial F|^{-3}, \quad M_1(q) := \frac{M(q)}{(1 - k^2)^2}\left(1 + \frac{8}{1 - k^2}\right).$$

Similarly, we get

$$(4.30) \qquad |U(z)| \leqslant \frac{k}{1 - k^2}\,|\partial F|^{-1}, \quad |V(z)| \leqslant \frac{1}{1 - k^2}\,|\partial F|^{-1}$$

for all $z \in \mathbb{D}$, where $V(z) := \overline{\partial F(z)}/J_F(z)$.

Since $\bar{\partial} H = U \circ H$ and $\partial H = V \circ H$, for all $w \in D_{t_0}$ we have

$$\frac{\Delta H(w)}{4} = \partial \bar{\partial} H(w) = \partial U(z)\partial H(w) + \bar{\partial} U(z)\overline{\bar{\partial} H(w)}$$

$$= \partial U(z)\,V(z) + \bar{\partial} U(z)\,\overline{U(z)}, \quad \text{where } z := H(w).$$

Using (4.29) and (4.30), we see that $|\Delta H(w)| \leqslant M_1(q)(1 - k)^{-1}|\partial F(z)|^{-4}$ for all $w \in D_{t_0}$. To complete the proof, it is remains to apply the lower estimate for $|\partial F|$ given in Lemma 4.3. $\qquad \square$

4.4. Estimates for functions in $\Sigma(k)$ with smooth image domains. In what follows, $\mathsf{R_c}(z_0, D)$ will stand for the conformal radius of a domain $D \subset \mathbb{C}$ w.r.t. the point $z_0 \in D$. For a point $z \in \mathbb{C}$ and two sets $A, B \subset \mathbb{C}$, we define

$$\mathsf{dist}(z, B) := \inf\{|z - w| \colon w \in B\}, \quad \mathsf{dist}(A, B) := \inf\{|z - w| \colon z \in A, w \in B\}.$$

Furthermore, for a map $g \colon \Delta \to \overline{\mathbb{C}}$ and $d > 0$, we denote

$$\mathcal{A}_g(d) := \{w \in g(\Delta) \colon \mathsf{dist}(w, \partial g(\Delta)) \leqslant d\}.$$

According to a well-known result by Kühnau [23, Satz 4], for any $g \in \Sigma(k)$,

$$(4.31) \qquad (1 - |z|^{-2})^k \leqslant |g'(z)| \leqslant \frac{1}{(1 - |z|^{-2})^k}, \qquad z \in \Delta.$$

Proposition 4.5. *Let $R > 1$ and $k \in (0, 1)$. Then for any $g \in \Sigma(k)$,*

$$\mathcal{A}_g\big(d_1(k, R)\big) \subset g\big(\{z \colon 1 < |z| \leqslant R\}\big) \subset \mathcal{A}_g\big(d_2(k, R)\big),$$

where $d_1(k, R) := \frac{1}{4}R^{-2k}(R - 1)^{1+k}(R + 1)^k$ and $d_2(k, R) := 4R^{2k}(R - 1)^{1-k}(R + 1)^{-k}$.

Proof. Let us fix some $z_0 \in \Delta$. Denote by D the open disk of radius $\mathsf{dist}(g(z_0), \partial g(\Delta))$ centered at $g(z_0)$ and let $\Omega := g^{-1}(D)$. Then $\mathsf{dist}(z_0, \partial \Omega) \leqslant |z_0| - 1$. Therefore, with the help of the upper bound in (4.31) and Koebe's 1/4-Theorem we obtain:

$$(4.32) \qquad \begin{aligned} \mathsf{dist}(g(z_0), \partial g(\Delta)) &= \mathsf{R_c}(g(z_0), D) = |g'(z_0)|\,\mathsf{R_c}(z_0, \Omega) \\ &\leqslant 4|g'(z_0)|\,\mathsf{dist}(z_0, \partial \Omega) \leqslant d_2(k, |z_0|). \end{aligned}$$

Similarly, denoting $\Omega' := \{z \colon |z - z_0| < |z_0| - 1\} \subset \Delta$ and $D' := g(\Omega')$, we get:

(4.33)
$$\mathsf{dist}(g(z_0), \partial g(\Delta)) \geqslant \mathsf{dist}(g(z_0), \partial D') \geqslant \tfrac{1}{4}\mathsf{R_c}(g(z_0), D')$$
$$= \tfrac{1}{4}|g'(z_0)|\, \mathsf{R_c}(z_0, \Omega') \geqslant d_1(k, |z_0|).$$

To complete the proof, it remains to take into account that $d_1(k, \cdot)$ and $d_2(k, \cdot)$ are strictly increasing on $[1, +\infty)$. □

Remark 4.2. According to Proposition 4.5, the preimage of any ε-neighborhood of $\partial g(\Delta)$ contains an annulus $\{z \colon 1 < |z| < R\}$, where $R > 1$ depends on ε and k, but not on the choice of $g \in \Sigma(k)$. Note that this property does not hold for the whole class Σ. Indeed, for any $\delta \in (0, \pi)$ and suitable $r_\delta > 1$ there is a unique $g_\delta \in \Sigma$ that maps Δ onto $D_\delta := \overline{\mathbb{C}} \setminus \{r_\delta e^{i\theta} \colon |\theta| \leqslant \pi - \delta\}$. Using the Carathéodory Kernel Convergence Theorem, see e.g. [8, §3.1], we see that $g_\delta(z) \to z$ locally uniformly in Δ and hence $|g_\delta^{-1}(0)| \to 1$ as $\delta \to 0^+$, although $\mathsf{dist}(0, \partial g_\delta(\Delta)) = r_\delta > 1$ for all $\delta \in (0, \pi)$.

The following lemma can be used to estimate the derivative of the Riemann map on the unit circle via the geometric quantities describing the image domain. This idea is borrowed from [34].

Lemma 4.6. *Let g be a conformal map of Δ onto a domain $\Omega \ni \infty$ bounded by a C^2-smooth Jordan curve. Let $u \colon U \to \mathbb{R}$ be a C^1-smooth function on a neighbourhood U of $\partial\Omega$. Suppose that u vanishes on $\partial\Omega$ and that it is negative and subharmonic in $U \cap \Omega$. If the image of $\{z \colon 1 < |z| \leqslant R\}$ w.r.t. g lies in U for some $R > 1$, then*

(4.34)
$$|g'(z)| \geqslant -\frac{4u_0}{\pi(R-1)|\nabla u(g(z))|} \quad \text{for all } z \in \partial\Delta,$$

where $u_0 := \max_{z \in A_0} u(g(z)) < 0$ and $A_0 := \{z \colon \sqrt{(1 + R^2)/2} \leqslant |z| \leqslant R\}$.

Proof. By the hypothesis, $\partial\Omega$ is C^2-smooth. It follows that g', and hence the gradient of $v := u \circ g$, extend continuously to the unit circle, see e.g. [33, Theorem 3.5 on p. 48].

Fix some $\alpha \in [0, 2\pi]$. Since v is subharmonic in $A := \{z \colon 1 < |z| < R\}$ and since it is continuous and non-positive on the closure of A, for any $\rho \in (1, R)$ we have:

$$v(\rho e^{i\alpha}) \leqslant \frac{1}{2\pi} \int_0^{2\pi} \mathcal{P}_\alpha(\rho e^{i\alpha}, \theta)\, v(z_0 + r e^{i\theta})\, d\theta,$$

where

$$\mathcal{P}_\alpha(z, \theta) := \mathsf{Re}\, \frac{r e^{i\theta} + z - z_0}{r e^{i\theta} - (z - z_0)}, \quad z_0 := \frac{R+1}{2} e^{i\alpha}, \quad r := \frac{R-1}{2},$$

is the Poisson kernel for the disk $D_\alpha := \{z \colon |z - z_0| < r\}$. Note that the intersection of A with the ray $\{t e^{i\alpha} \colon t \geqslant 0\}$ is a diameter of D_α and that exactly one half of the circle ∂D_α lies in A_0. Therefore, denoting $I_\alpha := \{\theta \in [0, 2\pi] \colon z_0 + r e^{i\theta} \in A_0\}$, we get:

(4.35)
$$v(\rho e^{i\alpha}) \leqslant \frac{u_0}{2\pi} \int_{I_\alpha} \mathcal{P}_\alpha(\rho e^{i\alpha}, \theta)\, d\theta = \frac{u_0}{2\pi} \int_{\pi/2}^{3\pi/2} \mathsf{Re}\, \frac{e^{i\vartheta} + x}{e^{i\vartheta} - x}\, d\vartheta$$
$$= \frac{u_0}{2\pi} \int_{\pi/2}^{3\pi/2} \frac{e^{i\vartheta} + x}{e^{i\vartheta} - x}\, d\vartheta = u_0 \left(\frac{1}{2} - \frac{2}{\pi} \operatorname{arctg} x \right), \quad x := 1 - \frac{\rho - 1}{r}.$$

Recall that v vanishes on the unit circle. Therefore, using (4.35) we obtain

$$|\nabla v(e^{i\alpha})| = -\left.\frac{\partial v(\rho e^{i\alpha})}{\partial \rho}\right|_{\rho=1} \geqslant -\frac{4u_0}{\pi r}.$$

It remains to notice that $|g'(e^{i\alpha})| = |\nabla v(e^{i\alpha})|/|\nabla u(g(e^{i\alpha}))|$. □

Corollary 4.7. *In conditions of Lemma 4.6, suppose that $g \in \Sigma(k)$ and that $\mathcal{A}_g(\mu) \subset U$ for some $\mu \in (0, 8]$. Then*

$$(4.36) \qquad |g'(z)| \geqslant \frac{4\inf\left\{|u(w)|\colon w \in U \cap \mathcal{B}_g\big(\alpha(k)\mu^K\big)\right\}}{\pi(\mu/8)^{1/(1-k)}|\nabla u(g(z))|} \quad \text{for all } z \in \partial\Delta,$$

where $\mathcal{B}_g(d) := g(\Delta) \setminus \mathcal{A}_g(d)$, $K := (1+k)/(1-k)$, and $\alpha(k) := 1/\big(3^{1+k} \cdot 8^{K+1}\big)$.

Proof. We apply Lemma 4.6 with $R := 1 + (\mu/8)^{1/(1-k)} \leqslant 2$. To ensure that

$$g\big(\{z\colon 1 < |z| \leqslant R\}\big) \subset U,$$

we use Proposition 4.5 together with the elementary estimate

$$d_2(k, R) = 4\left(\frac{R^2}{R+1}\right)^k (R-1)^{1-k} \leqslant 4 \cdot 2^k(R-1)^{1-k} < \mu.$$

Similarly, in view of other two elementary estimates:

$$2 > R_* := \sqrt{\frac{R^2+1}{2}} > \sqrt{R} \geqslant 1 + (\sqrt{2}-1)(R-1) \geqslant 1 + \frac{(\mu/8)^{1/(1-k)}}{3},$$

$$d_1(k, R_*) = \frac{1}{4}\left(\frac{R_*+1}{R_*^2}\right)^k (R_*-1)^{1+k} > \frac{(R_*-1)^{1+k}}{4 \cdot 2^k} > \frac{(\mu/8)^K/3^{1+k}}{8} = \alpha(k)\mu^K,$$

Proposition 4.5 implies that $g(A_0) \subset U \cap \mathcal{B}_g\big(\alpha(k)\mu^K\big)$. Hence (4.36) follows from (4.34). □

We use a somewhat similar argument to estimate $|g'(z)|$ on $\partial\Delta$ from above. Note that in this case, we actually do not need to assume that the boundary is smooth.

Proposition 4.8. *There exists $\mathcal{M}\colon (0, +\infty) \times (0, 1) \to (0, +\infty)$ such that for any $\varepsilon > 0$ and any $k \in (0, 1)$, the following assertion holds for all $g \in \Sigma(k)$: if $g(\Delta)$ contains an open disk D of radius ε such that ∂D and $\partial g(\Delta)$ have a common point w_0, then*

$$(4.37) \qquad \big|\angle g'(\zeta_0)\big| \leqslant \mathcal{M}(\varepsilon, k),$$

where $\angle g'(\zeta_0)$ stands for the angular derivative of g at the landing point ζ_0 of the slit $g^{-1}(I)$ and I is the straight line segment joining w_0 with the center of D.

Proof. The fact that preimages of slits are slits is well-known, see e.g. [31, Theorem 9.2 on p. 267]. Existence of finite angular derivative follows from [31, Theorem 10.6 on p. 307].

If $\angle g'(\zeta_0) = 0$, then there is nothing to prove. So suppose that $\angle g'(\zeta_0) \neq 0$. In such a case, the slit $\gamma := g^{-1}(I)$ tends to $\partial\Delta$ non-tangentially. Hence, $\angle g'(\zeta_0) = \lim_{\gamma \ni z \to \zeta_0} g'(z)$. Therefore, it is sufficient to find an upper bound of $|g'(z)|$ for $z \in \gamma$ close to ζ_0.

Using translations and rotations, we may suppose that $D = \{w\colon |w| < \varepsilon\}$ and $w_0 = \varepsilon$. Consider the function $u(w) := \log|g^{-1}(w)|$. It is harmonic and positive in D. Moreover, it extends continuously to ∂D, with $u(w_0) = 0$. Fix $w \in (0, \varepsilon)$ and apply

the Poisson representation for u in the smaller disk $D_1 := \{w \colon |w - \varepsilon/2| < \varepsilon/2\}$. We have

$$u(w) = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{P}_1(w, \theta) \, u((1 + e^{i\theta})\varepsilon/2) \, \mathrm{d}\theta \geqslant \frac{u_0}{2\pi} \int_{\pi/2}^{3\pi/2} \mathcal{P}_1(w, \theta) \, \mathrm{d}\theta$$

(4.38)

$$= u_0 \left(\frac{1}{2} - \frac{2}{\pi} \operatorname{arctg}\left(\frac{2w}{\varepsilon} - 1 \right) \right), \quad u_0 := \min_{\zeta \in C} u(\zeta),$$

where $\mathcal{P}_1$ is the Poisson kernel in D_1 and $C := \{\zeta \in \partial D_1 \colon \operatorname{Re} \zeta \leqslant \varepsilon/2\}$.

We have $\mathsf{dist}(w, \partial g(\Delta)) = \varepsilon - w$, while

$$\mathsf{dist}(g^{-1}(w), \partial\Delta) = \exp(u(w)) - 1 \geqslant u(w) \geqslant \frac{2u_0}{\pi\varepsilon}(\varepsilon - w) + o(\varepsilon - w) \quad \text{as} \quad w \to \varepsilon.$$

As in the proof of Proposition 4.5, we have

$$\left| g'\big(g^{-1}(w) \big) \right| \leqslant \frac{4 \, \mathsf{dist}(w, \partial g(\Delta))}{\mathsf{dist}(g^{-1}(w), \partial\Delta)} \leqslant \frac{2\pi\varepsilon}{u_0} \big(1 + o(1) \big).$$

It remains to estimate u_0. To this end we use Proposition 4.5. By construction, $\mathsf{dist}(C, \partial g(\Delta)) \geqslant \varepsilon_1 := (1 - 1/\sqrt{2})\varepsilon$. Hence, $u_0 \geqslant \log R(\varepsilon, k)$, where $R = R(\varepsilon, k) > 1$ is the unique solution to the equation $d_2(k, R) = \varepsilon_1$. Existence and uniqueness of the solution follows form the fact that for any fixed $k \in (0, 1)$, $R \mapsto d_2(k, R)$ is a strictly increasing map of $[1, +\infty)$ onto $[0, +\infty)$. Since R depends only on k and ε, the proposition is now proved. $\qquad\square$

Remark 4.3. Note that the upper estimate $\mathcal{M}(\varepsilon, k) = 2\pi\varepsilon / \log R(\varepsilon, k)$ for $|\angle g'|$ obtained above explode to $+\infty$ both as $\varepsilon \to +0$ and as $\varepsilon \to +\infty$. In particular, there exists certain $\varepsilon^* > 0$ for which $\mathcal{M}(\varepsilon, k)$ takes its minimal value. This provides an upper bound, depending only on k, for the angular derivatives of functions $g \in \Sigma(k)$ such that $\mathbb{C} \setminus g(\Delta)$ is convex. It is curious enough to mention that the latter bound does not explode as $k \to 1$ and in fact, gives an absolute bound $|\angle g'| < 165$ for any $g \in \Sigma$ with convex $\mathbb{C} \setminus g(\Delta)$.

4.5. Distance, diameter, and curvature estimates. Recall that we defined the choice of $t_1 > t_0$ and of the Moebius transformation T at the beginning of Sect. 4.3. Moreover, we choose the smallest $t_2 \in (t_0, t_1)$ such that $T(e^{t_0 - t_2}\mathbb{D}) \subset \mathbb{D}$. Namely,

(4.39) $$t_2 = t_0 + \log\big((1 + |z_0|)/(1 + |z_0|^2)\big),$$

where as before $z_0 := e^{t_0} G^{-1}(0)$. Recall also that in Lemma 4.3 we defined $F(z) := G(e^{-t_0}z)$ for all $z \in \mathbb{D}$. Finally, let $\Gamma_t := \partial g_t(\Delta) = \partial D_t$.

In addition to $\mathsf{dist}(A, B)$, for two sets $A, B \subset \mathbb{C}$ we define

$$\mathsf{dist}^*(A, B) := \max\left\{ \sup_{z \in A} \mathsf{dist}(z, B), \, \sup_{w \in B} \mathsf{dist}(w, A) \right\}.$$

Note that the diffeomorphism $L = F \circ T \circ F^{-1} \colon D_{t_1} \to D_{t_1}$ extends to a diffeomorphic map of $\overline{D_{t_2}}$ into $\overline{D_{t_0}}$.

Lemma 4.9. *In the above notation, there exist positive constants $M_*(q)$ and $M^*(q)$ depending only on q such that*

(4.40) $$M_*(q) \leqslant \mathsf{dist}(\Gamma_{t_0}, \Gamma_{t_1}) \leqslant \mathsf{dist}^*(\Gamma_{t_0}, \Gamma_{t_1}) \leqslant M^*(q).$$

Moreover, for any $t > t_2$ and any $s \in [t_2, t)$, we have

(4.41) $$\mathsf{dist}^*\big(L(\Gamma_t), L(\Gamma_s)\big) \leqslant M_2(q)(e^{-s} - e^{-t})$$

for some constant $M_2(q)$ depending only on q.

Proof. For a C^1-map $V : D \to \mathbb{C}$ of an open set $D \subset \mathbb{C}$ and $\alpha \in \mathbb{R}$, we denote:

$$\mathsf{D}_\alpha V(z) := \tfrac{\mathrm{d}}{\mathrm{d}s} V(z + se^{i\alpha})|_{s=0}, \quad \mathsf{D}^* V(z) := \max_{\alpha \in \mathbb{R}} |\mathsf{D}_\alpha V(z)|, \quad \mathsf{D}_* V(z) := \min_{\alpha \in \mathbb{R}} |\mathsf{D}_\alpha V(z)|.$$

Note that F extends diffeomorphically to a disk larger than $\mathbb{D}$, namely to $e^{t_0}\mathbb{D}$. The curves Γ_{t_0} and Γ_{t_1} are images under F of $A := \{z : |z| = 1\}$ and $B := \{z : |z| = e^{t_0 - t_1}\}$, respectively. Fix a point $z \in A$ and let γ be the straight line segment joining z with the closest point of B. Then the length of $F(\gamma)$ does not exceed $(1 - e^{t_0 - t_1}) \max \mathsf{D}^* F(\zeta)$, where the maximum is taken over all $\zeta \in \gamma$. Applying this simple argument again, but with A and B swapped, we conclude that

$$\mathsf{dist}^*\big(\Gamma_{t_0}, \Gamma_{t_1}\big) \leqslant (1 - e^{t_0 - t_1}) \max_{\zeta \in \overline{\mathbb{D}}} \mathsf{D}^* F(z) < \max_{\zeta \in \overline{\mathbb{D}}} \mathsf{D}^* F(z).$$

Hence, the upper bound in (4.40) follows directly from Lemma 4.3.

In a similar way, we prove (4.41). The curves $L(\Gamma_t)$ and $L(\Gamma_s)$ are images under the map $L \circ F = F \circ T$ of $A := \{z : |z| \leqslant e^{t_0 - t}\}$ and $B := \{z : |z| \leqslant e^{t_0 - s}\}$, respectively. Clearly, $\mathsf{D}^*(F \circ T)(z) = \mathsf{D}^* F(T(z))|T'(z)|$. Since $s, t \geqslant t_2$, both $T(A)$ and $T(B)$ lie in $\overline{\mathbb{D}}$. Therefore, as above, we can use the upper for $\mathsf{D}^* F(T(z))$ given in Lemma 4.3, and

$$
(4.42) \qquad
\begin{aligned}
|T'(z)| &= \frac{1 - |z_0|^4}{|1 + |z_0|^2 + 2\bar{z}_0 z|^2} \leqslant \frac{1 - |z_0|^4}{(1 + |z_0|^2 - 2|z_0| |z|)^2} \\
&= (1 + |z_0|^2) \frac{1 + |z_0|}{(1 - |z_0|)^3} \leqslant (1 + 3q) \frac{1 + \sqrt{3q}}{(1 - \sqrt{3q})^3}
\end{aligned}
$$

for all $z \in \overline{\mathbb{D}}$. Now (4.41) follows easily with

$$M_2(q) := 1 / \big((1 - k)^{1+q}(1 - \sqrt{k})^4\big), \quad \text{where } k := 3q.$$

It remains to prove the lower estimate in (4.40). Since Γ_{t_0} and Γ_{t_1} are two nested Jordan curves, it is not difficult to see that there exists a straight line segment $I \subset E$ joining Γ_{t_0} with Γ_{t_1} whose length is exactly $\mathsf{dist}(\Gamma_{t_0}, \Gamma_{t_1})$. Here E stands for the closure of the doubly connected domain bounded by Γ_{t_0} and Γ_{t_1}. The length of $\gamma := F^{-1}(I)$ is at least $1 - e^{t_0 - t_1}$. Therefore,

$$1 - e^{t_0 - t_1} \leqslant \mathsf{dist}(\Gamma_{t_0}, \Gamma_{t_1}) \max_{w \in I} \mathsf{D}^*(F^{-1})(w) = \frac{\mathsf{dist}(\Gamma_{t_0}, \Gamma_{t_1})}{\min_{z \in \gamma} \mathsf{D}_* F(z)}.$$

To complete the proof it is sufficient to use the lower bound in (4.7) and take into account that $t_1 - t_0 \geqslant \frac{1}{2} \log \big(2/(1 + 3q)\big)$, see the proof of Proposition 4.1 in Sect. 4.3. $\qquad \square$

Lemma 4.10. *For any $t > t_2$,*

$$(4.43) \qquad M_3(q) e^{-t} \leqslant \mathsf{diam}(L(\Gamma_t)) \leqslant M_4(q) e^{-t},$$

where $M_3(q)$ and $M_4(q)$ are positive constants depending only on q.

Proof. The upper bound in (4.43) can be obtained using the same method as in the proof of Lemma 4.9. Let us obtain the lower bound in (4.43). Denote $A := \{z : |z| = e^{t_0 - t}\}$. By a direct computation using (4.25), we find that $B := T(A) \subset \mathbb{D}$ is a circle of radius

$$e^{t_0 - t} \frac{1 - |z_0|^4}{1 + |z_0|^4 + (2 - 4e^{2(t_0 - t)})|z_0|^2} \geqslant e^{t_0 - t} \frac{1 - |z_0|^2}{1 + |z_0|^2} \geqslant e^{t_0 - t} \frac{1 - 3q}{1 + 3q},$$

where the last inequality is due to (4.24).

Hence, $\mathsf{diam}\,B \geqslant 2e^{t_0-t}(1-k)/(1+k)$, where $k := 3q$, and our task reduces to finding a lower bound for $\mathsf{diam}(F(B))/\mathsf{diam}(B)$. The technique used to in Lemma 4.9 does not apply directly to this case, because the longest straight line segment with the end-points in $F(B) = L(\Gamma_t)$ does not have to lie in $F(\mathbb{D}) = D_{t_0}$. However, D_{t_0} is a k-quasidisk and hence by a result of Gehring and Osgood, see e.g. [9, §8.1], for any two points $w_1, w_2 \in D_{t_0}$ there is a smooth curve $\gamma \subset D_{t_0}$ joining w_1 and w_2 such that $\mathsf{length}(\gamma) \leqslant c(k)|w_2 - w_1|$, where $c(k) > 0$ is a constant depending only on k. At the same time

$$|F^{-1}(w_2) - F^{-1}(w_1)| \leqslant \mathsf{length}(F^{-1}(\gamma)) \leqslant \frac{\mathsf{length}(\gamma)}{\inf_{z\in\mathbb{D}} \mathsf{D}_*F(z)}.$$

By choosing $w_j := F(z_j)$, $j = 1, 2$, where z_1 and z_2 are the end-point of a diameter of B, we obtain

$$\mathsf{diam}(F(B)) \geqslant |w_2 - w_1| \geqslant \frac{\mathsf{length}(\gamma)}{c(k)}$$
$$\geqslant \frac{|z_2 - z_1|\inf_{z\in\mathbb{D}} \mathsf{D}_*F(z)}{c(k)} = \mathsf{diam}(B)\frac{\inf_{z\in\mathbb{D}} \mathsf{D}_*F(z)}{c(k)}.$$

Employing the lower estimate for D_*F given in Lemma 4.3 completes the proof. $\square$

Lemma 4.11. *There exist positive constants $M_5(q)$ and $M_6(q)$ depending only on q such that for all $t \geqslant t_2$ the curvature of $L(\Gamma_t)$ does not exceed $\kappa_0(q,t) := M_5(q)e^t + M_6(q)$.*

Proof. Denote $A := \{z \colon |z| = e^{t_0-t}\}$. Then $L(\Gamma_t) = F(T(A))$. As in the proof of Lemma 4.10, we see that $T(A) \subset \mathbb{D}$ is a circle of radius

$$(4.44) \qquad \rho \geqslant e^{t_0-t}\frac{1-k}{1+k}, \quad k := 3q.$$

Parameterize $T(A)$ by $z(\theta) = z_* + \rho e^{i\theta}$, $\theta \in [0, 2\pi]$. Then $w(\theta) = F(z(\theta))$ is a parametrization of $L(\Gamma_t)$. The (signed) curvature of $L(\Gamma_t)$ at the point $w = w(\theta)$ equals

$$\kappa(w) = \frac{1}{|w'(\theta)|}\,\mathtt{Im}\,\frac{w''(\theta)}{w'(\theta)}.$$

By Lemma 4.3,

$$(4.45) \qquad |w'(\theta)| \geqslant |z'(\theta)|\,\mathsf{D}_*F(z(\theta)) \geqslant \rho e^{-t_0}\frac{(1-k)^{1+q}}{(1+k^2)^2}, \quad k := 3q.$$

Moreover,

$$\frac{w''(\theta)}{w'(\theta)} = i\,\frac{e^{i\theta}\partial F + e^{-i\theta}\bar{\partial}F + \rho(e^{2i\theta}\partial^2 F - 2\partial\bar{\partial}F + e^{-2i\theta}\bar{\partial}^2 F)}{e^{i\theta}\partial F - e^{-i\theta}\bar{\partial}F},$$

where the partial derivatives of F are to be evaluated at $z(\theta)$. Taking into account that F is k-q.c. and using Lemma 4.4, we see that

$$\left|w''(\theta)/w'(\theta)\right| \leqslant K + \frac{4M(q)\rho}{1-k}, \quad K := \frac{1+k}{1-k}.$$

Combining the above inequality with (4.44) and (4.45) and taking into account that k and $M(q)$ depend only on q leads to the desired conclusion. $\square$

Lemma 4.12. *Let Γ be a convex C^2-smooth Jordan curve and κ_* the maximum of its curvature. If C is a circle of radius R not exceeding $1/\kappa_*$ internally tangent to Γ, then the open disk D bounded by C is a subset of the Jordan domain Ω bounded by Γ.*

Proof. Clearly, if we prove the statement of Lemma 4.12 for all $0 < R < 1/\kappa_*$, then it also holds with $R = 1/\kappa_*$. Suppose that it fails for some $R < 1/\kappa_*$. Then there exists $R_0 \in (0, 1/\kappa_*)$ and a circle C_0 of radius R_0 which is internally tangent to Γ at two distinct points z_1 and z_2. Using if necessary translations, rotations, and/or reflections, we may suppose that C_0 is centered at the origin and that $z_1 = R_0 e^{-i\theta}$ and $z_2 = R_0 e^{i\theta}$ for some $\theta \in (0, \pi/2]$.

Let Γ_0 be the arc of Γ going in the counterclockwise direction from z_1 to z_2. Let $z = \varphi(t)$, $t \in [-1, 1]$, be a C^2-parametrization of Γ_0, with $z_1 = \varphi(-1)$ and $z_2 = \varphi(1)$. Then the C^1-function $\beta(t) := \arg \varphi'(t)$ is non-decreasing, because Γ is convex, and $\beta(\pm 1) = \pi/2 \pm \theta$ because Γ is tangent to C at z_1 and z_2. It follows that $0 \leqslant \beta(t) \leqslant \pi$ for all $t \in [-1, 1]$.

Note also that $\beta'(t) \leqslant \kappa(t)|\varphi'(t)| \leqslant \kappa_*|\varphi'(t)|$, where $\kappa(t)$ stands for the curvature of Γ at $z = \varphi(t)$. Therefore,

$$2R_0 \sin \theta = \text{Im}(z_2 - z_1) = \int_{-1}^{1} |\varphi'(t)| \sin \beta(t) \, \mathrm{d}t \geqslant \int_{-1}^{1} \kappa_*^{-1} \beta'(t) \sin \beta(t) \, \mathrm{d}t$$
$$= \kappa_*^{-1} \big(\cos \beta(-1) - \cos \beta(1) \big) > 2R_0 \sin \theta.$$

The contradiction we have obtained proves the lemma. $\qquad \square$

Lemma 4.13. *There exists a constant $\varepsilon_0(q) > 0$ depending only on q such that for any $t > t_1$ and any $w_0 \in L(\Gamma_t)$ the open disk of radius $\varepsilon_0(q)$ with the boundary tangent to $L(\Gamma_t)$ at w_0 from outside is entirely contained in the unbounded component of $\mathbb{C} \setminus L(\Gamma_t)$.*

Proof. Fix $t > t_1$. Recall that $L \circ F = F \circ T$, with T mapping $e^{t_0 - t_1}\mathbb{D}$ onto itself. Therefore, $L(\Gamma_t)$ is the image of a circle $C(t) \subset e^{t_0 - t_1}\mathbb{D}$ w.r.t. F. It follows that for any $\rho \in (0, 1 - e^{t_0 - t_1}]$ and any point $z \in C(t)$, the unbounded component of $\mathbb{C} \setminus L(\Gamma_t)$ contains a smooth Jordan domain $U_t(z, \rho)$ with $L(\Gamma_t)$ and $\partial U_t(z, \rho)$ tangent at the point $F(z)$ and such that $F^{-1}(U_t(z, \rho))$ is a disk of radius ρ lying together with its boundary in $\mathbb{D}$.

Since F is a diffeomorphic k-q.c. map,

$$|\bar{\partial}F(z)| \leqslant k|\partial F(z)| \quad \text{and} \quad \text{Re} \frac{e^{i\theta}\partial F(z) + e^{-i\theta}\bar{\partial}F(z)}{e^{i\theta}\partial F(z) - e^{-i\theta}\bar{\partial}F(z)} \geqslant \frac{1 - k}{1 + k}$$

for all $z \in \mathbb{D}$ and all $\theta \in \mathbb{R}$. Arguing as in the proof of Lemma 4.11, we see that the domain $U_t(z, \rho)$ is convex whenever

$$\text{Re} \frac{e^{i\theta}\partial F + e^{-i\theta}\bar{\partial}F}{e^{i\theta}\partial F - e^{-i\theta}\bar{\partial}F} \geqslant \left| \frac{\rho\big(e^{2i\theta}\partial^2 F - 2\partial\bar{\partial}F + e^{-2i\theta}\bar{\partial}^2 F\big)}{e^{i\theta}\partial F - e^{-i\theta}\bar{\partial}F} \right|.$$

Thanks to Lemma 4.4, the latter inequality holds provided

$$\rho \leqslant \rho_1(q) := \frac{(1 - k)^2}{4(1 + k)M(q)}.$$

Note that by (4.26) and (4.24), we have $1 - e^{t_0 - t_1} \geqslant 1 - \sqrt{(1 + 3q)/2} =: \rho_2(q) > 0$. Now set $\rho = \rho_0(q) := \min\{\rho_1(q), \rho_2(q)\} > 0$. Using again the proof of Lemma 4.11,

we see that the curvature of $\partial U_t(z, \rho_0)$ does not exceed some $\kappa_*(q)$ depending only on q. Applying Lemma 4.12 for $\Gamma := \partial U_t(z, \rho_0)$ leads to the desired conclusion with $\varepsilon_0(q) := 1/\kappa_*(q)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

4.6. Proof of Theorem 1. We use the notation introduced in Sect. 4.1–4.5. In particular, for $t > t_1$ we denote by $\tilde{g}_t$ the conformal map of Δ, normalized as usual by $\tilde{g}_t(\infty) = \infty$, $\tilde{g}_t'(\infty) > 0$, onto the unbounded component of $\overline{\mathbb{C}} \setminus L(\Gamma_t)$, where $\Gamma_t := \partial G(e^{-t}\mathbb{D}) = \partial g_t(\Delta)$.

Step 1. Fix some $t > t_1$. Let us obtain a lower estimate of $|\tilde{g}_t'(z)|$ for $z \in \partial\Delta$. Denote $\rho(t) := \tilde{g}_t'(\infty)$. Note that $\partial\tilde{g}_t(\Delta) = L(\Gamma_t) = G\big(e^{-t_0}T(\{z : |z| = e^{t_0-t}\})\big)$. Hence $\partial\tilde{g}_t(\Delta)$ is a k-quasidisk with $k := 3q$. It follows, see e.g. [9, Lemma 8.1.1 on p. 100], that

$$g := \rho(t)^{-1}\tilde{g}_t \in \Sigma(k'), \quad \text{where } k' := 2k/(1 + k^2).$$

Therefore, we may apply Corollary 4.7 with k' substituted for k and with $u(w) := \varphi\big(\rho(t)w\big) - e^{a(q)(t-t_0)}$, where φ and $a(q)$ are defined in Proposition 4.1. The function u is subharmonic in $U := \rho(t)^{-1}D_{t_0}$.

Since the curve $L(\Gamma_t)$ lies inside $L(\Gamma_{t_1}) = \Gamma_{t_1}$, we have $\rho(t) < \rho(t_1) = e^{-t_1} < 1$ and

$$\mathsf{dist}(\partial U, \partial g(\Delta)) = \rho(t)^{-1}\mathsf{dist}(\Gamma_{t_0}, L(\Gamma_t)) > \mathsf{dist}(\Gamma_{t_0}, \Gamma_{t_1}) \geqslant M_*(q).$$

The last inequality holds by Lemma 4.9. It follows that the hypothesis of Corollary 4.7 is satisfied with $\mu := \min\{M_*(q), 8\}$.

Note that $\frac{1}{4}\mathsf{diam}(\tilde{g}_t(\Delta)) \leqslant \rho(t) \leqslant \frac{1}{2}\mathsf{diam}(\tilde{g}_t(\Delta))$, see e.g. [35, §5.2, §5.3]. Hence by Lemma 4.10,

$$(4.46) \qquad\qquad \frac{M_3(q)}{4}e^{-t} \leqslant \rho(t) \leqslant \frac{M_4(q)}{2}e^{-t}.$$

Now we can estimate the denominator in (4.36). Namely, we are going to obtain an upper bound for $|\nabla u|$ on $\partial g(\Delta) = \rho(t)^{-1}L(\Gamma_t)$.

Note that $\varphi(w) = \phi(w)^{-a(q)} - e^{a(q)(t-t_0)}$ for all $w \in D_{t_0}$, where $\phi := |(F \circ T)^{-1}|$ and $F(z) := G(e^{-t_0}z)$. Hence, for any $w \in \partial g(\Delta)$, we have:

$$|\nabla u(w)| = \rho(t)|\nabla\varphi(\rho(t)w)| = a(q)\rho(t)\phi(\rho(t)w)^{-a(q)-1}\left|\nabla\phi(\rho(t)w)\right|$$

$$\leqslant a(q)\rho(t)\phi(\rho(t)w)^{-a(q)-1}\frac{|(T^{-1})'(z)|}{\mathsf{D}_*F(z)},$$

where $z := F^{-1}\big(\rho(t)w\big) \in \mathbb{D}$. Note that by construction for any $w \in \partial g(\Delta)$, $\phi\big(\rho(t)w\big) = e^{t_0-t}$. Moreover, for all $z \in \mathbb{D}$,

$$\left|(T^{-1})'(z)\right| \leqslant (1 + |z_0|^2)\frac{1 + |z_0|}{(1 - |z_0|)^3} \leqslant (1 + 3q)\frac{1 + \sqrt{3q}}{(1 - \sqrt{3q})^3}.$$

Therefore, taking into account (4.46) and (4.7), we may conclude that there exists a constant $M_7(q)$ depending only on q such that

$$(4.47) \qquad\qquad \left|\nabla u(w)\right| \leqslant a(q)M_7(q)e^{a(q)(t-t_0)} \quad \text{for all } w \in \partial g(\Delta).$$

Let us now estimate the numerator in (4.36). According to Lemma 4.9 and the lower estimate in (4.46), if $s \in [t_2, t)$ and

$$(4.48) \qquad\qquad e^{t-s} - 1 \leqslant \frac{M_3(q)\alpha(k')\mu^{K'}}{4M_2(q)}, \quad K' := \frac{1 + k'}{1 - k'} = K^2,$$

where $\alpha(\cdot)$ is defined in the statement of Corollary 4.7, then $\mathcal{B}_g\big(\alpha(k')\mu^{K'}\big)$ lies in the unbounded component of $\mathbb{C}\backslash\rho(t)^{-1}L(\Gamma_s)$. Since on $\rho(t)^{-1}L(\Gamma_s)$ the function u is equal identically to $u_s := e^{a(q)(s-t_0)} - e^{a(q)(t-t_0)}$, we conclude that under condition (4.48) the infimum in (4.36) is greater or equal to $-u_s = e^{a(q)(t-t_0)}\big(1 - e^{-a(q)(t-s)}\big)$. Recall that

$$t - t_2 > t_1 - t_2 = \tfrac{1}{2}\log\frac{2(1+|z_0|^2)}{(1+|z_0|)^2} \geqslant \frac{1}{2}\log\frac{2(1+3q)}{(1+\sqrt{3q})^2} > 0,$$

see (4.26), (4.39), and (4.24). Choosing the smallest $s \in [t_2, t)$ satisfying (4.48) and taking into account that $a(q)$, $\mu = \min\{M_*(q), 8\}$, as well as k and k', depend only on q, we conclude that the numerator of (4.36) is bounded from below by $M_8(q)e^{a(q)(t-t_0)}$, where $M_8(q) > 0$ depends only on q. Thus, taking into account (4.47), from (4.36) we obtain

$$(4.49)\qquad |\tilde{g}'_t(z)| \geqslant \frac{M_9(q)}{a(q)}\tilde{g}'_t(\infty) \geqslant \frac{M_9(q)M_3(q)}{4a(q)}e^{-t} \quad \text{for all } z \in \partial\Delta \text{ and all } t > t_1,$$

where $M_9(q) > 0$ depends only on q.

Step 2. Now we obtain an upper estimate for $|\tilde{g}'_t|$ for $t > t_1$. Again consider

$$g := \rho(t)^{-1}\tilde{g}_t \in \Sigma(k'), \quad \rho(t) := \tilde{g}'_t(\infty).$$

By Lemma 4.13 and the right inequality in (4.46), the function g satisfied the hypothesis of Proposition 4.8 with any point $w_0 \in \partial g(\Delta)$ and with any $\varepsilon > 0$ not exceeding $\rho(t)^{-1}\varepsilon_0(q)$. By (4.46), $\rho(t)^{-1} \geqslant 2e^t/M_4(q) > 2/M_4(q)$. Therefore, we can take $\varepsilon := 2\varepsilon_0(q)/M_4(q)$. Taking into account that when g' exists on the boundary, it clearly coincides with $\angle g'$, by Proposition 4.8 we have

$$(4.50)\qquad |\tilde{g}'_t(z)| \leqslant \mathcal{M}(\varepsilon, k')\tilde{g}'_t(\infty) \leqslant M_{10}(q)e^{-t} \quad \text{for all } z \in \partial\Delta, \text{ all } t > t_1,$$

and some $M_{10}(q)$ depending only on q.

Step 3. Now we will see that $\tilde{g}_t$ satisfies the Loewner–Kufarev equation and find bounds for the real part of Herglotz function.

Applying Proposition 3.1 with $\zeta \mapsto 1/F(T(e^{t_0}/\zeta))$, $a := t_1 < |\zeta| < b := +\infty$, substituted for Ψ, we see that $\tilde{g}_t$ is differentiable w.r.t. t for all $t > t_1$ and that it satisfies the Loewner–Kufarev PDE in Δ,

$$\partial\tilde{g}_t(z)/\partial t = z\tilde{g}'_t(z)p(z,t), \quad t > t_1, \ z \in \Delta,$$

with the Herglotz fuction $p: \Delta \times [0, +\infty) \to \mathbb{C}$, $\operatorname{Im} p(\infty, t) = 0$, determined for each fixed $t > t_1$ by the real part its continuous extension to $\partial\Delta$:

$$\operatorname{Re} p(e^{i\theta}, t)$$
$$(4.51)$$
$$= \frac{1}{|\tilde{g}'_t(e^{i\theta})|} \operatorname{Im}\left(\frac{\partial F\big(T(e^{t_0-t}e^{i\tau})\big)}{\partial\tau} \frac{\partial F\big(T(e^{t_0-t}e^{i\tau})\big)}{\partial t}\right)\left|\frac{\partial F\big(T(e^{t_0-t}e^{i\tau})\big)}{\partial\tau}\right|^{-1}$$

for all $\theta \in [0, 2\pi]$, where $\tau = \tau(\theta)$ satisfies $F\big(T(e^{t_0-t}e^{i\tau})\big) = \tilde{g}_t(e^{i\theta})$. In particular,

$$\operatorname{Re} p(e^{i\theta}, t) \leqslant \frac{1}{|\tilde{g}'_t(e^{i\theta})|}\left|\frac{\partial F\big(T(e^{t_0-t}e^{i\tau})\big)}{\partial t}\right|.$$

It follows easily with the help of (4.49), (4.42), and Lemma 4.3 that

$$(4.52)\qquad \operatorname{Re} p(e^{i\theta}, t) \leqslant M_{11}(q) \quad \text{for all } t > t_1 \text{ and all } \theta \in [0, 2\pi],$$

where $M_{11}(q)$ is a constant depending only on q.

A lower estimate for $\operatorname{Re} p$ is a bit more tricky. We have to use the fact that F is a smooth k-q.c. map. In particular, if η_1 and η_2 are images of some $\xi \in \mathbb{C}$ and $i\xi$ w.r.t. the differential of F, respectively, then $\operatorname{Im}(\overline{\eta_1}\,\eta_2) \geqslant (1-k^2)(1+k^2)^{-1}|\eta_1|\,|\eta_2|$. Taking into account that

$$(4.53) \qquad |T'(z)| \geqslant (1+|z_0|^2)\frac{1-|z_0|}{(1+|z_0|)^3} \geqslant (1+3q)\frac{(1-\sqrt{3q})}{(1+\sqrt{3q})^3} \quad \text{for all } z \in \mathbb{D},$$

with the help of (4.50) and Lemma 4.3, we obtain:

$$(4.54) \qquad \operatorname{Re} p(e^{i\theta}) \geqslant \frac{1-k^2}{1+k^2}\,\frac{\mathsf{D}_* F\big(T(e^{t_0-t}e^{i\tau})\big)\,\big|e^{t_0-t}T'(e^{t_0-t}e^{i\tau})\big|}{|\tilde{g}_t'(e^{i\theta})|} \geqslant M_{12}(q) > 0$$

for all $t > t_1$ and all $\theta \in [0, 2\pi]$, where $M_{12}(q)$ depends only on q.

Step 4. Now we have to find an estimate for the modulus of continuity of $\operatorname{Re} p$ on $\partial \Delta$. Fix $t > t_1$. Denote $z(\tau) := e^{t_0-t}e^{i\tau}$. A simple calculation shows that

$$\Phi(\tau) := -ie^{t-t_0}\frac{\partial F\big(T(z(\tau))\big)}{\partial \tau} = e^{i\tau}T'(z(\tau))\,\partial F\big(T(z(\tau))\big) - \overline{e^{i\tau}T'(z(\tau))}\,\bar{\partial}F\big(T(z(\tau))\big).$$

From (4.51) we obtain

$$(4.55) \qquad \begin{aligned} &\log|\tilde{g}_t'(e^{i\theta})| + \log \operatorname{Re} p(e^{i\theta}, t) \\ &= \log J_F\big(T(z(\tau))\big) + t_0 - t + 2\operatorname{Re}\log T'(z(\tau)) - \operatorname{Re}\log \Phi(\tau) =: V(\tau), \end{aligned}$$

where τ and θ are related, as above, by the equality $F\big(T(z(\tau))\big) = \tilde{g}_t(e^{i\theta})$. With τ regarded as a function of θ, we have $\mathrm{d}\tau/\mathrm{d}\theta = -e^{t-t_0}|\tilde{g}_t'(e^{i\theta})|/|\Phi(\tau)|$. Hence the l.h.s. of (4.55) is differentiable function of θ and

$$\left|\frac{\partial}{\partial\theta}\big(\log|\tilde{g}_t'(e^{i\theta})| + \log \operatorname{Re} p(e^{i\theta}, t)\big)\right| = e^{t-t_0}\frac{|\tilde{g}_t'(e^{i\theta})|}{|\Phi(\tau)|}|V'(\tau)| \leqslant \frac{e^{t-t_0}|\tilde{g}_t'(e^{i\theta})|}{\mathsf{D}_* F\big(T(z(\tau))\big)}\,\frac{|V'(\tau)|}{|T'(z(\tau))|}$$

With the help of (4.50) and Lemma 4.3, it follows that

$$(4.56) \qquad \left|\frac{\partial}{\partial\theta}\big(\log|\tilde{g}_t'(e^{i\theta})| + \log \operatorname{Re} p(e^{i\theta}, t)\big)\right| \leqslant M_{13}(q)\frac{V'(\tau)}{|T'(z(\tau))|}$$

for some constant $M_{13}(q)$ depending only on q.

Now let us estimate $|V'(\tau)|/|T'(z(\tau))|$. Using inequality (4.13) in Lemma 4.4 and taking into account that

$$J_F = |\partial F|^2 - |\bar{\partial}F|^2 \geqslant (1-k^2)|\partial F|^2,$$

we obtain

$$(4.57) \qquad \frac{1}{|T'(z(\tau))|}\left|\frac{\mathrm{d}}{\mathrm{d}\tau}\log J_F\big(T(z(\tau))\big)\right| \leqslant \frac{8M(q)}{1-k^2}e^{t_0-t}.$$

Moreover, it is straightforward to check that

$$(4.58) \qquad \frac{1}{|T'(z(\tau))|}\left|\frac{\mathrm{d}}{\mathrm{d}\tau}\log T'(z(\tau))\right| = e^{t_0-t}\frac{|T''(z(\tau))|}{|T'(z(\tau))|^2}$$

and that

$$(4.59) \qquad \frac{|T''(z(\tau))|}{|T'(z(\tau))|^2} = 4|z_0|\frac{|1+|z_0|^2+2\bar{z}_0 z|}{1-|z_0|^4} \leqslant 4\frac{\sqrt{3q}(1+\sqrt{3q})}{(1-\sqrt{3q})^3}.$$

Again by a straightforward computation,

$$\Phi'(\tau) = ie^{i\tau}T'\partial F + i\overline{e^{i\tau}T'}\,\bar\partial F$$

$$+ e^{t-t_0}\left(e^{2i\tau}T''\partial F - \overline{e^{2i\tau}T''}\,\bar\partial F \;+\; \left(e^{i\tau}T'\right)^2\partial^2 F - \overline{\left(e^{i\tau}T'\right)^2}\,\bar\partial^2 F\right),$$

where T and its derivatives are calculated at the point $z(\tau)$, while the derivatives of F are calculated at $T(z(\tau))$. Using Lemma 4.4 and bearing in mind that $|\bar\partial F| \leqslant |\partial F|$, from the above formula we get:

$$(4.60) \qquad \frac{\Phi'(\tau)}{|T'|} \leqslant 2\,|\partial F|\left(1 + e^{t_0-t}\left(M(q)|T'| + |T''|/|T'|\right)\right).$$

Since $|\Phi(\tau)| \geqslant |T'|\big(|\partial F| - |\bar\partial F|\big) \geqslant (1-k)\,|T'|\,|\partial F|$, from (4.60) it follows that

$$(4.61) \qquad \begin{aligned} &\frac{1}{|T'(z(\tau))|}\left|\frac{\mathrm{d}}{\mathrm{d}\tau}\log\Phi(\tau)\right| \\ &\leqslant \frac{2}{1-k}\left(\frac{1}{|T'(z(\tau))|} + e^{t_0-t}\left(M(q) + \frac{|T''(z(\tau))|}{|T'(z(\tau))|^2}\right)\right). \end{aligned}$$

Combining (4.56)–(4.59), (4.61), and (4.53) and bearing in mind that $t > t_1 > t_0$, we see that there exists a constant $M_{14}(q) > 0$ depending only on q such that

$$(4.62) \qquad \left|\frac{\partial}{\partial\theta}\left(\log|\tilde g_t'(e^{i\theta})| + \log\operatorname{Re}p(e^{i\theta},t)\right)\right| \leqslant M_{14}(q) \quad \text{for all } t > t_1 \text{ and } \theta \in [0,2\pi].$$

To complete Step 4, it remains to estimate the modulus of continuity of $\theta \mapsto \log|\tilde g_t'(e^{i\theta})|$. Denote by ω the modulus of continuity of the tangent unit vector $\beta(s)$ to $\partial\tilde g_t(\Delta)$ regarded as a function of the length parameter s. Then $\omega(\delta) \leqslant \kappa_0(q,t)\delta$, where $\kappa_0(q,t)$ is the upper bound for the curvature of $\partial\tilde g_t(\Delta)$ given in Lemma 4.11. Following the argument from [39], for the modulus of continuity of $\theta \mapsto \arg\tilde g_t'(e^{i\theta})$ denoted by ω_0, we have

$$(4.63) \qquad \omega_0(\delta) \leqslant \delta + \omega(\delta\max|\tilde g_t'(e^{i\theta})|) \leqslant \delta\left(1 + \kappa_0(q,t)M_{10}(q)e^{-t}\right),$$

where the maximum is taken over all $\theta \in [0,2\pi]$ and the last inequality holds because of (4.50).

Notice that $\log|\tilde g_t'|$ is harmonic conjugate to $\arg\tilde g_t'$. Denote the modulus of continuity of $\theta \mapsto \log|\tilde g_t'(e^{i\theta})|$ by ω_0^*. Using the well-known inequality due to Zygmund [40], see e.g. [10, Theorem 1.3 in Chap. III], and Lemma 4.11, we obtain

$$\begin{aligned} \omega_0^*(\delta) &\leqslant A\left(\int_0^\delta \frac{\omega_0(x)}{x}\,\mathrm{d}x + \delta\int_\delta^\pi \frac{\omega_0(x)}{x^2}\,\mathrm{d}x\right) \\ &\leqslant A\left(1 + \kappa_0(q,t)M_{10}(q)e^{-t}\right)\delta(1 + \log(\pi/\delta)) \\ (4.64) \qquad &\leqslant A\left(1 + (M_5(q) + M_6(q))M_{10}(q)\right)\delta(1 + \log(\pi/\delta)), \quad 0 \leqslant \delta \leqslant \pi, \end{aligned}$$

where A is an absolute constant.

Denote by ω_1 and ω_2 the continuity moduli of $\theta \mapsto \log\operatorname{Re}p(e^{i\theta},t)$ and $\theta \mapsto \operatorname{Re}p(e^{i\theta},t)$, respectively. Then for a fixed $t > t_1$, with the help of (4.52), (4.62), and (4.64), we have

$$(4.65) \qquad \begin{aligned} \omega_2(\delta) &\leqslant \max_{\theta\in\mathbb{R}}\operatorname{Re}p(e^{i\theta})\,\omega_1(\delta) \leqslant M_{11}(q)\big(M_{14}(q)\delta + \omega_0^*(\delta)\big) \\ &\leqslant M_{15}(q)\delta + M_{16}\delta\log(\pi/\delta). \end{aligned}$$

for some constants $M_{15}(q)$ and $M_{16}(q)$ depending only on q.

Step 5. Finally we can estimate $\mathrm{Im}\, p$. Again we fix an arbitrary $t > t_1$. According to (4.65), $\mathrm{Re}\, p(\cdot, t)$ is Dini-continuous on $\partial\Delta$. Therefore, by Zygmund's result mentioned above, $\mathrm{Im}\, p(\cdot, t)$ is continuous up to the boundary and on $\partial\Delta$ it can be written via a suitable version of the Hilbert transform. It follows that for all $\theta \in [0, 2\pi]$,

$$\left| \mathrm{Im}\, p(e^{i\theta}, t) \right| = \frac{1}{2\pi} \left| \int_0^\pi \frac{\mathrm{Re}\, p(e^{i(\theta+x)}, t) - \mathrm{Re}\, p(e^{i(\theta-x)}, t)}{x} \frac{x}{\mathrm{tg}(x/2)} \, \mathrm{d}x \right|$$

$$\leqslant \frac{1}{2\pi} \int_0^\pi \frac{\omega_2(2x)}{x} \frac{x}{\mathrm{tg}(x/2)} \, \mathrm{d}x \leqslant \frac{2}{\pi} \int_0^\pi \frac{\omega_2(x)}{x} \, \mathrm{d}x,$$

where we took into account that $\mathrm{Im}\, p(\infty, t) = 0$.

From the above inequality and (4.65), we finally obtain

$$(4.66) \qquad \left| \mathrm{Im}\, p(e^{i\theta}, t) \right| \leqslant \frac{2}{\pi} M_{15}(q) + 2M_{16}(q).$$

Recalling (4.52) and (4.54), we see that there exists $k_1(q) \in (0, 1)$ depending only on $q \in (0, 1/3)$ such that for any $t > t_1$, $p(\Delta, t) \subset U\big(k_1(q)\big)$, where the disk $U(\cdot)$ is defined in Becker's Theorem A. In particular, it follows that $t \mapsto -\log \tilde{g}_t'(\infty)$ is differentiable and its derivative is bounded in $(t_1, +\infty)$ from above and below by two positive constants (depending only on q). Therefore, taking into account that by construction $\tilde{g}_{t_1}(\Delta) = g_{t_1}$ with $\tilde{g}_t(\Delta) \to g_{t_1}(\Delta)$ in Carathéodory's sense as $t \to t_1 + 0$, and setting $\tilde{g}_t := g_t$ for all $t \in [0, t_1)$, we obtain a Loewner chain $(\tilde{g}_t)$ in Δ.

By construction, $F(T(0)) = 0 \notin \tilde{g}_t(\Delta)$ for all $t \geqslant 0$. Hence, (f_t) defined by

$$f_t(\zeta) := 1/\tilde{g}_t(1/\zeta), \quad \zeta \in \mathbb{D}, \ t \geqslant 0,$$

is a Loewner chain in $\mathbb{D}$. The Herglotz function of (f_t) is simply $p_{\mathbb{D}}(\zeta, t) = p(1/\zeta, t)$ for all $t \geqslant 0$, $t \neq t_1$, and all $\zeta \in \mathbb{D}$. It follows that (f_t) satisfies Becker's condition (2.4) with $k := k_1(q)$ for all $t > t_1$ and with $k := 3q$ for all $t \in [0, t_1)$. Finally notice that $f_0(\zeta) = 1/\tilde{g}_0(1/\zeta) = 1/g_0(1/\zeta) = f(\zeta)$. Thus, f admits a Becker $k_0(q)$-q.c. extension to $\mathbb{C}$, where $k_0(q) := \max\{k_1(q), 3q\}$. The proof is complete. $\square$

References

[1] AHLFORS, L. V.: Sufficient conditions for quasiconformal extension. - In: Discontinuous groups and Riemann surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973), Ann. of Math. Studies, 79, Princeton Univ. Press, Princeton, NJ, 23–29.

[2] AHLFORS, L. V.: Lectures on quasiconformal mappings. Second edition. - Univ. Lecture Ser. 38, Amer. Math. Soc., Providence, RI, 2006.

[3] AHLFORS, L., and G. WEILL: A uniqueness theorem for Beltrami equations. - Proc. Amer. Math. Soc. 13, 1962, 975–978.

[4] BECKER, J.: Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen. - J. Reine Angew. Math. 255, 1972, 23–43.

[5] BECKER, J.: Über die Lösungsstruktur einer Differentialgleichung in der konformen Abbildung. - J. Reine Angew. Math. 285, 1976, 66–74.

[6] BECKER, J.: Conformal mappings with quasiconformal extensions. - In: Aspects of contemporary complex analysis (Univ. Durham, Durham, 1979), Proc. NATO Adv. Study Inst., Academic Press, London, 37–77.

[7] CONTRERAS, M. D., S. DÍAZ-MADRIGAL, and P. GUMENYUK: Local duality in Loewner equations. - J. Nonlinear Convex Anal. 15:2, 2014, 269–297.

[8] DUREN, P. L.: Univalent functions. - Grundlehren Math. Wiss. 259, Springer-Verlag, New York, 1983.

[9] GEHRING, F. W., and K. HAG: The ubiquitous quasidisk. - Math. Surveys Monogr. 184, Amer. Math. Soc., Providence, RI, 2012.

[10] GARNETT, J. B.: Bounded analytic functions. Revised first edition. - Grad. Texts in Math. 236, Springer, New York, 2007.

[11] GÖKTÜRK, Z.: Estimates for univalent functions with quasiconformal extensions. - Ann. Acad. Sci. Fenn. Ser. A I Math. 589, 1974, 1–21.

[12] GOLUSIN, G. M.: Zur Theorie der schlichten Funktionen. - Mat. Sb. N. S. 12:54, 1943, 48–55.

[13] GOLUZIN, G. M.: Geometric theory of functions of a complex variable. - Amer. Math. Soc., Providence, R.I.,1969.

[14] GUMENYUK, P., and I. HOTTA: Chordal Loewner chains with quasiconformal extensions. - Math. Z. 285:3-4, 2017, 1063–1089.

[15] GUMENYUK, P., and I. HOTTA: Univalent functions with quasiconformal extensions: Becker's class and estimates of the third coefficient. - Proc. Amer. Math. Soc. 148:9, 2020, 3927–3942.

[16] GUMENYUK, P., and I. PRAUSE: Quasiconformal extensions, Loewner chains, and the λ-lemma. - Anal. Math. Phys. 8:4, 2018, 621–635.

[17] GUTLJANSKIĬ, V. JA.: Parametric representation of univalent functions. - Dokl. Akad. Nauk SSSR 194, 1970, 750–753; English transl. in Soviet Math. Dokl. 11, 1970, 1273–1276.

[18] GUTLJANSKIĬ, V. JA.: Distortion theorems for univalent analytic functions with a quasiconformal extension. - Dokl. Akad. Nauk SSSR 240:3, 1978, 515–517.

[19] HOTTA, I.: Explicit quasiconformal extensions and Löwner chains. - Proc. Japan Acad. Ser. A Math. Sci. 85:8, 2009, 108–111.

[20] HOTTA, I.: Löwner chains with complex leading coefficient. - Monatsh. Math. 163:3, 2011, 315–325.

[21] HOTTA, I.: Loewner chains with quasiconformal extensions: an approximation approach. - J. Anal. Math. 143:1, 2021, 123–149.

[22] KRUSHKAL, S. L.: Univalent holomorphic functions with quasiconformal extensions (variational approach). - In: Handbook of complex analysis: geometric function theory. Vol. 2, Elsevier Sci. B. V., Amsterdam, 165–241.

[23] KÜHNAU, R.: Wertannahmeprobleme bei quasikonformen Abbildungen mit ortsabhängiger Dilatationsbeschränkung. - Math. Nachr. 40, 1969, 1–11.

[24] KÜHNAU, R.: Eine Verschärfung des Koebeschen Viertelsatzes für quasikonform fortsetzbare Abbildungen. - Ann. Acad. Sci. Fenn. Ser. A I Math. 1:1, 1975, 77–83.

[25] KÜHNAU, R., and W. NISKE: Abschätzung des dritten Koeffizienten bei den quasikonform fortsetzbaren schlichten Funktionen der Klasse S. - Math. Nachr. 78, 1977, 185–192.

[26] ŁAWRYNOWICZ, J.: Quasiconformal mappings in the plane. - Lecture Notes in Math. 978, Springer-Verlag, Berlin, 1983.

[27] LEHTO, O.: On univalent functions with quasiconformal extensions over the boundary. - J. Anal. Math. 30, 1976, 349–354.

[28] LEHTO, O.: Univalent functions and Teichmüller spaces. - Grad. Texts in Math. 109, Springer-Verlag, New York, 1987.

[29] LÖWNER, K.: Untersuchungen über schlichte konforme Abbildungen des Einheitskreises. I. - Math. Ann. 89:1-2, 1923, 103–121.

[30] POMMERENKE, CH.: Über die Subordination analytischer Funktionen. - J. Reine Angew. Math. 218, 1965, 159–173.

[31] POMMERENKE, CH.: Univalent functions. - Vandenhoeck & Ruprecht, Göttingen, 1975.

[32] POMMERENKE, CH.: On the Epstein univalence criterion. - Results Math. 10:1-2, 1986, 143–146.

[33] POMMERENKE, CH.: Boundary behaviour of conformal maps. - Grundlehren Math. Wiss. 299, Springer-Verlag, Berlin, 1992.

[34] RANGE, R. M.: On a Lipschitz estimate for conformal maps in the plane. - Proc. Amer. Math. Soc. 58, 1976, 375–376.

[35] RANSFORD, T.: Potential theory in the complex plane. - London Math. Soc. Stud. Texts 28, Cambridge Univ. Press, Cambridge, 1995.

[36] ROTH, O., and E. SCHIPPERS: The Loewner and Hadamard variations. - Illinois J. Math. 52:4, 2008, 1399–1415.

[37] SCHIPPERS, E.: Behaviour of kernel functions under homotopic variations of planar domains. - Comput. Methods Funct. Theory 4:2, 2004, 283–298.

[38] TAKHTAJAN, L. A., and L.-P. TEO: Weil–Petersson metric on the universal Teichmüller space. - Mem. Amer. Math. Soc. 183:861, 2006.

[39] WARSCHAWSKI, S. E.: On differentiability at the boundary in conformal mapping. - Proc. Amer. Math. Soc. 12, 1961, 614–620.

[40] ZYGMUND, A.: On the continuity modulus of the sum of the series conjugate to a Fourier series. (O module ciągłości sumy szeregu sprzężonego z szeregiem Fouriera.) - Prace mat.-fiz. 33, 1924, 125–132.

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