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There are no exotic ladder surfaces

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Abstract. It is an open problem to provide a characterization of quasiconformally homogeneous Riemann surfaces. We show that given the current literature, this problem can be broken into four open cases with respect to the topology of the underlying surface. The main result is a characterization in one of these open cases; in particular, we prove that every quasiconformally homogeneous ladder surface is quasiconformally equivalent to a regular cover of a closed surface (or, in other words, there are no exotic ladder surfaces).

Eksoottisia tikapuupintoja ei ole

Tiivistelmä. Kvasikonformisesti tasalaatuisten Riemannin pintojen kuvaileminen on avoin ongelma. Osoitamme, että olemassa olevia tuloksia käyttäen tämä kysymys voidaan hajottaa neljään avoimeen tapaukseen kyseessä olevan pinnan topologian suhteen. Päätuloksemme antaa yhdelle näistä avoimista tapauksista tarkan kuvailun; erityisesti osoitamme, että jokainen kvasikonformisesti tasalaatuinen tikapuupinta on kvasikonformisesti yhtäpitävä suljetun pinnan säännöllisen peitteen kanssa (eli, toisin sanoen, eksoottisia tikapuupintoja ei ole).

1. Introduction

A Riemann surface X is K-*quasiconformally homogeneous*, or K-QCH, if given any two points $x, y \in X$ there exists a K-quasiconformal homeomorphism $f\colon X \to X$ such that $f(x) = y$. We say a Riemann surface is *quasiconformally homogeneous*, or QCH, if it is K-QCH for some K (note: this definition diverges from the literature, where such a surface is usually referred to as *uniformly* quasiconformally homogeneous). For a survey of the work on QCH surfaces see [6].

For example, the Riemann sphere, the unit disk, and any Riemann surface whose universal cover is isomorphic to the complex plane are all 1-QCH, or *conformally homogeneous.* In fact, this is a complete characterization of conformally homogeneous Riemann surfaces, which leads us to the problem with which this paper is concerned:

Characterize all QCH Riemann surfaces.

Given the characterization of 1-QCH Riemann surfaces above, all the remaining cases to consider are hyperbolic Riemann surfaces (i.e. Riemann surfaces whose universal cover is isomorphic to the unit disk).

The starting point for such a characterization comes from higher dimensions. The notion of being K-QCH readily extends to the setting of hyperbolic manifolds of any dimension. In dimension at least three, it was shown in [4, Theorem 1.3] that a hyperbolic manifold is QCH if and only if it is a (geometric) regular cover of a closed hyperbolic orbifold. Naturally, such a result relies on rigidity phenomena in higher dimensions that do not occur in dimension two; in particular, as being

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QCH is invariant under quasiconformal deformations, it is not too difficult to find a hyperbolic QCH surface that does not regularly cover a closed hyperbolic orbifold (see [4, Lemma 5.1]).

This leads one to wonder—maybe naively—if every hyperbolic QCH surface is quasiconformally equivalent to a cover of a closed hyperbolic orbifold? Interestingly, this is not the case: in [5, Theorem 1.1] the existence of *quasiconformally exotic* QCH surfaces (i.e. QCH surfaces that are not quasiconformally equivalent to regular covers of closed orbifolds) is shown. However, all the exotic QCH surfaces constructed in [5] are homeomorphic; in particular, they are homeomorphic to the one-ended infinite-genus surface (affectionately referred to as the Loch Ness monster surface).

Our first theorem establishes that all QCH surfaces (and, in particular, exotic QCH surfaces) are topological regular covers of closed surfaces, or in other words, there are no *topologically exotic* QCH surfaces:

Theorem 5.1. *Every quasiconformally homogeneous Riemann surface topologically covers a closed surface.*

Note that every closed Riemann surface is QCH (see [4, Proposition 2.4] for a bound), so in the characterization of all QCH surfaces it is only left to consider non-compact surfaces. As a corollary to Theorem 5.1, we see that, up to homeomorphism, there are only a finite of number cases to consider. In particular, combining Theorem 5.1 with the classification of non-simply connected, infinite sheeted, regular covers of closed surfaces (Proposition 5.2 below), we have:

Corollary 1.1. *Up to homeomorphism, there are six non-compact QCH Riemann surfaces, namely the plane, the annulus, the Cantor tree surface, the blooming Cantor tree surface, the Loch Ness monster surface, and the ladder surface*[1].

As an immediate consequence of Corollary 1.1, we can strengthen a result of Kwakkel–Markovic [13, Proposition 2.6]:

Corollary 1.2. *A Riemann surface of positive, finite genus is quasiconformally homogeneous if and only if it is closed.*

Consider the non-hyperbolic cases in Corollary 1.1: we know that (1) every Riemann surface homeomorphic to the plane is QCH and (2) that a Riemann surface homeomorphic to the annulus is QCH if and only if its universal cover is isomorphic to $\mathbb{C}$ (this follows from the discussion of 1-QCH surfaces, Theorem 5.1, and the fact that the fundamental group of a closed hyperbolic Riemann surface does not have a cyclic normal subgroup—this also follows from [4, Theorem 1.1]). This leaves only four topological cases to consider.

In this article, we give a characterization in one of the four cases: the ladder surface, that is, the two-ended infinite-genus surface with no planar ends. In this case, our main theorem shows that there are no exotic QCH ladder surfaces, yielding a complete classification of QCH ladder surfaces:

Theorem 4.1. *A hyperbolic ladder surface is quasiconformally homogeneous if and only if it is quasiconformally equivalent to a regular cover of a closed hyperbolic surface.*

Given Theorem 4.1, it is natural to ask if the distance (in the Teichmüller metric) of a K-QCH ladder surface from a regular cover can be explicitly bounded as a function of K. With this in mind, all of our proofs are written with the goal of

[1]This nomenclature is explained in Proposition 5.2

providing explicit bounds in terms of K for all constants that appear; however, we are unable to do this in one location, namely the constant A appearing in Lemma 4.3. It would be interesting to find such a bound.

The first step in the proof of Theorem 4.1 is to choose a distance-minimizing geodesic (that is, a proper embedding of $\mathbb{R}$ minimizing the distance between any two of its points); however, to do so, we need to know such a geodesic exists. Our final theorem provides a sufficient topological condition for such a geodesic to exist; in addition, we show that there can be no topological condition that is both necessary and sufficient for a distance-minimizing geodesic to exist.

Theorem 3.2. *Every non-compact hyperbolic Riemann surface with at least two topological ends contains a distance-minimizing geodesic. Moreover, if an orientable, non-compact topological surface has a unique end, then it admits complete hyperbolic structures containing distance-minimizing geodesics as well as complete hyperbolic structures that do not admit such geodesics.*

Despite the narrative arc of the results above, in what follows, the proofs of the theorems will appear in reverse order.

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2. Preliminaries

Every Riemann surface is Hausdorff, orientable, and second countable; hence we will require these attributes of all topological surfaces in this note.

2.1. Hyperbolic geometry. We mention some facts in hyperbolic geometry that will be used in the sequel. For more detailed information, see [7, 8, 9, 12, 15].

A homeomorphism $f\colon U \to V$ between domains U and V in $\mathbb{C}$ is K-quasiconformal if

$$\frac{1}{K}\,\mathrm{Mod}(A) \leq \mathrm{Mod}(f(A)) \leq K\,\mathrm{Mod}(A)$$

for any annulus A in U, where $\mathrm{Mod}(A)$ is the modulus of A, that is, the unique real positive number M such that A is isomorphic to $\{z \in \mathbb{C}\colon 1 < |z| < e^{2\pi M}\}$. To extend to Riemann surfaces, we say a homeomorphism $f\colon X \to Y$ is K-quasiconformal if the restriction to any chart is K-quasiconformal.

A Riemann surface X is *hyperbolic* if its universal cover is isomorphic to the unit disk $\mathbb{D}$. We can then realize X as the quotient of $\mathbb{D}$ by the action of a Fuchsian group Γ.

If we equip $\mathbb{D}$ with its unique Riemannian metric of constant curvature -1, then Γ acts on $\mathbb{D}$ by isometries and this metric descends to a metric on X, which we will generally denote by ρ. Given a closed geodesic γ in a hyperbolic Riemann surface X, we let $\ell_X(\gamma)$ denote its length in (X, ρ).

We first recall some basic geometric properties of quasiconformal maps. Before doing so, we require some notation. Given two compact subsets C_1 and C_2 in a metric space (M, d), let $d(C_1, C_2)$ denote the distance between the two subsets, that is,

$$d(C_1, C_2) = \min\{d(x, y) \colon x \in C_1, y \in C_2\},$$

and let $H(C_1, C_2)$ denote the *Hausdorff distance* between C_1 and C_2, that is,

$$H(C_1, C_2) = \max\left\{\sup_{x\in C_1}\inf_{y\in C_2} d(x, y), \sup_{y\in C_2}\inf_{x\in C_1} d(y, x)\right\}.$$

Finally, given two metric spaces (M_1, d_1) and (M_2, d_2), a surjection $f\colon M \to N$ is an (A, B)-quasi-isometry if

$$\frac{1}{A}d_1(x, y) - B \leq d_2(f(x), f(y)) \leq Ad_1(x, y) + B$$

for all $x, y \in M_1$.

Lemma 2.1. *Let Z be a hyperbolic Riemann surface, let γ be a simple closed geodesic in Z, let $f\colon Z \to Z$ be a K-quasiconformal homeomorphism, and let δ be the geodesic homotopic to $f(\gamma)$. Then,*

(i) $\frac{1}{K}\ell_Z(\gamma) \leq \ell_Z(\delta) \leq K\ell_Z(\gamma)$,
(ii) *f is a $(K, K\log 4)$-quasi-isometry, and*
(iii) *there exists a constant R depending only on K so that $H(f(\gamma), \delta) < R$.*

Throughout our arguments, we will require the use of the collar lemma:

Theorem 2.2. *Let X be a hyperbolic Riemann surface and let $\eta\colon \mathbb{R} \to \mathbb{R}$ be given by*

$$\eta(\ell) = \operatorname{arcsinh}\left(\frac{1}{\sinh(\ell/2)}\right).$$

If γ_1 and γ_2 are disjoint simple closed geodesics of length ℓ_1 and ℓ_2, respectively, then the $\eta(\ell_i)$-neighborhood of γ_i, that is, the set

$$A_{\eta(\ell_i)}(\gamma_i) = \{x \in X \colon \rho(x, \gamma) < \eta(\ell)\}$$

is embedded in X and $A_{\eta(\ell_1)}(\gamma_1) \cap A_{\eta(\ell_2)}(\gamma_2) = \emptyset$.

We end with a special property of compact hyperbolic surfaces with totally geodesic boundary—that is, a compact surface arising as the quotient of a countable intersection of pairwise-disjoint closed half planes in $\mathbb{D}$ by the action of a Fuchsian group. In a hyperbolic surface with totally geodesic boundary, an *orthogeodesic* is a geodesic arc whose end points meet the boundary of the surface orthogonally.

A *pants decomposition* of a topological surface is a collection of pairwise-disjoint simple closed curves, called the *cuffs*, such that each complementary component of their union is homeomorphic to a thrice-punctured sphere. It follows from the classification of surfaces (see Theorem 2.4 below) that every orientable topological surface with non-abelian fundamental group has a pants decomposition. Moreover, in a compact hyperbolic surface (possibly with boundary), there always exists a pants decomposition in which the curves are of bounded length, with the bound depending only on the topology of the surface and the length of its boundary:

Theorem 2.3. (Bers pants decomposition theorem) *Given positive real numbers A and L, there exists a positive real number B—depending only on A and L—such* *that every compact hyperbolic surface with totally geodesic boundary whose boundary length is less than L and whose area is less than A admits a pants decomposition with cuff lengths bounded above by B.*

2.2. Topological ends. The notion of an end of a topological space was introduced by Freudenthal and, in essence, encodes the topologically distinct "directions" of going to infinity in a non-compact space.

More formally, for a non-compact second-countable surface S, fix an exhaustion $\{K_n\}_{n\in\mathbb{N}}$ of S by compact sets so that for each $n \in \mathbb{N}$, K_n lies in the interior of K_{n+1} and such that each component of the complement of K_n is unbounded. We then define a *(topological) end* of S to be a sequence $e = \{U_n\}_{n\in\mathbb{N}}$, where U_n is a complementary component of K_n and $U_n \supset U_{n+1}$.

The *space of ends* of S, denoted $\mathcal{E}(S)$, is the set of ends of S equipped with the topology generated by sets of the form $\hat{U}_n = \{e \in \mathcal{E}(S)\colon U_n \in e\}$. It is an exercise to check that, up to homeomorphism, the definition of $\mathcal{E}(S)$ given does not depend on the choice of compact exhaustion. We will say that an open subset V of S with compact boundary is a *neighborhood* of an end $e = \{U_n\}_{n\in\mathbb{N}}$ if there exists $N \in \mathbb{N}$ such that $U_N \subset V$. We say a sequence $\{x_m\}_{m\in\mathbb{N}}$ in S converges to an end $e = \{U_n\} \in \mathcal{E}(S)$ if, for each $n \in \mathbb{N}$, only finitely many of the x_m are contained in the set $S \smallsetminus U_n$.

We say an end $e = \{U_n\}_{n\in\mathbb{N}}$ is *planar* if, for some $N \in \mathbb{N}$, U_N is planar (i.e. homeomorphic to a subset of $\mathbb{R}^2$). We denote the set of non-planar ends by $\mathcal{E}_{np}(S)$, which is a closed subset of $\mathcal{E}(S)$. Note that $\mathcal{E}_{np}(S)$ is non-empty if and only if S has infinite genus.

Theorem 2.4. (Classification of surfaces (see [14])) *Two orientable surfaces without boundary, S_1 and S_2, of the same (possibly infinite) genus are homeomorphic if and only if there is a homeomorphism $\mathcal{E}(S_1) \to \mathcal{E}(S_2)$ sending $\mathcal{E}_{np}(S_1)$ onto $\mathcal{E}_{np}(S_2)$.*

3. Distance-minimizing geodesics and rays

In a hyperbolic surface X, a *distance-minimizing geodesic* is a unit-speed geodesic curve $\alpha\colon \mathbb{R} \to X$ such that $d_X(\gamma(a), \gamma(b)) = |b - a|$ for all $a, b \in \mathbb{R}$. Recall that a map is *proper* if the inverse image of a compact set is compact.

Lemma 3.1. *Every distance-minimizing geodesic in a hyperbolic Riemann surface is proper.*

Proof. Let X be a hyperbolic Riemann surface and let $\alpha\colon \mathbb{R} \to X$ be a continuous non-proper map. Then, there exists a compact set K such that $\alpha^{-1}(K)$ is closed and not compact; in particular, $\alpha^{-1}(K)$ is unbounded while K is bounded. Hence, α cannot be a distance-minimizing geodesic. □

As an easy consequence of Lemma 3.1, no compact hyperbolic Riemann surface can have a distance-minimizing geodesic. In Theorem 3.2, we give a topologically sufficient condition for distance-minimizing geodesics to exist in a non-compact hyperbolic Riemann surface; however, in addition, we see that there cannot be a necessary topological condition for the existence of such a geodesic.

Theorem 3.2. *Every non-compact hyperbolic Riemann surface with at least two topological ends contains a distance-minimizing geodesic. Moreover, if an orientable, non-compact non-planar topological surface has a unique end, then it admits complete hyperbolic structures containing distance-minimizing geodesics as well as complete hyperbolic structures that do not admit such geodesics.*

We split the proof into three lemmas covering the separate cases. Let us first consider the multi-ended case.

Lemma 3.3. *Let X be a hyperbolic Riemann surface. If X has at least two topological ends, then X contains a distance-minimizing geodesic with distinct ends.*

Proof. Let e_1 and e_2 be distinct topological ends of X and let η be a separating, simple, closed geodesic separating e_1 and e_2. Label the two components of $X \smallsetminus \eta$ by U_1 and U_2 so that U_i is a neighborhood of e_i. For $i \in \{1, 2\}$, choose a sequence $\{x_n^i\}_{n\in\mathbb{N}}$ in U_i such that $\lim x_n^i = e_i$. Observe that any minimal-length geodesic between x_n^1 and x_n^2 must intersect η exactly once. Let $\gamma_n \colon I_n \to X$ be the minimal-length unit-speed geodesic curve between x_n^1 and x_n^2 parameterized so that $\gamma_n(0) \in \eta$. By the compactness of the lift of η to the unit tangent bundle of X, the sequence $\{\gamma_n'(0)\}_{n\in\mathbb{N}}$ accumulates, and by passing to a subsequence we can assume it converges; let v be the limit. Let $\alpha\colon \mathbb{R} \to X$ be the unit-speed geodesic satisfying $\alpha'(0) = v$.

Now, by convergence in the unit tangent bundle, we have that for each $\epsilon > 0$ and $T > 0$, there exists $N \in \mathbb{N}$ such that $\gamma_n|_{[-T,T]}$ is contained in the ϵ-neighborhood of $\alpha(\mathbb{R})$ for all $n > N$. From this, together with the fact that the γ_n are distance-minimizing geodesic curves, we can deduce that α is proper (in a similar fashion as in Lemma 3.1). It follows that there exists (not necessarily distinct) ends e^+ and e^- of X such that $\lim_{t\to\pm\infty} \rho(t) = e^{\pm}$. But, since the x_n^i enter every neighborhood of e_i, the same must be true of $\alpha(\mathbb{R})$, and hence we can conclude that $\{e^+, e^-\} = \{e_1, e_2\}$.

We claim α is a distance-minimizing geodesic: assume not and let w and z be points on α such that there exists a distance-minimizing path δ of length strictly less than that of the segment of α connecting w and z. Let β denote the segment of α connecting w and z and let $\Delta = \ell(\beta) - \ell(\delta)$. Choose a positive real number ϵ so that $\epsilon < \Delta$ and such that the $\frac{\epsilon}{2}$-neighborhood Q of β (that is, $Q = \{x \in X \colon \rho(x, \beta) < \frac{\epsilon}{2}\}$) is isometric to the $\frac{\epsilon}{2}$-neighborhood of a geodesic segment in $\mathbb{H}$ of length $\ell(\beta)$. Let η_w and η_z be the geodesic segments of length $\frac{\epsilon}{2}$ that are orthogonal to β and pass through w and z, respectively, and note that both are embedded in Q.

Now there exists $N \in \mathbb{N}$ such that $\gamma_N \cap Q$ is connected and intersects each of η_w and η_z in a single point, which we label w_N and z_N, respectively. If δ_w and δ_z are the shortest curves in Q connecting w_N to w and z_N to z, respectively, then, as $\delta_w \cup \delta \cup \delta_z$ is a path connecting w_N and z_N, it follows that

$$\ell(\delta_w) + \ell(\delta) + \ell(\delta_z) > \ell(\gamma_N \cap Q) > \ell(\beta),$$

where the first inequality follows from the fact that γ_N is distance minimizing and the second follows from β being the orthogonal connecting the geodesic sides of Q. But, at the same time, we have

$$\ell(\delta_w) + \ell(\delta) + \ell(\delta_z) < \ell(\delta) + \epsilon < \ell(\delta) + \Delta = \ell(\beta).$$

However, this is a contradiction as both inequalities cannot hold. □

We now move to the one-ended case. For Lemma 3.4 and Lemma 3.5 below, we remind the reader that, up to homeomorphism, there is a unique one-ended, orientable surface whose end is non-planar, namely, the Loch Ness monster surface.

Lemma 3.4. *If S is a non-planar, one-ended, orientable surface, then there exists a hyperbolic Riemann surface X homeomorphic to S that does not contain a distance-minimizing geodesic.*

Proof. There are two cases: either the end of S is planar or not. Since S has positive genus, if the end of S is planar, we can choose a hyperbolic Riemann surface

X homeomorphic to S in which the end of S corresponds to a cusp on X. Let $\alpha\colon \mathbb{R} \to X$ be a continuous function. If α fails to be proper, then it is not distance minimizing by Lemma 3.1, so we may assume that α is proper. In this case, the two unbounded components of the intersection of α with a cusp neighborhood become arbitrarily close; hence, α cannot be a distance-minimizing geodesic.

Now suppose that the end of S is non-planar. Let c_1 be any separating, simple closed curve in S. We inductively build a sequence of disjoint, separating, simple closed curves $\{c_n\}_{n\in\mathbb{N}}$ by requiring that c_{n+1} separates c_n from the end of S. Let X be a hyperbolic Riemann surface such that there exists $L > 0$ so that the length of the geodesic representative γ_n of c_n has length less than L.

Now let $\alpha\colon \mathbb{R} \to X$ be a geodesic in X; as before, we may assume that α is proper. By the choice of the c_n, there exists $N \in \mathbb{N}$ such that $\alpha \cap \gamma_N \neq \varnothing$. Let $M \in \mathbb{N}$ such that $M > N + L/(2\eta(L)) + 1$. Then, by the collar lemma (Theorem 2.2), $\rho(\gamma_M, \gamma_N) > 2(M - N - 1)\eta(L) > L$. By construction, there must be a subsegment β of α with endpoints on γ_M that passes through γ_N. However, there is a geodesic segment in γ_M connecting the endpoints of β of length at most $\frac{L}{2}$; hence, α cannot be a distance-minimizing geodesic. □

Lemma 3.5. *If S is a borderless one-ended, orientable surface, then there exists a hyperbolic Riemann surface X homeomorphic to S containing a distance-minimizing geodesic.*

Proof. Again we split into two cases: first, suppose that the end of S is planar. In this case, let X be a hyperbolic Riemann surface homeomorphic to S such that the end of S corresponds to a funnel on X. All funnels have distance-minimizing geodesics.

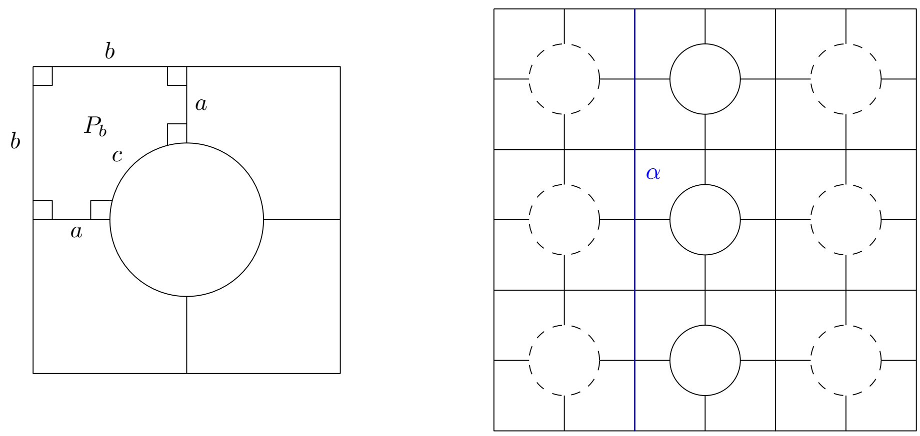


Figure 1. On the left, P is a hyperbolic right-angled pentagon and four copies of P are glued to form a square with a disk removed. On the right, these 1-holed squares are glued in a tiling extending in all directions; the extension of the vertical geodesic α forms a proper geodesic arc.

Let us continue to the case where S has a non-planar end. Given a positive number b, there exists a unique right-angled hyperbolic pentagon P_b having two consecutive sides of length b; let the consecutive sides of P_b have lengths b, b, a, c, a as in Figure 1. We can glue four copies of P_b to form a 1-holed squared with outer boundary having length $8b$ and inner boundary $4c$.

We now build a bordered hyperbolic surface inductively: let T_b^1 be a copy of the 1-holed square above. For $n \in \mathbb{N}$, we construct T_b^{n+1} by pasting eight copies of T_b^n to form a rectangle with 9^{n-1} holes. We identify T_b^n with the middle copy of T_b^n in

T_b^{n+1}. We then let T_b be the direct limit of the T_b^n. (Less formally, we obtain T_b by tiling the plane with copies of the 1-holed square, see Figure 1.)

Let R_b be the hyperbolic Riemann surface obtained by identifying the boundary components of T_b horizontally via an orientation-reversing isometry, so that a (dashed) circle is identified with the (solid) circle to its right in Figure 1. Note that R_b is infinite genus and one-ended and hence homeomorphic to S.

Now let α be a vertical geodesic as in Figure 1. We claim that α is a distance-minimizing geodesic in R_b. It is enough to prove that α minimizes distances between the corners of 1-holed squares for which it passes. Let x and y be two such corners and let γ a distance-minimizing path between them. By construction, the shortest path from one side of a 1-holed square to any other is at least $2b$. It follows that distance between two infinite horizontal geodesics in T_b is exactly $2b$. The same is true in R_b as the gluing of boundary components does not change height. Now if α crosses through $n-1$ horizontal geodesics from x to y, then γ must do the same and in particular the length of γ is at least $2nb$, which of course is the length of the segment of α connecting x and y. $\square$

Of course the difficulty in the one-ended case is that a proper arc needs to approach the unique end of the surface in both the forwards and backwards directions. To capture this, we prove the existence of a distance-minimizing ray. Here, a *ray* is the image of a continuous injective map of the half line $[0,\infty) \subset \mathbb{R}$.

Proposition 3.6. *Every point on a non-compact hyperbolic Riemann surface is the base point of some distance-minimizing ray.*

Proof. We provide the sketch of the proof as the details are nearly identical to those in the proof of Lemma 3.3. Let X be a non-compact hyperbolic Riemann surface and let e be a topological end of X. Fix a sequence $\{x_n\}_{n\in\mathbb{N}}$ that limits to e. Now let x be a point in X and, for $n \in \mathbb{N}$, let γ_n denote a unit-speed, minimal-length, geodesic curve starting at x and ending at x_n. Let $v_n = \gamma_n'(0)$, then we may choose a unit vector v in the accumulation set of the sequence $\{v_n\}_{n\in\mathbb{N}}$. Arguing as in Lemma 3.3, the geodesic ray based at x determined by v is distance minimizing. $\square$

4. QCH ladder surfaces are regular covers

In this section, we prove our main theorem:

Theorem 4.1. *A hyperbolic ladder surface is quasiconformally homogeneous if and only if it is quasiconformally equivalent to a regular cover of a closed hyperbolic surface.*

It is not difficult to see that being QCH is a quasi-conformal invariant and that every regular cover of a closed hyperbolic surface is QCH (see [4, Proposition 2.7]); hence, to prove Theorem 4.1, we only need to focus on the forwards direction.

The proof will be split into the lemmas in the subsections below. Throughout the subsections below X denotes a K-QCH ladder surface. Let $\mathcal{F}_K$ be the set of K-quasiconformal homeomorphisms $X \to X$. We say that a simple closed curve in X *separates the ends* of X if its complement consists of two unbounded components.

4.1. Shiga pants decomposition. A *Shiga pants decomposition* of a hyperbolic Riemann surface is a pants decompositions whose cuff lengths are uniformly bounded from above. The goal of this subsection is to show that every QCH ladder surface has a Shiga pants decomposition. It seems natural to expect a QCH surface to have such a pants decomposition, but, in fact, this is not always the case. For example, the surface R_b constructed in the proof of Lemma 3.5 is QCH (it is a regular cover of a closed hyperbolic surface), but does not have a Shiga pants decomposition [3].

The first step in the proof is to find a sequence of pairwise-disjoint simple closed geodesics that separate the ends of X, that are of uniformly bounded length, and that are "evenly" spaced throughout the surface (Lemma 4.2). We then show that the subsurfaces in the complement of these curves have bounded topology (this will follow from Lemma 4.3); the existence of a Shiga pants decomposition for X will follow by taking a Bers pants decomposition for each of these complementary subsurfaces.

Two real-valued functions $f(x)$ and $g(x)$ are said to be *comparable*, denoted $f \asymp g$, if there exists positive constants A and B so that $A \leq \frac{f(x)}{g(x)} \leq B$, for all x.

Lemma 4.2. *There exists a sequence of pairwise-disjoint simple separating geodesics $\{\gamma_n\}_{n\in\mathbb{Z}}$ such that each γ_n separates the ends of X and so that*

(1) $$\rho(\gamma_n, \gamma_m) \asymp H(\gamma_n, \gamma_m) \asymp |m - n|$$

for all $n, m \in \mathbb{Z}$. Moreover, the constants in the comparisons depend only on K and $L = \ell_X(\gamma_0)$.

Proof. Choose any simple closed geodesic separating the ends of X and label it γ_0. Set $L = \ell_X(\gamma_0)$. As X is two ended, by Lemma 3.3, we may choose a distance-minimizing geodesic β on X with distinct ends. Identify β with a unit-speed parameterization $\beta\colon \mathbb{R} \to X$ such that $\beta(0) \in \gamma_0$; set $x_0 = \beta(0)$.

Let $R = R(K)$ be as in Lemma 2.1 and, for $n \in \mathbb{Z}$, let $x_n = \beta(3n(R + KL))$. As X is K-QCH, we may choose $f_n \in \mathcal{F}_K$ such that $f_n(x_0) = x_n$. Finally, let γ_n be the geodesic in the homotopy class of $f_n(\gamma_0)$. (Recall $H(\gamma_n, f_n(\gamma_0)) \leq R$.)

We claim that the sequence $\{\gamma_n\}_{n\in\mathbb{Z}}$ has the desired properties. To see this, first observe, for every $n \in \mathbb{Z}$, that $\ell_X(\gamma_n) \leq KL$ and γ_n separates the ends of X. Next, we compute the distance between γ_n and γ_{n+1}. Observe that we can construct a path from x_n to x_{n+1} of length less than $\frac{KL}{2} + R + \rho(\gamma_n, \gamma_{n+1}) + R + \frac{KL}{2}$, which must have length at least $3(R + KL)$ as β is distance minimizing; hence,

$$\rho(\gamma_n, \gamma_{n+1}) \geq 3(R + KL) - 2R - KL = R + 2KL.$$

Regarding an upper bound, we have

$$\rho(\gamma_n, \gamma_{n+1}) \leq \rho(x_n, x_{n+1}) + 2R = 3(R + KL) + 2R = 5R + 3KL,$$

where the first inequality uses $H(f(\gamma_n), \gamma_n) < R$.

Assume $m > n$ and recall that any path from γ_n to γ_m must pass through γ_k for all $n < k < m$ and hence

(2) $$\rho(\gamma_n, \gamma_m) \geq \sum_{k=n}^{m-1} \rho(\gamma_k, \gamma_{k+1}) \geq (m - n)(R + 2KL).$$

It follows that $\rho(\gamma_n, \gamma_m) > 0$; in particular, $\gamma_n \cap \gamma_m = \emptyset$ for all distinct $n, m \in \mathbb{Z}$. Now, as γ_n and γ_m are disjoint, $\rho(\gamma_n, \gamma_m)$ is realized by an orthogeodesic between them. For the upper bound, using the fact that the orthogeodesic from γ_n to γ_m is shorter than the piecewise-continuous curve made up of orthogeodesics between successive γ_k and arcs along the γ_k we have

(3) $$\rho(\gamma_n, \gamma_m) \leq \sum_{k=n}^{m-1} \left[\rho(\gamma_k, \gamma_{k+1}) + \frac{\ell_X(\gamma_k)}{2}\right] \leq (m - n)\left[(5R + 3KL) + \frac{KL}{2}\right]$$

where the last inequality uses the fact that $\ell_X(\gamma_k) \leq KL$. Combining (2) and (3), we have shown

$$(R + 2KL) \leq \frac{\rho(\gamma_n, \gamma_m)}{|m-n|} \leq \left(5R + \frac{7KL}{2}\right) \tag{4}$$

implying $\rho(\gamma_n, \gamma_m) \asymp |n-m|$.

To show that $\rho(\gamma_n, \gamma_m)$ is comparable to $H(\gamma_n, \gamma_m)$, we first consider the following inequality:

$$\rho(\gamma_n, \gamma_m) \leq H(\gamma_n, \gamma_m) \leq \ell_X(\gamma_n)/2 + \rho(\gamma_n, \gamma_m) + \ell_X(\gamma_m)/2 \leq KL + \rho(\gamma_n, \gamma_m) \tag{5}$$

Dividing the above inequality by $\rho(\gamma_n, \gamma_m)$ we obtain

$$1 \leq \frac{H(\gamma_n, \gamma_m)}{\rho(\gamma_n, \gamma_m)} \leq \frac{KL}{\rho(\gamma_n, \gamma_m)} + 1 \tag{6}$$

Whenever $n \neq m$, we have by (2) that $\rho(\gamma_n, \gamma_m) \geq R + 2KL$, and hence,

$$1 \leq \frac{H(\gamma_n, \gamma_m)}{\rho(\gamma_n, \gamma_m)} \leq \frac{KL}{R + 2KL} + 1. \tag{7}$$

This finishes the proof of the lemma. □

We remark that putting together lines (4) and (7) yields the concrete comparison:

$$(R + 2KL) \leq \frac{H(\gamma_n, \gamma_m)}{|m-n|} \leq \left(\frac{KL}{2\eta(\frac{L}{K})} + 1\right)\left(5R + \frac{7KL}{2}\right) \tag{8}$$

Lemma 4.3. *Let $\{\gamma_n\}_{n\in\mathbb{N}}$ be the sequence of geodesics constructed in Lemma 4.2 and let Y_n be the compact subsurface co-bounded by γ_n and γ_{n+1}. There exists a positive real number A—depending on X—such that the area of Y_n is at most A.*

Proof. Denote the lower bound of (4) by a and the upper bound of (8) by b. Let $C = K \log 4$ and let $m, n \in \mathbb{N}$, then, as $f \in \mathcal{F}_K$ is a (K, C)-quasi-isometry,

$$\rho(f(z), f(w)) \geq \frac{1}{K}\rho(z, w) - C \geq \frac{1}{K}\rho(\gamma_n, \gamma_m) - C$$

for all $z \in \gamma_n$ and $w \in \gamma_m$. It follows that

$$\rho(f(\gamma_n), f(\gamma_m)) \geq \frac{1}{K}\rho(\gamma_n, \gamma_m) - C \tag{9}$$

for all $n, m \in \mathbb{Z}$.

Choose $m \in \mathbb{N}$ satisfying $m > \frac{K}{a}(b + C + R)$, where R is as in Lemma 2.1, and consider the geodesic subsurface Z bounded by the geodesics γ_{-m} and γ_m. We set A to be the area of Z.

Let $f_n \in \mathcal{F}_k$ be as defined as in the proof of Lemma 4.2. We claim that $Y_n \subset f_n(Z)$ for all $n \in \mathbb{Z}$. Before proving our claim, note that the area of $f_n(Z)$ is bounded above independent of $n \in \mathbb{Z}$—the bound only depends on K and the area of Z. This follows from [2], but we give a short (less precise) argument: let Z_n denote the subsurface with totally geodesic boundary that is homotopic to $f_n(Z)$, so that the area of Z_n agrees with that of Z. It must be that $f_n(Z)$ is in the R-neighborhood of Z_n. The area of the R-neighborhood of Z_n is bounded above by the area of Z_n and R. As R only depends on K, we see that the area of $f_n(Z)$ is bounded as a function of K and the area of Z.

Now to prove the claim, first note that using (9) we have:

$$\rho(\gamma_n, f_n(\gamma_m)) \geq \rho(f_n(\gamma_0), f_n(\gamma_m)) - R \geq \frac{1}{K}\rho(\gamma_0, \gamma_m) - C - R \geq \frac{ma}{K} - C - R > b, \tag{10}$$

where the last inequality follows from replacing m with the assumed lower bound in the choice of m.

On the other hand, $H(\gamma_n, \gamma_{n+1}) < b$ and thus, $\rho(\gamma_n, f_n(\gamma_m)) > H(\gamma_n, \gamma_{n+1})$; in particular, $f_n(\gamma_m)$ must be disjoint from Y_n. Observe that (10) holds with m replaced by $-m$; hence, Y_n and $f_n(\gamma_{-m})$ are disjoint. Thus $Y_n \subset f_n(Z)$ and hence the area of Y_n is less than A. □

Remark 4.4. Note that m can be explicitly chosen to be a function of K and L.

A Bers pants decomposition of each Y_n together with $\{\gamma_n\}_{n\in\mathbb{Z}}$ yields a pants decomposition for X with bounded cuff lengths, establishing:

Proposition 4.5. *Every QCH ladder surface admits a Shiga pants decomposition.* □

Aside: coarse geometry. We take a short tangent from the proof of Theorem 4.1 to discuss the coarse geometry of QCH ladder surfaces.

Note that as X is K-QCH, there is a lower bound on the injectivity radius of X depending only on K [4, Theorem 1.1]. In particular, the diameter of Y_n is bounded above independent of n. Now let β be the distance-minimizing geodesic from Lemma 4.2. Define $r\colon X \to \beta$ by sending a point x of X to any point $y \in \beta$ satisfying $\rho(x, y) = \rho(x, \beta)$. It then follows that r is a quasi-isometry and hence β is a *quasi-retract* of X. Note that in the proof of Lemma 4.2, we could have chosen any distance-minimizing curve, establishing:

Proposition 4.6. *If X is a QCH ladder surface, then every distance-minimizing geodesic in X is a quasi-retract of X.* □

Corollary 4.7. *Every QCH ladder surface is quasi-isometric to $\mathbb{R}$ equipped with the standard Euclidean metric.* □

Let us show that Proposition 4.5 and Proposition 4.6 do not characterize the property of being QCH amongst hyperbolic ladder surfaces. First, we note again that there exist QCH surfaces that do not admit a Shiga pants decomposition.

Proposition 4.8. *There exists a hyperbolic ladder surface R and distance-minimizing geodesic β in R such that β is a quasi-retract of R, R admits a Shiga pants decomposition, and R is not QCH.*

Proof. Let S be a topological ladder surface and let $\{a_n, b_n, c_n\}_{n\in\mathbb{Z}}$ be a pants decomposition for S as in Figure 2. Let R a hyperbolic surface such that $\ell(a_n) = \ell(b_n) = 1$ and $\ell(c_n) = \left|\frac{1}{n}\right|$ for all $n \in \mathbb{Z}$, and such that the geodesic arc in the homotopy class of β is distance minimizing.

By construction, R has a Shiga pants decomposition and moreover the nearest point projection $r\colon R \to \beta$ is a quasi-isometry. However, the injectivity radius of R goes to 0 and hence R cannot be QCH. □

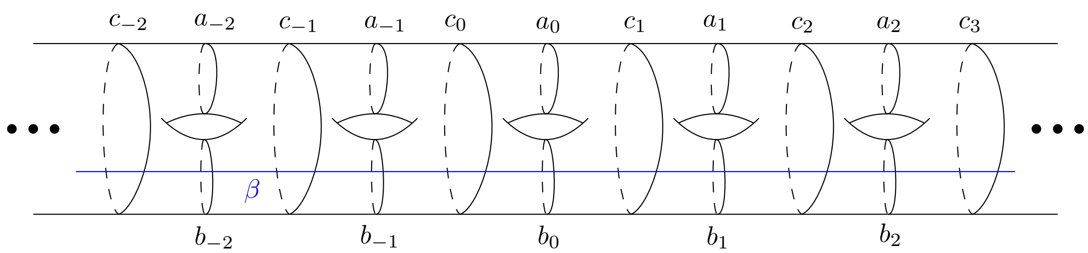
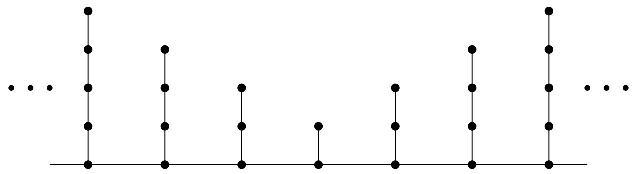


Figure 2. A pants decomposition for a (topological) ladder surface.

Proposition 4.9. *There exists a hyperbolic ladder surface with a Shiga pants decomposition that is not quasi-isometric to $\mathbb{Z}$ (and hence not QCH).*

Proof. Let Γ be the two-ended graph shown here:



For $i \in \{1, 2, 3\}$, let V_i be a hyperbolic i-holed torus with each boundary component of length 1. Let R be a hyperbolic surface obtained from Γ by taking a copy of V_i for each valence i vertex and identifying boundary components according to the edge relations in Γ. The resulting surface R is quasi-isometric to Γ; in particular, it is a ladder surface with a Shiga pants decomposition. However, Γ and hence R is not quasi-isometric to $\mathbb{Z}$. □

These propositions leads us to the follow question, which we end the aside with:

Question 4.10. If a hyperbolic ladder surface has positive injectivity radius and is quasi-isometric to $\mathbb{R}$, then is it QCH?

4.2. Preferred Shiga pants decomposition. In the previous section, we showed that X admits a Shiga pants decomposition; however, this is just an existence statement and does not give us enough information to directly construct the desired covering map. The goal of this subsection is to modify the Shiga pants decomposition from Proposition 4.5 into a (topological) form we can use to build a deck transformation (the desired form is shown in Figure 2).

It is not difficult to show the existence of the desired pants decomposition using a continuity and compactness argument in moduli space; however, it is not possible to extract explicit length bounds from such an argument. With a little extra effort, we proceed in a fashion allowing for effective constants—this is the content of Lemma 4.11.

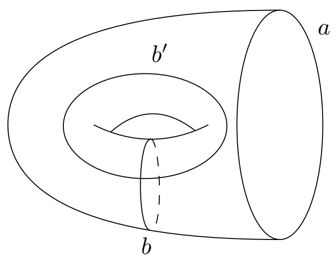


Figure 3. The elementary move up to homeomorphism in a once-punctured torus switching b and b'.

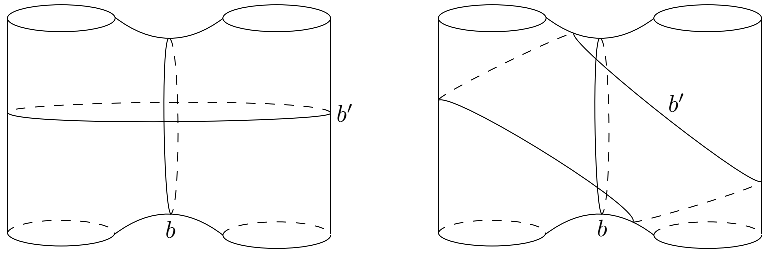


Figure 4. The two possible types of elementary moves up to homeomorphism in a 4-holed sphere.

Let Σ be a compact surface with non-abelian fundamental group. The *pants graph* associated to Σ, written $\mathcal{P}(\Sigma)$, is the graph whose vertices correspond to pants decompositions of Σ (up to isotopy) and where two vertices are adjacent if they differ by an elementary move. An elementary move corresponds to removing a single curve α from the pants decomposition and replacing α with a curve that is disjoint from all remaining curves of the pants decomposition and intersecting α minimally (see Figures 3 and 4).

Define an equivalence relation $\sim$ on the vertices of $\mathcal{P}(\Sigma)$ by setting two pants decompositions to be equivalent if they differ by a homeomorphism of Σ. The *modular pants graph*, written $\mathcal{MP}(\Sigma)$, is the graph whose vertices correspond to equivalence classes of pants decompositions of Σ; two vertices are connected by an edge if they have representatives in $\mathcal{P}(\Sigma)$ that are adjacent.

As $\mathcal{P}(\Sigma)$ is connected [10], we have that $\mathcal{MP}(\Sigma)$ is connected; moreover, $\mathcal{MP}(\Sigma)$ has finitely many vertices and hence finite diameter (in the graph metric). Observe that, up to homeomorphism, there are at most two ways to replace a single curve in a given pants decomposition; in particular, none of the edges in the modular pants graph correspond to the elementary move shown in Figure 3.

Let σ be a hyperbolic metric on Σ. Given a vertex $v \in \mathcal{MP}(\Sigma)$, define

$$M_\sigma(v) = \min\{M \colon \text{there exists } \{c_1, \ldots, c_\xi\} \in \mathcal{P}(\Sigma) \text{ such that } [\{c_1, \ldots, c_\xi\}] = v \text{ and } \ell_\sigma(c_i) \leq M \text{ for all } i \in \{1, \ldots, \xi\}\},$$

where $\xi = \xi(\Sigma) = 3g - 3 + b$ is the *topological complexity* of Σ (g is the genus of Σ and b the number of boundary components of Σ).

Lemma 4.11. *Let Σ be a compact surface possibly with boundary and with $\xi(\Sigma) > 0$. Let σ be a hyperbolic metric on Σ with injectivity radius m, let $v \in \mathcal{MP}(\Sigma)$, and let $M \in \mathbb{R}$ such that $M_\sigma(v) \leq M$. Then, for all $w \in \mathcal{MP}(\Sigma)$, $M_\sigma(w)$ is bounded above by a function of ξ, m, and M.*

Proof. Let P be a pair of pants representing v with cuff lengths all bounded above by M. If ℓ is the length of any orthogeodesic connecting a boundary component to itself, then

$$\sinh(\ell/2) \leq \frac{\cosh(M/2)}{\sinh(m/2)}$$

(this follows from standard formulas for right-angled hyperbolic pentagons, see [8, Formula 2.3.4(1)]).

Now choose a representative pants decomposition $\tilde{v} = \{c_1, \ldots, c_\xi\}$ for v such that $\ell_\sigma(c_i) \leq M$ for all $i \in \{c_1, \ldots, c_\xi\}$. Let $w \in \mathcal{MP}(\Sigma)$ be adjacent to v. Up to relabelling, we can assume that there exists a representative $\tilde{w} = \{c_1', c_2, \ldots, c_\xi\}$ of w adjacent to $\tilde{v}$.

Let R denote the 4-holed sphere component of $\Sigma \smallsetminus \bigcup_{i=2}^{\xi} c_i$ and let P_1 and P_2 be the two pairs of pants in R sharing c_1 as a common boundary component. Let α_1 and α_2 be the orthogeodesics in P_1 and P_2, respectively, connecting c_1 to itself. Up to Dehn twisting about c_1, there exists a curve homotopic to c_1' obtained by the taking the union of α_1, α_2, and two subarcs of c_1. It follows that

$$M_\sigma(w) \leq M + \operatorname{arccosh}\left(\frac{\cosh(\mathrm{M}/2)}{\sinh(\mathrm{m}/2)}\right).$$

The result now follows by the fact that $\mathcal{MP}(\Sigma)$ is connected with finite diameter, which only depends on ξ. □

Let S be a ladder surface and fix a pants decomposition $\mathcal{P} = \{a_k, b_k, c_k\}_{k\in\mathbb{Z}}$ as in Figure 2.

Lemma 4.12. *If X is a QCH ladder surface, then there exists a homeomorphism $f\colon S \to X$ such that $f(\mathcal{P})$ is a Shiga pants decomposition for X.*

Proof. Let $\{\gamma_n\}_{n\in\mathbb{Z}}$ be the collection of curves guaranteed by Lemma 4.2. By Lemma 4.3, the complexity of the surface bounded by γ_n and γ_{n+1}, denoted Y_n, is bounded. By Proposition 4.5, this guarantees the existence of a Shiga pants decomposition for X containing the collection of curves $\{\gamma_n\}$; let M be an upper bound for the lengths of the cuffs in this decomposition. Before continuing, we recall that there is a lower bound on the injectivity radius of any K-QCH surface, which only depends on K [4, Theorem 1.1]; so, let $K > 1$ such that X is K-QCH and let $m = m(K)$ denote this lower bound.

Fix a homeomorphism $f\colon S \to X$ such that for every $n \in \mathbb{Z}$ there exists $k_n \in \mathbb{Z}$ with $f(c_{k_n}) = \gamma_n$. Then, $\mathcal{P}_n = f(\mathcal{P}) \cap Y_n$ is a pants decomposition of Y_n. As the Y_n have bounded area and hence bounded topological complexity,

$$\xi = \max\{\xi(Y_n)\colon\ n \in \mathbb{Z}\},$$

is finite. Therefore, by Lemma 4.11, $M_{Y_n}([\mathcal{P}_n])$ is bounded above by a function of ξ, m, and M. In particular, by pre-composing f with a homeomorphism of S, we may assume that $f(\mathcal{P})$ is a Shiga pants decomposition for X. □

4.3. Proof of Theorem 4.1. We can now prove every QCH ladder surface is quasiconformally equivalent to a regular cover of a closed surface, finishing the proof of Theorem 4.1. Before giving the proof, we recall a definition from Teichmüller theory.

Let S be a (topological) ladder surface and let $\mathcal{P} = \{a_k, b_k, c_k\}_{k\in\mathbb{Z}}$ be the pants decomposition of S given in Figure 2. Given a hyperbolic surface Z and a homeomorphism $h\colon S \to Z$, define the Fenchel–Nielsen coordinates of the marked surface (S, h) be the collection of sextuplets

$$\mathrm{FN}((S,h)) = \{[\ell_Z(h(a_k)), \theta_{a_k}(h), \ell_Z(h(b_k)), \theta_{b_k}(h), \ell_Z(h(c_k)), \theta_{c_k}(h)]\}_{k\in\mathbb{Z}},$$

where $\theta_{a_k}, \theta_{b_k}$, and θ_{c_k} are the twist parameters associated to the curves a_k, b_k, and c_k, respectively, with respect to a collection of seams $\{d_k\}_{k\in\mathbb{Z}}$ as in Figure 5. The twist parameters are given as an angle (as opposed to an arc length).

Proof of Theorem 4.1. Let X be a QCH ladder surface and let $f\colon S \to X$ be the homeomorphism given by Lemma 4.12, so that $f(\mathcal{P})$ is a Shiga pants decomposition for X. By precomposing f with a (possibly infinite) product of Dehn twists about the cuffs of $\mathcal{P}$, we may assume that the twist parameters for $X = (S, f)$ are between 0 and 2π.

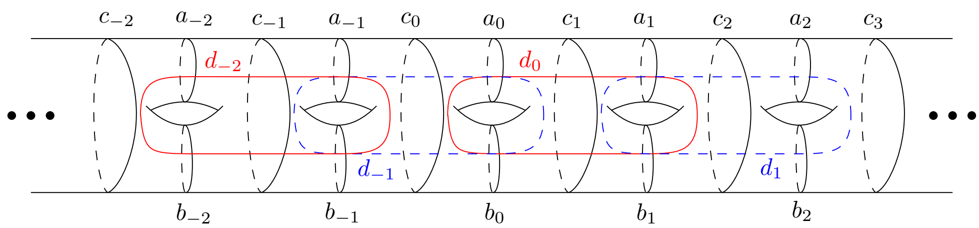


Figure 5. The pants decomposition $\mathcal{P}$ along with the seams $\{d_k\}_{k\in\mathbb{Z}}$ determine Fenchel–Nielsen coordinates for hyperbolic structures on S.

Now fix the marked surface $(S, h\colon S \to Z)$ such that

$$\mathrm{FN}((S,h)) = \{[1,0,1,0,1,0]\}_{k\in\mathbb{Z}}.$$

Since the cuff lengths and twist parameters of $f(\mathcal{P})$ are bounded from above and below, we can conclude that the map $h \circ f^{-1}\colon X \to Z$ is quasiconformal [1, Theorem 8.10].

To finish, we show that Z is a regular cover of a closed genus-3 hyperbolic surface: let $\tau\colon S \to S$ be the horizontal translation determined, up to isotopy, by requiring that

$$\tau((a_k, b_k, c_k, d_k)) = (a_{k+2}, b_{k+2}, c_{k+2}, d_{k+2})$$

for all $k \in \mathbb{Z}$. Observe that

$$\mathrm{FN}((S, h\circ\tau^{-1})) = \mathrm{FN}((S,h))$$

and hence $\tau^h = h \circ \tau \circ h^{-1}\colon Z \to Z$ is isotopic to an isometry of Z. It follows that $\langle\tau^h\rangle\backslash Z$ is a closed hyperbolic genus-3 surface. □

5. Topology of QCH surfaces

The goal of this section is to prove that there are no topologically exotic QCH surfaces, that is:

Theorem 5.1. *Every QCH surface is homeomorphic to a regular cover of a closed surface.*

Before proving Theorem 5.1, we need to understand the topology of a regular cover of a closed surface. The proposition below is stronger than we require, but with little extra work we state a more complete picture. Also, recall that, for the purpose of this article, surfaces are orientable and second countable.

Proposition 5.2. *A regular cover of a closed surface is either compact or homeomorphic to one of the following six surfaces:*

(1) $\mathbb{R}^2$,
(2) $\mathbb{R}^2 \smallsetminus \mathbf{0}$,
(3) *the Cantor tree surface, i.e. the planar surface whose space of ends is a Cantor space,*
(4) *the blooming Cantor tree surface, i.e. the infinite-genus surface with no planar ends and whose space of ends is a Cantor space,*
(5) *the Loch Ness Monster surface, i.e. the one-ended infinite-genus surface, or*
(6) *the ladder surface, i.e. the two-ended infinite-genus surface with no planar ends.*

Moreover, the torus is the only closed surface regularly covered by $\mathbb{R}^2 \smallsetminus \mathbf{0}$.

Proof. Let B be a closed surface and let $\pi\colon S \to B$ be a regular cover. It is not difficult to show that the end space of a regular cover of a closed manifold is either empty, discrete with 1 or 2 points, or a Cantor space (this is a classical theorem of Hopf [11]). If the end space of S is empty, then S is compact; in the other cases, S is non-compact. If S is non-planar, then the co-compactness of the action of the deck group associated to π on S will guarantee that every end of S is non-planar. Therefore, either S is compact; S is planar with 1, 2, or a Cantor space of ends; or S is infinite genus with 1, 2, or a Cantor space of ends, all of which are non-planar. Using the classification of surfaces, we see that—up to homeomorphism—there are only six non-compact surfaces that meet these criteria, namely the ones listed above.

We leave it as an exercise to show that, with the exception of $\mathbb{R}^2 \smallsetminus \mathbf{0}$, each of the listed surfaces covers a closed genus-2 surface.

Finally, $\mathbb{R}^2 \smallsetminus \mathbf{0}$ is a regular cover of the torus; moreover, no surface of genus at least two can be regularly covered by $\mathbb{R}^2 \smallsetminus \mathbf{0}$: indeed, $\pi_1(\mathbb{R}^2 \smallsetminus \mathbf{0})$ is cyclic and the fundamental group of a hyperbolic surface cannot have a normal cyclic subgroup. □

We can now prove Theorem 5.1.

Proof of Theorem 5.1. Let X be a K-QCH surface. Further, for the sake of arguing by contradiction, assume that X is not a regular cover of a closed surface. If X is closed, then it is trivially a regular cover of a closed surface, namely itself. So, we may assume that X is non-compact. Note that under these assumptions, X is necessarily hyperbolic.

First, assume that X has positive (possibly infinite) genus and has at least one planar end. We can then choose a non-planar compact subsurface Y of X and an unbounded planar subsurface U of X such that ∂U is compact. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in U such that every compact subset of X contains only finitely many of the x_n. Fix $x \in Y$ and let $f_n \colon X \to X$ be a K-quasiconformal map such that $f_n(x) = x_n$. Note that as f_n is a $(K, K\log 4)$-quasi-isometry (see Lemma 2.1), the diameter of $f_n(Y)$ is bounded as a function of K and the diameter of Y. In particular, as $\rho(x_n, \partial Y) \to \infty$ as $n \to \infty$, it must be that $f_n(Y) \subset U$ for large n, but this is impossible as every subsurface of a planar surface is planar.

We can now conclude that X is either (i) planar or (ii) has infinite genus and no planar ends. In either case, as we are assuming X is not a regular cover of a closed surface, by Proposition 5.2, X has at least three ends, one of which is isolated, call it e. (Note: if the end space does not contain an isolated point, then it is necessarily a Cantor space.) Let P be a compact subsurface in X with three boundary components, each of which is separating, and such that each component of $X \smallsetminus P$ is unbounded and such that there exists a component U of $X \smallsetminus P$ with U a neighborhood of e. The argument now proceeds nearly identically to the previous case. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in U such that every compact subset of X contains only finitely many of the x_n. Fix $x \in P$ and let $f_n \colon X \to X$ be a K-quasiconformal map such that $f_n(x) = x_n$. Again, as f_n is a $(K, K\log 4)$-quasi-isometry, the diameter of $f_n(P)$ is bounded as a function of K and the diameter of P. In particular, $\rho(x_n, P) \to \infty$ as $n \to \infty$, it must be that $f_n(P) \subset U$ for large n, but this is impossible as it would require $X \smallsetminus f_n(P)$ to have a bounded component. □

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