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Constructing uniform spaces

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Abstract. We exhibit geometric conditions that ensure a metric space is uniform.

Sisätieyhtenäisten avaruuksien rakentaminen

Tiivistelmä. Esitämme joukon geometrisia ehtoja, jotka takaavat metrisen avaruuden sisätieyhtenäisyyden.

1. Introduction

Throughout this section X is a rectifiably connected non-complete locally complete metric space. These are the minimal requirements for X to support a *quasihyperbolic distance* $k = k_X$, and we dub X a *quasihyperbolic metric space*; see Section 2 for precise definitions.

Roughly speaking, such an X is a *uniform metric space* provided each pair of points can be joined by a path that moves away from the boundary of X and whose length is comparable to the distance between the points. See §2.B for a precise definition.

The class of Euclidean uniform domains was introduced by Martio and Sarvas in [MS79] and has proven to be invaluable in geometric function theory, potential theory, geometric group theory, and especially for the "analysis in metric spaces" program; e.g., see [Geh87, Väi88, Jon81, Aik04, Aik06, BS07, CT95, CGN00, Gre01, BHK01, HSX08]. A finitely connected proper subdomain of the plane is uniform if and only if each boundary component is either a point or a quasicircle, but in general there are no such simple geometric criterion for uniformity.

Given their fundamental importance, it seems worthwhile to investigate two questions.

− *When can we "poke holes" in a uniform space and still have a uniform space?*
+ *When can we "fill in" some boundary points of a unifom space and keep uniformity?*

As a warm up, we have the following result, similar to [Her11, Prop. 2.3]. While surely not surprising to those well versed in uniform space theory, our discussion employs some tools that may not be well known. Again, see §2.B for definitions.

Theorem A. *Let X be a quasihyperbolic metric space. Let $o \in X$ be a fixed point and put $X_\star := X \setminus \{o\}$. If $X_\star$ is $C_\star$-uniform, then for any $C > C_\star$*

$$X \text{ is } C\text{-uniform and } C_1\text{-annular quasiconvex at } o \tag{1.1}$$

where $C_1 = 2(C_\star + 1)$. Conversely, if (1.1) holds, then $X_\star$ is $C_\star$-uniform with $C_\star = C_\star(C, C_1)$.

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An open subspace Ω of X is *uniformly collared* provided X is uniform and there are disjointed open sets U_i such that

$$B := X \setminus \Omega = \bigcup_i B_i \quad \text{where } B_i := B \cap U_i$$

and such that each *collar* $\Omega_i := \Omega \cap U_i$ is a uniform space.[1] This terminology was introduced in the Euclidean setting by Astala and Heinonen in [AH88]; see also [HK91]. We say that Ω is *uniformly collared with fat collars* provided it is uniformly collared and there is a positive constant Φ such that

$$\frac{1}{\Phi}\,\mathsf{diam}(B_i) \le \mathsf{dist}(B_i, \partial X) \le \Phi\;\mathsf{dist}(B_i, \mathsf{bd}\,U_i). \tag{1.2}$$

Theorem B. *Let Ω be an open subspace of a C_0-uniform space X. Suppose Ω is uniformly collared with C_1-uniform Φ-fat collars. Then Ω is C-uniform with $C = C(C_0, C_1, \Phi)$.*

As an application of the above, we establish the following.

Theorem C. *Let X be a quasihyperbolic metric space. Let $\Omega := X \setminus A$ where $A \subset X$. Assume there is a constant $\kappa \in (0, 1)$ such that*

$$\forall\, a \ne b \text{ in } A, \quad k(a, b) \ge \kappa. \tag{1.3}$$

If Ω is C-uniform, then for any $C_0 > C$

$$X \text{ is } C_0\text{-uniform and } C_1\text{-annular quasiconvex at each } a \in A \tag{1.4}$$

with $C_1 = 2(C + 1)$. Conversely, if both (1.4) *and*

$$\forall\, a \in A, \quad \mathsf{B}\Big(a; \frac{\kappa}{4C_0}\,\mathsf{dist}(a, \partial X)\Big) \text{ is } C_2\text{-uniform} \tag{1.5}$$

hold, then Ω is C-uniform with $C = C(C_0, C_1, C_2, \kappa)$.

A special case of Theorem C, with $\kappa = 1/2$ and $A \subset X$ a countable subset of a Banach space[2] uniform domain X, was proved in [HVW17]. Theorems B and C were established in the Euclidean setting in [Her89]; see also [Her87].

We can replace (1.3) by the (*a priori* stronger, but here equivalent) condition that for some positive constant υ,

$$\forall\, a \ne b \text{ in } A, \quad |a - b| \ge \upsilon\,\mathsf{dist}(a, \partial X).$$

The condition (1.5) can be relaxed, e.g., to asking that there exist an $\varepsilon \in (0, \kappa/4C_0]$ such that for each $a \in A$ there is a C_2-uniform ball $\mathsf{B}(a; r_a)$ with $r_a/\,\mathsf{dist}(a, \partial X) \in [\varepsilon, \kappa/4C_0]$.

Our results inspire some natural questions.

(A) When is a metric space annular quasiconvex?
(B) What properties of a metric space ensure that its balls are uniform spaces?
(C) Which properties of Euclidean uniformly collared spaces (e.g., see [HK91]) have metric space analogs?

Theorems A, B, C are established in §§3.A, 3.B, 3.C respectively.

[1] There should be a single uniformity constant for the collars.

[2] Banach spaces are annular quasiconvex at each point and balls are 2-uniform.

2. Metric space definitions

Our notation is relatively standard. We write $C = C(D, \dots)$ to indicate a constant C that depends only on the data $D, \dots$. For real numbers r and s,

$$r \wedge s := \min\{r, s\} \quad \text{and} \quad r \vee s := \max\{r, s\}.$$

2.A. Metric space notation and terminology. Throughout this section X is an arbitrary metric space with distance denoted $|x-y|$; this is not meant to imply that X possesses any sort of linear or group structure. In this setting, all topological notions refer to the metric topology; here $\mathsf{cl}(A), \mathsf{bd}(A), \mathsf{int}(A)$ are the topological closure, boundary, interior (respectively) of $A \subset X$.

The open ball, sphere, closed ball of radius r centered at the point $a \in X$ are

$$\mathsf{B}(a;r) := \{\, x\colon \; |\, x-a| < r \,\}, \quad \mathsf{S}(a;r) := \{\, x\colon \; |\, x-a| = r \,\},$$
$$\mathsf{B}[a;r] := \mathsf{B}(a;r) \cup \mathsf{S}(a;r).$$

The closed annular ring centered at a with inner radius r and outer radius s is

$$\mathsf{A}[a;r,s] := \mathsf{B}[a;s] \setminus \mathsf{B}(a;r) = \{\, x\colon r \le |x-a| \le s \,\}.$$

Recall that every metric space can be isometrically embedded into a complete metric space. We let $\bar{X}$ denote the metric completion of the metric space X; thus $\bar{X}$ is the closure of the image of X under such an isometric embedding. We call $\partial X := \bar{X} \setminus X$ the metric boundary of X. When X is non-complete, $\delta(x) = \delta_X(x) := \mathsf{dist}(x, \partial X)$ is the distance from a point $x \in X$ to the boundary ∂X of X. Note that ∂X is closed in $\bar{X}$ if and only if $\delta(x) > 0$ for all $x \in X$; e.g., this holds when X is locally compact.

When $A \subset X$, there is a natural embedding $\bar{A} \hookrightarrow \bar{X}$ and $\mathsf{bd}(A) \subset \partial A$. Here if $A \subset X$ is open and X complete, then $\partial A = \mathsf{bd}(A)$, but in general $\bar{A} = \overline{\mathsf{cl}}(A)$ and $\partial A = \overline{\mathsf{bd}}(A) \setminus A$ where $\overline{\mathsf{cl}}$ and $\overline{\mathsf{bd}}$ denote topological closure and boundary in $\bar{X}$

A metric space X is *locally complete* provided each point has an open neighborhood which is complete. When X is non-complete, this is equivalent to requiring that $\delta(x) > 0$ for all $x \in X$, or, ∂X is closed in $\bar{X}$, or, X is open in $\bar{X}$.

2.A.1. Paths, arcs, & length. A *path in* X is a continuous map $\mathsf{R} \supset I \xrightarrow{\gamma} X$ where $I = I_\gamma$ is an interval (called the *parameter interval for* γ) that may be closed or open or neither and finite or infinite. The *trajectory* of such a path γ is $|\gamma| := \gamma(I)$ which we call a *curve* and often denote by just γ. When I is closed and $I \neq \mathsf{R}$, $\partial\gamma := \gamma(\partial I)$ denotes the set of *endpoints of* γ and consists of one or two points depending on whether or not I is compact. For example, if $I_\gamma = [u, v] \subset \mathsf{R}$, then $\partial\gamma = \{\gamma(u), \gamma(v)\}$. When $\partial\gamma = \{a, b\}$, we write $\gamma\colon a \curvearrowright b$ (*in* X) to indicate that γ is a path (in X) with *initial point* a and *terminal point* b; this notation is meant to imply an orientation—a precedes b on γ.

We call γ a *compact path* if its parameter interval I is compact. An *arc* α is an injective compact path. Given points $a, b \in |\alpha|$, there are unique $u, v \in I$ with $\alpha(u) = a$, $\alpha(v) = b$ and we write $\alpha[a, b] := \alpha|_{[u,v]}$. We also use this notation for a general path γ, but here $\gamma[a, b]$ denotes the unique injective subpath of γ that joins a, b obtained by using the last time γ is at a up to the first time γ is at b.

When $\alpha\colon a \curvearrowright b$ and $\beta\colon b \curvearrowright c$ are paths that join a to b and b to c respectively, $\alpha \star \beta$ denotes the concatenation of α and β; so $\alpha \star \beta\colon a \curvearrowright c$. Of course, $|\alpha \star \beta| = |\alpha| \cup |\beta|$. Also, the *reverse of* γ is the path $\tilde{\gamma}$ defined by $\tilde{\gamma}(t) := \gamma(1-t)$ (when $I_\gamma = [0, 1]$) and going from $\gamma(1)$ to $\gamma(0)$. Of course, $|\tilde{\gamma}| = |\gamma|$.

Every compact path contains an arc with the same endpoints; see [Väi94].

The length of a compact path $[0,1] \xrightarrow{\gamma} X$ is defined in the usual way by

$$\ell(\gamma) := \sup \left\{ \sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})| \,\middle|\, 0 = t_0 < t_1 < \cdots < t_n = 1 \right\},$$

γ is *rectifiable* when $\ell(\gamma) < \infty$, and X is *rectifiably connected* provided each pair of points in X can be joined by a rectifiable path. Every rectifiable path can be parametrized with respect to its arclength [Väi71, p.5]. When γ is a rectifiable path, we tacitly assume its parameter interval is $I_\gamma = [0, \ell(\gamma)]$ unless specifically stated otherwise.

Every rectifiably connected metric space X admits a natural *intrinsic* distance, its so-called *(inner) length distance* given by

$$l(a,b) := \inf\{\ell(\gamma) \mid \gamma \colon a \curvearrowright b \text{ a rectifiable path in } X\}.$$

A metric space $(X, |\cdot|)$ is a *length* space provided for all points $a, b \in X$, $|a-b| = l(a,b)$; it is also common to call such a $|\cdot|$ an *intrinsic* distance function. Notice that an l-geodesic $[x,y]_l$ is a shortest curve joining x and y.

There are two useful properties of length spaces that we use repeatedly. First, for any open set U in a length space X, we always have $\mathsf{dist}(x, \mathsf{bd}\, U) = \mathsf{dist}(x, X \setminus U)$ for all points $x \in U$. Second, $\bar{X}$ is also a length space. In fact, for all $x \in X, \xi \in \partial X, \varepsilon > 0$ there is a path $\gamma \colon x \curvearrowright \xi$ in $X \cup \{\xi\}$ with $\ell(\gamma) < |x - \xi| + \varepsilon$.

2.A.2. Quasihyperbolic distance. Recall that X is a *quasihyperbolic metric space* if it is non-complete, locally complete, and rectifiably connected. In such a space, $\delta(x) = \delta_X(x) := \mathsf{dist}(x, \partial X) > 0$ for all $x \in X$, so $\delta^{-1} ds$ is a conformal metric that we call the *quasihyperbolic metric* on X. The length distance induced by the quasihyperbolic metric $\delta^{-1} ds$ is dubbed the *quasihyperbolic distance* $k = k_X$ in X. In a locally compact quasihyperbolic space, this is a geodesic distance: there are always k-geodesics joining any two points in X.

The following basic estimates for quasihyperbolic distance were first established for Euclidean domains by Gehring and Palka [GP76, 2.1]. For all $a, b \in X$ and any rectifiable path $\gamma \colon a \curvearrowright b$ in X

$$\text{(2.1a)} \qquad k(a,b) \geq \log\left(1 + \frac{l(a,b)}{\delta(a) \wedge \delta(b)}\right) \geq \log\left(1 + \frac{|a-b|}{\delta(a) \wedge \delta(b)}\right) \geq \left|\log \frac{\delta(a)}{\delta(b)}\right|$$

which is a special case of the more general inequality

$$\text{(2.1b)} \qquad \ell_k(\gamma) \geq \log\left(1 + \frac{\ell(\gamma)}{\mathsf{dist}(|\gamma|, \partial X)}\right).$$

2.B. Quasiconvex, annular quasiconvex, and uniform spaces. A rectifiable path $\gamma \colon a \curvearrowright b$ is *C-quasiconvex*, $C \geq 1$, if its length is at most C times the distance between its endpoints; i.e., if γ satisfies

$$\ell(\gamma) \leq C\,|a-b|.$$

A metric space is *C-quasiconvex* if each pair of points can be joined by a C-quasiconvex path. A 1-quasiconvex metric space is a *geodesic* space, and a space is a length space if and only if it is C-quasiconvex for all $C > 1$. By cutting out loops, we can always replace a C-quasiconvex path with a C-quasiconvex arc having the same endpoints; see [Väi94].

The inequalities in (2.1a) yield the following 'local' estimates for quasihyperbolic distances.

2.2. Fact. Let X be a C-quasiconvex quasihyperbolic metric space. Then for all $x, a \in X$,

$$k(x,a) \le 1 \text{ or } \frac{|x-a|}{\delta(a)} \le \frac{1}{2C} \implies \frac{1}{2}\frac{|x-a|}{\delta(a)} \le k(x,a) \le 2C\,\frac{|x-a|}{\delta(a)}.$$

2.B.1. Annular quasiconvexity. A metric space X is *C-annular quasiconvex at $p \in X$* provided it is connected and for all $r > 0$, points in $\mathsf{A}[p; r, 2r]$ can be joined by C-quasiconvex paths lying in $\mathsf{A}[p; r/C, 2Cr]$. We call X *C-annular quasiconvex* if it is C-annular quasiconvex at each point. Examples of quasiconvex and annular quasiconvex metric spaces include Banach spaces and upper regular Loewner spaces; the latter includes Carnot groups and certain Riemannian manifolds with non-negative Ricci curvature; see [HK98, 3.13,3.18, Section 6]. Korte [Kor07] proved that doubling metric measure spaces that support a $(1,p)$-Poincaré inequality with sufficiently small p are annular quasiconvex.

To the best of our knowledge, the notion of annular quasiconvexity was introduced in [Kor07] and [BHX08]; it was an essential ingredient in [HSX08]. A similar concept was employed in [Mac10].

2.B.2. Uniformity. Roughly speaking, a metric space is *uniform* when points in it can be joined by paths that are not "too long" and "move away" from the region's boundary. More precisely, a quasihyperbolic metric space X is *C-uniform* (for some constant $C \ge 1$) provided each pair of points can be joined by a C-uniform arc. Here a rectifiable $\gamma\colon a \curvearrowright b$ is a *C-uniform arc* if and only if it is both a C-quasiconvex arc and a *double C-cone arc*; this latter condition means that

$$\forall\, x \in |\gamma|, \quad \ell(\gamma[x,a]) \wedge \ell(\gamma[x,b]) \le C\delta(x). \tag{2.3}$$

Double cone arcs are often called *cigar arcs.* In [Väi88] Väisälä provides a description of various possible double cone conditions (which he calls *length cigars, diameter cigars, distance cigars,* and *Möbius cigars*). The work [Mar80] of Martio should also be mentioned.

To simplify an argument, we prevail upon the following characterization for uniform spaces established in [Her11, Prop. C].

2.4. Fact. A quasihyperbolic metric space is uniform if and only if it is plump and proximate points can be joined by uniform arcs. More precisely, if X is C-plump and $3C$-proximate points can be joined by B-uniform arcs, then X is $18B^2C$-uniform; conversely, if X is C-uniform, then it is $4C$-plump.

Two points x, y are *C-proximate*, for some constant $C > 0$, if $|x-y| \le C[\delta(x) \wedge \delta(y)]$. If this holds, then also $(C+1)^{-1} \le \delta(x)/\delta(y) \le C+1$. A non-complete locally complete metric space U is *C-plump*, $C \ge 1$, provided for each $x \in U$ and all $r \in (0, \mathsf{diam}\, U)$

$$\exists\, z \in \mathsf{B}[x; r] \text{ with } \mathsf{dist}(z, \partial U) \ge r/C\,. \tag{2.5}$$

This terminology was introduced by Väisälä in [Väi88] and perhaps is understood best when U is an open subspace of a length space X, for then (2.5) asserts that $\mathsf{dist}(z, X \setminus U) \ge r/C$, so the ball $\mathsf{B}(z; r/C)$, in X, is contained in U.

3. Proofs

Here we establish Theorems A, B, C as stated in the Introduction. In each of these, X is a given quasihyperbolic metric space.

3.A. Proof of Theorem A. Recall that $o \in X$ and $X_\star := X \setminus \{o\}$. We let $\delta_\star$ denote distance to $\partial X_\star$, so $\delta_\star(x) := |x| \wedge \delta(x)$ where $|x| := |x-o|$.

To utilize Fact 2.4, we first verify the following.

3.1. Lemma. *Suppose X is C-uniform. Then $X_\star := X \setminus \{o\}$ is $12C$-plump.*

Proof. Let $a \in X_\star$ and $r \in (0, \operatorname{diam} X_\star)$. We seek a point $z \in \mathsf{B}[a;r]$ with $\delta_\star(z) \ge r/12C$.

Pick $b \in X_\star$ with $|a-b| \ge \frac{1}{2}\operatorname{diam}(X_\star)$. Let $\gamma\colon a \curvearrowright b$ be a C-uniform arc in X. Let z_0 be the arclength midpoint of γ. Then

$$\delta(z_0) \ge \frac{\ell(\gamma)}{2C} \ge \frac{|a-b|}{2C} \ge \frac{r}{4C}.$$

Assume $|z_0-a| \le r/2$. If $|z_0| \ge r/12C$, then z_0 is the sought after point. Suppose $|z_0| < r/12C$. By examining paths from z_0 to ∂X, we obtain a point $z_1 \in \mathsf{S}(z_0; r/6C)$. It follows that $\delta_\star(z_1) \ge r/12C$, and that

$$|z_1-a| \le |z_1-z_0| + |z_0-a| \le \frac{r}{6C} + \frac{r}{2} \le r,$$

so z_1 is the sought after point.

Assume $|z_0-a| > r/2$. Pick $z_2 \in \gamma[z_0,a] \cap \mathsf{S}(a;r/2)$. Then $\delta(z_2) \ge r/2C$. Thus if $|z_2| \ge r/6C$, then z_2 is the sought after point. Suppose $|z_2| < r/6C$. By examining paths from z_2 to ∂X, we obtain a point $z_3 \in \mathsf{S}(z_2; r/3C)$. It follows that $\delta_\star(z_3) \ge r/6C$, and that

$$|z_3-a| \le |z_3-z_2| + |z_2-a| = \frac{r}{3C} + \frac{r}{2} \le r,$$

so z_3 is the sought after point. □

Now we establish Theorem A. When $X_\star$ is $C_\star$-uniform, it is not hard to check that X is C-uniform for any $C > C_\star$ (this also follows from Theorem C) and the proof of (c) $\implies$ (a) in [Her11, Prop. 2.3] shows that X is $2(C_\star+1)$-annular quasiconvex at o.

For the converse, assume X is C-uniform and C_1-annular quasiconvex at o. By Lemma 3.1 we know that $X_\star$ is $12C$-plump, so it suffices to show that there is a constant B such that $36C$-proximate points in $X_\star$ can be joined by B-uniform arcs; then Fact 2.4 asserts that $X_\star$ is $C_\star$-uniform with $C_\star = 216B^2C$.

Let $a, b \in X_\star$ be $36C$-proximate; so, $|a-b| \le 36C\big(\delta_\star(a) \wedge \delta_\star(b)\big)$. By relabeling, if necessary, we may assume $|a| \le |b|$. Now $|a-b| \le 36C|a|$. Let $\gamma_o\colon a \curvearrowright b$ be a C-uniform arc in X. Put $R := |a|/10CC_1$. Then $\ell(\gamma_o) \le C|a-b| \le 36C^2|a|$, so

$$R \ge \big(180C^3C_1\big)^{-1}\frac{\ell(\gamma_o)}{2}.$$

As $|b| \ge |a|$, $\{\, a,b \,\} \cap \mathsf{B}(o;|a|) = \emptyset$. Suppose $\gamma_o \cap \mathsf{B}(o;R) = \emptyset$. Then for each $x \in \gamma_o$, $|x| \ge R$ and we readily deduce that γ_o is a $180C^3C_1$-uniform arc in $X_\star$.

Assume $\gamma_o \cap \mathsf{B}(o;R) \ne \emptyset$. Let a_o, b_o be the first, last points (respectively) of γ_o in $\mathsf{S}(o;R)$. Put $\gamma := \alpha \star \sigma \star \beta$ where $\alpha := \gamma_o[a,a_o], \beta := \gamma_o[b_o,b]$ and where $\sigma\colon a_o \curvearrowright b_o$ is a C_1-quasiconvex arc in $\mathsf{A}[o; R/2C_1, C_1R]$.

Note that as α, β both join the spheres $\mathsf{S}(o;R), \mathsf{S}(o;|a|)$, they each have length at least $|a| - R = (10CC_1-1)R \ge 9CC_1R$. It follows that $\delta(a_o) \vee \delta(b_o) \ge 9C_1R$.

Evidently, $\ell(\gamma) \le C_1\ell(\gamma_o) \le CC_1|a-b|$, so to verify that γ is a uniform arc it remains to corroborate the double cone arc condition. To begin, let $x \in \sigma$. Then

$$|x| \ge \frac{R}{2C_1} \ge \big(360C^3C_1^2\big)^{-1}\frac{\ell(\gamma_o)}{2} \ge \big(360C^3C_1^3\big)^{-1}\frac{\ell(\gamma)}{2}.$$

Also,

$$9C_1R \le \delta(a_o) \le \delta(o) + |a_o| = \delta(o) + R$$

so

$$\delta(x) \ge \delta(o) - C_1R \ge \big(30C^3C_1\big)^{-1}\frac{\ell(\gamma)}{2}$$

and we see that the double cone condition holds for points in σ with constant $360C^3C_1^3$.

It remains to examine points $x \in \alpha\cup\beta$. Evidently, $|x| \ge R \ge (180C^3C_1^2)^{-1}\ell(\gamma)/2$. Let z_o be the arclength midpoint of γ_o; so, z_o lies in α or β or $\gamma_o \setminus \big(\alpha\cup\beta\big)$. If z_o lies in $\gamma_o \setminus \big(\alpha\cup\beta\big)$, then α, β are both "short subarcs of γ_o" and we readily see that

$$x \in \alpha \implies \ell(\gamma[x,a]) = \ell(\gamma_o[x,a]) \le C\delta(x)$$

and

$$x \in \beta \implies \ell(\gamma[x,b]) = \ell(\gamma_o[x,b]) \le C\delta(x)$$

and thus the double cone condition holds.

Suppose $z_o \in \beta$. Here α is again a "short subarc" and we precede exactly as above for points $x \in \alpha$. Assume $x \in \beta$. When $x \in \beta[z_o, b] = \gamma_o[z_o, b]$ we can–again–precede as above. So, assume $x \in \beta[b_o, z_o]$. Here

$$\delta(x) \ge \frac{\ell(\gamma_o[x,a])}{C} \ge \frac{\ell(\alpha)}{C} \ge 9C_1R \ge \big(20C^3C_1\big)^{-1}\frac{\ell(\gamma)}{2}.$$

Thus the double cone condition holds for points in $\alpha\cup\beta$ with constant $180C^3C_1^2$. It now follows that γ is a B-uniform arc with $B := 360C^3C_1^3$. □

3.B. Proof of Theorem B. Here we assume Ω is a uniformly collared subspace of a C_0-uniform space X with C_1-uniform Φ-fat[3] collars as described in the Introduction. Also, since the associated hypotheses and conclusions are all bi-Lipschitz invariant, we may and do assume that X is a length space.

To join points, we start with a uniform arc in X. If this arc gets near some B_i, we replace (by "cutting and pasting") an appropriate subarc with a uniform arc in Ω_i.

Before immersing ourselves in the details, we discuss some basic immediate properties. First, since the open sets U_i are disjointed, each B_i is closed and, e.g., $\mathsf{bd}(B) = \cup_i \mathsf{bd}(B_i)$. Also, while $\partial B \subset \partial X$ may be empty or non-empty, for each i,

$$d_i := \mathsf{dist}(B_i, \partial X) > 0.$$

It's not difficult to check that $\Omega_i := \Omega\cap U_i = U_i\setminus B_i$, and that $\mathsf{bd}(\Omega_i) = \mathsf{bd}(B_i)\cup\mathsf{bd}(U_i)$ and this is a disjoint union.

Next, as Ω is open in X and X is open in $\bar{X}$, we (eventually) see that

$$\partial\Omega = \mathsf{bd}(B) \cup \partial X \quad \text{(and this is a disjoint union)}$$

[3]We assume $\Phi \ge 2$.

from which we deduce that for each $x \in \Omega$,

$$\mathsf{dist}(x,\partial\Omega) = \mathsf{dist}(x,B) \wedge \mathsf{dist}(x,\partial X) = \inf_i \mathsf{dist}(x,B_i) \wedge \mathsf{dist}(x,\partial X).$$

It also follows that for each $x \in \Omega$:

(3.2a) If $\exists\, i$ with $\mathsf{dist}(x,B_i) \le d_i/2$, then $\mathsf{dist}(x,\partial\Omega) = \mathsf{dist}(x,B)$.

(3.2b) If $\exists\, i$ with $\mathsf{dist}(x,B_i) \le d_i/2\Phi$, then $\mathsf{dist}(x,\partial\Omega) = \mathsf{dist}(x,B_i)$.

(3.2c) If $x \in \Omega_i$, then $\mathsf{dist}(x,\partial\Omega) \ge \mathsf{dist}(x,\partial\Omega_i)$.

Now define

$$A_i := \left\{\, x \in X \setminus B_i \mid \mathsf{dist}(x,B_i) < \frac{d_i}{10\Phi} \,\right\} \quad \text{and} \quad A := \bigcup_i A_i.$$

From (3.2b) we see that

$$x \in \mathsf{cl}(A_i) \implies \mathsf{dist}(x,\partial\Omega) = \mathsf{dist}(x,B_i)$$

and similarly

$$\mathsf{dist}(x,\partial\Omega) = \mathsf{dist}(x,\partial X) \implies x \notin \mathsf{cl}(A).$$

Let $a,b \in \Omega$ and let $\gamma_o\colon a \curvearrowright b$ be a C_0-uniform arc in X. Suppose $\gamma_o \cap A = \emptyset$. Let $x \in \gamma_o$. Then for all i,

$$\begin{aligned}\mathsf{dist}(x,\partial X) &\le \mathsf{dist}(x,B_i) + \mathsf{diam}(B_i) + \mathsf{dist}(B_i,\partial X) \le \mathsf{dist}(x,B_i) + (\Phi+1)d_i \\ &\le \big(10\Phi(\Phi+1)+1\big)\,\mathsf{dist}(x,B_i) \le 20\Phi^2\,\mathsf{dist}(x,B_i),\end{aligned}$$

so $\mathsf{dist}(x,\partial\Omega) \ge (20\Phi^2)^{-1}\,\mathsf{dist}(x,\partial X)$ and we deduce that γ_o is a $20C_0\Phi^2$-uniform arc in Ω.

Suppose $\gamma_o \cap A \ne \emptyset$ and, for a moment, assume $\{\, a,b \,\} \cap A = \emptyset$. Let J denote the set of all indeces i with $\gamma_o \cap A_i \ne \emptyset$. For each $j \in J$: let a_j, b_j be the first, last points of γ_o in $\mathsf{bd}(A_j)$; let $\sigma_j\colon a_j \curvearrowright b_j$ be a C_1-uniform arc in Ω_j; and, replace each subarc $\gamma_o[a_j,b_j]$ with the corresponding σ_j.

If $a \in A$, say $a \in A_{i_o}$: let a_{i_o} be the last point of γ_o in $\mathsf{bd}(A_{i_o})$; let $\alpha\colon a \curvearrowright a_{i_o}$ be a C_1-uniform arc in Ω_{i_o}; and, replace the subarc $\gamma_o[a,a_{i_o}]$ with α. Similarly, if $b \in A_{j_o}$, we get a C_1-uniform $\beta\colon b_{j_o} \curvearrowright b$ in Ω_{j_o} that replaces $\gamma_o[b_{j_o},b]$, where b_{j_o} is the first point of γ_o in $\mathsf{bd}(A_{j_o})$.

We now have an arc $\gamma\colon a \curvearrowright b$ in Ω that has been obtained by replacing certain subarcs of γ_o with appropriate subarcs σ_j or α or β. As each of these new subarcs is C_1-quasiconvex, we see that

$$\ell(\gamma) \le C_1\ell(\gamma_o) \le C_0C_1|a-b|,$$

so γ is a C_0C_1-quasiconvex arc. It remains to verify the double cone condition.

Let $x \in \gamma$. Suppose $x \notin \alpha \cup \beta \cup \bigcup_j \sigma_j$. As above, where $\gamma_o \cap A = \emptyset$, we again see that $\mathsf{dist}(x,\partial\Omega) \ge \mathsf{dist}(x,\partial X)/20\Phi^2$ and the double cone condition holds with $C = 20C_0C_1\Phi^2$.

Suppose $x \in \alpha$. If $\ell(\gamma[x,a]) = \ell(\alpha[x,a]) \le \ell(\alpha[x,a_o])$, then $\mathsf{dist}(x,\partial\Omega) \ge \mathsf{dist}(x,\partial\Omega_{i_o}) \ge C_1^{-1}\ell(\gamma[x,a])$ and the double cone condition holds with constant $C = C_1$.

Assume $\ell(\alpha[x,a]) > \ell(\alpha[x,a_o])$. Now

$$\begin{aligned}\ell(\gamma[x,a]) &= \ell(\alpha[x,a]) \le \ell(\alpha) \le C_1|a-a_o| \\ &\le C_1\big(\mathsf{dist}(a,B_{i_o}) + \mathsf{diam}(B_{i_o}) + \mathsf{dist}(a_o,B_{i_o})\big) \\ &\le C_1(\Phi+1)d_{i_o} \le 2C_1\Phi d_{i_o}.\end{aligned}$$

If $\ell(\alpha[x,a_o]) \ge d_{i_o}/20\Phi$, then

$$\mathsf{dist}(x,\partial\Omega) \ge \mathsf{dist}(x,\partial\Omega_{i_o}) \ge C_1^{-1}\ell(\alpha[x,a_o]) \ge \frac{d_{i_o}}{20C_1\Phi} \ge \frac{\ell(\gamma[x,a])}{40C_1^2\Phi^2}.$$

On the other hand, if $\ell(\alpha[x,a_o]) < d_{i_o}/20\Phi$, then

$$\begin{aligned}\frac{d_{i_o}}{10\Phi} &= \mathsf{dist}(a_o,B_{i_o}) = \mathsf{dist}(a_o,\partial\Omega) \le |x-a_o| + \mathsf{dist}(x,\partial\Omega) \\ &\le \ell(\alpha[x,a_o]) + \mathsf{dist}(x,\partial\Omega)\end{aligned}$$

and so now

$$\mathsf{dist}(x,\partial\Omega) \ge \frac{d_{i_o}}{20\Phi} \ge \frac{\ell(\gamma[x,a])}{40C_1\Phi^2}.$$

Thus in all cases, when $x \in \alpha$, the double cone condition holds with constant $C = 40C_1^2\Phi^2$.

A similar argument applies if $x \in \beta$.

Finally, suppose $x \in \sigma_j$ for some $j \in J$. We demonstrate that

$$\ell(\gamma[x,a]) \wedge \ell(\gamma[x,b]) \le 3C_0C_1\Phi d_j \quad \text{and} \quad d_j \le 20C_1\Phi\,\mathsf{dist}(x,\partial\Omega)$$

which gives the double cone condition with constant $C = 60C_0C_1^2\Phi^2$.

First, if

$$\ell(\sigma_j[x,a_j]) \wedge \ell(\sigma_j[x,b_j]) \ge \frac{d_j}{20\Phi},$$

then $\mathsf{dist}(x,\partial\Omega) \ge \mathsf{dist}(x,\partial\Omega_j) \ge d_j/20C_1\Phi$. Suppose

$$\ell(\sigma_j[x,z]) < \frac{d_j}{20\Phi} \quad \text{for some } z \in \{\, a_j, b_j \,\}.$$

Then

$$\frac{d_j}{10\Phi} = \mathsf{dist}(z,B_j) = \mathsf{dist}(z,\partial\Omega) \le |x-z| + \mathsf{dist}(x,\partial\Omega) \le \ell(\sigma_j[x,z]) + \mathsf{dist}(x,\partial\Omega)$$

whence again $\mathsf{dist}(x,\partial\Omega) \ge d_j/20C_1\Phi$.

Next, $\ell(\gamma[x,a]) \le \ell(\sigma_j) + \ell(\gamma[a_j,a])$ and $\ell(\gamma[x,b]) \le \ell(\sigma_j) + \ell(\gamma[b_j,b])$, so

$$\begin{aligned}\ell(\gamma[x,a]) \wedge \ell(\gamma[x,b]) &\le \ell(\sigma_j) + \ell(\gamma[a_j,a]) \wedge \ell(\gamma[b_j,b]) \\ &\le \ell(\sigma_j) + C_1\big(\ell(\gamma_o[a_j,a]) \wedge \ell(\gamma_o[b_j,b])\big).\end{aligned}$$

Now

$$\ell(\sigma_j) \le C_1|a_j-b_j| \le C_1\big(\mathsf{dist}(a_j,B_j) + \mathsf{diam}(B_j) + \mathsf{dist}(b_j,B_j)\big) \le C_1\left(\Phi + \frac{1}{5\Phi}\right)d_j.$$

Also, for $z \in \{\, a_j, b_j \,\}$,

$$\mathsf{dist}(z,\partial X) \le \mathsf{dist}(z,B_j) + \mathsf{diam}(B_j) + \mathsf{dist}(B_j,\partial X) \le \left(\Phi + 1 + \frac{1}{10\Phi}\right)d_j.$$

If $\ell(\gamma_o[a_j, a]) \le \ell(\gamma_o[b_j, b])$, then

$$\ell(\gamma_o[a_j, a]) = \ell(\gamma_o[a_j, a]) \wedge \ell(\gamma_o[a_j, b]) \le C_0 \operatorname{dist}(a_j, \partial X)$$

and similarly if $\ell(\gamma_o[b_j, b]) \le \ell(\gamma_o[a_j, a])$, then

$$\ell(\gamma_o[b_j, b]) = \ell(\gamma_o[b_j, a]) \wedge \ell(\gamma_o[b_j, b]) \le C_0 \operatorname{dist}(b_j, \partial X).$$

Therefore

$$\ell(\gamma_o[a_j, a]) \wedge \ell(\gamma_o[b_j, b]) \le C_0 \left(\Phi + 1 + \frac{1}{10\Phi} \right) d_j.$$

It now follows that

$$\ell(\gamma[x, a]) \wedge \ell(\gamma[x, b]) \le C_1 \left(\Phi + \frac{1}{5\Phi} \right) d_j + C_0 C_1 \left(\Phi + 1 + \frac{1}{10\Phi} \right) d_j \le 3C_0 C_1 \Phi d_j.$$

Examining all our various constants we see that $\gamma\colon a \curvearrowright b$ is a C_0C_1-quasiconvex double $60C_0C_1^2\Phi^2$-cone arc in Ω. □

3.C. Proof of Theorem C. Here $\Omega = X \setminus A$ and (1.3) holds for some constant $\kappa \in (0, 1)$.

First, suppose Ω is C-uniform. The proof of (c) $\implies$ (a) in [Her11, Prop. 2.3] shows that X is $2(C + 1)$-annular quasiconvex at all points $a \in A$. Let $\varepsilon \in (0, 1)$. We verify that X is $(C + \varepsilon)$-uniform.

Since $A \subset \partial\Omega$, $X = \Omega \cup A$ is C'-quasiconvex for any $C' > C$ which permits use of Fact 2.2.

Let $a, b \in X$. Since we can join points in Ω with C-uniform arcs, we may assume $a, b \in A$; the case where one point lies in A and one in Ω is similar and easier. We select $a_o, b_o \in \Omega$ sufficiently near a, b, pick quasiconvex arcs $\alpha\colon a \curvearrowright a_o$, $\beta\colon b_o \curvearrowright b$ and a uniform arc $\gamma_o\colon a_o \curvearrowright b_o$ in Ω, and then check that $\gamma := \alpha \star \gamma_o \star \beta$ is $(C+\varepsilon)$-uniform.

For each $a \in A$, set $\rho_a := \upsilon\delta(a)$ where $\upsilon := \kappa/10C$. Employing (1.3) in conjunction with Fact 2.2 we see that that the balls $\mathsf{B}(a; \rho_a)$, with $a \in A$, are disjointed.

Fix points $a_o \in \mathsf{B}(a; \varepsilon\rho_a/10C^2)$, $b_o \in \mathsf{B}(b; \varepsilon\rho_b/10C^2)$ and let $\alpha\colon a \curvearrowright a_o$, $\beta\colon b_o \curvearrowright b$ be $(C + \varepsilon)$-quasiconvex arcs. Note that

$$\ell(\alpha) \le (C + \varepsilon)|a - a_o| \le \frac{C + \varepsilon}{10C^2} \varepsilon\rho_a \le \frac{\varepsilon}{5C} \rho_a$$

and similarly $\ell(\beta) \le \varepsilon\rho_b/5C$.

Let $\gamma_o\colon a_o \curvearrowright b_o$ be a C-uniform arc in Ω. Put $\gamma := \alpha \star \gamma_o \star \beta$; then $\gamma\colon a \curvearrowright b$ in $\Omega \cup \{ a, b \}$. Since $|a - b| \ge \rho_a + \rho_b$ and $|a_o - b_o| \le |a - a_o| + |a - b| + |b - b_o|$, we see that

$$\begin{aligned}
\ell(\gamma) &\le (C + \varepsilon)\big(|a - a_o| + |b - b_o|\big) + C|a_o - b_o| \\
&\le (2C + \varepsilon)\big(|a - a_o| + |b - b_o|\big) + C|a - b| \\
&\le \varepsilon \frac{2C + \varepsilon}{10C^2} \big(\rho_a + \rho_b\big) + C|a - b| \\
&\le (C + \varepsilon)|a - b|
\end{aligned}$$

and so γ is $(C + \varepsilon)$-quasiconvex.

To establish the double cone condition, let $x \in \gamma$, and let z_o be the arclength midpoint of γ_o. If $x \in \alpha$, it is not difficult to check that $\delta(x) \ge \ell(\alpha) \ge \ell(\gamma([x, a])$;

similarly for $x \in \beta$. Assume $x \in \gamma_o$, say $x \in \gamma_o[a_o, z_o]$. Here

$$\delta(x) \geq \mathsf{dist}(x, \partial\Omega) \geq C^{-1}\ell(\gamma_o[x, a_o]).$$

If $\ell(\gamma_o[x, a_o]) \geq \rho_a/5$, then $\ell(\alpha) \leq (C/\varepsilon)\ell(\gamma_o[x, a_o])$, so

$$\ell(\gamma[x, a]) = \ell(\gamma_o[x, a_o]) + \ell(\alpha) \leq \left(1 + \frac{C}{\varepsilon}\right)\ell(\gamma_o[x, a_o]) \leq (C + \varepsilon)\delta(x).$$

Finally, if $\ell(\gamma_o[x, a_o]) \leq \rho_a/5$, then

$$|x - a| \leq \frac{\rho_a}{5} + \frac{\varepsilon\rho_a}{10C^2} \leq \frac{2C^2 + \varepsilon}{10C^2}\upsilon\delta(a) \leq \frac{3\upsilon}{10}\delta(a)$$

so

$$\delta(x) \geq \left(1 - \frac{3\upsilon}{10}\right)\delta(a)$$

whence

$$\ell(\gamma[x, a]) \leq \frac{\rho_a}{5} + \frac{\varepsilon\rho_a}{5C} = \frac{C + \varepsilon}{5C}\upsilon\delta(a) \leq \frac{C + \varepsilon}{C}\frac{2\upsilon}{10 - 3\upsilon} \leq (1 + \varepsilon)\delta(x).$$

For the converse, suppose X is C_0-uniform and C_1-annular quasiconvex at each point of A. We show that $\Omega := X \setminus A$ is uniformly collared with fat collars and appeal to Theorem B.

For each $a \in A$, set $\rho_a := \upsilon_0\delta(a)$ where $\upsilon_0 := \kappa/4C_0$. Employing Fact 2.2 and (1.3) we see that the balls $\mathsf{B}(a; \rho_a)$, with $a \in A$, are disjointed. By assumption, each $\mathsf{B}(a; \rho_a)$ is C_2-uniform and X is C_1-annularly quasiconvex at a; by *the proof* of Theorem A, $\mathsf{B}_\star(a; \rho_a) := \mathsf{B}(a; \rho_a) \setminus \{a\}$ is $C_\star$-uniform. Since $A = X \setminus \Omega = \bigcup_{a \in A}\{a\}$ with $\{a\} = \{a\} \cap \mathsf{B}(a; \rho_a)$, we see that Ω is uniformly collared. Evidently, the collars $\mathsf{B}_\star(a; \rho_a)$ are Φ-fat with $\Phi := 1/\upsilon_0 = 4C_0/\kappa$. □

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