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Quasiconformality to quasisymmetry via weak (L, M)-quasisymmetry

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Abstract. This paper is devoted to the study of a fundamental problem in the theory of quasiconformal analysis: under what conditions local quasiconformality of a homeomorphism implies its global quasisymmetry. In particular, we introduce the concept of weak (L, M)-quasisymmetry, serving as a bridge between local quasiconformality and global quasisymmetry. We show that in general metric spaces local regularity and some connectivity together with the Loewner condition are sufficient for a quasiconformal map to be weakly (L, M)-quasisymmetric, and subsequently, quasisymmetric with respect to the internal metrics.

Kvasikonformisuudesta kvasisymmetriaan heikon (L, M)-kvasisymmetrian kautta

Tiivistelmä. Tämä työ on omistettu kvasikonformisen analyysin peruskysymykselle: millä ehdoilla homeomorfismin paikallisesta kvasikonformisuudesta seuraa sen globaali kvasisymmetria. Erityisesti otamme käyttöön heikon (L, M)-kvasisymmetrian käsitteen, joka toimii siltana paikallisen kvasikonformisuuden ja globaalin kvasisymmetrian välillä. Osoitamme, että yleisissä metrisissä avaruuksissa paikallinen säännöllisyys, tietty yhtenäisyys sekä Loewnerin ehto yhdessä riittävät takaamaan, että kvasikonforminen kuvaus on heikosti (L, M)-kvasisymmetrinen ja tämän seurauksena kvasisymmetrinen sisäisten metriikoiden suhteen.

1. Introduction and preliminaries

The concept of quasiconformality has evolved from a solution of the Beltrami equation in a domain on the complex plane to general metric space settings. Its theory in $\mathbb{R}^n$ is well established and well known (see [1, 10, 25]) while, in the setting of non-abelian Carnot groups, quasiconformal maps first appeared in [20]. Later, Väisälä (see [27, 28, 29, 30]) developed a dimension-free theory of quasiconformal mappings in infinite dimensional Banach spaces. In general setting, let X and Y be metric spaces with metrics d_X and d_Y, respectively. Following [14], a homeomorphism $f\colon X \to Y$ is said to be K-*quasiconformal* (with $K \geq 1$) if for each $x \in X$,

$$\limsup_{r\to 0} \frac{\max\{d_Y(f(a), f(x))\colon d_X(a, x) \leq r\}}{\min\{d_Y(f(b), f(x))\colon d_X(b, x) \geq r\}} \leq K. \tag{1.1}$$

The above limit, often denoted by $H_f(x)$, if referred to as the local dilatation of f at x.

Quasisymmetric homeomorphisms were introduced by Beurling and Ahlfors (see [1, 4]) as the boundary values of quasiconformal homeomorphisms of the upper half plane onto itself. Later, Tukia and Väisälä (see [22]) generalized the definition

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of quasisymmetry to general metric spaces. Following [22], for a homeomorphism $f\colon X \to Y$, if there exists a homeomorphism $\eta\colon [0, +\infty) \to [0, +\infty)$ such that

$$d_X(x, a) \le t d_X(x, b) \Rightarrow d_Y(f(x), f(a)) \le \eta(t) d_Y(f(x), f(b)) \tag{1.2}$$

for each $t > 0$ and for any $x, a, b \in X$, then f is said to be *η-quasisymmetric* (or η-QS). In particular, if there is a constant $H \ge 1$ such that

$$d_X(x, a) \le d_X(x, b) \Rightarrow d_Y(f(x), f(a)) \le H d_Y(f(x), f(b)) \tag{1.3}$$

for all $x, a, b \in X$, then f is said to be *weakly H-quasisymmetric* or simply H-quasisymmetric. It follows easily that

$$\eta\text{-QS} \Rightarrow H\text{-QS} \Rightarrow K\text{-QC} \tag{1.4}$$

with $H = \eta(1)$ and $K = H$.

1.1. Motivation. In the study of quasiconformal analysis, a fundamental problem is to investigate under what conditions the above implications can be reversed. In the case when $X = Y = \mathbb{R}^n$ ($n \ge 2$), it is a well known and celebrated fact that quasiconformality always induces quasisymmetry (and hence H-quasisymmetry) (see [8] for $n = 2$ and [25] for $n \ge 3$) while for $\mathbb{R}$, the statement is false (see [14]). Much attention has been drawn to the circumstance other than (in general, sufficiently distinct from) $\mathbb{R}^n$. In this circumstance, things become much more complicated even when X and Y are subdomains in $\mathbb{R}^n$ (see previous results mentioned below). The difficulty lies in the lack of appropriate ingredients that make the transition possible from the local infinitesimal condition of quasiconformality to the global condition of quasisymmetry.

There are two lines of investigation in pursuit of reversing the implication in (1.4). One is to show that, under certain geometric conditions on X and Y, a QC map f is H-QS. Then Väisälä's theory on HTB spaces (homogeneously totally bounded, see [26]) induces that such a map is indeed η-QS. For example, in 1995 Heinonen and Koskela [13, Theorem 1.7] showed that if X is a Carnot group of homogeneous dimension Q and Y is a c-LLC metric space that carries a Q-regular measure μ, then a K-QC map $f\colon X \to Y$ is η-QS. Later, they introduced the crucial concept of Loewner space and established the following results [14, Corollaries 4.8 & 4.10, Theorem 4.9].

Theorem A. *Suppose that X and Y are Q-regular metric spaces with $Q > 1$, that X is a Loewner space. Assume X and Y are simultaneously bounded or unbounded. Then a quasiconformal map f from X onto Y that maps bounded sets to bounded sets is η-QS if and only if Y is LLC.*

Under the same assumptions as above for metric spaces X and Y, Balogh and Koskela [3, Theorem 3.1] showed that a homeomorphism $f\colon X \to Y$ with a weaker local dilatation property than (1.1) is also H-QS (and hence η-QS). Tyson [24, Theorem 10.9] established a local version of this result for maps between domains in X and Y.

All the above mentioned results deal with QS and QC maps under the same original metrics of X and Y. A different line of investigation in the study of the reversion of the implications in (1.4) is to determine under what conditions a QC map (with respect to the original metrics) is QS with respect to the *internal metrics*. Of course, in this case, connectivity is needed for the definition of internal metric (see [26]). Along this line, the following result was established by Väisälä [26].

Theorem B. *Suppose that $f\colon D \to D'$ is a K-quasiconformal map between domains in $\mathbb{R}^n$, where D is φ-broad. Suppose also that $A \subset D$ is a pathwise connected set and that $f(A)$ has the c_1-carrot property in D' with center $y_0 \in \overline{D'}$. If $y_0 \neq \infty$ and hence $y_0 \in D'$, we assume that $d(A) \leq c_2 \cdot d(f^{-1}(y_0), \partial D)$. If $y_0 = \infty$, we assume that f extends to a homeomorphism $D \cup \{\infty\} \to D' \cup \{\infty\}$. Then $f|_A$ is η-QS with respect to the internal metrics δ_D and $\delta_{D'}$ with η depending only on the data $(c_1, c_2, K, \varphi, n)$.*

Another result along this line is due to Heinonen [11, Theorem 6.1 & 6.5] who weakened the above carrot condition for domains to LLC_2 (see definitions in the next subsection) and proved the following theorem.

Theorem C. *Suppose that $f\colon D \to D'$ is K-quasiconformal where D and D' are bounded domains (or unbounded domains with $f(\infty) = \infty \in D \cap D'$) in $\mathbb{R}^n$, and that D is $\varphi-$broad. If $A \subset D$ (and $\infty \in A$ for unbounded domains) is such that $f(A)$ is b-LLC_2 with respect to $\delta_{D'}$ in D', then the restriction $f|_A\colon A \to f(A)$ is weakly H-quasisymmetric in the metrics δ_D and $\delta_{D'}$ with H depending only on some associated data.*

More recently, to complete the reversion of (1.4) in this setting, in [16] and [6] the authors derived quasisymmetry from weak H-quasisymmetry and, at the same time, eliminated the above LLC_2 condition on $f(A)$. More specifically, the following theorem is established in [16].

Theorem D. *Suppose that $f\colon D \to D'$ is K-quasiconformal where D and D' are domains in $\mathbb{R}^n$, and that D is φ-broad. For an arcwise connected set A in D, if the restriction $f|_A : A \to f(A)$ is weakly H-quasisymmetric in the metrics δ_D and $\delta_{D'}$, then $f|_A\colon A \to f(A)$ is η-quasisymmetric in the metrics δ_D and $\delta_{D'}$ with η depending only on the data (n, K, H, φ).*

Note that in general the inverse of a weakly quasisymmetric homeomorphism is not necessarily weakly quasisymmetric. In [6, Theorem 1.1], using a different approach, the authors proved that the above theorem remains true if the target domain D', instead of the source domain D, is assumed to be φ-broad.

We want to point out that the crucial step in the proof of both Theorem A and Theorem B is to show that under given conditions a QC map f is weakly H-QS (with respect to original metrics in Theorem A and internal metrics in Theorem B). Then the results follow from [26, Theorem 2.9] by observing that the metric spaces involved have the HTB property. From the proofs of Theorems A and B and similar results, one can see that H-quasisymmetry plays a crucial transitioning role in proving a QC map is η-QS and the HTB property is key to go from H-QS to η-QS. To be able to apply to more general settings, we propose to introduce a more general class, called weakly (L, M)-quasisymmetric maps, as a new transition between QC and QS.

The main goal of this paper is to establish an approach to this second line of investigation and to push it to general metric space settings while weakening the global regularity condition on the Loewner space, which has always been assumed in the first line of investigation discussed above, to just local regularity.

1.2. Statement of main results. One of the main results is about the (L, M)-quasisymmetry in the internal metrics of a QC map in metric spaces, which can be stated as follows.

Theorem 1.1. *Suppose that X and Y are arcwise connected metric spaces which are simultaneously bounded or unbounded, that X is locally Q-regular with $Q > 1$, that Y is upper half Q-regular. Assume that X is a φ-Loewner space, Y is c-LLC$_2$ and locally $\widehat{c}$-BT, f from X onto Y is a K-quasiconformal homeomorphism which maps bounded sets to bounded sets. Then for any $L > 0$, there exists a constant $M > 0$ such that f is weakly (L, M)-QS in the internal metrics, where M depends on L and on the regularity constants and the data $(Q, K, c, \widehat{c}, \varphi)$, and possibly on f if both X and Y are bounded. In particular, f is weakly H-QS in the internal metrics where H depends on the regularity constants and the data $(Q, K, c, \widehat{c}, \varphi)$, and possibly on f if both X and Y are bounded.*

To the knowledge of the authors, this is the first result in general metric space setting on the H-quasisymmetry (in the internal metrics) of a QC map. Restricting to domains in $\mathbb{R}^n$, one immediately obtains the following corollary, which is a stronger version of Theorem C for the case $A = D$.

Corollary 1.2. *Suppose that $D, D' \subset \mathbb{R}^n$ are domains simultaneously bounded or unbounded and $f\colon D \to D'$ is a K-quasiconformal homeomorphism which maps bounded sets to bounded sets. Furthermore, let D be φ-Loewner and D' be c-LLC$_2$. Then for any $L > 0$, there exists a constant $M > 0$ such that f is weakly (L, M)-QS in the internal metrics, where (L, M) depends on (n, K, c, φ) and possibly on f if both D and D' are bounded. In particular, f is weakly H-QS in the internal metrics, where H depends only on (n, K, c, φ) and possibly on f if both D and D' are bounded.*

Built on Theorem 1.1, another main result is about the η-quasisymmetry (in the internal metrics) of a QC map.

Theorem 1.3. *Suppose that X and Y are arcwise connected metric spaces which are simultaneously bounded or unbounded, that X is locally Q-regular with $Q > 1$, that Y is Q-regular. Assume that X is a φ-Loewner space, Y is c-LLC, and f from X onto Y is a K-quasiconformal homeomorphism which maps bounded sets to bounded sets. Then f is η-QS in the internal metrics where η depends on the regularity constants and the data Q, K, φ, c, and possibly on f if both X and Y are bounded.*

Note that LLC condition implies LLC$_2$ and locally BT while the converse may not be true. Thus Theorem 1.3 requires a stronger geometric condition on the space Y than in Theorem 1.1, but arrives at a stronger conclusion on the map f that it is η-QS.

1.3. Basic concepts. In this subsection, we give some general notation and definitions in a metric measure space. Here and in what follows, all the measures will be assumed to be Borel regular and to assign a finite measure to every bounded measurable set. In a metric space, we will denote the open ball centered at x with radius r by $B(x, r)$ (or simply B_r). For any ball B, let $\overline{B}$ denote its closure and lB the ball with the same center as B of radius l times the original radius. For a homeomorphism $f\colon X \to Y$ between metric spaces, the prime symbol always indicates the image under f. For instance, $x' = f(x)$ for $x \in X$ and $E = f^{-1}(E')$ for a point or a set E' in Y.

1.3.1. Line integral. A curve γ is a continuous map of an interval $I \subset \mathbb{R}$ into a metric space X. For a curve γ defined on a compact interval $[a, b]$, which also called a compact curve, its length is defined by

$$l(\gamma) = \sup_T \sum_{i=1}^{n-1} d_X(\gamma_{t_{i+1}}, \gamma_{t_i}),$$

where the supremum is taken over all partitions $T = \{a = t_1 < t_2 < \cdots < t_{n-1} < t_n = b\}$ of $[a, b]$. In general, the length of a curve γ is defined to be the supremum of the length over all compact subcurve. We call γ rectifiable or locally rectifiable if $l(\gamma) < +\infty$ or any compact subcurve is rectifiable, respectively.

For any rectifiable compact curve γ in X, the line integral over γ of each non-negative Borel function $\rho\colon X \to [0, +\infty]$ is defined as

$$\int_\gamma \rho\, ds = \int_0^{l(\gamma)} \rho \circ \gamma_s(t)\, dt,$$

where $\gamma_s\colon [0, l(\gamma)] \to X$ is the arc length parametrization of γ. If γ is locally rectifiable, the line integral is defined to be the supremum of the values of line integrals over all rectifiable compact subcurves.

1.3.2. Modulus. Let (X, μ) be a metric measure space and Γ be a curve family in X. A Borel function $\rho\colon X \to [0, +\infty]$ is said to be admissible if $\int_\gamma \rho\, ds \geq 1$ for any locally rectifiable curve $\gamma \in \Gamma$. For any $p > 0$, the p-modulus of Γ in X is defined as

$$\operatorname{mod}_p(\Gamma) = \inf \int_X \rho^p\, d\mu,$$

where the infimum is taken over all non-negative admissible function ρ for curve family Γ. The classic definition and basic properties of modulus in the plane and in space $\mathbb{R}^n$ can be fund in [1, 2, 25]. Modulus is very closely related to extremal length and capacity (see [14, 15, 21, 31]), and they play crucial rules in geometric function theory and PDE theory.

1.3.3. Loewner space.

Definition 1.4. Suppose that (X, μ) is a metric measure space of Hausdorff dimension Q. X is called a Loewner (or φ-Loewner) space if there exists a non-increasing function $\varphi\colon (0, +\infty) \to (0, +\infty)$ such that

$$\operatorname{mod}_Q(E, F; X) \geq \varphi(t)$$

for any two disjoint nondegenerate continua E and F in X with

$$t \geq \Delta(E, F) = \frac{dist(E, F)}{\min\{\operatorname{diam}(E), \operatorname{diam}(F)\}}.$$

Heinonen and Koskela (see [14]) first introduced the concept of Loewner space for it was Loewner (see [19]) who first noticed and established the qualitative form for modulus in Euclidean n-space with $n \geq 3$, while Gehring (see [9]) proved the quantitative version (see also [25]). In $\mathbb{R}^2$, this classic estimate can be deduced from Grötzsch extremal problem and Teichmüller extremal problem (see [1, 2, 18]).

In the original definition of Loewner space in [14], the function φ is not assumed to be non-increasing. Heinonen and Koskela [14, Theorem 3.6] proved that Q-regularity implies there exists a decreasing homeomorphism $\psi\colon (0, +\infty) \to (0, +\infty)$ such that $\operatorname{mod}_Q(E, F; X) \geq \psi(\Delta(E, F))$. Furthermore, $\psi(t)$ has the asymptotic property $\log \frac{1}{t}$ and $(\log t)^{1-Q}$ as t tending to 0 and $+\infty$, respectively. Note that such a function ψ may not exist if globally Q-regular condition is not satisfied (see [14, Remark 3.28]).

In this paper, for the lack of global regularity, we assume in the definition that φ is non-increasing (see also [24]).

Euclidean space $\mathbb{R}^n$ and Heisenberg groups are Loewner spaces while a subdomain of $\mathbb{R}^n$ need not be. Loewner space is intimately connected with Poincaré inequality. It is shown in [14] that, under some connectivity condition, a proper Q-regular space is Loewner if and only if it supports some Poincaré inequality. This concept plays a central role in modern geometric analysis on metric spaces.

1.3.4. Regularity conditions. A metric space X endowed with a Borel measure μ (or write (X,μ)) is called an Ahlfors–David regular space of dimension Q (or *Q-regular space*) if there exists a constant $C_0 \geq 1$ such that

$$\frac{1}{C_0} r^Q \leq \mu(B_r) \leq C_0 r^Q \tag{1.5}$$

for each ball B_r in X with $0 < r < \operatorname{diam}(X)$. If only the left (or right) side of the Q-regular condition is satisfied, we say that μ satisfies the *lower (or upper) mass bound* with constant $\frac{1}{C_0}$ (or C_0), respectively.

A metric measure space (X,μ) is called *locally Q-regular* if there exists a constant $C_0 \geq 1$ such that for any point $x \in X$, there exist positive number $r_x < \operatorname{diam}(X)$ such that the lower bound in (1.5) is satisfied for all balls $B_r = B(x,r)$ with $r \leq r_x$ and the upper bound in (1.5) is satisfied for all balls $B(y,r) \subset B(x,r_x)$. The supremum of such r_x, denoted by $\widetilde{r}_x$, is called the *radius of local regularity* at x. X is called *upper half Q-regular* if it is locally Q-regular and μ satisfies the global upper mass bound with constant C_0.

As pointed out in [23], in a locally compact Q-regular metric space (X,μ), any measure satisfying the above inequality, possibly with different constant C_0, is comparable to μ. Thus one often says a metric space is Q-regular without specifying the measure. A metric measure space (X,μ) is said to be doubling with constant C_μ if $\mu(B_{2r}) \leq C_\mu \cdot \mu(B_r)$ for all balls $B_r \subset X$ with $0 < r < \operatorname{diam}(X)$. Clearly, if (X,μ) is Q-regular then it is doubling. The converse, as pointed out in [14], need not be true. Also the converse of the implications Q-regular $\Rightarrow$ upper half Q-regular $\Rightarrow$ locally Q-regular need not be true as well. It is also worth noting that our definition of upper half Q-regular is exactly the same as Tyson's (nonstandard) definition of local Q-regular [24]. Furthermore, Loewner condition in a metric space X with Hausdorff dimension greater than one implies the lower mass bound condition (see [14, Theorem 3.6]).

Finally, Euclidean space $\mathbb{R}^n$, Carnot groups and Heisenberg groups are all Q-regular spaces. As seen in [14], a measure μ with $d\mu(x) = P(|x|)\,dx$, where $P(x)$ is some positive function which grows sufficiently fast as $|x| \to \infty$, may not have upper mass bound. It is worth pointing out that Q ($Q > 1$) is not necessarily an integer for a Q-regular Loewner space (see [5, 17]). Throughout this paper, we always assume that $Q > 1$.

1.3.5. Internal metric. For an arcwise connected metric space X, the *internal metric* δ_X between $x, y \in X$ is defined by

$$\delta_X(x,y) = \inf \operatorname{diam}(\gamma),$$

where the infimum is taken over all arcs γ joining x and y in X. The internal diameter of a set $E \subset X$ is defined by

$$\delta_X(E) = \sup\{\delta_X(x,y) \colon x, y \in E\}.$$

The internal distance between two sets E and F is defined as

$$\delta_X(E,F)=\inf\{\delta_X(x,y)\colon x\in E, y\in F\}.$$

It is obvious and will be used later that $d_X(x,y)\le \delta_X(x,y)$ for any $x,y\in X$.

1.3.6. Linear local connectivity and bounded turning. Suppose that A is a subset of a metric space X and $c\ge 1$. We say that A is c-LLC$_1$ in X if for all $x\in A$ and $r>0$, the points in $A\cap B(x,r)$ can be joined in $A\cap B(x,cr)$. We say that A is c-LLC$_2$ in X if for all $x\in A$ and $r>0$, the points in $A\backslash\overline{B}(x,cr)$ can be joined in $A\backslash\overline{B}(x,r)$. If A is both c-LLC$_1$ and c-LLC$_2$, we say A is c-LLC. The corresponding LLC (linear local connectivity) properties *with respect to the internal metric* δ_X are defined similarly with the balls replaced by internal metric balls such as $B_{\delta_X}(x,r)$.

It is known (see [11]) that if X is LLC$_2$, then it is LLC$_2$ with respect to the internal metric δ_X. However, the converse is not true. It is also worth noting that A need not be connected even if it is b-LLC$_2$ in X. It is shown (see [14]) that Q-regular φ-Loewner space is c-LLC with c depending only on Q,φ and the regularity constant. However, as pointed out in [14, Theorem 3.6], Loewner space without upper mass bound (for example, only locally Q-regular), may not have the LLC property.

Finally, we say that a metric space X is of b-bounded turning (or b-BT) if each pair of points $x,y\in X$ can be joined by an arc $E\subset X$ such that $d(E)\le b\cdot d_X(x,y)$. X is called locally b-BT if for each $x_0\in X$, there exists a neighborhood $U(x_0)$ such that each pair of points $x,y\in U(x_0)$ can be joined by an arc $E\subset X$ with $d(E)\le b\cdot d_X(x,y)$.

1.3.7. Homogeneously totally bounded. As in [22], a metric space X is k-homogeneously totally bounded (or k-HTB) if there exists an increasing function $k\colon[\frac{1}{2},+\infty)\to[1,+\infty)$ such that for any $\alpha\ge\frac{1}{2}$, every closed ball $\overline{B}(x,r)$ in X can be covered by sets $A_1,A_2,\cdots,A_s$ so that $s\le k(\alpha)$ and $d(A_j)<\frac{r}{\alpha}$ for any j. As shown in [26], if D is a bounded k-HTB set, $a_1,a_2,\cdots,a_n$ are distinct points in D which have mutual distances at least $t>0$, then $n\le k\big(\frac{\mathrm{diam}(D)}{t}\big)$. It is obvious that a Q-regular space is k-HTB (in the original metric) where k depends on Q. However, a Q-regular space may not be HTB in the internal metric.

1.4. Outline. This paper is organized as follows. In Section 2 we give some preliminary results and examples to sort out the basic relations amongst the classes of (L,M)-QS, H-QS and η-QS maps. In Section 3, local (L,M)-quasisymmetry of the inverse of a QC map is derived, while in Section 4 it is shown that the same inverse map is globally (L,M)-QS in the internal metrics, completing the proof of Theorem 1.1. Finally, in Section 5 we complete the journey from quasiconformality to η-quasisymmetry by establishing Theorem 1.3. Along the way, a crucial transition is established from (L,M)-quasisymmetry to η-quasisymmetry (Theorem 5.1).

2. Weakly (L,M)-QS homeomorphisms

In an attempt to provide a unified approach in the pursuit from local to global properties, we introduce the concept of weakly (L,M) quasisymmetric homeomorphisms which plays a crucial role in this paper. This is a class of maps more general than the traditional class of weakly H-quasisymmetric homeomorphisms introduced by Tukia and Väisälä [22]. Yet, many results about weakly H-QS homeomorphisms remain true for weakly (L,M)-QS homeomorphisms. As illustrated through this paper, this concept serves as a bridge from local infinitesimal quasiconformality to global quasisymmetry.

2.1. Weak (L, M)-quasisymmetry. Note that when the quasisymmetry condition (1.2) is restricted to the case $t = 1$, one obtains the (weak) H-quasisymmetry condition (1.3). By allowing the parameter t to take on other specific values, we can introduce the class of (weakly) (L, M)-quasisymmetric homeomorphisms as follows.

Definition 2.1. Let X and Y be metric spaces with distance functions $d_X(x, y)$ and $d_Y(x, y)$, respectively, and f be a homeomorphism from X to Y. If there exist constants L and M such that

$$\frac{d_X(a, x)}{d_X(b, x)} \le L \Rightarrow \frac{d_Y(f(a), f(x))}{d_Y(f(b), f(x))} \le M \tag{2.1}$$

for all distinct points $a, b, x \in X$, then f is said to be weakly (L, M)-quasisymmetric (or weakly (L, M)-QS). Furthermore, f is called locally weakly (L, M)-QS if for any point in X, there exists a neighbourhood in which f is weakly (L, M)-QS.

Note that weak H-QS is the same as weak (L, M)-QS with $L = 1$ and $M = H$. It also follows easily from the definitions that if a homeomorphism $f\colon X \to Y$ is η-QS, then it is weakly (L, M)-QS for any $L > 0$ and $M = \eta(L)$, and that if f is weakly (L, M)-QS for some $L \ge 1$, then it is H-QS and hence K-QC with $K = M$. But the converses are not true in general. Furthermore, the *locally* weak H-quasisymmetry or locally weak (L, M)-quasisymmetry may not imply the weak H-quasisymmetry or weak (L, M)-quasisymmetry (see section 2.2 below for examples). In this section, we explore some basic properties of weakly (L, M)-QS homeomorphisms and their relations to other classes of QS maps.

To clarify and simplify the usage of terminologies, we note that the terms (L, M)-QS and weakly (L, M)-QS are interchangeable and mean that (2.1) is satisfied. The same goes with H-QS and weakly H-QS. The adverb weakly is only used sometimes to emphasize that these conditions are indeed *weaker* than the general η-QS condition (1.2).

2.2. The inverse of an (L, M)-QS homeomorphism. It is shown by Tukia and Väisälä (see [22]) that the inverse of an η-QS map is η'-QS with $\eta'(t) = (\eta^{-1}(t^{-1}))^{-1}$, and that the inverse of an H-QS map may not be H'-QS for any H'. For the inverse of an (L, M)-QS homeomorphism, we have the following result.

Lemma 2.2. *Let $f\colon X \to Y$ be weakly (L, M)-quasisymmetric for some $L > 0$ and $M > 0$. Then f^{-1} is weakly (L', M')-quasisymmetric for any $L' < 1/M$ and $M' = 1/L$.*

Proof. Fix three distinct points $a, b, x \in X$ and consider their respective images $a', b', x' \in Y$. Assume that $d_Y(a', x') \le L' d_Y(b', x')$. Need to show that $d_X(a, x) \le M' d_X(b, x)$. Suppose this is not the case. Then, by weak (L, M)-quasisymmetry of f,

$$\frac{d_X(b, x)}{d_X(a, x)} < \frac{1}{M'} = L \;\Rightarrow\; \frac{d_Y(b', x')}{d_Y(a', x')} \le M \quad\Rightarrow\; \frac{d_Y(a', x')}{d_Y(b', x')} \ge \frac{1}{M} > L',$$

which contradicts the assumption that $d_Y(a', x') \le L' d_Y(b', x')$. This proves that f^{-1} is weakly (L', M')-quasisymmetric for any $L' < 1/M$ with $M' = 1/L$. □

2.3. Relation between H-QS and (L, M)-QS. It follows immediately from the definitions that H-quasisymmetry implies (L, M)-quasisymmetry for all $L \le 1$ with $M = H$ and, conversely, (L, M)-quasisymmetry for $L \ge 1$ implies H-quasisymmetry with $H = M$. Other than these apparent containment relations, the following examples show that there are no other general containment relations between these two classes of QS maps.

Example 1. In $\mathbb{R}^2$ consider the points $a_n = a'_n = (n, 0)$, $b_n = (n, 1)$, and $b'_n = \left(n, \frac{1}{2^n}\right)$ $(n = 1, 2, \cdots)$. Let $X = \bigcup_{n=1}^{\infty}\{a_n, b_n\}$ and $Y = \bigcup_{n=1}^{\infty}\{a'_n, b'_n\}$. Define homeomorphism $f\colon X \to Y$ by $f(a_n) = a'_n$ and $f(b_n) = b'_n$. One can verify that f is weakly (L, M)-quasisymmetric for any $L < 1$. On the other hand, since

$$\frac{|a_{n+1} - a_n|}{|a_n - b_n|} = 1 \quad \text{while} \quad \frac{|a'_{n+1} - a'_n|}{|a'_n - b'_n|} = 2^n \to +\infty \quad \text{as } n \to +\infty,$$

f is not weakly H-quasisymmetric for any H.

Example 2. Fix any $H > 1$ and for $n = 1, 2, \cdots$, let

$$a_n = \frac{1}{H^2} + \frac{1}{H^{2^2}} + \cdots + \frac{1}{H^{2^n}},$$

$$a'_n = a_n \text{ for } n \neq 5k + 3 \text{ and } a'_{5k+3} = \frac{1}{2}(a'_{5k+2} + a'_{5k+4}) \ (k = 0, 1, 2, \cdots).$$

Then the map $f\colon \{a_n\} \to \{a'_n\}$ given by $f(a_n) = a'_n$ is H-QS.

On the other hand, for any given $L < 1$ and $M < 1$,

$$\frac{|a_{5n+4} - a_{5n+3}|}{|a_{5n+3} - a_{5n+2}|} = \frac{1}{H^{2^{5n+3}}} < L$$

for large n, while

$$\frac{|a'_{5n+4} - a'_{5n+3}|}{|a'_{5n+3} - a'_{5n+2}|} = 1 > M.$$

This shows that f is not weakly (L, M)-QS for any $L < 1$ and $M < 1$.

If one regards these as discrete examples, the following are continuous ones and more sophisticated.

Example 3. For any positive integer n, let $z_n = \left(\frac{1}{n}, (-1)^n\right) \in \mathbb{R}^2$ and $I_{z_n, z_{n+1}}$ be the closed line segment in $\mathbb{R}^2$ connecting z_n and z_{n+1}. Define curve γ as the union of these line segments:

$$\gamma = \bigcup_{n=1}^{+\infty} I_{z_n, z_{n+1}}.$$

Let $\gamma_{p,w}$ denote the subarc in γ between two points p and w and $l(\cdot)$ denote the arclength. It is easy to see that $l(\gamma) = +\infty$ and that $2 < l(\gamma_{z_n, z_{n+1}}) \to 2$ as $n \to +\infty$. Let $A = [0, +\infty)$ and define a mapping $f\colon \gamma \to A$ by means of the arclength parameterization of γ, that is,

$$f\colon \quad z_1 \mapsto 0, \quad z \mapsto l(\gamma_{z_1, z}) \text{ for any } z \in \gamma.$$

It is obvious that f is a homeomorphism from γ onto A. We claim that, under the internal metrics δ_γ and δ_A, f is weakly $(L, 2L)$-QS for any $L \le 2/\sqrt{5}$, but not weakly H-QS for any H.

To show that f is weakly $(L, 2L)$-QS, note that γ is contained in the rectangle $R = [0, 1] \times [-1, 1]$. Thus, for any points $p, w \in \gamma$,

$$\delta_\gamma(p, w) = \operatorname{diam}(\gamma_{p,w}) < \operatorname{diam}(R) = \sqrt{5}.$$

Fix any three distinct points $p_1, p_2, p_3 \in \gamma$ with

$$\frac{\delta_\gamma(p_1, p_3)}{\delta_\gamma(p_1, p_2)} \le L.$$

Without loss of generality, we may assume that $p_1 \in I_{z_n,z_{n+1}}$. Then

$$\delta_\gamma(p_1,p_3) \le L\delta_\gamma(p_1,p_2) < L\sqrt{5}.$$

It follows that p_3 must lie in one of the three line segments $I_{z_n,z_{n+1}}$, $I_{z_{n+1},z_{n+2}}$ or I_{z_{n-1},z_n}. Otherwise $\delta_\gamma(p_1,p_3) \ge 2$, contradicting to the inequality $\delta_\gamma(p_1,p_3) < L\sqrt{5} \le 2$. Therefore,

$$l(\gamma_{p_1,p_3}) \le 2\delta_\gamma(p_1,p_3) \le 2L\delta_\gamma(p_1,p_2) \le 2Ll(\gamma_{p_1,p_2}).$$

By the definition of f, we have

$$\frac{\delta_A(p_1',p_3')}{\delta_A(p_1',p_2')} = \frac{l(\gamma_{p_1,p_3})}{l(\gamma_{p_1,p_2})} \le 2L.$$

This shows that f is weakly $(L,2L)$-QS provided that $L \le 2/\sqrt{5}$.

Next we will show that f is not weakly H-QS for any $H > 0$. In fact, we will show that

$$\frac{\delta_\gamma(z_{4n},z_{2n^2})}{\delta_\gamma(z_{4n},z_{2n})} \le 1 \quad \text{while} \quad \frac{\delta_A(z_{4n}',z_{2n^2}')}{\delta_A(z_{4n}',z_{2n}')} \to \infty \quad \text{as } n \to \infty.$$

To this end, we note that

$$\delta_\gamma(z_{4n},z_{2n^2}) \le \sqrt{2^2 + |z_{4n} - z_{2n^2}|^2} = \sqrt{4 + \left(\frac{n-2}{4n^2}\right)^2}$$

and that

$$\delta_\gamma(z_{4n},z_{2n}) \ge \delta_\gamma(z_{4n-1},z_{2n}) = |z_{4n-1} - z_{2n}| = \sqrt{4 + \left(\frac{2n-1}{2n(4n-1)}\right)^2}.$$

Simple comparison yields that

$$\frac{\delta_\gamma(z_{4n},z_{2n^2})}{\delta_\gamma(z_{4n},z_{2n})} \le 1.$$

On the other hand, since $\gamma_{z_{4n},z_{2n^2}}$ consists of $(2n^2 - 4n)$ line segments of length no less than 2,

$$l(\gamma_{z_{4n},z_{2n^2}}) \ge 2(2n^2 - 4n) = 4n^2 - 8n.$$

Furthermore, since $\gamma_{z_{4n},z_{2n}}$ consists of $(4n - 2n)$ line segments of length no greater than $\sqrt{5}$,

$$l(\gamma_{z_{4n},z_{2n}}) \le 2n\sqrt{5}.$$

Thus it follows that

$$\frac{\delta_A(z_{4n}',z_{2n^2}')}{\delta_A(z_{4n}',z_{2n}')} \ge \frac{4n^2 - 8n}{2n\sqrt{5}} \to +\infty,$$

and hence f is not weakly H-QS for any H.

In summary, the above example shows that a weakly (L,M)-QS homeomorphism (with $L, M < 1$) may not be weakly H-QS. In the next example, we exhibit a map that is weakly H-QS, but not weakly (L,M)-QS for any $L < 1$ and $M < 1$.

Example 4. Let $g = f^{-1}\colon A \to \gamma$ where f is the homeomorphism in Example 3. By the above example, f is weakly $(L,2L)$-QS for any $L \le 2/\sqrt{5}$. Thus, for sufficiently small L, it follows from Lemma 2.2 that $g = f^{-1}$ is weakly H-QS with $H = 1/L$.

To show that g is not a weakly (L, M)-QS for any $L < 1, M < 1$, fix constant $L < 1$ and let

$$q_1 = 0,\ q_2 = g^{-1}(z_n) = f(z_n),\ q_3 = q_2\left(\frac{1}{L} + 1\right).$$

Here we use the same notation as in Example 3. Then

$$\frac{\delta_A(q_1, q_2)}{\delta_A(q_2, q_3)} = \frac{q_2 - q_1}{q_3 - q_2} = L.$$

On the other hand, routine estimates yield that

$$\frac{\delta_\gamma(g(q_1), g(q_2))}{\delta_\gamma(g(q_2), g(q_3))} = \frac{\delta_\gamma(z_1, z_n)}{\delta_\gamma(z_n, g(q_3))} > \frac{\sqrt{2^2 + \left(1 - \frac{1}{n-1}\right)^2}}{\sqrt{2^2 + \left(\frac{1}{n}\right)^2}} \to \frac{\sqrt{5}}{2}$$

as $n \to +\infty$. This shows that g is not weakly (L, M)-QS for any $L < 1$ and $M < \sqrt{5}/2$.

Remark 2.3. By Examples 1-4, we can see that the class of H-QS maps and the class of (L, M)-QS maps for $L < 1$ do not contain each other. Furthermore, Example 3 shows that local H-quasisymmetry may not imply global H'-quasisymmetry, while Example 4 shows that local $(L, 2L)$-QS for $L < \frac{1}{2}$ may not imply global (L', M')-QS for any $L' < 1$ and $M' < 1$.

2.4. Relation between (L, M)-QS and η-QS. It is known that, in general, an (L, M)-quasisymmetric or H-quasisymmetric homeomorphism may not be η-quasisymmetric. In this subsection, however, we show that their combination does induce η-quasisymmetry in pseudoconvex spaces. Both this result and its proof will be used later in the paper.

Following [22], a metric space X is *C-pseudoconvex* if there is an increasing function $C\colon [1, \infty) \to [1, \infty)$ with the following property: If $a, b \in X$ with $0 < r \le d_X(a, b)$, then there is a finite sequence of points $a = a_0, a_1, \cdots, a_s = b$ such that $s \le C(d_X(a, b)/r)$ and $d_X(a_{j+1}, a_j) \le d_X(a_j, a_{j-1}) \le r$ for $j = 1, \cdots, s - 1$.

Theorem 2.4. *Suppose that X is a pathwise connected C-pseudoconvex space. Let $f\colon X \to Y$ be an embedding such that*

(1) *f is weakly H-quasisymmetric;*
(2) *f is weakly (L, M)-quasisymmetric for some $L < 1$ and $M < 1$.*

Then f is η-quasisymmetric with $\eta = \eta(C, H, L, M)$.

Proof. Fix distinct points $a, x, b \in X$ and let

$$\rho = \frac{d_X(a, x)}{d_X(b, x)}, \quad \rho' = \frac{d_Y(a', x')}{d_Y(b', x')}.$$

Here and in what follows the prime indicates the image of the corresponding point or set under the map f. We need to show that $\rho' \le \eta(\rho)$ with $\eta(\rho) \to 0$ as $\rho \to 0$. For small values of ρ, we use the weak (L, M)-quasisymmetry and, for large values of ρ, we use the weak H-quasisymmetry and pseudoconvexity. Thus we shall divide the proof into two cases.

Case 1: $\rho \le L$. Let m be the unique nonnegative integer such that

$$L^2 < \frac{\rho}{L^m} \le L.$$

If $m = 0$, then the weak (L, M)-quasisymmetry of f implies that $\rho' \le M$. If $m > 0$, one can fix a path $\gamma_{b,x}$ connecting b and x in X and choose points $b_1, b_2, \cdots, b_m$ consecutively on $\gamma_{b,x}$ such that

$$d_X(x, b_1) = L d_X(x, b), \quad d_X(x, b_i) = L d_X(x, b_{i-1}) = L^i d_X(x, b)$$

for $i = 1, 2, \cdots, m$. Note that the last point b_m satisfies

$$\frac{d_X(a, x)}{d_X(x, b_m)} = \frac{\rho}{L^m} \le L.$$

By the weak (L, M)-quasisymmetry of f, it follows that

$$d_Y(x', a') \le M d_Y(x', b'_m), \quad d_Y(x', b'_i) \le M d_Y(x', b'_{i-1}).$$

Thus,

$$d_Y(x', a') \le M d_Y(x', b'_m) \le M^2 d_Y(x', b'_{m-1}) \le \cdots \le M^{m+1} d_Y(x', b').$$

Therefore, by the choice of the integer m above, we conclude that

$$\rho' = \frac{d_Y(a', x')}{d_Y(b', x')} \le M^{m+1} \le M^{\frac{\log \rho}{\log L} - 1}$$

with $\rho' \to 0$ when $\rho \to 0$.

Case 2: $\rho = \frac{d_X(x,a)}{d_X(x,b)} > L$. Let $r = L d_X(x, b)$. Since X is C-pseudoconvex and $d_X(x, a) > r$, there exist finite number of points $a_0 = x, a_1, \cdots, a_s = a$ such that

$$s \le C\left(\frac{d_X(x, a)}{r}\right) = C(\rho/L), \quad d_X(a_{j+1}, a_j) \le d_X(a_j, a_{j-1}) \le r$$

for $j = 1, 2, \cdots, s-1$. By the weak H-quasisymmetry of f, it follows that

$$d_Y(a'_{j+1}, a'_j) \le H d_Y(a'_j, a'_{j-1}) \le H^j d_Y(a'_1, a'_0) \le H^{j+1} d_Y(x', b').$$

Thus

$$\begin{aligned} d_Y(a', x') &= d_Y(a'_s, a'_0) \le d_Y(a'_s, a'_{s-1}) + \cdots + d_Y(a'_2, a'_1) + d_Y(a'_1, a'_0) \\ &\le (H^s + \cdots + H^2 + H) d_Y(x', b') = \frac{H^{s+1} - H}{H - 1} d_Y(x', b'), \end{aligned}$$

which yields that

$$\rho' = \frac{d_Y(a', x')}{d_Y(x', b')} \le \frac{H^{s+1} - H}{H - 1} = \frac{H^{C(\rho/L)+1} - H}{H - 1}$$

as desired. This completes the proof of Theorem 2.4. $\square$

Remark 2.5. To compare Theorem 2.4 with previously known results, we note that, in [22, Theorem 3.10], Tukia and Väisälä arrived at the same conclusion with a weaker condition on the space X, but stronger conditions on the map f than in Theorem 2.4 above. On the other hand, in [22, Theorem 2.15], the condition on the target space Y is stronger while the condition on the map f is weaker than in Theorem 2.4, also reaching the same conclusion of f being quasisymmetric. Note also that, by the example mentioned in [22, Remark 2.13], condition (2) in Theorem 2.4 is necessary.

We conclude this section with two corollaries which follow directly from Theorem 2.4.

Corollary 2.6. *Suppose that X is a pathwise connected C-pseudoconvex space. If an embedding $f\colon X\to Y$ is both (L,M)-QS for some $L<1$ and $M<1$, and (L',M')-QS for some $L'\geq 1$ and $M'<\infty$, then f is η-quasisymmetric with $\eta=\eta(C,L,M,L',M')$.*

Corollary 2.7. *Suppose that D is a domain in $\mathbb{R}^n$ which is C-pseudoconvex with respect to the internal metric δ_D. Let $f\colon D\to D'$ be an embedding such that*

(1) *f is weakly H-quasisymmetric in metrics δ_D and $\delta_{D'}$;*
(2) *f is weakly (L,M)-quasisymmetric for some $L<1$ and $M<1$ in metrics δ_D and $\delta_{D'}$.*

Then f is η-quasisymmetric in metrics δ_D and $\delta_{D'}$ with $\eta=\eta(C,H,L,M)$.

3. Locally weak (L,M)-quasisymmetry

As the first step towards to the proof of Theorems 1.1 and 1.3, in this section we show that the inverse of a QC map as described in Theorem 1.1 is locally weakly (L,M)-quasisymmetric for certain parameters $L<1$ and $M<1$.

3.1. Distortion of modulus of a ring domain. Going from local QC to global QS, one of the key Lemmas established in [14] is the following distortion result of modulus of a ring like domain under a QC map in metric spaces.

Lemma 3.1. [14, Lemma 4.12] *Suppose that X and Y are Q-regular metric spaces with $Q>1$ and that f is K-QC from X onto Y. If E and F are two continua in X such that $f(E)\subset\overline{B}(y,r)$ and that $f(F)\subset Y\backslash B(y,R)$ for some $y\in Y$ and some $R>2r$, then*

$$\operatorname{mod}_Q(E,F;X)\leq C\left(\log\frac{R}{r}\right)^{1-Q}.$$

The constant $C\geq 1$ only depends on K and on the data associated with X and Y.

In [3], Balogh and Koskela obtained the same estimate under a weaker dilatation condition on the mapping (see [3, Lemma 3.3]). To fit the requirement in our main theorem (Theorem 1.1), we establish the following similar modulus estimates under weaker regularity conditions on the spaces. These estimates are of independent interests.

Lemma 3.2. *Suppose that X is locally Q-regular with $Q>1$ and Y is upper half Q-regular metric spaces with regularity constants C_X and C_Y, respectively, and that f is K-QC from X onto Y. Let E and F be two continua in X.*

(1) *If $f(E)\subset\overline{B}(f(x_0),r)$ and $f(F)\subset Y\backslash B(f(x_0),R)$ for some $x_0\in X$ and some $R>2r$, then*

$$\operatorname{mod}_Q(E,F;X)\leq C_1\left(\log\frac{R}{r}\right)^{1-Q}. \tag{3.1}$$

The constant $C_1\geq 1$ depends only on the data Q,K,C_X,C_Y.

(2) *If $f(E)\subset\overline{B}_{\delta_Y}(f(x_0),r)$ and $f(F)\subset Y\backslash B_{\delta_Y}(f(x_0),R)$ for some $x_0\in X$ and some $R>2r$, then*

$$\operatorname{mod}_Q(E,F;X)\leq C_2\left(\log\frac{R}{r}\right)^{1-Q}. \tag{3.2}$$

The constant $C_2\geq 1$ depends only on the data Q,K,C_X,C_Y.

Remark 3.3. Due to the weaker regularity conditions in Lemma 3.2, one can not take Lemma 3.2(1) as a direct corollary of Lemma 3.1. Nevertheless, the method and procedures of its proof are the same as in [13, Theorem 1.7] and [14, Lemma 4.12] except for some subtle differences. For completeness, we outline the main steps in the proof.

Proof. For the proof of Lemma 3.2 Part (1), we may assume that

$$E = f^{-1}(\overline{B}(f(x_0), r)) \quad \text{and} \quad F = f^{-1}(Y \backslash B(f(x_0), R)).$$

We divide the proof into two steps. The first step is to show that there exists a constant C depending on Q, C_X such that

$$\mathrm{mod}_Q(E, F; X) \leq C \text{ d-mod}_{Q,2}(E, F),$$

where the definition of *discrete* $(Q, 2)$*-modulus*, denoted d-mod$_{Q,2}(E, F)$, is the same as in [13, Section 2.5], with the only modification that the radius of each ball B in a cover $\mathscr{B}$ is required to be less than the radius of local regularity at its center. With this modification for the definition of discrete modulus, the proof of the above inequality is exactly the same as in [13, Proposition 2.9].

The second step is to show that the discrete modulus d-mod$_{Q,2}(E, F)$ has a desired upper bound. To this end, we need to construct an appropriate admissible pair $(\nu, \mathscr{B})$ for the given continua E and F. For simplicity of notation and without loss of generality, we assume that the strict inequality in (1.1) holds. For a fixed $\delta > 0$, partition $X \backslash (E \cup F)$ into two disjoint subsets $\mathscr{H}$ and $\mathscr{P}$, where $\mathscr{H}$ consists of those points $x \in X \backslash (E \cup F)$ for which

$$\limsup_{r \to 0} \frac{\mu(f(B(x, 2r)))}{\mu(f(B(x, r)))} < D = 2(5K^2)^Q C_Y^2.$$

For each $x \in \mathscr{H} \cup \mathscr{P}$, choose a radius r_x $(0 < r_x < \delta)$ satisfying

(1) $B(x, 4r_x) \subset X \backslash (E \cup F)$;
(2) $4r_x < \widetilde{r}_x$ if $x \in \mathscr{H}$, or $f(B(x, 4r_x)) \subset B(x', \widetilde{r}_{x'})$, where $\widetilde{r}_x$ denotes the radius of local regularity at a point x;
(3) for each $0 < r < r_x$,

$$\frac{\max\{d_Y(f(a), f(x)) \colon d_X(a, x) \leq r\}}{\min\{d_Y(f(b), f(x)) \colon d_X(b, x) \geq r\}} \leq K,$$

and

$$\frac{\mu(f(B(x, 2r)))}{\mu(f(B(x, r)))} < D \ \text{ if } x \in \mathscr{H}, \text{ or } \ \frac{\mu(f(B(x, 2r)))}{\mu(f(B(x, r)))} \geq \frac{D}{2} \ \text{ if } x \in \mathscr{P}.$$

Following the argument in [13, Section 3.5], one can construct covers $\mathscr{B}_1$ and $\mathscr{B}_2$ for $\mathscr{H}$ and $\mathscr{P}$, respectively. Specifically, for the set $\mathscr{H}$, by using a general Besicovitch covering theorem (see [12, Theorem 1.16] or [7, Page 53]), a countable subfamily $\mathscr{B}_1$ can be extracted from the family $\{B(x, r_x) \colon x \in \mathscr{H}\}$ such that $\mathscr{H} \subset \bigcup_{B \in \mathscr{B}_1} B$ and $\frac{1}{5}B \cap \frac{1}{5}B' = \emptyset$ whenever $B, B' \in \mathscr{B}_1$ with $B \neq B'$. Similarly, for the set $\mathscr{P}$, same argument as in [13, Page 71] shows that there exist a countable subfamily $\mathscr{B}_2 = \{B(x_i, r_{x_i}) \colon x_i \in \mathscr{P},\ i = 1, 2, \cdots\}$ and $V_i \subset B(x_i, r_{x_i})$ such that $\mathscr{P} \subset \bigcup_{B \in \mathscr{B}_2} B$, $V_i \cap V_j = \emptyset$ whenever $i \neq j$, and

$$\mu(f(B(x_i, r_i))) \leq C_Y^2 5^Q K^Q \mu(f(V_i)).$$

Then $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$ is a desired cover of $X \backslash (E \cup F)$.

Next, define set function $\nu\colon \mathscr{B} \to (0,\infty)$ by

$$\nu(B) = \left(\log \frac{R}{r}\right)^{-1} \frac{\operatorname{diam}(f(B))}{\operatorname{dist}(f(B), f(x_0))}.$$

Now proceeding as in [13, Pages 72–73], and taking into account of the fact that Y has the global upper Q-mass bound, one can deduce that $(c\nu, \mathscr{B})$ is 2-admissible for some appropriate constant c, and consequently that

$$\text{d-mod}_{Q,2}(E,F;X) \le C' \left(\log \frac{R}{r}\right)^{1-Q},$$

where C' depends on K, Q, C_Y. This completes the proof of Part (1).

For the prove of Part (2), we may assume that

$$E = f^{-1}(\overline{B}_{\delta_Y}(f(x_0), r)) \quad \text{and} \quad F = f^{-1}(Y \backslash B_{\delta_Y}(f(x_0), R)).$$

As in the proof of Part (1), we only need to find an appropriate upper bound for the discrete modulus $\text{d-mod}_{Q,2}(E,F;X)$. For this, construct ball covering $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$ of $X \backslash (E \cup F)$ as above. The only modification one needs to make is the definition of set function $\nu\colon \mathscr{B} \to (0,\infty)$, which is given by

$$\nu(B) = \left(\log \frac{R}{r}\right)^{-1} \frac{\operatorname{diam}(f(B))}{\delta_Y(f(B), f(x_0))}$$

for $B \in \mathscr{B}$. Then the same procedure as in [13, Page 72] shows that there exists a constant $c = c(Q, K, C_Y)$ such that $(c\nu, \mathscr{B})$ is 2-admissible for the configuration $(E, F; X)$. Finally, one derives the desired upper bound

$$\text{d-mod}_{Q,2}(E,F;X) \le C'' \left(\log \frac{R}{r}\right)^{1-Q},$$

by following [13, Page 73] and taking into consideration of the fact that

$$\mu(B_{\delta_Y}(f(x_0), 2^{-j-1}R)) \le \mu(B(f(x_0), 2^{-j-1}R)).$$

This completes the proof of Lemma 3.2. □

From the proof one can see that a local version of Lemma 3.2 is also valid. More precisely, if both X and Y are only assumed to be locally Q-regular, then for each $x_0 \in X$, there exists R_{x_0} such that for any $R < R_{x_0}$ and $r < \frac{R}{2}$, the inequality (3.1) holds for any continua E and F in X with $f(E) \subset \overline{B}(f(x_0), r)$ and $f(F) \subset Y \backslash B(f(x_0), R)$. A similar statement is true for Lemma 3.2 Part (2).

3.2. Local (L,M)-quasisymmetry.

Proposition 3.4. *Suppose that X and Y are arcwise connected metric spaces, that X is upper half Q-regular with $Q > 1$, and that Y is locally Q-regular. Assume X is c-LLC$_2$ and locally $\widehat{c}$-BT, Y is φ-Loewner, and f^{-1} from Y onto X is a K-quasiconformal homeomorphism which maps bounded sets to bounded sets. Then f is locally weakly (L,M)-QS for all $L \le L_0 < 1$ with $M < 1$ depending on L, where L_0 is a constant depending on the regularity constants and the data $(Q, K, c, \widehat{c}, \varphi)$. Furthermore,*

$$\lim_{L \to 0} M = 0.$$

Proof. Fix any $x_0 \in X$ and let r' to be sufficiently small such that $X\backslash\overline{B}(x_0, 2L_{f^{-1}}(x_0', 2r')) \neq \emptyset$, where

$$L_{f^{-1}}(x_0', 2r') = \sup_{d_Y(x', x_0') \leq 2r'} d_X(f^{-1}(x'), f^{-1}(x_0')).$$

Next choose $r > 0$ sufficiently small such that $B(x_0, r)$ is $\widehat{c}$-BT in X and that

$$f(B(x_0, (1 + 2\widehat{c})r)) \subset B(x_0', r'),$$

which implies that

$$B(x_0, (1 + 2\widehat{c})r) \subset B(x_0, L_{f^{-1}}(x_0', r')) \subset B(x_0, L_{f^{-1}}(x_0', 2r')).$$

To show that f is weakly (L, M)-QS in $B(x_0, r)$, let

$$L_0 = \frac{1}{4c\widehat{c}e^{\left(\frac{C}{\varphi(2)}\right)^{\frac{1}{Q-1}}}},$$

where C is the constant in Lemma 3.2(1). For any $x, a, b \in B(x_0, r)$ with $\frac{d_X(x,a)}{d_X(x,b)} \leq L < L_0$, we will show that

$$\frac{d_Y(x', a')}{d_Y(x', b')} \leq M = \frac{2}{\varphi^{-1}\left(C\left(\log\frac{1}{2c\widehat{c}L}\right)^{1-Q}\right)}. \tag{3.3}$$

This will be done through modulus comparison. First, by $\widehat{c}$-BT property of $B(x_0, r)$, there exists an arc $E \subset X$ connecting x and a such that $d(E) \leq \widehat{c}d_X(x, a)$. Next fix $w \in X$ with $d_X(w, x_0) > 2L_{f^{-1}}(x_0', 2r')$. By the definition of $L_{f^{-1}}(x_0', 2r')$, one deduces that $f(w) = w' \in Y\backslash B(x_0', 2r')$. Since

$$2L_{f^{-1}}(x_0', 2r') < d_X(w, x) + d_X(x, x_0) \leq d_X(w, x) + L_{f^{-1}}(x_0', 2r'),$$

it follows that

$$d_X(w, x) \geq L_{f^{-1}}(x_0', 2r') > 2r > d_X(x, b).$$

By the c-LLC$_2$ property of X, there is a continua F, connecting b and w in X, such that

$$F \subset X\backslash\overline{B}\left(x, \frac{d_X(x, b)}{2c}\right).$$

Therefore, by Lemma 3.2 (1) (with the roles of X and Y switched), there exists a constant C, depending on Q, K, C_X, C_Y, such that

$$\operatorname{mod}_Q(E', F'; Y) \leq C\left(\log\frac{\frac{d_X(x,b)}{2c}}{\widehat{c}d_X(x, a)}\right)^{1-Q} \leq C\left(\log\frac{1}{2c\widehat{c}L}\right)^{1-Q}.$$

To find a lower bound for the modulus $\operatorname{mod}_Q(E', F'; Y)$, one observes that

$$\operatorname{dist}(E', F') \leq d_Y(x', b'),$$

$$\min\{\operatorname{diam}(E'), \operatorname{diam}(F')\} \geq \min\{d_Y(x', a'), r'\} > \frac{d_Y(x', a')}{2}.$$

The last two inequalities result from the fact that

$$w' \in Y\backslash\overline{B}(x_0', 2r') \quad \text{and} \quad b' \in f(B(x_0, r)) \subset B(x_0', r').$$

Thus, by the φ-Loewner condition on Y, we obtain

$$\operatorname{mod}_Q(E', F'; Y) \geq \varphi\left(\frac{dist(E', F')}{\min\{\operatorname{diam}(E'), \operatorname{diam}(F')\}}\right) \geq \varphi\left(\frac{2d_Y(x', b')}{d_Y(x', a')}\right).$$

Combining this with the upper bound of $\mathrm{mod}_Q(E', F', Y)$ above yields (3.3) as desired. Finally, it is easy to see that $M \to 0$ as $L \to 0$. $\square$

4. Global (L, M)-quasisymmetry in internal metrics

In the previous section, we showed that the inverse of the QC map described in Theorem 1.1 is locally (L, M)-QS with respect to the original metrics. In this section, we further establish its global (L, M)-quasisymmetry with respect to the internal metrics.

4.1. Bounded turning property of a locally regular Loewner space. In [14, Theorem 3.13], Heinonen and Koskela proved that, in a Q-regular space, the Loewner condition implies quasiconvexity, and hence the bounded turning property. Using the same idea, we establish a local version of this result, which will be needed in this paper.

Lemma 4.1. *Let Y be a connected locally Q-regular φ-Loewner space with $Q > 1$. Then it is locally $\widehat{c}$-BT with $\widehat{c}$ depending only on the regularity constant C_Y, Q and φ.*

Before proving Lemma 4.1, we quote some estimates for modulus from [14], which will be needed in the proof. We point out that the original results and proofs in [14] were formulated for Q-regular spaces. However, as noted in [14], they are valid for spaces that only satisfy the upper mass bound condition.

Lemma 4.2. [14, Lemma 3.15 & 3.14] *Let X be a metric space with a Borel measure μ satisfying the global upper mass bound condition with constant C_0 and dimension $Q > 1$. If Γ is a family of curves in a ball $B_R = B(x, R)$ such that $l(\gamma) \geq L > 0$ for each $\gamma \in \Gamma$, then*

$$\mathrm{mod}_Q(\Gamma) \leq \frac{\mu(B_R)}{L^Q}.$$

Furthermore, if $0 < 2r < R$, then there is a constant $\overline{C} = \overline{C}(Q, C_0)$ such that

$$\mathrm{mod}_Q(\overline{B}(x, r), X \backslash B(x, R); X) \leq \overline{C} \left(\log \frac{R}{r} \right)^{1-Q}.$$

Remark 4.3. We need to point out that, by further examining the proofs in [14], one can see that a local version of Lemma 4.2 holds for a locally regular space X. More precisely, if X is only assumed to satisfy a *local upper mass bound condition* (in particular, if X is locally regular), then the modulus estimates in the above lemma remain valid if the curve family under consideration lies in a neighborhood of local regularity. This will be needed in the proof of Lemma 4.1.

Proof of Lemma 4.1. For $z \in Y$, fix $\varepsilon > 0$ no greater than the radius of local regularity of Y at z. Let

$$\varepsilon^* = \frac{\varepsilon}{1 + 2p} \quad \text{with} \quad p = \frac{1}{10} e^{\left(\frac{2\overline{C}}{\varphi(10)} \right)^{\frac{1}{Q-1}}},$$

where $\overline{C}$ is the constant in Lemma 4.2. We shall show that $B(z, \varepsilon^*) \cap B(z, \frac{1}{2})$ is BT in Y.

For any distinct pair $x, y \in B(z, \varepsilon^*) \cap B(z, \frac{1}{2})$, let $r = d_Y(x, y) < 1$. By the connectivity of Y, there exist continua $E \subset \overline{B}(x, \frac{r}{10})$ connecting x and $Y \backslash B\left(x, \frac{r}{10}\right)$, and $F \subset \overline{B}(y, \frac{r}{10})$ connecting y and $Y \backslash B\left(y, \frac{r}{10}\right)$.

Let Γ be the curve family connecting E and F in Y and write $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_1 consists of the curves in Γ that are contained in $B(x, pr)$ and $\Gamma_2 = \Gamma \backslash \Gamma_1$. The φ-Loewner property induces that

$$\operatorname{mod}_Q(\Gamma) \geq \varphi\left(\frac{r}{\frac{1}{10}r}\right) = \varphi(10).$$

On the other hand, since $\overline{B}(x, pr) \subset B(z, \varepsilon)$ by the choices of parameters above, it follows from Lemma 4.2 (or its local version as mentioned in Remark 4.3) that

$$\operatorname{mod}_Q(\Gamma_2) \leq \overline{C}\left(\log \frac{pr}{\frac{1}{10}r}\right)^{1-Q} = \frac{\varphi(10)}{2}.$$

Therefore,

$$\operatorname{mod}_Q(\Gamma_1) \geq \operatorname{mod}_Q(\Gamma) - \operatorname{mod}_Q(\Gamma_2) = \frac{\varphi(10)}{2}.$$

This, together with local Q-regularity and the local version of Lemma 4.2, yields that there exists a curve $\gamma_0 \in \Gamma_1$ whose length l_0 satisfies

$$l_0^Q \leq \frac{2C_Y(pr)^Q}{\varphi(10)},$$

where C_Y is the regularity constant of Y. Thus one can construct a path $\gamma \subset E \cup \gamma_0 \cup F$ connecting x and y with diameter satisfying

$$\begin{aligned} d(\gamma) &\leq d(E) + d(F) + l_0 \leq \frac{2}{5} d_Y(x, y) + \left(\frac{2C_Y}{\varphi(10)}\right)^{\frac{1}{Q}} p d_Y(x, y) \\ &= d_Y(x, y)\left(\frac{2}{5} + \left(\frac{2C_Y}{\varphi(10)}\right)^{\frac{1}{Q}} p\right), \end{aligned}$$

and the desired result follows. $\square$

4.2. Proof of Theorem 1.1. In light of Lemma 2.2 on the inverse of an (L, M)-QS map, to prove Theorem 1.1, it suffices to prove the following proposition.

Proposition 4.4. *Suppose that X and Y are arcwise connected metric spaces which are simultaneously bounded or unbounded, that X is upper half Q-regular with $Q > 1$, that Y is locally Q-regular. Assume that X is c-LLC$_2$ and locally $\widehat{c}$-BT, Y is φ-Loewner, f^{-1} from Y onto X is a K-quasiconformal homeomorphism. Then there is a constant $L_0 < 1$ such that f is weakly (L, M)-QS in the internal metrics for all $L \leq L_0$ with $M = M(L)$ such that*

$$\lim_{L \to 0} M = 0.$$

Furthermore, the constant L_0 depends only on the regularity constants and the data $(Q, K, c, \widehat{c}, \varphi)$, and possibly on f if both X and Y are bounded.

Note that one obtains Theorem 1.1 by reversing the roles of X and Y in the above proposition and by applying Lemma 2.2.

4.3. Proof of Proposition 4.4: unbounded case. We start the proof by considering the easier case when X and Y are unbounded. Since X is c-LLC$_2$ with respect to the original metric, it is c_1-LLC$_2$ with respect to the internal metric δ for some constant c_1 (see [11]). We will show that f is (L, M)-QS with respect to the internal metrics for all $L \leq L_0$ with $M = M(L)$, where

$$L_0^{-1} = 4c_1 e^{\left(\frac{\widetilde{C}}{\varphi(1)}\right)^{\frac{1}{Q-1}}},$$

and $\widetilde{C}$ is the constant in (3.2) of Lemma 3.2.

Fix three distinct points $x, a, b \in X$, let

$$\rho = \frac{\delta_X(x,a)}{\delta_X(x,b)} \leq L < L_0, \quad \rho' = \frac{\delta_Y(x',a')}{\delta_Y(x',b')}$$

and let

$$L_f(x, \delta_X(x,b)) = \sup\{\delta_Y(x',z') \colon z \in B(x, \delta_X(x,b))\}.$$

For $w' \in Y \backslash \overline{B}(x', \max\{L_f(x, \delta_X(x,b)), 2\delta_Y(x',b')\})$, we have

$$w \in X \backslash \overline{B}(x, \delta_X(x,b)) \subset X \backslash \overline{B}_\delta(x, \delta_X(x,b)).$$

Since X is c_1-LLC$_2$ with respect to the internal metric δ_X, there exists an arc β connecting b and w such that

$$\beta \subset X \backslash B_\delta\left(x, \frac{\delta_X(x,b)}{c_1}\right).$$

Next choose an arc α connecting x and a such that $d(\alpha) \leq 2\delta_X(x,a)$. Since

$$\frac{\frac{\delta_X(x,b)}{c_1}}{2\delta_X(x,a)} > \frac{1}{2c_1 L_0} > 2,$$

by Lemma 3.2 Part (2) (with the roles of X and Y switched), we have

$$\operatorname{mod}_Q(\alpha', \beta'; Y) \leq \widetilde{C}\left(\log \frac{\frac{\delta_X(x,b)}{c_1}}{2\delta_X(x,a)}\right)^{1-Q}. \tag{4.1}$$

On the other hand, since $\delta_Y(w',b') \geq \delta_Y(w',x') - \delta_Y(x',b') \geq \delta_Y(x',b')$, one can deduce that

$$\frac{\operatorname{dist}(\alpha', \beta')}{\min\{d(\alpha'), d(\beta')\}} \leq \frac{\delta_Y(x',b')}{\min\{\delta_Y(x',a'), \delta_Y(x',b')\}}.$$

If $\delta_Y(x',a') > \delta_Y(x',b')$, then $\operatorname{mod}_Q(\alpha', \beta'; Y) \geq \varphi(1)$ by the Loewner condition. This together with (4.1) yields that

$$\varphi(1) \leq \widetilde{C}\left(\log \frac{1}{2c_1\rho}\right)^{1-Q},$$

which implies that

$$\frac{1}{\rho} \leq 2c_1 e^{\left(\frac{\widetilde{C}}{\varphi(1)}\right)^{\frac{1}{Q-1}}},$$

Contradicting with the assumption that $\rho < L_0$. Thus we must have $\delta_Y(x',a') \leq \delta_Y(x',b')$, and it follows that

$$\operatorname{mod}_Q(\alpha', \beta'; Y) \geq \varphi\left(\frac{\delta_Y(x',b')}{\delta_Y(x',a')}\right) = \varphi\left(\frac{1}{\rho'}\right).$$

Appealing to (4.1), one derives that

$$\varphi\left(\frac{1}{\rho'}\right) \leq \widetilde{C}\left(\log \frac{\frac{\delta_X(x,b)}{c_1}}{2\delta_X(x,a)}\right)^{1-Q}.$$

And hence

$$\frac{1}{\rho'} \geq \varphi^{-1}\left(\widetilde{C}\left(\log\frac{1}{2c_1\rho}\right)^{1-Q}\right).$$

Therefore, f is (L,M)-QS with

$$M = M(L) = \frac{1}{\varphi^{-1}\left(\widetilde{C}\left(\log\frac{1}{2c_1 L}\right)^{1-Q}\right)},$$

which converges to zero as $L \to 0$.

4.4. Proof of Proposition 4.4: for bounded $\boldsymbol{X}$ and $\boldsymbol{Y}$. To show that f has the claimed quasisymmetry property, in light of the proof of Case 1 in Theorem 2.4, we only need to show that there exist constants $L_0 < 1$ and $M < 1$, depending only on the given parameters associated with X, Y and f, such that for any distinct points $x, a, b \in X$,

$$\rho = \frac{\delta_X(x,a)}{\delta_X(x,b)} \leq L_0 \;\Rightarrow\; \rho' = \frac{\delta_Y(x',a')}{\delta_Y(x',b')} \leq M. \tag{4.2}$$

By appealing to Lemma 4.1, we may assume that Y is locally $\widehat{c}'$-BT. According to Proposition 3.4, one can fix constants $L^* < 1$ and $M^* < \frac{1}{2\widehat{c}'}$ such that f is locally weakly (L^*, M^*)-QS (in the original metrics). To establish (4.2), we fix $x_0 \in X$ and choose small positive parameters $\widetilde{r}, r, r'$ as follows. First, fix $\widetilde{r} > 0$ such that f is weakly (L^*, M^*)-QS in $f^{-1}(B(x_0', (1+\frac{1}{M^*})\widetilde{r})$ and that $f^{-1}(B(x_0', (1+\frac{1}{M^*})\widetilde{r}) \subset X$ and $B(x_0', \widetilde{r}) \subset Y$ are $\widehat{c}$-BT and $\widehat{c}'$-BT, respectively. Next choose r such that

$$f(B(x_0, (2\widehat{c}^2+1)r)) \subset B(x_0', \widetilde{r}).$$

Finally, fix $r' < d(Y)$ with

$$f^{-1}(\overline{B}(x_0', r')) \subset \overline{B}\left(x_0, \frac{r}{2}\right),\; f^{-1}(\partial\overline{B}(x_0', r')) \cap \partial B\left(x_0, \frac{r}{2}\right) \neq \emptyset.$$

We shall divide the proof of (4.2) into two cases by using the following constant:

$$C_0 = 4ce^{\left(\frac{\widetilde{C}}{\varphi(\frac{1}{N})}\right)^{\frac{1}{Q-1}}} \quad\text{with } N = \frac{r'}{2\,d(Y)} \leq \frac{1}{2},$$

where $\widetilde{C}$ is the constant in Lemma 3.2 Part (2) as above.

4.4.1. Case 1: $\boldsymbol{\delta_X(x,x_0) \leq C_0\delta_X(x,a)}$. In this case the idea is to show that x, a, b lie in a neighborhood that is BT and in which f is (L^*, M^*)-QS in original metric. The BT property makes the internal metric comparable with the original metric, which enables the transition from (L^*, M^*)-quasisymmetry in original metric to quasisymmetry in internal metric. More specifically, we show that (4.2) holds with

$$L_0 = \min\left\{\frac{L^*}{\widehat{c}}, \frac{r}{(C_0+1)d(X)}\right\} \quad\text{and}\quad M = 2\widehat{c}'M^*.$$

Note that, since

$$\begin{aligned}\delta_X(x,x_0) &\leq C_0\delta_X(x,a) = C_0\delta_X(x,b)\rho \leq C_0 d(X)L_0 < r,\\ \delta_X(a,x_0) &\leq \delta_X(x,a) + \delta_X(x,x_0) \leq (C_0+1)d(X)L_0 \leq r,\end{aligned}$$

it follows that $x, a \in \overline{B}(x_0, r)$ and that $x', a' \in f(\overline{B}(x_0, r)) \subset B(x_0', \widetilde{r})$. By the BT-property, $\delta_Y(x', a') \leq \widehat{c}' d_Y(x', a')$.

To show that $\rho' \leq M$, we assume, on the contrary, that

$$\rho' = \frac{\delta_Y(x', a')}{\delta_Y(x', b')} > M = 2\widehat{c}\widetilde{c}M^*.$$

One deduces that

$$d_Y(x', b') \leq \delta_Y(x', b') < \frac{\widetilde{c}\, d_Y(x', a')}{2\widehat{c}\widetilde{c}M^*} < \frac{\widetilde{r}}{M^*},$$

which yields

$$b' \in B\left(x_0', \left(1 + \frac{1}{M^*}\right)\widetilde{r}\right) \quad \text{and} \quad x, a, b \in f^{-1}\left(B\left(x_0', \left(1 + \frac{1}{M^*}\right)\widetilde{r}\right)\right).$$

Using the comparative relation between internal metric and original metric and the (L^*, M^*)-quasisymmetry, one can derive that

$$\frac{\delta_Y(x', a')}{\delta_Y(x', b')} > M \Rightarrow \frac{d_Y(x', a')}{d_Y(x', b')} > M^* \Rightarrow \frac{d_X(x, a)}{d_X(x, b)} > L^* \Rightarrow \frac{\delta_X(x, a)}{\delta_X(x, b)} > \frac{L^*}{\widehat{c}},$$

a contradiction with $\rho \leq L_0$. This verifies (4.2) in Case 1.

4.4.2. Case 2: $\boldsymbol{\delta_X(x, x_0) > C_0\delta_X(x, a)}$. Comparing to Case 1 above and the unbounded case, this is the most complicated case and we describe the main idea first as follows. One can use Lemma 3.2 to obtain an upper bound for the modulus of a curve family $\Delta(\alpha', \beta'; Y)$ and use the Loewner condition to obtain a lower bound for the modulus of the same curve family in Y. Bring these two estimates together will yields the desired relation between ρ' and ρ. More specifically, we show that (4.2) holds for

$$L_0 = \min\left\{\frac{2}{4\widehat{c}^2 + \widehat{c}}, \frac{L^*}{\widehat{c}}, \frac{(2\widehat{c} - 1)r}{2C_0 d(X)}\right\} \quad \text{and} \quad M = \max\{N, \widetilde{c}M^*\}.$$

To this end, fix an arc α joining x and a in X with $d(\alpha) \leq 2\delta_X(a, x)$. The choice of β (an arc joining b to a point) is more sophisticated and depends on the subcases considered below.

Subcase 2.1: $b' \notin B(x_0', \frac{r'}{2})$. In this case, we establish (4.2) by assuming, on the contrary, that $\rho \leq L_0$ but $\frac{\delta_Y(x', a')}{\delta_Y(x', b')} > M \geq N$.

By the c-LLC$_2$ property (in original metric, and hence in internal metric by [11]), one can fix an arc β joining b to x_0 with

$$\beta \subset X \backslash \overline{B}_\delta\left(x, \frac{C_0}{c}\delta_X(a, x)\right).$$

Combining this with the fact that $\alpha \subset \overline{B}_{\delta_X}(x, 2\delta_X(a, x))$ and $\frac{C_0}{2c} > 2$, it follows from Lemma 3.2 Part (2) that

$$\operatorname{mod}_Q(\alpha', \beta'; Y) \leq \widetilde{C}\left(\log \frac{\frac{C_0}{c}\delta_X(a, x)}{2\delta_X(a, x)}\right)^{1-Q} = \widetilde{C}\left(\log \frac{C_0}{2c}\right)^{1-Q}. \tag{4.3}$$

Next, to obtain a lower bound for the modulus $\operatorname{mod}_Q(\alpha', \beta'; Y)$, we note that $d(\beta') \geq \frac{r'}{2}$ and

$$\operatorname{dist}(\alpha', \beta') \leq \delta_Y(x', b') < \frac{\delta_Y(x', a')}{N} \leq \frac{d(\alpha')}{N} \quad \text{and} \quad \frac{\operatorname{dist}(\alpha', \beta')}{d(\beta')} \leq \frac{2d(Y)}{r'}.$$

Thus,

$$\frac{\operatorname{dist}(\alpha',\beta')}{\min\{d(\alpha'),d(\beta')\}} \le \max\left\{\frac{1}{N},\frac{2d(Y)}{r'}\right\} = \frac{1}{N}.$$

And it follows from φ-Loewner condition that

$$\operatorname{mod}_Q(\alpha',\beta';Y) \ge \varphi\left(\frac{1}{N}\right).$$

This together with (4.3) yields that

$$\varphi\left(\frac{1}{N}\right) \le \widetilde{C}\left(\log\frac{C_0}{2c}\right)^{1-Q},$$

which leads to a contradiction with the choice of the constant C_0. This contradiction proves (4.2) as desired in this subcase. Note that this inequality is the reason why C_0 was chosen as above.

Subcase 2.2: $b' \in B(x_0', \frac{r'}{2})$. By the choices of r and r', we have $b \in B(x_0, \frac{r}{2})$. To treat this case, we first assume that $x \in B(x_0, 2\widehat{c}r)$. By the local BT-property, $\delta_X(x,b) \le \widehat{c}d_X(x,b) < \widehat{c}(2\widehat{c}+\frac{1}{2})r$. Thus

$$\rho = \frac{\delta_X(x,a)}{\delta_X(x,b)} \le L_0 \le \frac{2}{4\widehat{c}^2+\widehat{c}} \Rightarrow \delta_X(x,a) < r.$$

Thus it follows that $x,a,b \in B(x_0,(2\widehat{c}+1)r)$. Using the (L^*,M^*)-quasisymmetry of f in this neighborhood and BT-property again, it is not difficult to derive that

$$\begin{aligned}
\rho = \frac{\delta_X(x,a)}{\delta_X(x,b)} \le L_0 \le \frac{L^*}{\widehat{c}} &\Rightarrow \frac{d_X(x,a)}{d_X(x,b)} \le \frac{\delta_X(x,a)}{\frac{\delta_X(x,b)}{\widehat{c}}} \le L^* \\
&\Rightarrow \frac{d_Y(x',a')}{d_Y(x',b')} \le M^* \Rightarrow \frac{\delta_Y(x',a')}{\delta_Y(x',b')} \le \widehat{c}'\frac{d_Y(x',a')}{d_Y(x',b')} \le \widehat{c}'M^* < M,
\end{aligned}$$

which verifies (4.2) under the assumption that $x \in B(x_0, 2\widehat{c}r)$.

Next assume that $x \notin B(x_0, 2\widehat{c}r)$. In this case, one can choose $z' \in \partial\overline{B}(x_0', r')$ such that $d_X(z,x_0) = \frac{r}{2}$. By the locally $\widehat{c}$-BT property, there exists an arc β connecting b and z with diameter $d(\beta) \le \widehat{c}d_X(b,z)$.

If $\beta \cap \overline{B}_{\delta_X}(x, C_0\delta_X(x,a)) \neq \emptyset$, then there exists a point w with

$$w \in \beta \subset B\left(x_0,\left(\widehat{c}+\frac{1}{2}\right)r\right) \quad \text{and} \quad \delta_X(x,w) \le C_0\delta_X(x,a).$$

On the other hand, $\delta_X(x,w) \ge d(x,x_0) - d(w,x_0) > (\widehat{c}-\frac{1}{2})r$. This yields

$$\frac{\delta_X(x,a)}{\delta_X(x,b)} \ge \frac{\delta_X(x,a)}{d(X)} > \frac{(2\widehat{c}-1)r}{2C_0d(X)},$$

a contradiction with the assumption in (4.2). Thus, to verify (4.2) in this case, we may assume that $\beta \cap \overline{B}_{\delta_X}(x, C_0\delta_X(x,a)) = \emptyset$.

To deal with this final case, we again establish (4.2) by assuming, on the contrary, that $\rho' > M \ge N$. The contradiction will result from estimates of the lower and upper bound of $\operatorname{mod}_Q(\alpha',\beta';Y)$, respectively.

To obtain an upper bound for $\operatorname{mod}_Q(\alpha',\beta';Y)$, we note that

$$\alpha \subset \overline{B}_\delta(x, 2\delta_X(a,x)), \quad \beta \subset X\backslash\overline{B}_\delta(x, C_0\delta_X(a,x)).$$

By Lemma 3.2 Part (2) (again with the roles of X and Y switched as above), we have

$$\operatorname{mod}_Q(\alpha', \beta'; Y) \leq \widetilde{C}\left(\log \frac{C_0 \delta_X(a,x)}{2\delta_X(a,x)}\right)^{1-Q} = \widetilde{C}\left(\log \frac{C_0}{2}\right)^{1-Q}. \tag{4.4}$$

To obtain a lower bound for the modulus $\operatorname{mod}_Q(\alpha', \beta'; Y)$, as in the Subcase 2.1 above (details omitted), one can verify that

$$\frac{\operatorname{dist}(\alpha', \beta')}{\min\{d(\alpha'), d(\beta')\}} \leq \max\left\{\frac{1}{N}, \frac{2d(Y)}{r'}\right\} = \frac{1}{N}.$$

By φ-Loewner condition on Y,

$$\operatorname{mod}_Q(\alpha', \beta'; Y) \geq \varphi\left(\frac{1}{N}\right).$$

Combining this together with (4.4), one concludes that

$$\varphi\left(\frac{1}{N}\right) \leq \widetilde{C}\left(\log \frac{C_0}{2}\right)^{1-Q},$$

which leads to a contradiction with the definition of the constant C_0. This completes the proof of Proposition 4.4.

5. Proof of Theorem 1.3: from QC to η-quasisymmetry

We finally complete the journey from QC to η-quasisymmetry by showing that in the internal metrics an (L, M)-QS map is η-QS.

5.1. From weakly (L, M)-QS to η-QS. The following theorem serves as a bridge going from weakly (L, M)-QS to η-QS in an HTB metric space, just like the bridge established by Väisälä' going from weakly H-QS to η-QS [26, Theorem 2.9]. However, the main difference is that in the following result no extra condition is imposed on Y. This weakened condition will make the result widely applicable. As an application, we shall use it to establish Theorem 1.3.

Theorem 5.1. *Suppose that (X, d_X) and (Y, d_Y) are arcwise connected metric spaces and that X is k-HTB. If $f\colon X \to Y$ is weakly (L,M)-QS for some $L < 1$ and $M < 1$, then f is η-QS with η depending only on k, L, M.*

Proof. Fix distinct points $a, x, b \in X$ and let

$$\rho = \frac{d_X(a,x)}{d_X(b,x)}, \qquad \rho' = \frac{d_Y(a',x')}{d_Y(b',x')}.$$

Here and in what follows the prime indicates the image of the corresponding point or set under the map f. We need to show that $\rho' \leq \eta(\rho)$ with $\eta(\rho) \to 0$ as $\rho \to 0$.

Case 1: $\rho \leq L$. The proof of this case is the same as Theorem 2.4, Case 1.

Case 2: $\rho = \frac{d_X(x,a)}{d_X(x,b)} > L$. Fix an arc $\gamma_{a,x} \subset X$ connecting a and x. Let $a_0 = x$ and let a_1 be the last point on $\gamma_{a,x}$ along the direction from x to a such that

$$d_X(a_0, a_1) = L d_X(x, b).$$

Inductively, one can choose consecutive points $a_2, \cdots, a_n$ on $\gamma_{a,x}$ along the direction from x to a such that a_i is the last point after a_{i-1} with

$$d_X(a_{i-1}, a_i) = L d_X(a_0, a_{i-1}) \quad (i = 2, \cdots, n)$$

and that a_n is the first of these points outside $B(x, \frac{1}{L} d_X(x,a))$.

Observe that, for any $i \neq j$ with $0 < i < j \leq n$, by the choice of a_{i+1} we have

$$d_X(a_i, a_j) \geq d_X(a_i, a_{i+1}) = L d_X(a_0, a_i) \geq L^2 d_X(x, b)$$

and that $d_X(a_0, a_j) \geq d_X(a_0, a_1) = L d_X(x, b)$ for $j > 0$. Combining this with the fact that $a_i \in B(x, \frac{1}{L} d_X(x, a))$ for each $i = 1, 2, \cdots, n-1$, the k-HTB property of X implies that

$$n \leq k\left(\frac{\frac{1}{L} d_X(x, a)}{L^2 d_X(x, b)}\right) = k\left(\frac{1}{L^3}\rho\right).$$

Note that this argument also shows that the above process of choosing consecutive points a_i on $\gamma_{a,x}$ terminates after a finte number of steps due to the HTB property. Furthermore, by (L, M)-quasisymmetry, one can inductively deduce that

$$\begin{aligned} d_Y(x', a') &\leq M d_Y(a'_0, a'_n) \leq M(d_Y(a'_0, a'_{n-1}) + d_Y(a'_{n-1}, a'_n)) \\ &\leq M(1+M) d_Y(a'_0, a'_{n-1}) \leq M(1+M)^2 d_Y(a'_0, a'_{n-2}) \\ &\leq \cdots \leq M(1+M)^{n-1} d_Y(a'_0, a'_1) \leq M^2(1+M)^{n-1} d_Y(x', b'). \end{aligned}$$

Therefore, it yields that

$$\rho' = \frac{d_Y(a', x')}{d_Y(b', x')} \leq \eta(\rho) \quad \text{with } \eta(\rho) = M^2(1+M)^{k(\frac{1}{L^3}\rho)-1}. \qquad \square$$

Since an η-QS map is always weakly H-QS, Theorem 5.1 and Example 3 in Section 2 immediately yield the following corollary.

Corollary 5.2. *Suppose that X and Y are arcwise connected metric spaces and that X is k-HTB. If $f\colon X \to Y$ is weakly (L,M)-QS for some $L < 1$ and $M < 1$, then f is weakly H-QS. The converse is not true unless Y is also HTB.*

Interestingly, it follows from Theorem 5.1 and [26, Theorem 2.8] that these three classes of maps are equivalent if both X and Y are HTB.

Corollary 5.3. *Suppose that X and Y are arcwise connected HTB metric spaces. Let $f\colon X \to Y$ be a homeomorphism. Then the following are equivalent.*

(1) *f is weakly H-quasisymmetric;*
(2) *f is weakly (L, M)-quasisymmetric for some $L < 1$ and $M < 1$;*
(3) *f is η-quasisymmetric.*

5.2. Proof of Theorem 1.3. For the proof of Theorem 1.3, we need a lemma which can be regarded as a generalization of [26, Lemma 2.14] from the Euclidean space to metric spaces.

Lemma 5.4. *Suppose that X is an arcwise connected metric space with k-HTB and c-LLC$_1$ properties. Then X is k'-HTB in the internal metric.*

Proof. It suffices to show that there is a constant $k' = k'(k, c)$ (depending only on k, c) such that for any $x_0 \in X$ and $r > 0$, if $x_1, x_2, \cdots, x_n \in \overline{B}_{\delta_X}(x_0, r)$ with $\delta_X(x_i, x_j) \geq \frac{r}{2}$ for any $i \neq j$, then $n \leq k'$.

Fix $x_0 \in X$, $r > 0$, and $x_1, x_2, \cdots, x_n \in \overline{B}_{\delta_X}(x_0, r)$ with above properties. For $j = 2, \cdots, n$, choose an arc γ_j connecting x_j and x_1, and then fix a point $\widetilde{x}_j \in \gamma_j$ such that $\delta_X(x_j, \widetilde{x}_j) = \frac{r}{2^m}$, where m is a fixed constant with $m > 2 + \frac{\log(1+2c)}{\log 2}$. Obviously, $\delta_X(\widetilde{x}_j, x_1) \geq \frac{r}{2^m}$.

If $B\left(\widetilde{x}_i, \frac{r}{2^m}\right) \cap B\left(\widetilde{x}_j, \frac{r}{2^m}\right) \neq \emptyset$ for some $i \neq j$, the c-LLC$_1$ property implies that there exists an arc $\widetilde{\gamma}_{i,j}$ connecting $\widetilde{x}_i$ and $\widetilde{x}_j$ such that

$$\widetilde{\gamma}_{i,j} \subset B\left(\widetilde{x}_i, \frac{cr}{2^{m-1}}\right).$$

Thus, by the choice of the constant m above, it follows that

$$\delta_X(x_i, x_j) \leq \delta_X(x_i, \widetilde{x}_i) + \delta_X(\widetilde{x}_i, \widetilde{x}_j) + \delta_X(\widetilde{x}_j, x_j) \leq \frac{2r}{2^m} + \frac{2cr}{2^{m-1}} = \frac{(1+2c)r}{2^{m-1}} < \frac{r}{2},$$

which contradicts the assumption that $\delta_X(x_i, x_j) \geq \frac{r}{2}$. Therefore, it follows that $B\left(\widetilde{x}_i, \frac{r}{2^m}\right) \cap B\left(\widetilde{x}_j, \frac{r}{2^m}\right) = \emptyset$ for distinct $i, j \in \{2, 3, \cdots, n\}$.

Furthermore, since

$$d_X(\widetilde{x}_i, x_0) \leq \delta_X(\widetilde{x}_i, x_0) \leq \delta_X(\widetilde{x}_i, x_i) + \delta_X(x_i, x_0) \leq \left(\frac{1}{2^m} + 1\right) r$$

for $i = 2, \cdots, n$, the k-HTB property (in original metric) yields

$$n - 1 \leq k\left(\frac{\left(\frac{1}{2^m} + 1\right) r}{\frac{r}{2^m}}\right) = k(1 + 2^m)$$

as desired. □

Proof of Theorem 1.3. By Theorem 1.1 (or by Proposition 4.4 more precisely), we can see that $f^{-1}\colon Y \to X$ is weakly (L, M)-QS in the internal metrics for some $L < 1, M < 1$. Then, since Q-regular spaces are HTB (in original metric), Lemma 5.4 implies that Y is k'-HTB in the internal metric. Applying Theorem 5.1 to the internal metric spaces (Y, δ_Y) and (X, δ_X) and the map f^{-1}, it follows that f^{-1} is η'-QS in the internal metrics. And therefore f is η-QS in the internal metric for some η. □

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