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Littlewood–Paley inequalities for fractional derivative on Bergman spaces

José Ángel Peláez and Elena de la Rosa

Abstract. For any pair (n,p), $n\in\mathbb{N}$ and $0<p<\infty$, it has been recently proved by Peláez and Rättyä (2021) that a radial weight ω on the unit disc of the complex plane $\mathbb{D}$ satisfies the Littlewood–Paley equivalence

$$\int_{\mathbb{D}}|f(z)|^p\,\omega(z)\,dA(z)\asymp\int_{\mathbb{D}}|f^{(n)}(z)|^p(1-|z|)^{np}\omega(z)\,dA(z)+\sum_{j=0}^{n-1}|f^{(j)}(0)|^p,$$

for any analytic function f in $\mathbb{D}$, if and only if $\omega\in\mathcal{D}=\widehat{\mathcal{D}}\cap\check{\mathcal{D}}$. A radial weight ω belongs to the class $\widehat{\mathcal{D}}$ if $\sup_{0\le r<1}\frac{\int_r^1\omega(s)\,ds}{\int_{\frac{1+r}{2}}^1\omega(s)\,ds}<\infty$, and $\omega\in\check{\mathcal{D}}$ if there exists $k>1$ such that $\inf_{0\le r<1}\frac{\int_r^1\omega(s)\,ds}{\int_{1-\frac{1-r}{k}}^1\omega(s)\,ds}>1$.

In this paper we extend this result to the setting of fractional derivatives. Being precise, for an analytic function $f(z)=\sum_{n=0}^\infty\widehat{f}(n)z^n$ we consider the fractional derivative $D^\mu(f)(z)=\sum_{n=0}^\infty\frac{\widehat{f}(n)}{\mu_{2n+1}}z^n$ induced by a radial weight $\mu\in\mathcal{D}$ where $\mu_{2n+1}=\int_0^1r^{2n+1}\mu(r)\,dr$. Then, we prove that for any $p\in(0,\infty)$, the Littlewood–Paley equivalence

$$\int_{\mathbb{D}}|f(z)|^p\omega(z)\,dA(z)\asymp\int_{\mathbb{D}}|D^\mu(f)(z)|^p\left[\int_{|z|}^1\mu(s)\,ds\right]^p\omega(z)\,dA(z)$$

holds for any analytic function f in $\mathbb{D}$ if and only if $\omega\in\mathcal{D}$. We also prove that for any $p\in(0,\infty)$, the inequality

$$\int_{\mathbb{D}}|D^\mu(f)(z)|^p\left[\int_{|z|}^1\mu(s)\,ds\right]^p\omega(z)\,dA(z)\lesssim\int_{\mathbb{D}}|f(z)|^p\omega(z)\,dA(z)$$

holds for any analytic function f in $\mathbb{D}$ if and only if $\omega\in\widehat{\mathcal{D}}$.

Bergmanin avaruuksien murtoasteisen derivaatan Littlewoodin–Paleyn epäyhtälöt

Tiivistelmä. Peláez ja Rättyä (2021) osoittivat hiljattain, että jokaisella parilla (n,p), missä $n\in\mathbb{N}$ ja $0<p<\infty$, kompleksitason yksikkökiekon $\mathbb{D}$ säteittäinen paino ω toteuttaa jokaisella kiekon $\mathbb{D}$ analyyttisellä funktiolla f Littlewoodin–Paleyn verrannon

$$\int_{\mathbb{D}}|f(z)|^p\,\omega(z)\,dA(z)\asymp\int_{\mathbb{D}}|f^{(n)}(z)|^p(1-|z|)^{np}\omega(z)\,dA(z)+\sum_{j=0}^{n-1}|f^{(j)}(0)|^p,$$

jos ja vain jos $\omega\in\mathcal{D}=\widehat{\mathcal{D}}\cap\check{\mathcal{D}}$. Säteittäinen paino ω kuuluu luokkaan $\widehat{\mathcal{D}}$, mikäli $\sup_{0\le r<1}\frac{\int_r^1\omega(s)\,ds}{\int_{\frac{1+r}{2}}^1\omega(s)\,ds}<\infty$, ja $\omega\in\check{\mathcal{D}}$ jos on olemassa $k>1$, joka toteuttaa $\inf_{0\le r<1}\frac{\int_r^1\omega(s)\,ds}{\int_{1-\frac{1-r}{k}}^1\omega(s)\,ds}>1$.

Tässä työssä yleistämme tämän tuloksen murtoasteisille derivaatoille. Tarkemmin sanottuna, jos $f(z)=\sum_{n=0}^\infty\widehat{f}(n)z^n$ on analyyttinen funktio ja $\mu\in\mathcal{D}$ on säteittäinen paino, tarkastelemme

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funktion f painoon μ liittyvää murtoasteista derivaattaa $D^\mu(f)(z) = \sum_{n=0}^\infty \frac{\widehat{f}(n)}{\mu_{2n+1}} z^n$, missä $\mu_{2n+1} = \int_0^1 r^{2n+1}\mu(r)\,dr$. Osoitamme, että jokaisella $p \in (0,\infty)$ on voimassa Littlewoodin–Paleyn verranto

$$\int_{\mathbb{D}} |f(z)|^p \omega(z)\,dA(z) \asymp \int_{\mathbb{D}} |D^\mu(f)(z)|^p \left[\int_{|z|}^1 \mu(s)\,ds\right]^p \omega(z)\,dA(z)$$

jokaisella kiekon $\mathbb{D}$ analyyttisellä funktiolla f, jos ja vain jos $\omega \in \mathcal{D}$. Osoitamme lisäksi, että jokaisella $p \in (0,\infty)$ on voimassa epäyhtälö

$$\int_{\mathbb{D}} |D^\mu(f)(z)|^p \left[\int_{|z|}^1 \mu(s)\,ds\right]^p \omega(z)\,dA(z) \lesssim \int_{\mathbb{D}} |f(z)|^p \omega(z)\,dA(z)$$

jokaisella kiekon $\mathbb{D}$ analyyttisellä funktiolla f, jos ja vain jos $\omega \in \widehat{\mathcal{D}}$.

1. Introduction

Let $\mathcal{H}(\mathbb{D})$ denote the space of analytic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$. For $f \in \mathcal{H}(\mathbb{D})$ and $0 < r < 1$, set

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p\,dt\right)^{\frac{1}{p}}, \quad 0 < p < \infty,$$

and $M_\infty(r,f) = \max_{|z|=r} |f(z)|$. For $0 < p \le \infty$, the Hardy space H^p consists of $f \in \mathcal{H}(\mathbb{D})$ such that $\|f\|_{H^p} = \sup_{0<r<1} M_p(r,f) < \infty$. For a nonnegative function $\omega \in L^1([0,1))$, the extension to $\mathbb{D}$, defined by $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$, is called a radial weight. For $0 < p < \infty$ and such an ω, the Lebesgue space L^p_ω consists of complex-valued measurable functions f on $\mathbb{D}$ such that

$$\|f\|^p_{L^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \omega(z)\,dA(z) < \infty,$$

where $dA(z) = \frac{dx\,dy}{\pi}$ is the normalized Lebesgue area measure on $\mathbb{D}$. The corresponding weighted Bergman space is $A^p_\omega = L^p_\omega \cap \mathcal{H}(\mathbb{D})$. Throughout this paper we assume $\widehat{\omega}(z) = \int_{|z|}^1 \omega(s)\,ds > 0$ for all $z \in \mathbb{D}$, for otherwise $A^p_\omega = \mathcal{H}(\mathbb{D})$.

A well-known formula ensures that for each $n \in \mathbb{N}$ and $0 < p < \infty$

$$\|f\|^p_{A^p_\omega} \asymp \int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|)^{np} \omega(z)\,dA(z) + \sum_{j=0}^{n-1} |f^{(j)}(0)|^p, \quad f \in \mathcal{H}(\mathbb{D}), \tag{1.1}$$

if ω is a standard radial weight, that is, $\omega(z) = (\alpha+1)(1-|z|^2)^\alpha$ for some $-1 < \alpha < \infty$. Generalizations of this Littlewood–Paley formula have been obtained in [3, 14, 23] for different classes of radial weights. However, the question for which radial weights the above equivalence (1.1) is valid has been a known open problem for decades. This question has been recently solved in [20, Theorem 5], in fact (1.1) holds for a radial weight ω if and only if $\omega \in \mathcal{D} = \widehat{\mathcal{D}} \cap \check{\mathcal{D}}$. Recall that a radial weight ω belongs to $\widehat{\mathcal{D}}$ if there exists a constant $C = C(\omega) > 1$ such that $\widehat{\omega}(r) \le C\widehat{\omega}(\frac{1+r}{2})$ for all $0 \le r < 1$. Further, a radial weight ω belongs to $\check{\mathcal{D}}$ if there exist constants $k = k(\omega) > 1$ and $C = C(\omega) > 1$ such that $\widehat{\omega}(r) \ge C\widehat{\omega}(1 - \frac{1-r}{k})$ for all $0 \le r < 1$.

It is also worth mentioning that the inequality

$$\int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|)^{np} \omega(z)\,dA(z) + \sum_{j=0}^{n-1} |f^{(j)}(0)|^p \lesssim \|f\|^p_{A^p_\omega}, \quad f \in \mathcal{H}(\mathbb{D}), \tag{1.2}$$

holds for a radial weight ω and each pair (n,p), $n \in \mathbb{N}$ and $0<p<\infty$, if and only if $\omega \in \widehat{\mathcal{D}}$ [20, Theorem 6].

Throughout the next few lines we offer a brief insight to the classes of weights $\mathcal{D}$, $\widehat{\mathcal{D}}$ and $\check{\mathcal{D}}$. Each standard radial weight belongs to $\mathcal{D}$, while $\check{\mathcal{D}} \setminus \mathcal{D}$ contains weights that tend to zero exponentially. The class of rapidly increasing weights, introduced in [17], lies entirely within $\widehat{\mathcal{D}} \setminus \mathcal{D}$, and a typical example of such a weight is $\omega(z) = (1-|z|^2)^{-1}\left(\log \frac{e}{1-|z|^2}\right)^{-\alpha}$, where $\alpha > 1$. However we emphasize that the containment in $\widehat{\mathcal{D}}$ or $\check{\mathcal{D}}$ does not require continuity neither positivity. In fact, weights in these classes may vanish on a relatively large part of each outer annulus $\{z\colon r \le |z| < 1\}$ of $\mathbb{D}$. For basic properties of the aforementioned classes, concrete nontrivial examples and more, see [15, 17, 20] and the relevant references therein.

The theory of weighted Bergman spaces A^p_ω induced by non-radial weights is at its early stages, and plenty of essential properties have not been described yet. However there have been developments towards different directions during the last decades [1, 8]. As for Littlewood–Paley formulas for derivatives, we recall that (1.1) holds if ω is a Bekollé–Bonami weight [1, 2], see also [4, 21] for related results.

On the other hand, (1.1) can be extended to the setting of fractional derivatives when ω is a standard weight. Indeed, for $f(z) = \sum_{n=0}^\infty \widehat{f}(n) z^n \in \mathcal{H}(\mathbb{D})$ and $\beta > 0$, consider the operator

$$D^\beta(f)(z) = \frac{2}{\Gamma(\beta+1)} \sum_{n=1}^\infty \frac{\Gamma(n+\beta+1)}{\Gamma(n+1)} \widehat{f}(n) z^n, \quad z \in \mathbb{D}, \tag{1.3}$$

which basically coincides with the fractional derivative of order $\beta > 0$ introduced by Hardy and Littlewood in [7, p. 409]. The differences between (1.3) and [7, (3.13)] are in the multiplicative factor $\frac{2}{\Gamma(\beta+1)}$ and the inessential factor z^β. A floklore result states that

$$\|f\|^p_{A^p_\omega} \asymp \int_{\mathbb{D}} |D^\beta(f)(z)|^p (1-|z|)^{\beta p} \omega(z)\, dA(z), \quad f \in \mathcal{H}(\mathbb{D}), \tag{1.4}$$

for any $\beta, p > 0$ and any standard radial weight ω, see [5, Theorem A] for the range $p \ge 1$. Moreover, Flett proved in [6, Theorem 6] that (1.4) remains true for any $\beta, p > 0$ if $D^\beta f$ is replaced by the multiplier transformation $f^{[\beta]}(z) = \sum_{n=0}^\infty (n+1)^\beta \widehat{f}(n) z^n$, which may also be regarded as fractional derivative of order $\beta > 0$.

For a radial weight $\mu \in \widehat{\mathcal{D}}$, we define the fractional derivative of f induced by μ

$$D^\mu(f)(z) = \sum_{n=0}^\infty \frac{\widehat{f}(n)}{\mu_{2n+1}} z^n, \ z \in \mathbb{D}.$$

Here μ_{2n+1} are the odd moments of μ, and in general from now on we write $\mu_x = \int_0^1 r^x \mu(r)\, dr$ for μ a radial weight and $x \ge 0$. It is clear that $D^\mu(f)$ is a polynomial if f is a polynomial and $D^\mu(f) \in \mathcal{H}(\mathbb{D})$ for each $f \in \mathcal{H}(\mathbb{D})$, by Lemma 5 below. See [22, 24] for related definitions or reformulations of classical and generalized fractional derivative, and observe that $D^\mu = D^\beta$ if μ is the standard weight $\mu(z) = \beta(1-|z|^2)^{\beta-1}$, $\beta > 0$.

The primary purpose of this paper is twofold: extending the Littlewood–Paley formulas (1.1) and (1.2) replacing the higher order derivative $f^{(n)}$ by the fractional derivative $D^\mu(f)$ induced by $\mu \in \mathcal{D}$, and describing the radial weights such that the arising formulas hold. With this aim, observe that $\widehat{\mu}(z) \asymp (1-|z|)^\beta$ when

$\mu(z) = \beta(1-|z|^2)^{\beta-1}$, $\beta > 0$, so an appropriate interpretation of the Littlewood–Paley estimate (1.4) is

$$\|f\|_{A^p_\omega}^p \asymp \int_{\mathbb{D}} |D^\mu(f)(z)|^p \widehat{\mu}(z)^p \omega(z)\, dA(z), \quad f \in \mathcal{H}(\mathbb{D}),$$

when μ is a standard weight.

Our main result shows that the discussion above regarding standard weights actually describes a general phenomenon rather than a particular case, and moreover describes the radial weights such that the formula holds.

Theorem 1. *Let ω be a radial weight, $0 < p < \infty$ and $\mu \in \mathcal{D}$. Then*

$$\|f\|_{A^p_\omega}^p \asymp \int_{\mathbb{D}} |D^\mu(f)(z)|^p \widehat{\mu}(z)^p \omega(z)\, dA(z), \quad f \in \mathcal{H}(\mathbb{D}), \tag{1.5}$$

if and only if $\omega \in \mathcal{D}$.

In particular, as a byproduct of Theorem 1 we obtain a proof of the folklore result

$$\|f\|_{A^p_\alpha}^p \asymp \int_{\mathbb{D}} |D^\beta(f)(z)|^p (1-|z|)^{\beta p+\alpha}\, dA(z), \quad f \in \mathcal{H}(\mathbb{D}),$$

for any $\beta, p > 0$, and $\alpha > -1$. Here and throughout the paper A^p_α stands for the classical weighted Bergman spaces induced by the standard radial weight $\omega(z) = (\alpha+1)(1-|z|^2)^\alpha$.

En route to the proof of Theorem 1 we will establish the following result, which generalizes [20, Theorem 5] to the setting of fractional derivatives induced by radial doubling weights.

Theorem 2. *Let ω be a radial weight, $0 < p < \infty$ and $\mu \in \mathcal{D}$. Then, there exists a constant $C = C(\omega, \mu, p) > 0$ such that*

$$\int_{\mathbb{D}} |D^\mu(f)(z)|^p \widehat{\mu}(z)^p \omega(z)\, dA(z) \le C\|f\|_{A^p_\omega}^p, \quad f \in \mathcal{H}(\mathbb{D}), \tag{1.6}$$

if and only if $\omega \in \widehat{\mathcal{D}}$.

The proof of (1.6) of is strongly based on the following inequality between the integral means of order p of $D^\mu f$ and f,

$$M_p(r, D^\mu f) \le C\frac{M_p(r,f)}{\widehat{\mu}\left(\frac{r}{\rho}\right)}, \quad 0 < r < \rho < 1,\ 0 < p < \infty. \tag{1.7}$$

The inequality (1.7) is proved in Proposition 7 below and it is a natural extension of [14, Lemma 3.1]. The proof of this last result employes the Cauchy formula for f', Minkowski's inequality for the case $p \ge 1$ and factorization results of H^p functions when $0 < p < 1$. However, the proof of (1.7) is strongly based on smooth properties of universal Cesáro basis of polynomials introduced by Jevtić and Pavlović [9].

Reciprocally, the other implication in the proof of Theorem 2 uses ideas from [20, Theorem 6] and some technicalities. In particular, the proof reveals that (1.6) holds if and only if the inequality there holds for all monomials only.

As for the proof of Theorem 1 we show, in Theorem 11 below, that the inequality

$$\|f\|_{A^p_\omega}^p \le C\int_{\mathbb{D}} |D^\mu(f)(z)|^p \widehat{\mu}(z)^p \omega(z)\, dA(z), \quad f \in \mathcal{H}(\mathbb{D}), \tag{1.8}$$

implies that $\omega \in \mathcal{M}$. Recall that $\omega \in \mathcal{M}$ if there exist constants $C = C(\omega) > 1$ and $k = k(\omega) > 1$ such that $\omega_x \ge C\omega_{kx}$ for all $x \ge 1$. It is known that $\check{\mathcal{D}} \subset \mathcal{M}$ [20, Proof of Theorem 3] but $\check{\mathcal{D}} \not\subset \mathcal{M}$ [20, Proposition 14]. However, [20, Theorem 3] ensures that $\mathcal{D} = \widehat{\mathcal{D}} \cap \check{\mathcal{D}} = \widehat{\mathcal{D}} \cap \mathcal{M}$, so Theorem 11 together with Theorem 2 yields that $\omega \in \mathcal{D}$ when (1.5) holds.

Concerning the reverse implication in the proof of Theorem 1, we construct ad hoc norms for the weighted Bergman spaces A^p_ω and $A^p_{\omega\widehat{\mu}^p}$, in the spirit of the decomposition results [12, Theorem 7.5.8] and [16, Theorem 4]. These two results are valid for $1 < p < \infty$ and their proofs employ the boundedness of the Riesz projection on $L^p(\partial\mathbb{D})$ to deal with the H^p-norm of polynomials of the type $\Delta_{n_1,n_2} z = \sum_{k=n_1}^{n_2} z^k$. However we use universal Cesáro basis of polynomials instead of the polynomyals Δ_{n_1,n_2}, their smooth properties allow us to get equivalent norms for any $0 < p < \infty$. Being precise, we prove that there is $k > 1$ and a shared universal Cesáro basis of polynomials $\{V_{n,k}\}_{n=0}^\infty$ such that $\|f\|_{A^p_\omega} \asymp \sum_{n=0}^\infty \omega_{k^n} \|V_{n,k} * f\|^p_{H^p}$ and $\|D^\mu(f)\|_{A^p_{\omega\widehat{\mu}^p}} \asymp \sum_{n=0}^\infty (\omega\widehat{\mu}^p)_{k^n} \|V_{n,k} * D^\mu(f)\|^p_{H^p}$ for any $p \in (0,\infty)$, where $*$ denotes the convolution.

Finally, we introduce the following notation that has already been used above in the introduction. The letter $C = C(\cdot)$ will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation $a \lesssim b$ if there exists a constant $C = C(\cdot) > 0$ such that $a \le Cb$, and $a \gtrsim b$ is understood in an analogous manner. In particular, if $a \lesssim b$ and $a \gtrsim b$, then we write $a \asymp b$ and say that a and b are comparable.

2. Preliminary results

2.1. Radial weights. In this section we provide several characterizations of the classes of radial weight $\widehat{\mathcal{D}}, \check{\mathcal{D}}$ and $\mathcal{M}$, which will be used in the proofs of the main results of this paper.

For each $\beta > 0$ and ω a radial weight, let us denote $\omega_{[\beta]}(s) = (1-s)^\beta \omega(s)$. The next result gathers descriptions of the class $\widehat{\mathcal{D}}$.

Lemma A. *Let ω be a radial weight. Then, the following statements are equivalent:*

(i) $\omega \in \widehat{\mathcal{D}}$;

(ii) *There exist $C = C(\omega) \ge 1$ and $\alpha_0 = \alpha_0(\omega) > 0$ such that*

$$\widehat{\omega}(s) \le C \left(\frac{1-s}{1-t}\right)^\alpha \widehat{\omega}(t), \quad 0 \le s \le t < 1;$$

for all $\alpha \ge \alpha_0$;

(iii)

$$\omega_x = \int_0^1 s^x \omega(s)\,ds \asymp \widehat{\omega}\left(1 - \frac{1}{x}\right), \quad x \in [1,\infty);$$

(iv) *There exists $C(\omega) > 0$ such that $\omega_n \le C\omega_{2n}$, for any $n \in \mathbb{N}$;*

(v) *There exist $C(\omega) > 0$ and $\eta(\omega) > 0$ such that*

$$\omega_x \le C \left(\frac{y}{x}\right)^\eta \omega_y, \quad 0 < x \le y < \infty;$$

(vi) *For some (equivalently for each) $\beta > 0$ there exists a constant $C = C(\omega,\beta) > 0$ such that $x^\beta (\omega_{[\beta]})_x \le C\omega_x$, $0 < x < \infty$.*

Proof. The equivalences (i)–(v) can be found in [15, Lemma 2.1] and (i) $\Leftrightarrow$ (vi) is proved in [20, Theorem 6], where it is provided a direct proof of (vi) $\Rightarrow$ (i), but (i) $\Rightarrow$ (vi) is obtained by using the Littlewood–Paley inequality [20, (1.5)]. So, here we give a detailed direct proof of (i) $\Rightarrow$ (vi) for the convenience of the reader and the sake of completeness.

Let $\beta > 0$, $\omega \in \widehat{\mathcal{D}}$ and $x > 0$. Observe that

$$x^\beta (\omega_{[\beta]})_x = x^\beta \int_0^{1-\frac{1}{x}} r^x (1-r)^\beta \omega(r)\, dr + x^\beta \int_{1-\frac{1}{x}}^1 r^x (1-r)^\beta \omega(r)\, dr = I + II.$$

By Fubini's theorem and Lemma A (ii)

$$\begin{aligned} I &= x^{\beta+1} \int_0^{1-\frac{1}{x}} (1-r)^\beta \left(\int_0^r s^{x-1}\, ds \right) \omega(r)\, dr \\ &= x^{\beta+1} \int_0^{1-\frac{1}{x}} s^{x-1} \left(\int_s^{1-\frac{1}{x}} (1-r)^\beta \omega(r)\, dr \right) ds \le x^{\beta+1} \int_0^{1-\frac{1}{x}} s^{x-1} (1-s)^\beta \widehat{\omega}(s)\, ds \\ &\lesssim x^{\beta+\alpha+1} \widehat{\omega}\left(1-\frac{1}{x}\right) \int_0^{1-\frac{1}{x}} s^{x-1}(1-s)^{\beta+\alpha}\, ds \lesssim \widehat{\omega}\left(1-\frac{1}{x}\right) \le \omega_x. \end{aligned}$$

Moreover, it is easy to observe that $II \le \omega_x$. This finishes the proof. □

We will also need the following characterizations of the class $\check{\mathcal{D}}$.

Lemma B. *Let ω be a radial weight. The following statements are equivalent:*

(i) $\omega \in \check{\mathcal{D}}$;

(ii) *There exist $C = C(\omega) > 0$ and $\beta = \beta(\omega) > 0$ such that*

$$\widehat{\omega}(s) \le C \left(\frac{1-s}{1-t} \right)^\beta \widehat{\omega}(t), \quad 0 \le t \le s < 1;$$

(iii) *There exist $k = k(\omega) > 1$ and $C = C(\omega) > 0$ such that*

$$\int_r^{1-\frac{1-r}{k}} \omega(s)\, ds \ge C \widehat{\omega}(r), \quad 0 \le r < 1. \tag{2.1}$$

Proof. The condition (iii) is just a reformulation of the definition of the class $\check{\mathcal{D}}$, so we omit the proof of (i) $\Leftrightarrow$ (iii). Next, assume that (i) holds and consider the sequence $\{r_n\}_{n=0}^\infty = \{1 - \frac{1}{k^n}\}_{n=0}^\infty$. If $0 \le t \le s < 1$, there exist $m, n \in \mathbb{N} \cup \{0\}$, $m \ge n$ such that $r_n \le t < r_{n+1}$ and $r_m \le s < r_{m+1}$. If $n + 1 \le m$, then

$$\begin{aligned} \widehat{\omega}(s) &\le \widehat{\omega}(r_m) \le \frac{1}{C} \widehat{\omega}(r_{m-1}) \le \cdots \le \frac{1}{C^{m-n-1}} \widehat{\omega}(r_{n+1}) \le \frac{C^2}{k^{(m-n+1)\log_k C}} \widehat{\omega}(t) \\ &\le C^2 \left(\frac{1-s}{1-t} \right)^{\log_k C} \widehat{\omega}(t). \end{aligned}$$

Next, if $m = n$, then for any constant $C_1 \ge k^\beta$

$$\frac{\widehat{\omega}(s)}{\widehat{\omega}(t)} \le 1 \le C_1 \frac{1}{k^\beta} \le C_1 \left(\frac{1-r_{n+1}}{1-r_n} \right)^\beta \le C_1 \left(\frac{1-s}{1-t} \right)^\beta,$$

so (ii) holds for any exponent $\beta \in (0, \log_k C]$. Reciprocally, assume (ii) and let be $k > 1$. By (ii) there exist $C = C(\omega) > 0$ and $\beta = \beta(\omega) > 0$ such that $\frac{C}{k^\beta} \widehat{\omega}(r) \ge \widehat{\omega}\left(1 - \frac{1-r}{k}\right)$. So taking $k > C^{\frac{1}{\beta}}$, (i) holds. This finishes the proof. □

A proof of the following description of the weights $\omega \in \mathcal{M}$, in terms of the moments of ω, can be found in [20, Theorem 2].

Lemma C. *Let ω be a radial weight. The following statements are equivalent:*

(i) $\omega \in \mathcal{M}$;

(ii) *For each $\beta > 0$, there is $C = C(\beta, \omega)$ such that*

$$\omega_x \le C x^{\beta} \left(\omega_{[\beta]}\right)_x, \quad x \ge 1.$$

The next result is an essential tool to deal with the inequality

$$\|f\|_{A^p_\omega}^p \lesssim \int_{\mathbb{D}} |D^\mu(f)(z)|^p \widehat{\mu}(z)^p \omega(z)\, dA(z), \quad f \in \mathcal{H}(\mathbb{D}),$$

(see Theorem 10 below), so to prove Theorem 1.

Lemma 3. *Let ω be a radial weight. Then, the following statements holds:*

(i) *If $\omega \in \check{\mathcal{D}}$ and $\varphi\colon [0,1) \to (0,\infty)$ is decreasing, then $\omega\varphi \in \check{\mathcal{D}}$. Furthermore, if there exist $k = k(\omega) > 1$ and $C = C(\omega) > 0$ such that*

$$C \int_{1-\frac{1-r}{k}}^{1} \omega(s)\, ds \le \int_r^{1-\frac{1-r}{k}} \omega(s)\, ds, \quad 0 \le r < 1, \tag{2.2}$$

then

$$C \int_{1-\frac{1-r}{k}}^{1} \omega(s)\varphi(s)\, ds \le \int_r^{1-\frac{1-r}{k}} \omega(s)\varphi(s)\, ds, \quad 0 \le r < 1. \tag{2.3}$$

(ii) *If $\omega \in \mathcal{D}$, $\mu \in \widehat{\mathcal{D}}$ and $0 < p < \infty$, then $\omega\widehat{\mu}^p \in \mathcal{D}$.*

Proof. By Lemma B (iii), the proof of (i) follows from the proof of (2.2)⇒(2.3). Indeed, since φ is decreasing

$$\begin{aligned}\int_r^{1-\frac{1-r}{k}} \omega(s)\varphi(s)\, ds &\ge \varphi\left(1 - \frac{1-r}{k}\right) \int_r^{1-\frac{1-r}{k}} \omega(s)\, ds \\ &\ge C\varphi\left(1 - \frac{1-r}{k}\right) \int_{1-\frac{1-r}{k}}^{1} \omega(s)\, ds \ge C \int_{1-\frac{1-r}{k}}^{1} \omega(s)\varphi(s)\, ds,\end{aligned}$$

and (2.3) holds.

Throughout the rest of the proof let us denote $\nu = \omega\widehat{\mu}^p$. Since $\omega \in \mathcal{D}$, by Lemma B (iii) there exists $k = k(\omega) > 1$ such that

$$\widehat{\nu}(r) \le \widehat{\mu}(r)^p \widehat{\omega}(r) \lesssim \widehat{\omega}\left(\frac{1+r}{2}\right) \widehat{\mu}(r)^p \lesssim \widehat{\mu}(r)^p \int_{\frac{1+r}{2}}^{1-\frac{1-r}{2k}} \omega(s)\, ds.$$

Moreover, by Lemma A (ii) there exist $C = C(\mu) > 0$ and $\alpha = \alpha(\mu) > 0$ such that $\widehat{\mu}(r) \le 2^\alpha k^\alpha C \widehat{\mu}\left(1 - \frac{1-r}{2k}\right)$, so

$$\widehat{\nu}(r) \lesssim \int_{\frac{1+r}{2}}^{1-\frac{1-r}{2k}} \omega(s)\widehat{\mu}(s)^p\, ds \le \widehat{\nu}\left(\frac{1+r}{2}\right),$$

that is $\nu \in \widehat{\mathcal{D}}$. Moreover, $\nu \in \check{\mathcal{D}}$ by (i). This finishes the proof. □

2.2. Universal Cesáro basis of polynomials. In this section, we establish some notation and previous results on universal Cesáro basis of polynomials, which will be strongly used in the proofs of Theorem 1 and Theorem 2.

The Hadamard product of a polynomial $W(z)=\sum_{k\in J} b_k z^k$, where J denotes a finite subset of $\mathbb{N}$ and $f(z)=\sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}(\mathbb{D})$ is

$$(W*f)(z)=\sum_{k=0}^{\infty} a_k b_k z^k, \quad z\in\mathbb{D}.$$

Furthermore, it is easy to observe that

$$(W*f)(e^{it})=\frac{1}{2\pi}\int_{-\pi}^{\pi} W(e^{i(t-\theta)})f(e^{i\theta})\,d\theta.$$

For a given C^∞-function $\Phi\colon \mathbb{R}\to\mathbb{C}$ with compact support, set

$$A_{\Phi,m}=\max_{x\in\mathbb{R}}|\Phi(x)|+m\max_{x\in\mathbb{R}}|\Phi^{(m)}(x)|, \quad m\in\mathbb{N}\cup\{0\},$$

and define the polynomials

$$W_n^\Phi(z)=\sum_{k\in\mathbb{Z}}\Phi\left(\frac{k}{n}\right)z^k, \quad n\in\mathbb{N}.$$

The next result can be found in [12, pp. 111–113].

Theorem D. *Let $\Phi\colon \mathbb{R}\to\mathbb{C}$ be a compactly supported C^∞-function. Then the following statements hold:*

(i) *There exists a constant $C>0$ such that*

$$|W_n^\Phi(e^{i\theta})|\le C\min\left\{n\max_{s\in\mathbb{R}}|\Phi(s)|, n^{1-m}|\theta|^{-m}\max_{s\in\mathbb{R}}|\Phi^{(m)}(s)|\right\}$$

for all $m\in\mathbb{N}\cup\{0\}$, $n\in\mathbb{N}$ and $0<|\theta|<\pi$.

(ii) *If $0<p\le 1$ and $m\in\mathbb{N}$ with $mp>1$, there exists a constant $C=C(p)>0$ such that*

$$\left(\sup_n |(W_n^\Phi * f)(e^{i\theta})|\right)^p \le CA^p_{\Phi,m}M(|f|^p)(e^{i\theta})$$

for all $f\in H^p$, where M denotes the Hardy–Littlewood maximal-operator

$$M(f)(e^{i\theta})=\sup_{0<h<\pi}\frac{1}{2h}\int_{\theta-h}^{\theta+h}|f(e^{it})|dt.$$

(iii) *For each $0<p<\infty$ and $m\in\mathbb{N}$ with $mp>1$, there exists a constant $C=C(p)>0$ such that*

$$\|W_n^\Phi * f\|_{H^p}\le CA_{\Phi,m}\|f\|_{H^p}$$

for all $f\in H^p$ and $n\in\mathbb{N}$.

The property (iii) shows that the polynomials $\{W_n^\Phi\}_{n\in\mathbb{N}}$ can be seen as a universal Césaro basis for H^p for any $0<p<\infty$. In the statement of the next result, we consider a particular family of polynomials $\{W_n^\Phi\}_{n\in\mathbb{N}}$ which play a key role in this manuscript.

Proposition 4. *Let $k\in\mathbb{N}$, $k>1$ and $\Psi\colon\mathbb{R}\to\mathbb{R}$ be a C^∞-function such that $\Psi\equiv 1$ on $(-\infty,1]$, $\Psi\equiv 0$ on $[k,\infty)$ and Ψ is decreasing and positive on $(1,k)$. Set $\psi(t)=\Psi\left(\frac{t}{k}\right)-\Psi(t)$ for all $t\in\mathbb{R}$. Let $V_{0,k}(z)=\sum_{j=0}^{k-1}\Psi(j)z^j$ and*

$$V_{n,k}(z)=W^\psi_{k^{n-1}}(z)=\sum_{j=0}^{\infty}\psi\left(\frac{j}{k^{n-1}}\right)z^j=\sum_{j=k^{n-1}}^{k^{n+1}-1}\psi\left(\frac{j}{k^{n-1}}\right)z^j, \quad n\in\mathbb{N}.$$

Then,

$$f(z)=\sum_{n=0}^{\infty}(V_{n,k}*f)(z),\quad z\in\mathbb{D},\quad f\in\mathcal{H}(\mathbb{D}), \tag{2.4}$$

and for each $0<p<\infty$ *there exists a constant* $C=C(p,\Psi,k)>0$ *such that*

$$\|V_{n,k}*f\|_{H^p}\le C\|f\|_{H^p},\quad f\in H^p,\quad n\in\mathbb{N}. \tag{2.5}$$

If $k=2$ we simply denote $V_{n,2}=V_n$. A proof of Proposition 4 for this choice appears in [9, pp. 175–177] or [13, pp. 143–144]. For the convenience of the reader and the sake of completeness, we present a proof of Proposition 4 following the ideas in [9, 13].

Proof of Proposition 4. Let us denote by $\{\widehat{V_{n,k}}(j)\}_j$ the sequence of Taylor coefficients of $V_{n,k}$. Since $\sum_{j=0}^{\infty}|\widehat{f}(j)||z|^j$ converges for each $z\in\mathbb{D}$, $\operatorname{supp}\widehat{V_{n,k}}\subset\mathbb{N}\cap[k^{n-1},k^{n+1})$ and $|\widehat{V_{n,k}}(j)|\le 2$ for all $n\in\mathbb{N}$ and $j\in\mathbb{N}$,

$$\left|\sum_{n=1}^{\infty}(V_{n,k}*f)(z)\right|\le 2\sum_{n=1}^{\infty}\sum_{j=k^{n-1}}^{k^{n+1}}|\widehat{f}(j)||z|^j\le 4\sum_{j=0}^{\infty}|\widehat{f}(j)||z|^j,$$

that is, $\sum_{n=0}^{\infty}(V_{n,k}*f)(z)$ converges for each $z\in\mathbb{D}$. Let us prove that $\sum_{n=0}^{\infty}\widehat{V_{n,k}}(j)=1$, for each $j=0,1,2,\dots$.

If $0\le j\le k-1$, $\sum_{n=0}^{\infty}\widehat{V_{n,k}}(j)=\widehat{V_{0,k}}(j)+\widehat{V_{1,k}}(j)=\Psi\left(\frac{j}{k}\right)=1$. On the other hand, if $j\ge k$ then $\Psi(j)=0$ and $\sum_{n=0}^{\infty}\widehat{V_{n,k}}(j)=\lim_{m\to\infty}\sum_{n=1}^{m}\psi\left(\frac{j}{k^{n-1}}\right)=\lim_{m\to\infty}\Psi\left(\frac{j}{k^m}\right)=1$.

Therefore, it is clear that (2.4) holds for polynomials. Let us show that (2.4) holds for each $f\in\mathcal{H}(\mathbb{D})$. Let be $S_nf(z)=\sum_{l=0}^{n}\widehat{f}(l)z^l$, the n-th partial sum of f. Fixed $z\in\mathbb{D}$,

$$\begin{aligned}\left|f(z)-\sum_{n=0}^{\infty}(V_{n,k}*f)(z)\right|&\le|f(z)-S_{k^m}f(z)|+\left|S_{k^m}f(z)-\sum_{n=0}^{\infty}(V_{n,k}*f)(z)\right|\\&=I(f,m,z)+II(f,m,z),\end{aligned}$$

where $I(f,m,z)=|f(z)-S_{k^m}f(z)|$ and $II(f,m,z)=|S_{k^m}f(z)-\sum_{n=0}^{\infty}(V_{n,k}*f)(z)|$. We have that $\lim_{m\to\infty}I(f,m,z)=0$, and using (2.4) for $S_{k^m}f$,

$$\begin{aligned}II(f,m,z)&=\left|\sum_{n=0}^{\infty}(V_{n,k}*S_{k^m}f)(z)-\sum_{n=0}^{\infty}(V_{n,k}*f)(z)\right|\\&=\left|\sum_{n=0}^{m+1}(V_{n,k}*f)(z)-\sum_{n=0}^{\infty}(V_{n,k}*f)(z)\right|\lesssim\sum_{j=k^{m+1}}^{\infty}|\widehat{f}(j)||z|^j,\end{aligned}$$

so $\lim_{m\to\infty}II(f,m,z)=0$. Consequently (2.4) holds for any $f\in\mathcal{H}(\mathbb{D})$.

Finally, (2.5) follows from Theorem D (iii). □

3. Proof of Theorem 2

To begin with, we will prove some technical lemmas. The first one ensures that the definition of D^μ makes sense when $\mu\in\widehat{\mathcal{D}}$.

Lemma 5. *Let* $\mu\in\widehat{\mathcal{D}}$ *and* $f\in\mathcal{H}(\mathbb{D})$. *Then, the fractional derivative* $D^\mu f\in\mathcal{H}(\mathbb{D})$.

Proof. By Lemma A (ii), there exist $C = C(\mu) > 0$ and $\alpha = \alpha(\mu) > 0$ such that

$$\mu_{2k+1} \geq C\widehat{\mu}\left(1 - \frac{1}{2(k+1)}\right) \geq \frac{C\widehat{\mu}(0)}{2^\alpha}\frac{1}{(k+1)^\alpha} = \frac{C}{(k+1)^\alpha}, \quad k \in \mathbb{N} \cup \{0\}.$$

So, for each $k \in \mathbb{N} \cup \{0\}$, $\sqrt[k]{\frac{|\widehat{f}(k)|}{\mu_{2k+1}}} \leq \sqrt[k]{\frac{(k+1)^\alpha}{C}}\sqrt[k]{|\widehat{f}(k)|}$. Then, it follows that

$$\limsup_{k\to\infty} \sqrt[k]{\frac{|\widehat{f}(k)|}{\mu_{2k+1}}} \leq 1,$$

and therefore $D^\mu(f) \in \mathcal{H}(\mathbb{D})$. □

Lemma 6. *Let $\mu \in \widehat{\mathcal{D}}$, $\gamma > 0$ and $k \in \mathbb{N} \setminus \{1\}$. Then,*

$$\sum_{n=0}^{\infty} \frac{r^{k^n}}{\mu_{k^n}^\gamma} \asymp \int_0^r \frac{dt}{(1-t)\widehat{\mu}(t)^\gamma}, \quad 0 \leq r < 1. \tag{3.1}$$

Moreover, if $\mu \in \mathcal{D}$

$$1 + \sum_{n=0}^{\infty} \frac{r^{k^n}}{\mu_{k^n}^\gamma} \asymp \frac{1}{\widehat{\mu}(r)^\gamma}, \quad 0 \leq r < 1. \tag{3.2}$$

Proof. Since $\mu \in \widehat{\mathcal{D}}$,

$$\sum_{n=0}^{\infty} \frac{r^{k^n}}{\mu_{k^n}^\gamma} \asymp \frac{r}{\mu_1^\gamma} + \sum_{n=1}^{\infty} \frac{r^{k^n}}{\mu_{k^n}^\gamma}\frac{1}{k^n}\sum_{j=k^{n-1}}^{k^n-1} 1 \lesssim \frac{r}{\mu_1^\gamma} + \sum_{n=1}^{\infty}\sum_{j=k^{n-1}}^{k^n-1} \frac{r^j}{(j+1)\mu_j^\gamma}$$
$$\asymp \sum_{j=1}^{\infty} \frac{r^j}{(j+1)\mu_{2j+1}^\gamma}, \quad 0 \leq r < 1.$$

Analogously, it can be proved that

$$\sum_{n=0}^{\infty} \frac{r^{k^n}}{\mu_{k^n}^\gamma} \gtrsim \sum_{j=1}^{\infty} \frac{r^j}{(j+1)\mu_{2j+1}^\gamma}, \quad 0 \leq r < 1.$$

Now, arguing as in [18, (2.9)], it follows

$$\sum_{j=1}^{\infty} \frac{r^j}{(j+1)\mu_{2j+1}^\gamma} \asymp \int_0^r \frac{dt}{(1-t)\widehat{\mu}(t)^\gamma}, \quad 0 \leq r < 1,$$

and we get (3.1).

Next, bearing in mind Lemma B(ii),

$$1 + \int_0^r \frac{dt}{(1-t)\widehat{\mu}(t)^\gamma} \leq 1 + C^\gamma \frac{(1-r)^{\beta\gamma}}{\widehat{\mu}(r)^\gamma}\int_0^r \frac{dt}{(1-t)^{1+\beta\gamma}} \lesssim \frac{1}{\widehat{\mu}(r)^\gamma}, \quad 0 \leq r < 1. \tag{3.3}$$

On the other hand, it is clear that

$$1 + \int_0^r \frac{dt}{(1-t)\widehat{\mu}(t)^\gamma} \gtrsim \frac{1}{\widehat{\mu}(r)^\gamma}, \quad 0 \leq r \leq \frac{1}{2}, \tag{3.4}$$

and because $\mu \in \widehat{\mathcal{D}}$,

$$1 + \int_0^r \frac{dt}{(1-t)\widehat{\mu}(t)^\gamma} \geq \int_{2r-1}^r \frac{dt}{(1-t)\widehat{\mu}(t)^\gamma} \gtrsim \frac{1}{\widehat{\mu}(2r-1)^\gamma} \gtrsim \frac{1}{\widehat{\mu}(r)^\gamma}, \quad \frac{1}{2} < r < 1. \tag{3.5}$$

Consequently, (3.1) together with (3.3), (3.4) and (3.5) implies (3.2). This finishes the proof. □

Now we will prove a generalization of [14, Lemma 3.1] to the setting of fractional derivatives induced by doubling weights, which is interesting on its own right.

Proposition 7. *Let $0 < p < \infty$ and $\mu \in \mathcal{D}$. Then, there exists a constant $C = C(p, \mu) > 0$ such that*

$$M_p(r, D^\mu(f)) \le C\frac{M_p(\rho, f)}{\widehat{\mu}\left(\frac{r}{\rho}\right)}, \quad 0 \le r < \rho < 1, \quad f \in \mathcal{H}(\mathbb{D}). \tag{3.6}$$

Proof. We will split the proof in two cases according to the value of r and ρ.

Case 1. $\frac{1}{2} \le \frac{r}{\rho} < 1$. Bearing in mind (2.4),

$$M_p(r, D^\mu(f)) \le \sum_{n=0}^{1} \|V_n * (D^\mu f)_r\|_{H^p} + \sum_{n=2}^{\infty} \|V_n * (D^\mu f)_r\|_{H^p}, \quad f \in \mathcal{H}(\mathbb{D}), \tag{3.7}$$

for all $1 < p < \infty$, and

$$M_p^p(r, D^\mu(f)) \le \sum_{n=0}^{1} \|V_n * (D^\mu f)_r\|_{H^p}^p + \sum_{n=2}^{\infty} \|V_n * (D^\mu f)_r\|_{H^p}^p, \quad f \in \mathcal{H}(\mathbb{D}), \tag{3.8}$$

for $0 < p \le 1$, where $V_n = V_{n,2}$ are the polynomials defined in the statement of Proposition 4.

Firstly, [14, Lemma 3.1] yields

$$\begin{aligned}\|V_0 * (D^\mu f)_r\|_{H^p} &\le \frac{|\widehat{f}(0)|}{\mu_1} + r\frac{|\widehat{f}(1)|}{\mu_3} \le \frac{M_p(r, f)}{\mu_1} + r\frac{M_p(\frac{\rho}{2}, f')}{\mu_3} \\ &\le \frac{M_p(r, f)}{\mu_1} + C(p)2r\frac{M_p(\rho, f)}{\rho\mu_3} \le C(p)\frac{M_p(\rho, f)}{\widehat{\mu}(\frac{r}{\rho})}, \quad \frac{1}{2} \le \frac{r}{\rho} < 1. \end{aligned} \tag{3.9}$$

The inequality

$$\|V_1 * (D^\mu f)_r\|_{H^p} \le C(p)\frac{M_p(\rho, f)}{\widehat{\mu}(\frac{r}{\rho})}, \quad \frac{1}{2} \le \frac{r}{\rho} < 1, \tag{3.10}$$

can be proved analogously. Next, we will estimate from above the series in (3.7) and (3.8). For each $n \in \mathbb{N}$, $n \ge 2$, let us consider the function

$$\varphi_n(x) = \frac{\left(\frac{r}{\rho}\right)^x}{\mu_{2x+1}}\chi_{[2^{n-1}, 2^{n+1}-1]}(x), \quad \frac{1}{2} \le \frac{r}{\rho} < 1,$$

and fix $m \in \mathbb{N}$ such that $mp > 1$. Observe that for each $k \in \mathbb{N}$, there is $C = C(k) > 0$ such that

$$\int_0^1 s^x \left(\log\frac{1}{s}\right)^k \mu(s)\,ds \le C\mu_{x-1} \le C\mu_x, \quad \text{for any } x \ge 2. \tag{3.11}$$

Since

$$\varphi_n'(x) = -\frac{2\int_0^1 s^{2x+1}\log\frac{1}{s}\mu(s)\,ds}{(\mu_{2x+1})^2}\left(\frac{r}{\rho}\right)^x + \frac{\left(\frac{r}{\rho}\right)^x}{\mu_{2x+1}}\log\frac{r}{\rho}, \quad x \in (2^{n-1}, 2^{n+1}-1),$$

using (3.11) and the inequality $\frac{1}{2} \le \frac{r}{\rho}$, it follows that there is an absolute constant $C > 0$ such that

$$|\varphi_n'(x)| \le C\frac{\left(\frac{r}{\rho}\right)^x}{\mu_{2x+1}}, \quad x \in (2^{n-1}, 2^{n+1}-1), \quad \frac{1}{2} \le \frac{r}{\rho} < 1.$$

By this way, using (3.11) and an induction process on m, there is $C = C(m)$ such that

$$|\varphi_n^{(m)}(x)| \le C\frac{\left(\frac{r}{\rho}\right)^x}{\mu_{2x+1}}, \quad x \in (2^{n-1}, 2^{n+1}-1), \quad \frac{1}{2} \le \frac{r}{\rho} < 1.$$

Now, for each $n \in \mathbb{N} \setminus \{1\}$, choose a C^∞-function Φ_n with compact support contained in $[2^{n-2}, 2^{n+2}]$ such that $\Phi_n = \varphi_n$ in $[2^{n-1}, 2^{n+1}-1]$. Then, bearing in mind Lemma A (v), there is $C = C(p,\mu) > 0$ such that

$$A_{\Phi_n,m} \le C \max_{x\in[2^{n-1},2^{n+1}-1]} \frac{\left(\frac{r}{\rho}\right)^x}{\mu_{2x+1}} \le C\frac{\left(\frac{r}{\rho}\right)^{2^{n-1}}}{\mu_{2^{n+2}+1}} \le C\frac{\left(\frac{r}{\rho}\right)^{2^{n-1}}}{\mu_{2^n}}, \quad \frac{1}{2} \le \frac{r}{\rho} < 1. \tag{3.12}$$

Moreover,

$$\begin{aligned} V_n * (D^\mu f)_r(z) &= \sum_{k=2^{n-1}}^{2^{n+1}-1} \psi\left(\frac{k}{2^{n-1}}\right) \frac{\left(\frac{r}{\rho}\right)^k}{\mu_{2k+1}} \widehat{f}(k)\rho^k z^k = \sum_{k=2^{n-1}}^{2^{n+1}-1} \psi\left(\frac{k}{2^{n-1}}\right) \Phi_n(k)\widehat{f}(k)\rho^k z^k \\ &= (W_1^{\Phi_n} * V_n * f_\rho)(z). \end{aligned}$$

So, Theorem D(iii), (3.12) and (2.5) imply that for each $n \in \mathbb{N} \setminus \{1\}$

$$\begin{aligned} \|V_n * (D^\mu f)_r\|_{H^p} &\le CA_{\Phi_n,m}\|V_n * f_\rho\|_{H^p} \\ &\le C\frac{\left(\frac{r}{\rho}\right)^{2^{n-1}}}{\mu_{2^n}} \|V_n * f_\rho\|_{H^p} \le C\frac{\left(\frac{r}{\rho}\right)^{2^{n-1}}}{\mu_{2^n}} M_p(\rho, f), \quad \frac{1}{2} \le \frac{r}{\rho} < 1, \end{aligned}$$

where $C = C(p,\mu) > 0$. So, by Lemma 6, there is $C = C(\mu, p) > 0$ such that

$$\begin{aligned} \sum_{n=2}^\infty \|V_n * (D^\mu f)_r\|_{H^p} &\le CM_p(\rho, f)\left(\sum_{n=2}^\infty \frac{\left(\frac{r}{\rho}\right)^{2^{n-1}}}{\mu_{2^n}}\right) \\ &\le C\frac{M_p(\rho,f)}{\widehat{\mu}\left(\frac{r}{\rho}\right)}, \quad \frac{1}{2} \le \frac{r}{\rho} < 1,\ f \in \mathcal{H}(\mathbb{D}),\ 1 < p < \infty. \end{aligned} \tag{3.13}$$

Analogously, using Lemma 6 again

$$\begin{aligned} \sum_{n=2}^\infty \|V_n * (D^\mu f)_r\|_{H^p}^p &\le CM_p^p(\rho, f)\left(\sum_{n=2}^\infty \frac{\left(\frac{r}{\rho}\right)^{p2^{n-1}}}{\mu_{2^n}^p}\right) \le C\frac{M_p^p(\rho,f)}{\left[\widehat{\mu}\left(\left(\frac{r}{\rho}\right)^p\right)\right]^p} \\ &\le C\frac{M_p^p(\rho,f)}{\left(\widehat{\mu}\left(\frac{r}{\rho}\right)\right)^p}, \quad \frac{1}{2} \le \frac{r}{\rho} < 1,\ f \in \mathcal{H}(\mathbb{D}),\ 0 < p \le 1, \end{aligned} \tag{3.14}$$

where in the last inequality we have used $\mu \in \widehat{\mathcal{D}}$. Finally, joining (3.7), (3.9), (3.10) and (3.13) we obtain (3.6) for $p > 1$, and in the case $0 < p \le 1$ (3.6) follows from (3.8), (3.9), (3.10) and (3.14).

Case 2. $0 \le \frac{r}{\rho} < \frac{1}{2}$. Observe that (3.6) has already been proved for any $\rho > 0$ and $r = \frac{\rho}{2}$. So,

$$M_p(r, D^\mu f) \le M_p\left(\frac{\rho}{2}, D^\mu f\right) \le C(p,\mu)\frac{M_p(\rho, f)}{\widehat{\mu}\left(\frac{1}{2}\right)} \le C(p,\mu)\frac{M_p(\rho,f)}{\widehat{\mu}\left(\frac{r}{\rho}\right)}, \quad 0 \le \frac{r}{\rho} < \frac{1}{2}.$$

This finishes the proof. □

Proof of Theorem 2. Assume $\omega \in \widehat{\mathcal{D}}$. Without loss of generality we may assume that $f \in A^p_\omega$. So, using that $\mu \in \widehat{\mathcal{D}}$ and Proposition 7, there is $C = C(\omega, \mu, p) > 0$ such that

$$\begin{aligned}\|D^\mu f\|^p_{A^p_{\widehat{\mu}^p\omega}} &\le C\int_{\frac{1}{2}}^1 M_p^p(r, D^\mu(f))\omega(r)\widehat{\mu}(r)^p\,dr \\ &\le C\int_{\frac{1}{2}}^1 M_p^p(\sqrt{r}, f)\frac{\widehat{\mu}(r)^p}{\widehat{\mu}(\sqrt{r})^p}\omega(r)\,dr \\ &\le C\int_{\frac{1}{2}}^1 M_p^p(\sqrt{r}, f)\omega(r)\,dr = 2C\int_{\frac{1}{\sqrt{2}}}^1 rM_p^p(r, f)\omega(r^2)\,dr.\end{aligned} \tag{3.15}$$

Next, since $f \in A^p_\omega$,

$$\|f\|^p_{A^p_\omega} \ge \int_r^1 M_p^p(s, f)\omega(s)\,ds \ge M_p^p(r, f)\widehat{\omega}(r) \to 0 \quad \text{as } r \to 1^-.$$

So, two integration by parts and an application of Lemma A (ii) yield

$$\begin{aligned}2\int_{\frac{1}{\sqrt{2}}}^1 rM_p^p(r, f)\omega(r^2)\,dr &= M_p^p\left(\frac{1}{\sqrt{2}}, f\right)\widehat{\omega}\left(\frac{1}{2}\right) + \int_{\frac{1}{\sqrt{2}}}^1\left[\frac{d}{dr}M_p^p(r, f)\right]\widehat{\omega}(r^2)\,dr \\ &\le M_p^p\left(\frac{1}{\sqrt{2}}, f\right)\widehat{\omega}\left(\frac{1}{2}\right) + C\int_{\frac{1}{\sqrt{2}}}^1\left[\frac{d}{dr}M_p^p(r, f)\right]\widehat{\omega}(r)\,dr, \\ &\le M_p^p\left(\frac{1}{\sqrt{2}}, f\right)\widehat{\omega}\left(\frac{1}{2}\right) + C\int_{\frac{1}{2}}^1 rM_p^p(r, f)\omega(r)\,dr \le C\|f\|^p_{A^p_\omega},\end{aligned}$$

which together with (3.15) implies (1.6).

Reciprocally, assume that (1.6) holds. By choosing $f_n(z) = z^n$, $n \in \mathbb{N}\cup\{0\}$, in (1.6) we obtain

$$\int_{\mathbb{D}}\frac{|z|^{np}}{\mu^p_{2n+1}}\widehat{\mu}(z)^p\omega(z)\,dA(z) \le C^p\int_{\mathbb{D}}|z|^{np}\omega(z)\,dA(z),\ n \in \mathbb{N}\cup\{0\}.$$

Since $\mu \in \widehat{\mathcal{D}}$, by Lemma A (iii) there exists $C = C(\mu) > 0$ such that $\mu_{2n+1} \le C\widehat{\mu}\left(1 - \frac{1}{n+1}\right)$, $n \in \mathbb{N}\cup\{0\}$. Therefore,

$$\int_{\mathbb{D}}\frac{|z|^{np}}{\widehat{\mu}\left(1 - \frac{1}{n+1}\right)^p}\widehat{\mu}(z)^p\omega(z)\,dA(z) \le C^p\int_{\mathbb{D}}|z|^{np}\omega(z)\,dA(z), \quad n \in \mathbb{N}\cup\{0\}.$$

If $x \ge 1$, we can find $m \in \mathbb{N}$ such that $m \le x < m+1$. By applying the previous inequality to $n = m+1$,

$$\int_0^1\frac{s^{(m+1)p+1}}{\widehat{\mu}\left(1 - \frac{1}{m+2}\right)^p}\widehat{\mu}(s)^p\omega(s)\,ds \le C^p\int_0^1 s^{(m+1)p+1}\omega(s)\,ds \le C^p\int_0^1 s^{xp+1}\omega(s)\,ds,$$

Moreover, bearing in mind the monotonicity of s^x and $\widehat{\mu}(s)$ there exist $C = C(\omega, \mu, p) > 0$ such that

$$\begin{aligned}\int_0^1 \frac{s^{(m+1)p+1}}{\widehat{\mu}\left(1-\frac{1}{m+2}\right)^p}\widehat{\mu}(s)^p\omega(s)\,ds &\geq \int_0^1 \frac{s^{mp+p+1}}{\widehat{\mu}\left(1-\frac{1}{x}\right)^p}\widehat{\mu}(s)^p\omega(s)\,ds\\ &\geq C\int_0^1 \frac{s^{mp+1}}{\widehat{\mu}\left(1-\frac{1}{x}\right)^p}\widehat{\mu}(s)^p\omega(s)\,ds\\ &\geq C\int_0^1 \frac{s^{xp+1}}{\widehat{\mu}\left(1-\frac{1}{x}\right)^p}\widehat{\mu}(s)^p\omega(s)\,ds.\end{aligned}$$

Therefore, there exists $C = C(\omega, \mu, p) > 1$ such that

$$\int_0^1 \frac{s^{xp}}{\widehat{\mu}\left(1-\frac{1}{x}\right)^p}\widehat{\mu}(s)^p\omega(s)\,ds \leq C^p\int_0^1 s^{xp}\omega(s)\,ds, \quad \text{for all } x \geq 1.$$

That is,

$$\int_0^1 s^{xp}\omega(s)\left(\left(\frac{\widehat{\mu}(s)}{\widehat{\mu}\left(1-\frac{1}{x}\right)}\right)^p - C^p\right)ds \leq 0, \qquad \text{for all } x \geq 1. \tag{3.16}$$

Take $k_1 \geq 1$ such that $\widehat{\mu}\left(1-\frac{1}{x}\right) < \frac{\widehat{\mu}(0)}{C}$ for $x \geq k_1$. Then, for any $x \geq k_1$ there exists $s_x = s_x(x, C, \mu) \in (0, 1-\frac{1}{x})$, the infimum of the points $s \in (0, 1-\frac{1}{x})$ such that $\frac{\widehat{\mu}(s)}{\widehat{\mu}\left(1-\frac{1}{x}\right)} = C$. By (3.16),

$$\begin{aligned}\int_0^{s_x} s^{xp}\omega(s)\left(\left(\frac{\widehat{\mu}(s)}{\widehat{\mu}\left(1-\frac{1}{x}\right)}\right)^p - C^p\right)ds &\leq \int_{s_x}^1 s^{xp}\omega(s)\left(C^p - \left(\frac{\widehat{\mu}(s)}{\widehat{\mu}\left(1-\frac{1}{x}\right)}\right)^p\right)ds\\ &\leq C^p\widehat{\omega}(s_x), \quad \text{for all } x \geq k_1.\end{aligned}$$

So,

$$\widehat{\omega}(s_x) \geq C^{-p}\int_0^{s_x} s^{xp}\omega(s)\left(\left(\frac{\widehat{\mu}(s)}{\widehat{\mu}\left(1-\frac{1}{x}\right)}\right)^p - C^p\right)ds.$$

Next, choose $k_2 \geq k_1$ such that $\widehat{\mu}\left(1-\frac{1}{x}\right) < \frac{\widehat{\mu}(0)}{\left(\frac{3}{2}\right)^{1/p}C}$ if $x \geq k_2$. So, for any $x \geq k_2$, there exists $r_x = r_x(x, C, \mu) \in (0, 1-\frac{1}{x})$, the infimum of the points $r \in (0, 1-\frac{1}{x})$ such that $\frac{\widehat{\mu}(r)}{\widehat{\mu}\left(1-\frac{1}{x}\right)} = \left(\frac{3}{2}\right)^{1/p}C$. Then, $r_x < s_x < 1-\frac{1}{x}$ and

$$\widehat{\omega}(s_x) \geq C^{-p}\int_0^{s_x} s^{xp}\omega(s)\left(\left(\frac{\widehat{\mu}(s)}{\widehat{\mu}\left(1-\frac{1}{x}\right)}\right)^p - C^p\right)ds \geq \frac{1}{2}\int_0^{r_x} s^{xp}\omega(s)\,ds,$$

for any $x \geq k_2$. By Fubini's theorem,

$$\begin{aligned}2\widehat{\omega}(s_x) &\geq \int_0^{r_x} s^{xp}\omega(s)\,ds = \int_0^{r_x}\omega(s)\left(\int_0^s pxt^{xp-1}\,dt\right)ds\\ &= \int_0^{r_x} pxs^{xp-1}(\widehat{\omega}(s) - \widehat{\omega}(r_x))\,ds\\ &\geq \int_0^r pxs^{xp-1}\widehat{\omega}(s)\,ds - \widehat{\omega}(r_x)\int_0^{r_x} pxs^{xp-1}\,ds\\ &\geq \widehat{\omega}(r)r^{px} - \widehat{\omega}(r_x)r_x^{px}, \quad 0 < r < r_x < 1.\end{aligned}$$

Then, for any $x \geq k_2$ and $0 < r < r_x < 1$,

$$\widehat{\omega}(r)r^{px} \leq 2\widehat{\omega}(s_x) + \widehat{\omega}(r_x)r_x^{px} \leq (2 + r_x^{px})\widehat{\omega}(r_x) \leq 3\widehat{\omega}(r_x).$$

It is clear that $r_x > \frac{1}{2}$ if k_2 is large enough. In this case, take $r = 2r_x - 1$ in the previous inequality, that is

$$\widehat{\omega}(r) \leq 3r^{-px}\widehat{\omega}\left(\frac{1+r}{2}\right), \quad 0 < r = 2r_x - 1 < r_x < 1, \quad x \geq k_2. \tag{3.17}$$

Since $\mu \in \check{\mathcal{D}}$, by Lemma B (ii) there exist $C_2 = C_2(\mu) > 0$ and $\beta = \beta(\mu) > 0$ such that

$$\left(\frac{3}{2}\right)^{1/p} C = \frac{\widehat{\mu}(r_x)}{\widehat{\mu}\left(1 - \frac{1}{x}\right)} \geq C_2(x(1 - r_x))^{\beta}.$$

So,

$$x \leq \frac{\left(\frac{(\frac{3}{2})^{\frac{1}{p}} C}{C_2}\right)^{\frac{1}{\beta}}}{1 - r_x} = \frac{2\left(\frac{(\frac{3}{2})^{\frac{1}{p}} C}{C_2}\right)^{\frac{1}{\beta}}}{1 - r}.$$

Then, for $r \geq r_0 = 1 - \frac{2}{k_2}\left(\frac{(\frac{3}{2})^{\frac{1}{p}} C}{C_2}\right)^{\frac{1}{\beta}}$,

$$r^{px} \geq \left(\inf_{r_0 \leq r < 1} r^{\frac{1}{1-r}}\right)^{2\left(\frac{(\frac{3}{2})^{\frac{1}{p}} C}{C_2}\right)^{\frac{1}{\beta}} p} = C_3 = C_3(\omega, \mu, p) > 0,$$

which together with (3.17) yields $\widehat{\omega}(r) \leq 3C_3\widehat{\omega}(\frac{1+r}{2})$, for $r_0 \leq r < 1$. Therefore $\omega \in \widehat{\mathcal{D}}$. □

4. Proof of Theorem 1

We begin this section proving a technical result on L^p-integrability of power series with nonnegative coefficients. We use ideas from the proofs of [10, Theorem 6] and [16, Proposition 9].

Proposition 8. *Let $0 < p < \infty$, $\eta \in \mathcal{D}$ and $k \in \mathbb{N} \setminus \{1\}$ such that* (2.1) *holds for η. Let be $f(r) = \sum_{j=0}^{\infty} a_j r^j$ where $a_j \geq 0$ for all $j \in \mathbb{N} \cup \{0\}$. If $t_0 = \sum_{j=0}^{k-1} a_j$ and $t_n = \sum_{j=k^n}^{k^{n+1}-1} a_j$, $n \in \mathbb{N}$. Then, there exist positive constants $C_1 = C_1(p, \eta) > 0$ and $C_2 = C_2(p, \eta) > 0$ such that*

$$C_1 \sum_{n=0}^{\infty} \eta_{k^n} t_n^p \leq \int_0^1 f(s)^p \eta(s)\, ds \leq C_2 \sum_{n=0}^{\infty} \eta_{k^n} t_n^p.$$

Proof. First, we show the lower estimate

$$\begin{aligned}
\int_0^1 f(s)^p \eta(s)\, ds &\geq \sum_{n=0}^{\infty} \int_{1-\frac{1}{k^{n+1}}}^{1-\frac{1}{k^{n+2}}} \left(\sum_{m=0}^{k-1} a_m s^m + \sum_{j=1}^{\infty} \sum_{m=k^j}^{k^{j+1}-1} a_m s^m\right)^p \eta(s)\, ds \\
&\geq \sum_{n=0}^{\infty} \int_{1-\frac{1}{k^{n+1}}}^{1-\frac{1}{k^{n+2}}} s^{pk^{n+1}} t_n^p \eta(s)\, ds \geq C(p, \eta) \sum_{n=0}^{\infty} t_n^p \int_{1-\frac{1}{k^{n+1}}}^{1-\frac{1}{k^{n+2}}} \eta(s)\, ds \\
&= C(p, \eta) \sum_{n=0}^{\infty} t_n^p \left(\widehat{\eta}\left(1 - \frac{1}{k^{n+1}}\right) - \widehat{\eta}\left(1 - \frac{1}{k^{n+2}}\right)\right).
\end{aligned}$$

Since (2.1) holds for k and η, there exists a constant $C = C(\eta) > 1$ such that $\widehat{\eta}\left(1 - \frac{1}{k^{n+1}}\right) \geq C\widehat{\eta}\left(1 - \frac{1}{k^{n+2}}\right)$. This together with Lemma A(iii) yields

$$\int_0^1 f(s)^p \eta(s)\, ds \geq C(p,\eta) \sum_{n=0}^{\infty} t_n^p \widehat{\eta}\left(1 - \frac{1}{k^{n+2}}\right) \geq C(p,\eta) \sum_{n=0}^{\infty} \eta_{k^n} t_n^p.$$

In order to show the upper bound, observe that

$$f(s) \leq a_0 + \sum_{n=0}^{\infty} s^{k^n} t_n. \tag{4.1}$$

If $0 < p \leq 1$, by Lemma A (v), there is $C = C(p,\eta) > 0$ such that

$$\begin{aligned}\int_0^1 f(s)^p \eta(s)\, ds &\leq a_0^p \widehat{\eta}(0) + \int_0^1 \sum_{n=0}^{\infty} s^{k^n p} t_n^p \eta(s)\, ds = a_0^p \widehat{\eta}(0) + \sum_{n=0}^{\infty} t_n^p \eta_{k^n p} \\ &\leq C\left(t_0^p \eta_{k^0} + \sum_{n=0}^{\infty} t_n^p \eta_{k^n}\right) \leq C \sum_{n=0}^{\infty} t_n^p \eta_{k^n}.\end{aligned}$$

If $1 < p < \infty$, take γ such that $0 < \frac{\gamma p}{p'} < 1$. Then, by (4.1), Hölder's inequality and Lemma 6, we obtain

$$f(s) \leq a_0 + \left(\sum_{n=0}^{\infty} \frac{s^{k^n}}{\eta_{k^n}^{\gamma}}\right)^{\frac{1}{p'}} \left(\sum_{n=0}^{\infty} s^{k^n} t_n^p \eta_{k^n}^{\frac{\gamma p}{p'}}\right)^{\frac{1}{p}} \lesssim a_0 + \frac{1}{\widehat{\eta}(s)^{\frac{\gamma}{p'}}} \left(\sum_{n=0}^{\infty} s^{k^n} t_n^p \eta_{k^n}^{\frac{\gamma p}{p'}}\right)^{\frac{1}{p}}$$

which yields

$$\begin{aligned}\int_0^1 f(s)^p \eta(s)\, ds &\lesssim a_0^p \widehat{\eta}(0) + \sum_{n=0}^{\infty} t_n^p \eta_{k^n}^{\frac{\gamma p}{p'}} \int_0^1 s^{k^n} \frac{\eta(s)}{\widehat{\eta}(s)^{\frac{\gamma p}{p'}}}\, ds \\ &\lesssim t_0^p \eta_{k^0} + \sum_{n=0}^{\infty} t_n^p \eta_{k^n}^{\frac{\gamma p}{p'}} \int_0^1 s^{k^n} \frac{\eta(s)}{\widehat{\eta}(s)^{\frac{\gamma p}{p'}}}\, ds.\end{aligned} \tag{4.2}$$

Next, let us prove that

$$\eta_{k^n}^{\frac{\gamma p}{p'}} \int_0^1 s^{k^n} \frac{\eta(s)}{\widehat{\eta}(s)^{\frac{\gamma p}{p'}}}\, ds \lesssim \eta_{k^n},$$

which together with (4.2) finishes the proof. Indeed, by Lemma A (iii)

$$\eta_{k^n}^{\frac{\gamma p}{p'}} \int_0^{1-\frac{1}{k^n}} s^{k^n} \frac{\eta(s)}{\widehat{\eta}(s)^{\frac{\gamma p}{p'}}}\, ds \leq \frac{\eta_{k^n}^{\frac{\gamma p}{p'}}}{\widehat{\eta}\left(1 - \frac{1}{k^n}\right)^{\frac{\gamma p}{p'}}} \int_0^1 s^{k^n} \eta(s)\, ds \asymp \eta_{k^n},$$

Morever, an integration and another application of Lemma A (iii) imply

$$\eta_{k^n}^{\frac{\gamma p}{p'}} \int_{1-\frac{1}{k^n}}^1 s^{k^n} \frac{\eta(s)}{\widehat{\eta}(s)^{\frac{\gamma p}{p'}}}\, ds \leq \eta_{k^n}^{\frac{\gamma p}{p'}} \int_{1-\frac{1}{k^n}}^1 \frac{\eta(s)}{\widehat{\eta}(s)^{\frac{\gamma p}{p'}}}\, ds \asymp \eta_{k^n}^{\frac{\gamma p}{p'}} \widehat{\eta}\left(1 - \frac{1}{k^n}\right)^{1-\frac{\gamma p}{p'}} \lesssim \eta_{k^n}.$$

This finishes the proof. □

The right choice of the norm used is in many cases a key to a good understanding of how a concrete operator acts in a given space. Here, the following decomposition result provides an effective tool for the study of the fractional derivative D^{μ}.

Proposition 9. *Let $0 < p < \infty$, $\eta \in \mathcal{D}$ and $k = k(\eta) > 1$, $k \in \mathbb{N}$ such that (2.1) holds for η and k. If $\{V_{n,k}\}_{n=0}^{\infty}$ is a sequence of polynomials considered in Proposition 4, then there are constants $C_1 = C_1(p,\eta) > 0$ and $C_2 = C_2(p,\eta) > 0$ such that*

$$C_1 \sum_{n=0}^{\infty} \eta_{k^n} \|V_{n,k} * f\|_{H^p}^p \le \|f\|_{A^p_\eta}^p \le C_2 \sum_{n=0}^{\infty} \eta_{k^n} \|V_{n,k} * f\|_{H^p}^p, \quad f \in \mathcal{H}(\mathbb{D}).$$

Proof. By (2.5) and [11, Lemma 3.1], there is $C > 0$ such that

$$\|f_r\|_{H^p} \ge C^{-1} \|V_{n,k} * f_r\|_{H^p} \ge C^{-1} r^{k^{n+1}} \|V_{n,k} * f\|_{H^p}$$

for any $0 \le r < 1$ and $n \in \mathbb{N}$. So,

$$\|f_r\|_{H^p} \ge C^{-1} \sup_{n \in \mathbb{N}} r^{k^{n+1}} \|V_{n,k} * f\|_{H^p},$$

which implies

$$\begin{aligned} \|f\|_{A^p_\eta}^p &\asymp \int_0^1 \|f_r\|_{H^p}^p \eta(r)\,dr \ge C \sum_{n=0}^{\infty} \int_{1-\frac{1}{k^{n+1}}}^{1-\frac{1}{k^{n+2}}} \left(\sup_{j \in \mathbb{N}} r^{k^{j+1}} \|V_{j,k} * f\|_{H^p} \right)^p \eta(r)\,dr \\ &\ge C \sum_{n=0}^{\infty} \|V_{n,k} * f\|_{H^p}^p \int_{1-\frac{1}{k^{n+1}}}^{1-\frac{1}{k^{n+2}}} r^{k^{n+1}p} \eta(r)\,dr \qquad (4.3) \\ &\ge C \sum_{n=0}^{\infty} \|V_{n,k} * f\|_{H^p}^p \left(\widehat{\eta}\left(1 - \frac{1}{k^{n+1}}\right) - \widehat{\eta}\left(1 - \frac{1}{k^{n+2}}\right) \right), \quad f \in \mathcal{H}(\mathbb{D}). \end{aligned}$$

Since (2.1) holds for k and η, there exists a constant $C = C(\eta) > 1$ such that $\widehat{\eta}\left(1 - \frac{1}{k^{n+1}}\right) \ge C \widehat{\eta}\left(1 - \frac{1}{k^{n+2}}\right)$, which together with (4.3) and Lemma A(iii), yields

$$\|f\|_{A^p_\eta}^p \gtrsim \sum_{n=0}^{\infty} \|V_{n,k} * f\|_{H^p}^p \widehat{\eta}\left(1 - \frac{1}{k^{n+2}}\right) \asymp \sum_{n=0}^{\infty} \|V_{n,k} * f\|_{H^p}^p \eta_{k^n}, \quad f \in \mathcal{H}(\mathbb{D}). \qquad (4.4)$$

In order to show the reverse inequality, we distinguish two cases according to the range of p. If $0 < p \le 1$, by using (2.4) and [11, Lemma 3.1] we obtain

$$\begin{aligned} \|f_r\|_{H^p}^p = \left\| \sum_{n=0}^{\infty} V_{n,k} * f_r \right\|_{H^p}^p &\lesssim \sum_{n=0}^{\infty} \|V_{n,k} * f_r\|_{H^p}^p \\ &\lesssim \|V_{0,k} * f\|_{H^p}^p + \sum_{n=1}^{\infty} \|V_{n,k} * f\|_{H^p}^p \, r^{k^{n-1}p}, \quad f \in \mathcal{H}(\mathbb{D}), \end{aligned}$$

and therefore by Lemma A (v),

$$\begin{aligned} \|f\|_{A^p_\eta}^p &\lesssim \|V_{0,k} * f\|_{H^p}^p \widehat{\eta}(0) + \sum_{n=1}^{\infty} \|V_{n,k} * f\|_{H^p}^p \int_0^1 r^{k^{n-1}p} \eta(r)\,dr \\ &\lesssim \sum_{n=0}^{\infty} \|V_{n,k} * f\|_{H^p}^p \, \eta_{k^n}, \quad f \in \mathcal{H}(\mathbb{D}),\ 0 < p \le 1. \end{aligned} \qquad (4.5)$$

On the other hand, if $1<p<\infty$, by (2.4) and [11, Lemma 3.1], we obtain

$$\begin{aligned}\|f_r\|_{H^p}^p &= \left\|\sum_{n=0}^{\infty} V_{n,k}*f_r\right\|_{H^p}^p \le \left(\sum_{n=0}^{\infty}\|V_{n,k}*f_r\|_{H^p}\right)^p \\ &\lesssim \left(\|V_{0,k}*f\|_{H^p}+\sum_{n=1}^{\infty}\|V_{n,k}*f\|_{H^p} r^{k^{n-1}}\right)^p \\ &\lesssim \|V_{0,k}*f\|_{H^p}^p+\left(\sum_{n=1}^{\infty}\|V_{n,k}*f\|_{H^p} r^{k^{n-1}}\right)^p, \quad f\in\mathcal{H}(\mathbb{D}).\end{aligned}$$

The above chain of inequalities together with Proposition 8 yields

$$\begin{aligned}\|f\|_{A^p_\eta}^p &\lesssim \|V_{0,k}*f\|_{H^p}^p\widehat{\eta}(0)+\int_0^1\left(\sum_{n=1}^{\infty}\|V_{n,k}*f\|_{H^p} r^{k^{n-1}}\right)^p\eta(r)\,dr \\ &\lesssim \sum_{n=0}^{\infty}\|V_{n,k}*f\|_{H^p}^p\,\eta_{k^n}, \quad f\in\mathcal{H}(\mathbb{D}),\ 1<p<\infty.\end{aligned} \tag{4.6}$$

Consequently, joining (4.4), (4.5) and (4.6), the proof is finished. □

With Proposition 9 in hand, we are able to prove that the space of analytic functions $D^p_{\omega,\widehat{\mu}}=\{f\in\mathcal{H}(\mathbb{D})\colon \int_{\mathbb{D}}|D^\mu(f)(z)|^p\widehat{\mu}(z)^p\omega(z)\,dA(z)<\infty\}$ is continuuosly embedded into A^p_ω when $\omega\in\mathcal{D}$ and $\mu\in\widehat{\mathcal{D}}$. This result together with Theorem 2 proves that (1.5) holds when $\omega\in\mathcal{D}$.

Theorem 10. *Let $\omega\in\mathcal{D}$, $0<p<\infty$ and $\mu\in\widehat{\mathcal{D}}$. Then there exists $C=C(\omega,\mu,p)>0$ such that*

$$\|f\|_{A^p_\omega}\le C\|D^\mu(f)\|_{A^p_{\omega\widehat{\mu}^p}}, \quad f\in\mathcal{H}(\mathbb{D}).$$

Proof. By Lemma B(iii) there exists $k=k(\omega)>1$, $k\in\mathbb{N}$ such that (2.1) holds for k and ω. Next, Lemma 3 (ii) ensures that $\omega\widehat{\mu}^p\in\mathcal{D}$ and Lemma 3 (i) implies that $\omega\widehat{\mu}^p$ satisfies (2.1) with the same k as ω does. Therefore, we can apply Proposition 9 to the weights $\omega,\omega\widehat{\mu}^p\in\mathcal{D}$ and the choosen k. That is, there are positive constants $C_j(\omega,p)>0$, $j=1,2$ such that

$$C_1(\omega,p)\sum_{n=0}^{\infty}\omega_{k^n}\|V_{n,k}*f\|_{H^p}^p\le\|f\|_{A^p_\omega}^p\le C_2(\omega,p)\sum_{n=0}^{\infty}\omega_{k^n}\|V_{n,k}*f\|_{H^p}^p, \quad f\in\mathcal{H}(\mathbb{D}), \tag{4.7}$$

and $C_j(\omega,\mu,p)>0$, $j=3,4$ such that

$$\begin{aligned}C_3(\omega,\mu,p)\sum_{n=0}^{\infty}(\omega\widehat{\mu}^p)_{k^n}\|V_{n,k}*D^\mu f\|_{H^p}^p &\le \|D^\mu f\|_{A^p_{\omega\widehat{\mu}^p}}^p \\ &\le C_4(\omega,\mu,p)\sum_{n=0}^{\infty}(\omega\widehat{\mu}^p)_{k^n}\|V_{n,k}*D^\mu f\|_{H^p}^p,\end{aligned} \tag{4.8}$$

for all $f\in\mathcal{H}(\mathbb{D})$.

Observe that for each $n\in\mathbb{N}$, $(V_{n,k}*f)(z)=\sum_{j=k^{n-1}}^{k^{n+1}-1}\widehat{f}(j)\widehat{V_{n,k}}(j)z^j$ and

$$(V_{n,k}*D^\mu f)(z)=\sum_{j=k^{n-1}}^{k^{n+1}-1}\frac{\widehat{f}(j)}{\mu_{2j+1}}\widehat{V_{n,k}}(j)z^j.$$

So, applying [19, Lemma 9(i)] to $g = V_{n,k} * D^\mu(f)$, $h = V_{n,k} * f$ and $S_{k^{n-1},k^{n+1}-1}h = V_{n,k} * f$, there exists a constant $C = C(p) > 0$ such that

$$\|V_{n,k} * f\|_{H^p} \le C\mu_{k^{n-1}}\|V_{n,k} * D^\mu(f)\|_{H^p}, \quad f \in \mathcal{H}(\mathbb{D}). \tag{4.9}$$

A similar argument shows that

$$\|V_{0,k} * f\|_{H^p} \le C\mu_0\|V_{0,k} * D^\mu(f)\|_{H^p}, \quad f \in \mathcal{H}(\mathbb{D}). \tag{4.10}$$

Moreover, by Lemma A and (2.1) (for k and ω), there is $C = (\omega, \mu, p) > 0$ such that

$$\begin{aligned}\omega_{k^n}\mu^p_{k^{n-1}} &\le C\omega_{k^n}\mu^p_{k^{n+1}} \le C\widehat{\omega}\left(1-\frac{1}{k^n}\right)\widehat{\mu}\left(1-\frac{1}{k^{n+1}}\right)^p\\ &\le C\widehat{\mu}\left(1-\frac{1}{k^{n+1}}\right)^p\int_{1-\frac{1}{k^n}}^{1-\frac{1}{k^{n+1}}}\omega(s)\,ds \le C\int_{1-\frac{1}{k^n}}^{1-\frac{1}{k^{n+1}}}\omega(s)\widehat{\mu}(s)^p\,ds\\ &\le C(\omega\widehat{\mu}^p)_{k^n}, \quad \text{for all } n \in \mathbb{N}.\end{aligned} \tag{4.11}$$

Then, by joining (4.7), (4.9), (4.10), (4.11) and (4.8),

$$\begin{aligned}\|f\|^p_{A^p_\omega} &\le C\sum_{n=0}^\infty \omega_{k^n}\|V_{n,k} * f\|^p_{H^p}\\ &\le C\left[\omega_1\mu_0^p\|V_{0,k} * D^\mu(f)\|^p_{H^p} + \sum_{n=1}^\infty \omega_{k^n}\mu^p_{k^{n-1}}\|V_{n,k} * D^\mu(f)\|^p_{H^p}\right]\\ &\le C\sum_{n=0}^\infty(\omega\widehat{\mu}^p)_{k^n}\|V_{n,k} * D^\mu(f)\|^p_{H^p} \le C\|D^\mu(f)\|^p_{A^p_{\omega\widehat{\mu}^p}}, \quad f \in \mathcal{H}(\mathbb{D}).\end{aligned}$$

This finishes the proof. □

In order to complete a proof of Theorem 1 we have to show that $\omega \in \mathcal{D}$ is a necessary condition so that (1.5) holds. This implication will follow from Theorem 2 and the next result.

Theorem 11. *Let ω be a radial weight, $0 < p < \infty$ and $\mu \in \mathcal{D}$. If there exists $C = C(\omega, \mu, p) > 0$ such that*

$$\|f\|^p_{A^p_\omega} \le C\int_{\mathbb{D}} |D^\mu(f)(z)|^p\omega(z)\widehat{\mu}(z)^p\,dA(z), \quad f \in \mathcal{H}(\mathbb{D}), \tag{4.12}$$

then $\omega \in \mathcal{M}$.

Proof. By choosing $f_n(z) = z^n \in \mathcal{H}(\mathbb{D})$, $n \in \mathbb{N}$ in (4.12), we deduce

$$\begin{aligned}\int_{\mathbb{D}} |z|^{np}\omega(z)\,dA(z) &\le C^p\int_{\mathbb{D}}\frac{|z|^{np}}{\mu^p_{2n+1}}\widehat{\mu}(z)^p\omega(z)\,dA(z)\\ &\le C^p\int_{\mathbb{D}}\frac{|z|^{np}}{\widehat{\mu}\left(1-\frac{1}{n+1}\right)^p}\widehat{\mu}(z)^p\omega(z)\,dA(z), \quad n \in \mathbb{N}.\end{aligned}$$

Now, if $x \geq 1$, we can find $m \in \mathbb{N}$ such that $m \leq x < m+1$. Then, bearing in mind the monotonicity of s^x and $\widehat{\mu}(s)$

$$\begin{aligned}\int_0^1 s^{xp+1}\omega(s)\,ds &\leq \int_0^1 s^{mp+1}\omega(s)\,ds \leq C^p \int_0^1 \frac{s^{mp+1}}{\widehat{\mu}\left(1-\frac{1}{m+1}\right)^p}\widehat{\mu}(s)^p\omega(s)\,ds \\ &\leq C^p \int_0^1 \frac{s^{(m+1)p+1}}{\widehat{\mu}\left(1-\frac{1}{m+1}\right)^p}\widehat{\mu}(s)^p\omega(s)\,ds \\ &\leq C^p \int_0^1 \frac{s^{xp+1}}{\widehat{\mu}\left(1-\frac{1}{x+1}\right)^p}\widehat{\mu}(s)^p\omega(s)\,ds, \quad x \geq 1.\end{aligned}$$

That is, there exists $C = C(p,\omega,\mu) > 1$ such that

$$\int_0^1 s^{xp+1}\omega(s)\left(\frac{1}{C^p} - \left(\frac{\widehat{\mu}(s)}{\widehat{\mu}\left(1-\frac{1}{x+1}\right)}\right)^p\right)ds \leq 0.$$

Since $\widehat{\mu}\left(1-\frac{1}{x+1}\right) < C\widehat{\mu}(0)$ for all $x \geq 1$, there exists $s_x = s_x(x,C,\mu)$, the supremum of the points $s \in (1-\frac{1}{x},1)$ such that $\frac{\widehat{\mu}(s)}{\widehat{\mu}\left(1-\frac{1}{x+1}\right)} = \frac{1}{C}$. Then,

$$\begin{aligned}&\int_{s_x}^1 s^{xp+1}\omega(s)\left(\frac{1}{C^p} - \left(\frac{\widehat{\mu}(s)}{\widehat{\mu}\left(1-\frac{1}{x+1}\right)}\right)^p\right)ds \\ &\leq \int_0^{s_x} s^{xp+1}\omega(s)\left(\left(\frac{\widehat{\mu}(s)}{\widehat{\mu}\left(1-\frac{1}{x+1}\right)}\right)^p - \frac{1}{C^p}\right)ds, \quad x \geq 1.\end{aligned}$$

There also exists $r_x = r_x(x,C,\mu) \in (s_x,1)$ the supremum of the points $r \in (1-\frac{1}{x},1)$ such that $\frac{\widehat{\mu}(r)}{\widehat{\mu}\left(1-\frac{1}{x+1}\right)} = \frac{1}{C}\left(\frac{1}{2}\right)^{\frac{1}{p}}$. So,

$$\begin{aligned}&\int_{s_x}^1 s^{xp+1}\omega(s)\left(\frac{1}{C^p} - \left(\frac{\widehat{\mu}(s)}{\widehat{\mu}\left(1-\frac{1}{x+1}\right)}\right)^p\right)ds \\ &\geq \frac{1}{2C^p}\int_{r_x}^1 s^{xp+1}\omega(s)\,ds = \frac{1}{2C^p}\omega_{xp+1} - \frac{1}{2C^p}\int_0^{r_x} s^{xp+1}\omega(s)\,ds, \quad x \geq 1.\end{aligned}$$

Therefore,

$$\begin{aligned}\omega_{xp+1} &\leq 2C^p \int_0^{s_x} s^{xp+1}\omega(s)\left(\left(\frac{\widehat{\mu}(s)}{\widehat{\mu}\left(1-\frac{1}{x+1}\right)}\right)^p - \frac{1}{C^p}\right)ds + \int_0^{r_x} s^{xp+1}\omega(s)\,ds \\ &\leq 2C^p \int_0^{r_x} s^{xp+1}\omega(s)\left(\frac{\widehat{\mu}(s)}{\widehat{\mu}\left(1-\frac{1}{x+1}\right)}\right)^p ds + \int_0^{r_x} s^{xp+1}\omega(s)\,ds, \quad x \geq 1.\end{aligned} \tag{4.13}$$

Next, by Lemma A(ii), there exist $C_1 = C_1(\mu) > 1$ and $\alpha = \alpha(\mu) > 0$ such that

$$\begin{aligned}\widehat{\mu}(s) &\leq C_1\widehat{\mu}(r_x)\left(\frac{1-s}{1-r_x}\right)^\alpha \\ &= C_1\left(\frac{1}{2}\right)^{\frac{1}{p}}\frac{1}{C}\widehat{\mu}\left(1-\frac{1}{x+1}\right)\left(\frac{1-s}{1-r_x}\right)^\alpha, \quad 0 < s \leq r_x,\end{aligned} \tag{4.14}$$

where in the last identity we have used the definition of r_x. Consequently, putting together (4.13) and (4.14)

$$\omega_{xp+1} \le \frac{C_1^p}{(1-r_x)^{\alpha p}} (\omega_{[\alpha p]})_{xp+1}, \quad x \ge 1. \tag{4.15}$$

On the other hand, by Lemma B (ii) there exist $C_2 = C_2(\mu) > 0$ and $\beta = \beta(\mu) > 0$ such that

$$\left(\frac{1}{2}\right)^{\frac{1}{p}} \frac{1}{C} = \frac{\widehat{\mu}(r_x)}{\widehat{\mu}\left(1-\frac{1}{x+1}\right)} \le C_2(2x(1-r_x))^\beta,$$

so there is $C_3 = C_3(p, \omega, \mu) > 0$ such that $\frac{1}{1-r_x} \le C_3 x$. This together with (4.15) implies that there is $C = C(\omega, \mu, p) > 0$ such that

$$\omega_{xp+1} \le C x^{\alpha p} (\omega_{[\alpha p]})_{xp+1}, \quad x \ge 1.$$

That is,

$$\omega_y \le C \left(\frac{y-1}{p}\right)^{\alpha p} (\omega_{[\alpha p]})_y \le C \left(\frac{1}{p}\right)^{\alpha p} y^{\alpha p} (\omega_{[\alpha p]})_y, \quad y \ge p+1,$$

which together with Lemma C implies that $\omega \in \mathcal{M}$. This finishes the proof. □

Finally, we are ready to prove Theorem 1.

Proof of Theorem 1. If $\omega \in \mathcal{D}$, putting together Theorem 2 and Theorem 10, we get (1.5). Reciprocally, if (1.5) holds, $\omega \in \widehat{\mathcal{D}}$ by Theorem 2 and $\omega \in \mathcal{M}$ by Theorem 11. Then, it follows from [20, Theorem 3] that $\omega \in \widehat{\mathcal{D}} \cap \mathcal{M} = \mathcal{D}$.
This finishes the proof. □

We would like to point out that it would be interesting to obtain some progress about Littlewood–Paley inequalities for fractional derivatives on Bergman spaces A^p_ω induced by a non-radial weight ω. For instance, to know whether or not (1.5) ($\mu \in \mathcal{D}$) remains true for Bekollé–Bonami weights.

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José Ángel Peláez
Universidad de Málaga
Departamento de Análisis Matemático
Campus de Teatinos, 29071 Málaga, Spain
japelaez@uma.es

Elena de la Rosa
Universidad de Málaga
Departamento de Análisis Matemático
Campus de Teatinos, 29071 Málaga, Spain
elena.rosa@uma.es