Isomorphisms on interpolation spaces generated by the method of means

MIECZYSŁAW MASTYŁO

Abstract. We investigate the stability of isomorphisms acting between interpolation spaces generated by the method of means. We focus on the methods which are determined by balanced sequences for non-degenerate quasi-concave functions. The key point for our investigation is that these methods have orbital description by a single element generated by a special ideal of operators between Banach couples. We prove that if an operator is invertible in one orbit it is also invertible by nearby orbits provided that the corresponding indices of quasi-concave functions generated these orbits are close to each other. In particular, these results apply to the real method of interpolation.

1. Introduction

One of the major problems in interpolation theory is the study of stability properties of operators acting between interpolation Banach spaces. Studying these types of properties spins off applications in various areas of analysis. Development of the theory of Fredholm operators and a general recognition of the importance of the subject in applications to the solvability of partial differential equations motivates the study of stability of the Fredholm properties under interpolation. The first result on the stability of Fredholm property is due to Shneiberg [20]; it states that, if $T : (X_0, X_1) \to (Y_0, Y_1)$ is an operator between compatible couples of complex Banach spaces, then the set of all $\theta \in (0, 1)$, for which the operator $T : [X_0, X_1]_\theta \to [Y_0, Y_1]_\theta$ is Fredholm between Calderón interpolation spaces is open. This result was overlooked at first, but after a while it became crucial for further research (see [1, 3, 5, 10, 13, 21]) and references given there. It is perhaps appropriate to remark that the stability of Fredholm properties of operators between interpolation scales of Banach spaces is deeply connected with the stability of interpolated isomorphisms. This phenomenon was discussed in [4] for interpolation scales constructed by using vector-valued analytic functions introduced in [11], which recover, up to equivalence of norms, the real and the complex methods of interpolation. These results were used in [5] to study

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the stability of the inverses of isomorphisms acting on interpolation scales of Banach spaces.

The main purpose of the present paper is to investigate the stability of invertibility property for operators between interpolation spaces generated by the method of means. It is a natural continuation of previous work in the literature related to the problem mentioned above (see, e.g., [1, 4, 5, 15]). We focus on methods of means determined by quasi concave-functions which proved to play a key role in the theory of interpolation of linear operators. As we will see, the key to our study is that these methods have an orbital description by a single element generated by a special ideal of operators. We prove that if an operator is invertible in one orbit it is also invertible in nearby orbits provided that the corresponding indices of quasi-concave functions generated these orbits are close to each other. In particular, these results apply to methods of means. It is a natural continuation of previous work in the literature related to the real method of interpolation.

Throughout the paper we will use standard notation. As usual, for a given Banach space \( X \) we denote by \( L(X) \) the Banach space of all bounded linear operators on \( X \) equipped with the uniform norm. If \( X \) and \( Y \) are Banach spaces such that \( X \subset Y \) and the inclusion map \( \text{id} : X \to Y \) is bounded, then we write \( X \hookrightarrow Y \). We write \( X \cong Y \) whenever \( X = Y \), with equality of norms.

2. Notation and main results

First, we introduce some essential definitions and notation. For basic notation of interpolation theory, we refer to [7] and [8]. We recall that a mapping \( F : \mathcal{B} \to \mathcal{B} \), from the category \( \mathcal{B} \) of all couples of Banach spaces into the category \( \mathcal{B} \) of all Banach spaces is said to be an interpolation functor (or method) if, for any couple \( \vec{X} := (X_0, X_1) \), the Banach space \( F(X_0, X_1) \) is intermediate with respect to \( \vec{X} \) (i.e., \( X_0 \cap X_1 \subset F(\vec{X}) \subset X_0 + X_1 \)), and \( T : F(X_0, X_1) \to F(Y_0, Y_1) \) for all \( T : (X_0, X_1) \to (Y_0, Y_1) \); here, as usual, the notation \( T : (X_0, X_1) \to (Y_0, Y_1) \) means that \( T : X_0 + X_1 \to Y_0 + Y_1 \) is a linear operator, such that the restrictions of \( T \) to the space \( X_j \) is a bounded operator from \( X_j \) to \( Y_j \), for \( j = 0 \) and \( j = 1 \). Notice that by the closed graph theorem, for any Banach couples \( \vec{X} \) and \( \vec{Y} \) one has

\[
\|T\|_{F(\vec{X}) \to F(\vec{Y})} \leq C \|T\|_{\vec{X} \to \vec{Y}} := C \max_{j=0,1} \|T\|_{X_j \to Y_j}.
\]

If \( C \) may be chosen independently of \( \vec{X} \) and \( \vec{Y} \), then \( F \) is called a bounded interpolation functor and it is called exact if \( C = 1 \).

An operator \( T : (X_0, X_1) \to (Y_0, Y_1) \) between Banach couples is said to be invertible whenever the restriction \( T|_{X_j} : X_j \to Y_j \) is invertible (i.e., \( T \) is an isomorphism of \( X_j \) onto \( Y_j \)) for \( j = 0 \) and \( j = 1 \). In what follows we will often omit the domain of the restricted operator. We will use Peetre’s \( K \)-functional. Let \( \vec{X} = (X_0, X_1) \) be a Banach couple. For \( t > 0 \)

\[
K(t, x; \vec{X}) = K(t, x; X_0, X_1) := \inf \{ \|x_0\|_{X_0} + t\|x_1\|_{X_1} : x_0 + x_1 = x \}, \quad x \in X_0 + X_1.
\]

Following [15] we will consider a special class of operators between Banach couples. For a fixed Banach couple \( \vec{A} \), we define a left operator ideal \( \mathcal{I} \) (from \( \vec{A} \)) as a subclass of \( L(\vec{A}, \cdot) \) such that for any Banach couple \( \vec{X} \) its components

\[
\mathcal{I}(\vec{A}, \vec{X}) := L(\vec{A}, \vec{X}) \cap \mathcal{I}
\]

are linear spaces which satisfy the following properties for all Banach couples \( \vec{X} \) and \( \vec{Y} \):
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(i) \( \mathcal{I}(\vec{A}, \vec{X}) \) equipped with a norm \( \| \cdot \|_I \) is a Banach space such that
\[
\gamma(\vec{A}) := \sup_{\vec{x} \in \vec{B}} \| \text{id}: \mathcal{I}(\vec{A}, \vec{X}) \to L(\vec{A}, \vec{X}) \| \leq \infty.
\]

(ii) \( \mathcal{I}(\vec{A}, \vec{X}) \) contains all one rank operators from \( \vec{A} \) to \( \vec{X} \).

(iii) (The left ideal property) If \( T \in \mathcal{I}(\vec{A}, \vec{X}) \), \( S \in L(\vec{X}, \vec{Y}) \), then \( ST \in \mathcal{I}(\vec{A}, \vec{Y}) \) and
\[
\| ST \|_I \leq \| S \|_{X \to Y} \| T \|_I.
\]

Let \( \vec{A} \) be a Banach couple and let \( \mathcal{I}(\vec{A}, \cdot) \) be a left operator ideal. For an arbitrary element \( a \neq 0 \) in \( A_0 + A_1 \), we define the orbit \( \text{Orb}^\vec{T}_a(\vec{X}) \) to be the space of all elements in the form \( Ta \), where \( T \in \mathcal{I}(\vec{A}, \vec{X}) \). This space is equipped with the norm
\[
\| x \| = \inf \{ \| T \|_I; \ x = Ta \}.
\]

Since
\[
\| Ta \|_{X_0 + X_1} \leq \| T \|_{\vec{A} \to \vec{X}} \| a \|_{A_0 + A_1} \leq \gamma(\vec{A}) \| T \|_I \| a \|_{A_0 + A_1},
\]
the map \( \delta_a: \mathcal{I}(\vec{A}, \vec{X}) \to X_0 + X_1 \) given by
\[
\delta_a(T) := Ta, \quad T \in \mathcal{I}(\vec{A}, \vec{X}),
\]
is continuous. This implies that \( \text{Orb}^\vec{T}_a(\vec{X}) \) is isometrically isomorphic to the quotient space \( \mathcal{I}(\vec{A}, \vec{X})/\ker^\vec{T}_a(\vec{X}) \), where
\[
\ker^\vec{T}_a(\vec{X}) := \ker(\delta_a) = \{ T \in \mathcal{I}(\vec{A}, \vec{X}); \ Ta = 0 \}
\]
and so \( \text{Orb}^\vec{T}_a(\vec{X}) \) is a Banach space. Clearly, the left ideal property yields that \( \vec{B} \ni \vec{X} \to \text{Orb}^\vec{T}_a(\vec{X}) \) is an exact interpolation functor.

We will use the theorem on stability of invertible operators acting between interpolation orbits generated by the left ideal operator. We need first some notation and collect some results which will be used extensively in the remainder.

Given a Banach space \( U \) and any closed subspaces \( U_0, U_1 \) of \( U \) we let
\[
\rho(U_0, U_1) := \sup_{\| u \|_U = 1} |\text{dist}(u, U_0) - \text{dist}(u, U_1)|,
\]
where \( \text{dist}(u, U_j) \) is the distance from \( u \in U \) to \( U_j \), that is,
\[
\text{dist}(u, U_j) := \inf_{u_j \in U_j} \| u - u_j \|_U, \quad j \in \{0, 1\}.
\]

Let \( U, V \) be Banach spaces and \( U_0, U_1 \) and \( V_0, V_1 \) be closed subspaces of \( U \) and \( V \), respectively. Suppose that \( H \) is a linear bounded operator from \( U \) to \( V \) which maps \( U_0 \) to \( V_0 \) and \( U_1 \) to \( V_1 \). Since for \( j = 0 \) and \( j = 1 \) one has \( H(u + u_j) = H(u) + H(u_j) \in H(u) + V_j \) for all \( u_j \in U_j \), we can define the quotient operators \( H_j: U/U_j \to V/V_j \) by
\[
H_j(u + U_j) := H(u) + V_j, \quad u \in U.
\]

The key result we need is the following theorem from [3, Theorem 3.4] which is a slightly modified variant Theorem 9 from [15].

**Theorem 2.1.** Suppose that \( H: U \to V \) maps \( U_j \) to \( V_j \) for each \( j \in \{0, 1\} \), and the quotient operator \( H_0: U/U_0 \to V/V_0 \) is invertible. If
\[
\max\{\text{dist}(U_0, U_1), \text{dist}(V_0, V_1)\} < \frac{1}{2(1 + \|H\|_{U \to V} \|H^{-1}_0\|_{V/V_0 \to U/U_0})},
\]
then
\[
\|H(u + U_j) - H(u) - V_j\|_V < \|H\|_{U \to V} \|H^{-1}_0\|_{V/V_0 \to U/U_0} \|u + U_j - u - U_0\|_U.
\]
then the quotient operator $H_1: U/U_1 \to V/V_1$ is invertible. Moreover, an upper estimate for the norm of $H_1$ is given by

$$\|H_1^{-1}\|_{V/V_1 \to U/U_1} \leq 2\|H_0^{-1}\|_{V/V_0 \to U/U_0}.$$  

Let $\mathcal{I}(\bar{A}, \cdot)$ be the left ideal of operators. Following [15] in the case of an ideal $L$ of all operators, let $\rho$ be a distance defined on closed subspaces of the spaces of $\mathcal{I}(\bar{A}, \bar{X})$, and let $a_0, a_1 \in A_0 + A_1$. Then we let

$$\rho(\text{Orb}_{a_0}(\bar{X}), \text{Orb}_{a_1}(\bar{X})) := \rho(\ker a_0, \ker a_1).$$

We define a distance between orbits $\text{Orb}_{a_0}$ and $\text{Orb}_{a_1}$ by the following pseudo-metric on $A_0 + A_1$:

$$\rho_I(a_0, a_1) := \rho(\text{Orb}_{a_0}, \text{Orb}_{a_1}) := \sup_{\bar{X} \in \mathcal{B}} \rho(\text{Orb}_{a_0}(\bar{X}), \text{Orb}_{a_1}(\bar{X})).$$

We start with an obvious observation which follows from the definition.

**Lemma 2.2.** For every $a_0, a_1 \in A_0 + A_1$ and any left ideal $\mathcal{I}$ one has

$$\rho_I(a_0, a_1) = \sup \left| \|T a_0\|_{\text{Orb}^2_{a_0}(\bar{X})} - \|T a_1\|_{\text{Orb}^2_{a_1}(\bar{X})} \right|,$$

where the supremum is taken over all $T: \bar{A} \to \bar{X}$ such that $\|T\|_{\mathcal{I}(\bar{A}, \bar{X})} \leq 1$.

Applying Lemma 2.2 we get estimates of the metric $\rho_I$ using operators which allow a suitable cancellations. We omit the proof since it is an inessential modification of the proof of Proposition 3 in [15].

**Proposition 2.3.** Let $\bar{A}$ be a Banach couple, and let $\mathcal{I}(\bar{A}, \cdot)$ be a left ideal of operators. Assume that $\varepsilon > 0$ and $a_0, a_1 \in A_0 + A_1$ are such that the following conditions are satisfied:

(i) For any Banach couple $\bar{X}$ and any operator $T \in \mathcal{I}(\bar{A}, \bar{X})$ with $T a_0 = 0$, it follows that

$$\|T a_0\|_{\text{Orb}^2_{a_0}(\bar{X})} \leq \varepsilon \|T\|_{\mathcal{I}(\bar{A}, \bar{X})}.$$  

(ii) For any Banach couple $\bar{X}$ and any operator $T \in \mathcal{I}(\bar{A}, \bar{X})$ with $T a_1 = 0$, it follows that

$$\|T a_0\|_{\text{Orb}^2_{a_1}(\bar{X})} \leq \varepsilon \|T\|_{\mathcal{I}(\bar{A}, \bar{X})}.$$  

Then one has

$$\rho_I(a_0, a_1) \leq 2\varepsilon.$$  

The theorem which follows now will be useful in proving results on stability of invertible operators acting between spaces generated by method of means.

**Theorem 2.4.** Let $\bar{A}, \bar{X}, \bar{Y}$ be Banach couples, and let $\mathcal{I}(\bar{A}, \cdot)$ be a left ideal of operators. Assume that for some $a_0 \in A_0 + A_1$ an operator $T: \bar{X} \to \bar{Y}$ is such that $T: \text{Orb}^2_{a_0}(\bar{X}) \to \text{Orb}^2_{a_0}(\bar{Y})$ is invertible. Then $T': \text{Orb}^2_{a}(\bar{X}) \to \text{Orb}^2_{a}(\bar{Y})$ is also invertible for all $a \in A_0 + A_1$ such that

$$\rho_I(a_0, a) \leq \left[2(1 + \|T\|_{\bar{X} \to \bar{Y}} \|T^{-1}\|_{\text{Orb}^2_{a}(\bar{Y}) \to \text{Orb}^2_{a}(\bar{X})})\right]^{-1}.$$  

**Proof.** The argument is similar to the proof in [15, Theorem 10(c)] for $\mathcal{I} = L$. We include the proof for completeness. We apply Theorem 2.1 for Banach spaces
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$U := \mathcal{I}(\tilde{A}, \tilde{X})$, $V := \mathcal{I}(\tilde{A}, \tilde{Y})$, and subspaces $U_0 := \ker a_\emptyset (\tilde{X})$, $V_0 := \ker a_\emptyset (\tilde{Y})$ and $U_1 := \ker a_1 (\tilde{X})$, $V_1 := \ker a_1 (\tilde{Y})$ and $H: U \to V$ given by

$H(S) := T \circ S$, $S \in L(\tilde{A}, \tilde{X})$.

Observe that for all $a \in A_0 + A_1$, we have $H: \ker a_\emptyset (\tilde{X}) \to \ker a_\emptyset (\tilde{X})$. Thus using $H$, we can define the quotient operators $H_j: \mathcal{I}(\tilde{A}, \tilde{X})/U_j \to \mathcal{I}(\tilde{A}, \tilde{Y})/V_j$ by

$H_j(S + U_j) := H(S) + V_j$, $S \in \mathcal{I}(\tilde{A}, \tilde{X})$, $j \in \{0, 1\}$.

Clearly, for all $a \in A_0 + A_1$, we have $H: \ker a_\emptyset (\tilde{X}) \to \ker a_\emptyset (\tilde{X})$. This implies that $U: U_j \to V_j$ for $j = 0$ and $j = 1$. Since for any $B \in B$ and for any operator $S \in L(\tilde{A}, \tilde{B})$ one has

$\|Sa\|_{\operatorname{Orb}_E(B)} = \|[S]\|_{\mathcal{I}(\tilde{A}, \tilde{B})/\ker a_\emptyset (\tilde{B})}$, $[S] := S + \ker a_\emptyset (\tilde{B})$,

it follows that

$\|H_j\|_{U_j \to V_j} = \|T\|_{\operatorname{Orb}_j, X(\tilde{X}) \to \operatorname{Orb}_j, Y(\tilde{Y})}$, $j \in \{0, 1\}$.

Finally, applying this and an obvious estimate $\|H\|_{U \to V} \leq \|T\|_{X \to Y}$, we obtain the desired statement from Theorem 2.1. \hfill \square

We consider Banach sequence lattices on $Z$. For any such lattice $E$ and any positive sequence $w = \{w_n\} := \{w_n\}_{n \in Z}$, we define the weighted lattice $E(\{w_n\})$ on $Z$ to be the Banach space of all scalar sequences $\xi = \{\xi_n\}$ such that $\xi w := \{\xi_n w_n\} \in E$. The norm on $E(\{w_n\})$ is defined in the usual way, i.e., $\|x\|_{E(\{w_n\})} = \|xw\|_E$. In the following, we briefly write $E(\{w_n\})$ instead of $E(\{w_n\})$.

Let $E$ be a Banach sequence lattice on $Z$ and let $X$ be a Banach space. The vector sequence $x = \{x_n\}_{n \in Z}$ in $X$ is called strongly $E$-summable if the scalar sequence $\{|x_n|_X\}$ is in $E$. We denote by $E(X)$ the set of all such sequences in $X$. This forms a Banach space under pointwise operations, and a natural norm in $E(X)$ is given by

$\|x\|_{E(X)} := \left\{\|x_n\|_X\right\}_E$.

A pair $\Phi = (\Phi_0, \Phi_1)$ of Banach sequence lattices on $Z$ is called a parameter of the method of means if $\Phi_0 \cap \Phi_1 \subset \ell_1$. The space $J_\Phi(\tilde{X}) = J_{\Phi_0, \Phi_1}(\tilde{X})$ built by the method of means consists of all $x \in X_0 + X_1$ which may be represented in the form

$x = \sum_{n \in Z} u_n \quad (\text{convergence in } X_0 + X_1)$

with $\{u_n\} \in \Phi_0(X_0) \cap \Phi_1(X_1)$. It is well known that $J_\Phi$ is an exact interpolation functor when the Banach space $J_\Phi(\tilde{X})$ is equipped with the norm

$\|x\|_{J_\Phi(\tilde{X})} = \inf \max \left\{\|\{u_n\}\|_{\Phi_0(X_0)}, \|\{u_n\}\|_{\Phi_1(X_1)}\right\}$,

where the infimum is taken over all the above representations of $x$ (see, e.g., [8, 14]).

As usual for a given quasi-concave function $\varphi$, we define $\varphi^*$ by $\varphi^*(t) := t/\varphi(t)$ for all $t > 0$ and $\varphi^*(0) := 0$. We are interested in the special method of means generated by weighted Banach sequence spaces determined by nondegenerate quasi-concave functions, i.e., such quasi-concave functions $\varphi$ that the images of the semi-infinite interval $(0, \infty)$ under $\varphi$ and $\varphi^*$ are $(0, \infty)$. In what follows we consider some important sequences in $(0, \infty)$ connected with quasi-concave functions. The importance of these sequences in the study of abstract real interpolation spaces was discovered independently by Brudnyi and Kruglyak, and Janson (see [8, 12]).
Following [9, Definitions 3.1, 3.2], a positive sequence \( \{ t_k \}_{k \in I} \) is said to be a partition of \( \mathbb{R}_+ := (0, \infty) \) if

(i) \( I \) is a nonempty interval of integers, that is, \( \emptyset \neq I = \mathbb{Z} \cap [\inf I, \sup I] \),
(ii) \( t_{k-1} < t_k \) for each \( k \) such that \( k-1, k \in I \),
(iii) \( \mathbb{R}_+ \subset \bigcup_{k-1, k \in I} [t_{k-1}, t_k] \).

Let \( \varphi \) be a quasi-concave function. A partition \( \{ t_k \}_{k \in I} \) is called a balanced sequence for \( \varphi \) if there exists a constant \( \gamma \geq 1 \) and positive integer \( N \) such that

(i) For each \( k \in I \) such that \( k-1 \in I \), at least one of the inequalities

\[
\varphi(t_k) \leq \gamma \varphi(t_{k-1}), \quad \varphi^*(t_k) \leq \gamma \varphi^*(t_{k-1})
\]

holds,
(ii) for every \( a > 0 \),

\[
\text{card}\{ k \in I; a < \varphi(t_k) \leq 2a \} \leq N, \quad \text{card}\{ k \in I; a < \varphi^*(t_k) \leq 2a \} \leq N.
\]

It is well known that every quasi-concave function \( \varphi \) has a balanced sequence. Note that such sequences can be constructed by induction (see [9, Proposition 3.4]).

From the point of view of applications such functions \( \varphi \) are interesting only when they have a balanced sequence \( \{ t_k \}_{k \in I} \) in which \( I \) is infinite. For this reason, we will focus below only on this case. The proofs presented in this section apply to corresponding Banach sequence lattices \( E \) and quasi-concave functions having balanced sequences modelled on \( I \in \{ \mathbb{Z}, \mathbb{Z}_+, \mathbb{Z}_- \} \), where \( I = \mathbb{Z}_- \) (resp., \( I = \mathbb{Z}_+ \)) in the setting of the subclass of ordered Banach couples \((X_0, X_1)\) with \( X_1 \hookrightarrow X_0 \) (resp., \( X_0 \hookrightarrow X_1 \)) and \( I = \mathbb{Z} \) in the general case of all Banach couples. For simplicity of presentation, we consider only the case when \( I = \mathbb{Z} \) is the whole set of integers, which corresponds to nondegenerate quasi-concave functions \( \varphi \). Notice that it follows from [9, Proposition 3.5] that if \( \{ t_k \}_{k \in \mathbb{Z}} \) is a balanced sequence for a nondegenerate quasi-concave function \( \varphi \), then there exists a positive constant \( C \) such that

\[
C^{-1} \sum_{k \in \mathbb{Z}} \varphi(t_k) \min \left\{ 1, \frac{t}{t_k} \right\} \leq \varphi(t) \leq C \sup_{k \in \mathbb{Z}} \varphi(t_k) \min \left\{ 1, \frac{t}{t_k} \right\}, \quad t > 0.
\]

In terms of the \( K \)-functional of Peetre the above inequalities are equivalent to

\[
C^{-1} K(t, \{ \varphi(t_k) \}; \ell_1, \ell_1(1/t_k)) \leq \varphi(t) \leq C K(t, \{ \varphi(t_k) \}; \ell_\infty, \ell_\infty(1/t_k)).
\]

We point out that a balanced sequence \( \{ t_k \}_{k \in \mathbb{Z}} \) can be constructed by induction so that the above inequalities hold with \( C = 4 \) (see [9, Remark 3.7]). It is straightforward to check that \( \{ 2^n \} \) is the balanced sequence for \( \varphi_\theta \) given by \( \varphi_\theta(t) = t^\theta \) for all \( t \geq 0 \).

Let \( \{ t_n \}_{n \in \mathbb{Z}} \) be a balanced sequence for a nondegenerate quasi-concave function \( \varphi \). From the above inequalities, it follows that \( \{ \varphi(t_n) \} \in \ell_1 + \ell_1(1/t_n) \). This implies that for any couple \((E_0, E_1)\) of Banach sequence lattices on \( \mathbb{Z} \) such that \( E_j \hookrightarrow \ell_\infty \) for \( j = 0, 1 \) one has

\[
\ell_\infty \hookrightarrow E_0'(\varphi(t_n)) + E_1'(\varphi(t_n)/t_n),
\]

where \( E' \) denotes the Köthe dual space of a Banach sequence lattice \( E \) on \( \mathbb{Z} \). Thus, by Köthe duality, it follows that \((\Phi_0, \Phi_1) := (E_0(1/\varphi(t_n)), E_1(t_n/\varphi(t_n)))\) is a parameter of the method of means. In this case the space \( J_\theta(\vec{X}) \) is denoted by \( \vec{X}_{\varphi,E_0,E_1} \). If \( 1 \leq p_0, p_1 \leq \infty, \varphi(t) = t^\theta \) for all \( t \geq 0 \) with \( 0 < \theta < 1 \) and \( \{ t_n \} = \{ 2^n \} \), then \( \vec{X}_{\varphi,E_0,E_1} \) is the classical Lions–Peetre method of means denoted by \( J_{\theta,p_0,p_1}(\vec{X}) \) (see [16]). If \( E_0 = E_1 = E \), then \( J_{\theta,p_0,p_1}(\vec{X}) \) is the classical abstract \( J \)-space, which is denoted by \( J_E(\vec{X}) \) (see [8, 14]).
We will give an orbital description of the spaces \((X_0, X_1)_{\varphi, E_0, E_1}\). In order to do this we need to introduce some additional notation. Suppose we are given a balanced sequence \(\{t_n\}_{n \in \mathbb{Z}}\) for a nondegenerate quasi-concave function \(\varphi\), and a couple \(\vec{E} = (E_0, E_1)\) of Banach sequence lattices on \(\mathbb{Z}\) such that \(E_j \hookrightarrow \ell_\infty\). For any Banach couple \(\vec{X} = (X_0, X_1)\), we define \(\mathcal{I}_{\varphi, \vec{E}}((\ell_1, \ell_1(1/t_n)), \vec{X})\) to be the space of all operators \(T: (\ell_1, \ell_1(1/t_n)) \to (X_0, X_1)\) such that \(\{T(e_n)\} \in E_0(X_0)\) and \(\{T(t_n e_n)\} \in E_1(X_1)\), equipped with the norm given by

\[
\|T\|_{\mathcal{I}_{\varphi, \vec{E}}}(\vec{X}) = \max \left\{ \|\{T(e_n)\}\|_{E_0(X_0)}, \|\{T(t_n e_n)\}\|_{E_1(X_1)} \right\}.
\]

Frequently, for simplicity, we write \(\mathcal{I}_{\varphi, \vec{E}}(\vec{X})\) instead of \(\mathcal{I}_{\varphi, \vec{E}}((\ell_1, \ell_1(1/t_n)), \vec{X})\).

It can be easily verified that \(\mathcal{I}_{\varphi, \vec{E}}\) is a left ideal of the Banach couple \((\ell_1, \ell_1(1/t_n))\). If \(1 \leq q \leq \infty, 0 < \theta < 1\) and \(\varphi(t) = t^\theta\) for all \(t \geq 0\) and \(E_0 = E_1 = \ell_q\), then we recover the left ideal \(\mathcal{I}_q := \mathcal{I}_{\varphi, \ell_q}\) introduced in [15].

Note that under the above assumptions, we have \(a_{\varphi} := \{\varphi(t_n)\}_{n \in \mathbb{Z}} \in \ell_q + \ell_1(1/t_n)\) and so the orbit \(\text{Orb}_{a_{\varphi}}(\vec{X})\) from the couple \((\ell_1, \ell_1(1/t_n))\) to \(\vec{X}\) generated by an ideal \(\mathcal{I} := \mathcal{I}_{\varphi, \vec{E}}\) is well defined. It is denoted by \(\text{Orb}_{a_{\varphi}, \vec{E}}(\vec{X})\). In the case \(E_0 = E_1 = E\), we write \(\mathcal{I}_{\varphi, E}\) and \(\text{Orb}_{a_{\varphi}, \vec{E}}(\vec{X})\) for short instead of \(\mathcal{I}_{\varphi, \vec{E}}\) and \(\text{Orb}_{a_{\varphi}, \vec{E}}\). Moreover if \(\mathcal{I} := \mathcal{I}_{\varphi, \vec{E}}\) we write \(\rho_{\vec{E}}\) instead of \(\rho_{\vec{E}}\).

The starting point for our investigation is the following result. We omit the proof which is an inessential modification of the proof Theorem 12 in the work by Kruglyak-Milman [15] in the case of the left ideal \(\mathcal{I}_q = \mathcal{I}_{\varphi, \ell_q}\) with \(\varphi(t) = t^\theta\) for all \(t \geq 0\) and for some \(\theta \in (0, 1)\).

**Theorem 2.5.** For every Banach couple \(\vec{X}\) the following isometrical formula holds

\[
\text{Orb}_{a_{\varphi}, \vec{E}}(\vec{X}) \cong J_{\varphi, \vec{E}}(\vec{X}).
\]

The following variant of Lemma 2 from [15] is the essential ingredient in the proofs of the main results.

**Lemma 2.6.** Let \(\vec{X} = (X_0, X_1)\) be a Banach couple and let \(x \in X_0 + X_1\). Assume that \(w = \{w_k\}_{k \in \mathbb{Z}}\) is a positive sequence and \(\{x_k\}_{k \in \mathbb{Z}}\) and \(\{x_k^1\}_{k \in \mathbb{Z}}\) are sequences in \(X_0\) and \(X_1\), respectively such that \(x_k^0 + x_k^1 = x\) for each \(k \in \mathbb{Z}\) and the series \(\sum_{k \in \mathbb{Z}} w_k (x_k^0 - x_{k-1}^0)\) converges in \(X_0 + X_1\). If for a given \(n \in \mathbb{Z}\) the series \(\sum_{k < n} |w_{k+1} - w_k| \|x_k^0\|_{X_0}\) and \(\sum_{k \geq n} |w_{k+1} - w_k| \|x_k^1\|_{X_1}\) converge absolutely in \(X_0\) and \(X_1\), respectively, then for every \(t > 0\) one has

\[
K(t, x_w - w_n x) \leq \sum_{k < n} |w_{k+1} - w_k| \|x_k^0\|_{X_0} + t \sum_{k \geq n} |w_{k+1} - w_k| \|x_k^1\|_{X_1},
\]

where \(x_w := \sum_{k \in \mathbb{Z}} w_k (x_k^0 - x_{k-1}^0)\).

**Proof.** Clearly, our hypotheses imply that the following series converge in \(X_0 + X_1\) and it is follows readily that

\[
x_w - w_n x = \sum_{k \leq n} w_k (x_k^0 - x_{k-1}^0) - w_n x_n^0 + \sum_{k > n} w_k (x_{k-1}^0 - x_k^0) - w_n x_n^1
\]

\[
= \sum_{k < n} (w_k - w_{k+1}) x_k^0 + \sum_{k \geq n} (w_{k+1} - w_k) x_k^1.
\]

Since \(\sum_{k < n} (w_k - w_{k+1}) x_k^0 \in X_0\) and \(\sum_{k \geq n} (w_{k+1} - w_k) x_k^1 \in X_1\), the desired estimate follows. \(\square\)
We need to introduce the following notion. Fix pair \((\varphi_0, \varphi_1)\) of nondegenerate quasi-concave functions. A positive sequence \(\{t_k\}_{k \in I}\) is said to be admissible of \(\varphi_0\) and \(\varphi_1\) if it is a common balanced sequence for \(\varphi_0\) and \(\varphi_1\) such that

\[
\sup_{k, k+1 \in I} \frac{w_{k+1}}{w_k} < \infty,
\]

where \(w_k := \frac{\varphi_1(t_k)}{\varphi_0(t_k)}\) for each \(k \in I\). A pair \((\varphi_0, \varphi_1)\) is said to be admissible whenever there exists an admissible sequence \(\{t_k\}_{k \in I}\) of \(\varphi_0\) and \(\varphi_1\) such that

\[
\inf_{k, k+1 \in I} \frac{w_{k+1}}{w_k} > 0.
\]

For any balanced sequence \(t := \{t_n\}_{n \in \mathbb{Z}}\) for a nondegenerate quasi-concave function \(\varphi\), the Calderón operator \(S^t\) defined on \(\ell_1 + \ell_1(1/t)\) is given by

\[
S^t(\xi) := \left\{ \sum_{k \leq n} \xi_k + t_n \sum_{k > n} \frac{\xi_k}{t_k} \right\}_{n \in \mathbb{Z}}, \quad \xi = \{\xi_n\} \in \ell_1 + \ell_1(1/t).
\]

Now we are ready to state the following result.

**Theorem 2.7.** Let \(t = \{t_n\}_{n \in \mathbb{Z}}\) be a common balanced sequence for nondegenerate quasi-concave functions \(\varphi_0\) and \(\varphi_1\). Assume that the Calderón operator \(S := S^t\) is bounded in spaces \(F_0\) and \(F_1\), where \(F_j := E(1/\varphi_j(t_n))\) for \(j \in \{0, 1\}\) and \(E\) is a Banach sequence lattice on \(\mathbb{Z}\) with \(E \hookrightarrow \ell_\infty\). Then for any Banach couple \(\tilde{X} = (X_0, X_1)\) and any operator \(T: (\ell_1, \ell_1(1/t)) \to (X_0, X_1)\) with \(T \in \mathcal{I}_{\varphi_0,E}(\tilde{X})\), one has

\[
\left\| K(t_n, T \left( \frac{a_{\varphi_1}}{\varphi_1(t_n)} - \frac{a_{\varphi_0}}{\varphi_0(t_n)} ; \tilde{X} \right) \right\|_E \\
\leq 2 \sup_{n \in \mathbb{Z}} \frac{w_{n+1}}{w_n} - 1 \|S\|_{F_0 \to F_0} \|S\|_{F_1 \to F_1} \|T\|_{\mathcal{I}_{\varphi_0,E}(\tilde{X})},
\]

where \(w_n = \frac{\varphi_1(t_n)}{\varphi_0(t_n)}\) for each \(n \in \mathbb{Z}\).

**Proof.** Fix an operator \(T: (\ell_1, \ell_1(1/t)) \to (X_0, X_1)\). Since for both \(j = 0\) and \(j = 1\) one has \(a_{\varphi_j} = \sum_{k \in \mathbb{Z}} \varphi_j(t_k)e_k\) (convergence in \(\ell_1 + \ell_1(1/t)\)),

\[
T(a_{\varphi_j}) = \sum_{k \in \mathbb{Z}} T(\varphi_j(t_k)e_k) \quad \text{(convergence in } X_0 + X_1)\]

Let \(x := T(a_{\varphi_0})\). Since for each \(k \in \mathbb{Z}\) the series \(\sum_{i < k} \varphi_0(t_i)e_i\) and \(\sum_{i \geq k} \varphi_0(t_i)e_i\) converge absolutely in \(\ell_1\) and \(\ell_1(1/t)\) respectively, it follows that \(x = x_k^0 + x_k^1\), where

\[
x_k^0 := \sum_{i < k} T(\varphi_0(t_i)e_i) \in X_0, \quad x_k^1 := \sum_{i \geq k} T(\varphi_0(t_i)e_i) \in X_1.
\]

The above formulas yield

\[
T(a_{\varphi_1}) = \sum_{k \in \mathbb{Z}} T(\varphi_1(t_k)e_k) = \sum_{k \in \mathbb{Z}} \frac{\varphi_1(t_k)}{\varphi_0(t_k)} T(\varphi_0(t_k)e_k) = \sum_{k \in \mathbb{Z}} w_k(x_k^0 - x_k^1).
\]

We set \(\gamma(w) := \sup_{n \in \mathbb{Z}} \left| \frac{w_{n+1}}{w_n} - 1 \right|\) and \(\widetilde{K}(t_k, x) := \|x_k^0\|_{X_0} + t_k \|x_k^1\|_{X_1}\) for each \(k \in \mathbb{Z}\).

Applying Lemma 2.6 with \(x = T(a_{\varphi_0})\) and \(x_w = T(a_{\varphi_1})\), we conclude that for each
$n \in \mathbb{Z}$ one has

\[
K(t_n, T(a_{\varphi_1}) - w_n T(a_{\varphi_0})) \leq \sum_{k<n} |w_{k+1} - w_k| \|x^0_k\|_X + t_n \sum_{k\geq n} |w_{k+1} - w_k| \|x^1_k\|_X
\]

\[
\leq \left( \sup_{k \in \mathbb{Z}} \left| \frac{w_{k+1}}{w_k} - 1 \right| \right) \left( \sum_{k<n} w_k \|x^0_k\|_X + t_n \sum_{k \geq n} w_k \|x^1_k\|_X \right)
\]

\[
\leq \gamma(w) \left( \sum_{k<n} w_k K(t_k, x) + t_n \sum_{k \geq n} \frac{w_k}{t_k} K(t_k, x) \right) = \gamma(w) S(\{w_kK(t_k, x)\}).
\]

Consequently

\[
K(t_n, T\left( \frac{a_{\varphi_1}}{\varphi_1(t_n)} - \frac{a_{\varphi_0}}{\varphi_0(t_n)} \right)) \leq \gamma(w) \frac{S(\{w_kK(t_k, x)\})}{\varphi_1(t_n)}, \quad n \in \mathbb{Z},
\]

whence

\[
\left\| \left\{ K(t_n, T\left( \frac{a_{\varphi_1}}{\varphi_1(t_n)} - \frac{a_{\varphi_0}}{\varphi_0(t_n)} \right); \bar{X} \right\} \right\|_{E(X_o)} \leq \gamma(w) \left\| S(\{w_kK(t_k, x)\}) \right\|_{F_0}.
\]

The hypothesis $T \in I := I_{\varphi_0, E}(\bar{X})$ yields

\[
\left\| \{ \varphi_0(t_k)\|T(e_k)\|_X \} \right\|_{F_0} = \left\| \{ T(e_k) \} \right\|_{E(X_o)} \leq \|T\|_{I},
\]

and

\[
\left\| \{ \varphi_0(t_k)\|T(t_k e_k)\|_X \} \right\|_{F_0} = \left\| \{ T(t_k e_k) \} \right\|_{E(X)} \leq \|T\|_{I}.
\]

Now observe that

\[
\bar{K}(t_n, x) = \|x^0_n\|_X + t_n \|x^1_n\|_X \leq \sum_{k<n} \varphi_0(t_k) \|T(e_k)\|_X + t_n \sum_{k \geq n} \frac{\varphi_0(t_k)}{t_k} \|T(t_k e_k)\|_X
\]

\[
\leq S(\{\varphi_0(t_k)\|T(e_k)\|_X\}) + S(\{\varphi_0(t_k)\|T(t_k e_k)\|_X\}).
\]

Combining this with the estimates above gives

\[
\left\| \{ \bar{K}(t_n, x) \} \right\|_{F_0} \leq 2 \left\| S(\{\varphi_0(t_k)\|T\|_{I} + \|T\|_{I}) \right\|_{F_0},
\]

so the required estimate follows from the inequality (*). \qed

To present applications of Theorem 2.7 we need a connection between the special method of means and the $K$-method. Suppose we are given a positive sequence $s = \{s_n\} \in c_0(\mathbb{Z})$ (that is $\lim_{|n| \to \infty} s_n = 0$) and a Banach sequence lattice on $\mathbb{Z}$ such that

\[
\ell_\infty \cap \ell_\infty(1/s_n \in \Phi \subset \ell_1 + \ell_1(1/s_n).
\]

Note that $\Phi \subset \ell_1 + \ell_1(1/s_n)$ implies that $\Phi \cap \Phi(s) \subset \ell_1$, so the method of means $J_{\Phi, \Phi(s)}$ is well defined. For any Banach couple $\bar{X} = (X_0, X_1)$, we define $K^*_\Phi(\bar{X})$ to be a Banach space of all $x \in X_0 + X_1$ equipped with the norm

\[
\|x\|_K^\Phi(\bar{X}) := \left\| \{K(s_n, x; \bar{X})\} \right\|_{\Phi},
\]

The proof of the following lemma is similar to the equivalence theorem for the classical Lions-Peetre $J_{\Phi, \Phi}$- and $K_{\Phi, \Phi}$-methods. For the sake of completeness we include a proof.

**Lemma 2.8.** Let $\{s_n\} \in c_0(\mathbb{Z})$ be a positive sequence. Then under the above conditions on $\Phi$ the following hold for any Banach couple $\bar{X} = (X_0, X_1)$:

(i) $K^*_\Phi(\bar{X}) \hookrightarrow J_{\Phi, \Phi(s)}(\bar{X})$ with the norm less than or equal to 4.
(ii) If the Calderón operator $S = S^*$ is bounded in $\Phi$, then $J_{\Phi, \Phi(s)}(\bar{X}) \hookrightarrow K^*_\Phi(\bar{X})$ with norm less than or equal to $2 \|S\|_{\Phi \to \Phi}$. 

Isomorphisms on interpolation spaces generated by the method of means
Proof. (i). For any \( x \in K^*_\varphi(\vec{X}) \) one has \( \{K(s_k, x; \vec{X})\} \in \ell_1 + \ell_1(1/s_k) \). Hence
\[
\min \{1, 1/s_k\} K(s_k, x; \vec{X}) \to 0 \quad \text{as} \quad |k| \to \infty.
\]
Then it follows from the proof of a fundamental lemma for interpolation (see [7, Lemma 3.3.2]) that there exists a representation of \( x \), such that
\[
x = \sum_{k \in \mathbb{Z}} u_k \quad \text{(convergence in} \ X_0 + X_1)\]
and
\[
\max\{\|u_k\|_{X_0}, s_k \|u_k\|_{X_1}\} \leq 4K(s_k, x; \vec{X}), \quad k \in \mathbb{Z}.
\]
This shows that
\[
\|\text{id}: K^*_\varphi(\vec{X}) \to J_{\Phi, \Phi(\cdot)}(\vec{X})\| \leq 4.
\]
(ii). For given \( \varepsilon > 0 \) and \( x \in X := J_{\Phi, \Phi(\cdot)}(\vec{X}) \) there exists a representation of \( x \), such that
\[
x = \sum_{k \in \mathbb{Z}} u_k \quad \text{(convergence in} \ X_0 + X_1)\]
and
\[
\max\{\|\{u_k\}\|_{\Phi(X_0)} , \|\{s_k u_k\}\|_{\Phi(X_1)}\} \leq \|x\|_X + \varepsilon.
\]
Hence, letting \( \xi = \{\xi_k\} \) with \( \xi_k := \max\{\|u_k\|_{X_0}, s_k \|u_k\|_{X_1}\} \) for each \( k \in \mathbb{Z} \), we get
\[
K(s_n, x; \vec{X}) \leq \sum_{k \in \mathbb{Z}} K(s_n, u_k; \vec{X}) \leq \sum_{k \in \mathbb{Z}} \min\{\|u_k\|_{X_0}, s_n \|u_k\|_{X_1}\}
\]
\[
\leq \sum_{k \in \mathbb{Z}} \min\left\{1, \frac{s_n}{s_k}\right\} \max\{\|u_k\|_{X_0}, s_k \|u_k\|_{X_1}\} = (S\xi)_n.
\]
The above estimate combined with
\[
\|\xi\|_{\Phi} \leq 2 \max\{\|\{u_k\}\|_{\Phi(X_0)} , \|\{s_k u_k\}\|_{\Phi(X_1)}\} \leq 2 \|x\|_X + \varepsilon
\]
yields
\[
\|x\|_{K^*_\varphi(\vec{X})} \leq 2\|S\|_{\Phi \to \Phi}(\|x\|_X + \varepsilon).
\]
Since \( \varepsilon > 0 \) is arbitrary, the required statement follows. \( \square \)

To state the next result we note that in what follows for a given balanced sequence \( s = \{s_n\} \) for a nondegenerate quasi-concave function \( \varphi \) and a Banach sequence lattice \( E \hookrightarrow \ell_\infty \), we let
\[
K_{\varphi, E}(\vec{X}) := K^*_\varphi(\vec{X}),
\]
for any Banach couple \( \vec{X} \) where \( \Phi := E(1/\varphi(s_n)) \).

**Theorem 2.9.** Assume that \( \varphi_0, \varphi_1 \) and \( E \) satisfy the conditions of Theorem 2.7, where \( E \) is a Banach sequence lattice on \( \mathbb{Z} \) intermediate between \( \ell_1 \) and \( \ell_\infty \). Then there exists a constant \( C > 0 \) such that for \( S := S^t \) with \( t = \{t_n\}_{n \in \mathbb{Z}} \) one has
\[
\rho_E(a_{\varphi_0}, a_{\varphi_1}) \leq C \sup_{k \in \mathbb{Z}} \left| \frac{w_{k+1}}{w_k} - 1 \right| ||S||_{F_0 \to F_0} ||S||_{F_1 \to F_1},
\]
where \( w_k = \frac{\varphi_1(u_k)}{\varphi_0(u_k)} \) for each \( k \in \mathbb{Z} \) and \( F_j := E(1/\varphi_j(t_n)) \) for \( j \in \{0,1\} \).
Proof. From Lemma 2.8 it follows that \( K_{\varphi,E}(\vec{X}) \hookrightarrow J_{\varphi,E}(\vec{X}) \) with the norm of the inclusion map less than or equal to 4.

By Theorem 2.7, we conclude that there exists a universal constant \( C > 0 \) such that
\[
\|\text{id} : K_{\varphi,E}(\vec{X}) \rightarrow \text{Orb}_{a,\varphi,E}(\vec{X}) \| < C/4.
\]

Now observe that if \( T : (\ell_1, \ell_1(1/t_{ij})) \) is any operator in \( \mathcal{I} := \mathcal{I}_{\varphi,E}(\vec{X}) \) such that \( \|T\|_\mathcal{I} \leq 1 \) and \( T(a_{\varphi_j}) = 0 \) for \( j = 0 \). The above continuous inclusion combined with Theorem 2.7 yields,
\[
\|Ta_{\varphi_j}\|_{\text{Orb}_{a,\varphi_j,E}(\vec{X})} \leq \frac{C}{2} \sup_{k \in \mathbb{Z}} \left| \frac{w_{k+1}}{w_k} - 1 \right| \|S\|_{F_0 \rightarrow F_0} \|S\|_{F_1 \rightarrow F_1}, \quad j = 0.
\]

Similar statement holds for \( j = 1 \). This completes the proof by Proposition 2.3. \( \square \)

We make applications to a special class of pairs of quasi-concave functions. Following Brudnyi and Shteinberg [9], quasi-concave functions \( \varphi_0 \) and \( \varphi_1 \) are said to be in the same scale if there exist quasi-concave functions \( \psi_0, \psi_1 \) and \( \theta_0, \theta_1 \in (0, 1) \) such that
\[
\varphi_0 = \psi_0^{1-\theta_0} \psi_1^{\theta_0}, \quad \varphi_1 = \psi_0^{1-\theta_1} \psi_1^{\theta_1}.
\]

We will use the following lemma.

**Lemma 2.10.** Let \( \varphi_0 \) and \( \varphi_1 \) be quasi-concave functions in the same scale. Then the pair \( (\varphi_0, \varphi_1) \) is admissible.

**Proof.** By [9, Proposition 3.9], it follows that if quasi-concave functions \( \varphi_0 \) and \( \varphi_1 \) are in the same scale, then \( \{t_k\}_{k \in I} \) is a balanced sequence for \( \varphi_0 \) if and only if it is a balanced sequence for \( \varphi_1 \) and, moreover for any common balanced sequence \( \{t_k\}_{k \in I} \) for \( \varphi_0 \) and \( \varphi_1 \) there exists a constant \( \gamma \) such that for each \( k \) such that \( k - 1, k \in I \) at least one of the following conditions holds:
\[
\varphi_j(t_k) < \varphi_j(t_{k-1}), \quad j \in \{0, 1\},
\]
or
\[
\varphi_j^*(t_k) < \gamma \varphi_j^*(t_{k-1}), \quad j \in \{0, 1\}.
\]

Now we assume that \( \{t_k\}_{k \in I} \) is a common balanced sequence for \( \varphi_0 \) and \( \varphi_1 \). We claim that this sequence satisfies the required conditions, which ensure that the pair \( (\varphi_0, \varphi_1) \) is admissible. To see this let \( \{w_k\}_{k \in I} \) denote the sequence given by \( w_k := \varphi_1(t_k)/\varphi_0(t_k) \) for each \( k \in I \). Recall that \( t_k < t_{k+1} \) for each pair \( (k, k+1) \in I \times I \). Since \( \varphi_j \) and \( \varphi_j^* \) are non-decreasing functions, for \( j = 0 \) and \( j = 1 \), the first (resp., the second) estimate above yields for each \( k \) such that \( k, k+1 \in I \),
\[
\frac{1}{\gamma} \leq \frac{\varphi_0(t_k)}{\varphi_0(t_{k+1})} \leq \frac{w_{k+1}}{w_k} = \frac{\varphi_1(t_{k+1})}{\varphi_1(t_k)} \frac{\varphi_0(t_k)}{\varphi_0(t_{k+1})} \leq \frac{\varphi_1(t_{k+1})}{\varphi_1(t_k)} \leq \gamma
\]
(resp.,
\[
\frac{1}{\gamma} \leq \frac{\varphi_1^*(t_k)}{\varphi_1^*(t_{k+1})} \leq \frac{w_{k+1}}{w_k} = \frac{\varphi_1^*(t_{k+1})}{\varphi_1^*(t_k)} \frac{\varphi_0(t_k)}{\varphi_0(t_{k+1})} \leq \frac{\varphi_1^*(t_{k+1})}{\varphi_1^*(t_k)} \leq \gamma.
\]

This completes the proof. \( \square \)

Recall that a Banach sequence lattice \( E \) on \( \mathbb{Z} \) is called rearrangement invariant (r.i. for short) if \( \xi = \{\xi_k\}_{k \in \mathbb{Z}} \in E \) and \( \pi \) is an arbitrary permutation of \( \mathbb{Z} \), then the sequence \( \xi_\pi = \{\xi_{\pi(k)}\}_{k \in \mathbb{Z}} \) belongs to \( E \) and \( \|\xi_\pi\|_E = \|\xi\|_E \).

We are ready to state the following result.
Theorem 2.11. Let nondegenerate quasi-concave functions $\varphi_0$ and $\varphi_1$ be given by $\varphi_0 = \psi_0^{1-\theta_0} \psi_1^{\theta_0}$ and $\varphi_1 = \psi_0^{1-\theta_1} \psi_1^{\theta_1}$, where $\psi_0$, $\psi_1$ are quasi-concave functions and $\theta_0, \theta_1 \in (0, 1)$ with $\theta_0 \neq \theta_1$. Suppose that there exist $C > 1$ and a balanced sequence $\{t_n\}_{n \in \mathbb{Z}}$ for $\varphi_0$ such that

$$\frac{1}{C} \leq \frac{\psi_1(t_{k+1})}{\psi_1(t_k)} \frac{\psi_0(t_k)}{\psi_0(t_{k+1})} \leq C, \quad k \in \mathbb{Z}.$$ 

Then for any r.i. Banach sequence lattice $E$ on $\mathbb{Z}$ one has

$$\rho_E(a_{\varphi_0}, a_{\varphi_1}) \leq (C^{1-\theta_1} - 1) \displaystyle{\|S\|_{F_0 \rightarrow F_0} \|S\|_{F_1 \rightarrow F_1}},$$

where $F_j := E(1/\varphi_j(t_n))$ for $j \in \{0, 1\}$.

Proof. Without loss of generality we may assume that $\theta_0 < \theta_1$. The result by Brudnyi and Shteinberg mentioned in the proof of Lemma 2.10 shows that $\{t_n\}$ is also a balanced sequence for $\varphi_1$. Since $E$ is a r.i. Banach sequence lattice, the analysis of the proof of Lemma 6 in [12] shows that the Calderón operator $S$ generated by a sequence $\{t_n\}$ is bounded in $E(1/\varphi_j(t_n))$ for $j = 0, 1$.

Let $\{w_n\}$ be given by $w_n = \frac{\varphi_0(t_n)}{\psi_0(t_n)}$ for each $n \in \mathbb{Z}$. Then we have

$$\frac{1}{\gamma} \geq \frac{\psi_0(t_k)}{\psi_0(t_{k+1})} = \frac{w_{k+1}}{w_k} = \left( \frac{\psi_1(t_{k+1})}{\psi_1(t_k)} \frac{\psi_0(t_k)}{\psi_0(t_{k+1})} \right)^{\theta_1 - \theta_0}, \quad k \in \mathbb{Z}.$$ 

The hypotheses on $\varphi_0$ and $\varphi_1$ combined with

$$\sup_{k \in \mathbb{Z}} \left| \frac{w_{k+1}}{w_k} - 1 \right| \leq \max \left\{ -1 + \sup_{k \in \mathbb{Z}} \frac{w_{k+1}}{w_k}, 1 - \inf_{k \in \mathbb{Z}} \frac{w_{k+1}}{w_k} \right\}$$

gives

$$\sup_{k \in \mathbb{Z}} \left| \frac{w_{k+1}}{w_k} - 1 \right| \leq \max \left\{ C^{\theta_1 - \theta_0} - 1, 1 - C^{\theta_0 - \theta_1} \right\} \leq C^{\theta_1 - \theta_0} - 1.$$ 

To finish we apply Theorem 2.9. \qed

We give applications to quasi-power functions. We first recall that the lower and upper dilation indices of a quasi-concave function $\phi: [0, a) \rightarrow [0, \infty)$ $(0 < a \leq \infty)$ are given by

$$\gamma_\phi := \lim_{t \rightarrow 0^+} \frac{\ln s_\phi(t)}{\ln t}, \quad \delta_\phi := \lim_{t \rightarrow \infty} \frac{\ln s_\phi(t)}{\ln t},$$

respectively, where $s_\phi(t) := \sup_{0 < s < a, 0 < t < a} \phi(s) \phi(t)^{\phi(s)t}$. We have $0 \leq \gamma_\phi \leq \delta_\phi \leq 1$.

A quasi-concave function $\phi(0, \infty) \rightarrow (0, \infty)$ is said to be quasi-power if $0 < \gamma_\phi \leq \delta_\phi < 1$. It is well-known that $\{2^n\}_{n \in \mathbb{Z}}$ is a balanced sequence for any quasi-power concave function $\varphi$. In what follows for any quasi-power function $\varphi$, we let $a_{\varphi} := \{\varphi(2^n)\}_{n \in \mathbb{Z}}$ and $S := S^t$ with $t := \{2^n\}$.

Corollary 2.12. Let quasi-power functions $\varphi_0$ and $\varphi_1$ be given by $\varphi_0 = \psi_0^{1-\theta_0} \psi_1^{\theta_0}$ and $\varphi_1 = \psi_0^{1-\theta_1} \psi_1^{\theta_1}$, where $\psi_0$, $\psi_1$ are quasi-concave functions and $\theta_0, \theta_1 \in (0, 1)$ with $\theta_0 \neq \theta_1$. Then for any r.i. Banach sequence lattice $E$ on $\mathbb{Z}$ one has

$$\rho_E(a_{\varphi_0}, a_{\varphi_1}) \leq |\theta_0 - \theta_1| \displaystyle{\|S\|_{F_0 \rightarrow F_0} \|S\|_{F_1 \rightarrow F_1}},$$

where $F_j := E(1/\varphi_j(2^n))$ for $j \in \{0, 1\}$.

Proof. Note that any quasi-power function $\rho$ is nondegenerate. By quasi-concavity of $\psi_0$ and $\psi_1$ we get

$$\frac{1}{2} \leq \frac{\psi_1(2^{k+1})}{\psi_1(2^k)} \frac{\psi_0(2^k)}{\psi_0(2^{k+1})} \leq 2, \quad k \in \mathbb{Z}.$$
We now apply Theorem 2.11 for \( \{t_k\}_{k \in \mathbb{Z}} := \{2^k\} \) and \( C = 2 \) to get by the obvious inequality \( 2^n - 1 \leq \alpha \) for all \( \alpha \in [0, 1] \), the required statement.

**Theorem 2.13.** Assume that quasi-power functions \( \varphi_0 \) and \( \varphi_1 \) are such that for some \( \theta_0, \theta_1 \in (0, 1) \) one of the following conditions is satisfied:

(i) \( \{\varphi_1(2^n)/\varphi_0(2^n)\}_{n \in \mathbb{Z}} \) is a nondecreasing sequence, \( t \mapsto \varphi_0(t)/t^{\theta_0} \) is a nondecreasing function on \( \mathbb{R}_+ \) and \( t \mapsto \varphi_1(t)/t^{\theta_1} \) is a nonincreasing function on \( \mathbb{R}_+ \).

(ii) \( \{\varphi_1(2^n)/\varphi_0(2^n)\}_{n \in \mathbb{Z}} \) is a nonincreasing sequence, \( t \mapsto \varphi_0(t)/t^{\theta_0} \) is a nonincreasing function on \( \mathbb{R}_+ \) and \( t \mapsto \varphi_1(t)/t^{\theta_1} \) is nondecreasing on \( \mathbb{R}_+ \).

Then for any translation-invariant Banach sequence lattice on \( \mathbb{Z} \) one has

\[
\rho_E(a_{\varphi_0}, a_{\varphi_1}) \leq C|\theta_0 - \theta_1| \|S\|_{F_0 \to F_0} \|S\|_{F_1 \to F_1},
\]

where \( F_j := E(1/\varphi_j(2^n)) \) for \( j \in \{0, 1\} \).

**Proof.** Since \( E \) is an r.i. Banach sequence lattice, it is easy to verify that the Calderón operator \( S \) generated by a sequence \( \{2^n\} \) is bounded in \( E(1/\varphi(2^n)) \) for any quasi-power function \( \varphi \).

Assume that the condition (i) (resp., (ii)) holds. Let \( w_n = \frac{\varphi_1(2^n)}{\varphi_0(2^n)} \) for each \( n \in \mathbb{Z} \). Then our hypotheses on \( \varphi_0 \) and \( \varphi_1 \) imply that for each \( k \in \mathbb{Z} \) one has

\[
1 \leq \frac{w_{k+1}}{w_k} = \frac{\varphi_1(2^{k+1})}{\varphi_1(2^k)} \cdot \frac{\varphi_0(2^k)}{\varphi_0(2^{k+1})} \leq 2^{\theta_1} \cdot 2^{-\theta_0} = 2^{\theta_1 - \theta_0}
\]

(resp., \( 2^{\theta_1 - \theta_0} \leq \frac{w_{k+1}}{w_k} \leq 1 \)); in particular, \( \theta_0 \leq \theta_1 \) (resp., \( \theta_1 \leq \theta_0 \)). These estimates combined with the above-mentioned inequality \( 2^n - 1 \leq \alpha \) for all \( \alpha \in [0, 1] \) yield

\[
\sup_{k \in \mathbb{Z}} \left| \frac{w_{k+1}}{w_k} - 1 \right| \leq 2^{\theta_0 - \theta_1} - 1 \leq |\theta_0 - \theta_1|.
\]

Thus the required statement follows from Theorem 2.9.

We give examples of non-power functions satisfying the conditions of Theorem 2.13. Fix a quasi-concave function \( \varphi \) and any \( \alpha_0, \alpha_1, \beta_0, \beta_1, \theta_0 \in (0, 1) \) such that \( 0 < \theta_0 < \alpha_1 < 1/2 \) and \( 0 < \beta_0 < \beta_1 < \alpha_1 \). For \( j \in \{0, 1\} \) we define

\[
\varphi_j(t) := t^{\alpha_j} \varphi(t)^{\beta_j}, \quad t \geq 0,
\]

and note that

\[
\gamma_{\varphi_j} = \alpha_j + \beta_j \gamma_{\varphi} \leq \delta_{\varphi_j} = \alpha_j + \beta_j \delta_{\varphi}.
\]

Since \( \varphi \) is quasi-concave, \( 0 \leq \gamma_{\varphi} \leq \delta_{\varphi} \leq 1 \). Thus our hypotheses imply that \( \varphi_0 \) and \( \varphi_1 \) are quasi-power functions. If we put \( \theta_0 := \alpha_0 \) and \( \theta_1 := 2\alpha_3 \), then the functions \( t \mapsto \varphi_1(t)/\varphi_0(t) \), \( t \mapsto \varphi_0(t)/t^{\theta_0} \) are nondecreasing and \( t \mapsto \varphi_1(t)/t^{\theta_1} \) is nonincreasing on \( (0, \infty) \). Thus the quasi-power functions \( \varphi_0, \varphi_1 \) satisfy the condition (i) of Theorem 2.13. In a similar way, we can construct an example of functions satisfying the second condition (ii).

Before we state the next result we introduce the following definition. A family \( \{\varphi_\theta\}_{\theta \in (0, 1)} \) of nondegenerate quasi-concave functions is said to be stable with respect to a Banach sequence lattice \( E \) on \( \mathbb{Z} \) with \( E \mapsto \ell_\infty \) if there exists a common balanced sequence \( \{t_n\}_{n \in \mathbb{Z}} \) for all functions of this family having the property: if \( \theta_0 \in (0, 1) \) is fixed, then for any \( \delta > 0 \) there exists \( \varepsilon > 0 \) such that if \( |\theta - \theta_0| < \varepsilon \), then

\[
\rho_E(a_{\varphi_\theta}, a_{\varphi_{\theta_0}}) < \delta.
\]
We are ready to formulate a variant of Shneiberg’s result on stability of bounded invertible operators acting on a corresponding scale of interpolation spaces generated by the method of means.

**Theorem 2.14.** Let \( \{\varphi_\theta\}_{\theta \in (0,1)} \) a family of nondegenerate quasi-concave functions which is stable with respect to a Banach sequence lattice \( E \hookrightarrow \ell_\infty \). If \( T : X \to Y \) is an operator between Banach couples such that \( T : J_{\varphi_{\theta_0},E}(X) \to J_{\varphi_{\theta_0},E}(Y) \) is invertible for some \( \theta_0 \in (0,1) \), then there exists \( \varepsilon > 0 \) such that \( T : J_{\varphi_{\theta},E}(X) \to J_{\varphi_{\theta},E}(Y) \) is invertible for all \( \theta \in (0,1) \) with \( |\theta - \theta_0| < \varepsilon \).

**Proof.** Applying Theorem 2.5 we infer that for every \( \theta \in (0,1) \) one has
\[
\text{Orb}_{n_{\varphi_{\theta}},E}(\cdot) \cong J_{\varphi_{\theta},E}(\cdot).
\]
Thus if we put \( X := \text{Orb}_{n_{\varphi_{\theta_0}},E}(X) \) and \( Y := \text{Orb}_{n_{\varphi_{\theta_0}},E}(Y) \), then \( T : X \to Y \) is invertible. Now choose \( \delta > 0 \) such that
\[
\delta < \left[ 2(1 + \|T\|_{X \to Y} \|T^{-1}\|_{Y \to X}) \right]^{-1}.
\]
By our hypothesis, it follows that there exists \( \varepsilon > 0 \) such that if \( |\theta - \theta_0| < \varepsilon \), then
\[
\rho_E(a_{\varphi_{\theta}}, a_{\varphi_{\theta_0}}) < \delta.
\]
Consequently, the desired result follows from the Theorem 2.4. \( \square \)

The following result provides examples of families which have the *stability property*.

**Lemma 2.15.** Let \( \psi_0 \) and \( \psi_1 \) be nondegenerate quasi-concave functions with the same balanced sequence \( \{t_n\}_{n \in \mathbb{Z}} \) generated by induction. Then the family \( \{\varphi_\theta\}_{\theta \in (0,1)} \) where \( \varphi_0 = \psi_0^{1-\theta} \psi_1^\theta \) for all \( \theta \in (0,1) \) is stable with respect to any r.i. Banach sequence lattice \( E \) on \( \mathbb{Z} \).

**Proof.** Let \( \{t_n\}_{n \in \mathbb{Z}} \) be a balanced sequence for both \( \psi_0 \) and \( \psi_1 \) generated by induction (via a fixed \( q > 1 \), that is, for \( j = 0 \) and \( j = 1 \),
\[
\min \left\{ \frac{\psi_j(t_{k+1})}{\psi_j(t_k)} , \frac{t_{k+1} \psi_j(t_k)}{t_k \psi_j(t_{k+1})} \right\} = q, \quad k \in \mathbb{Z}.
\]
Then for every \( \theta \in (0,1) \) and each \( k \in \mathbb{Z} \) one has
\[
\min \left\{ \frac{\varphi_\theta(t_{k+1})}{\varphi_\theta(t_k)}, \frac{t_{k+1} \varphi_\theta(t_k)}{t_k \varphi_\theta(t_{k+1})} \right\} = q
\]
and so \( \{t_n\} \) is a balanced sequence of \( \varphi_\theta \) for all \( \theta \in (0,1) \).

Fix an r.i. Banach sequence lattice \( E \) on \( \mathbb{Z} \). Then the Calderón operator \( S \) generated by \( \{t_n\} \) is bounded in \( E_j := E(1/\psi_j(t_n)) \) for \( j \in \{0,1\} \). Since \( S \) is positive operator, it follows from an interpolation theorem for positive operators between the Calderón product spaces \( (X(\theta) := X_0^{1-\theta} X_1^\theta) \), generated by Banach function lattices \( X_0 \) and \( X_1 \) on any \( \sigma \)-finite measure space) that \( S \) is bounded on \( E_0^{1-\theta} E_1^\theta \) with (see, e.g., [14, p. 246])
\[
\|S\|_{E(\theta) \to E(\theta)} \leq (\|S\|_{E_0 \to E_0})^{1-\theta} (\|S\|_{E_1 \to E_1})^\theta \leq \max_{j=0,1} \|S\|_{E_j \to E_j}.
\]
We shall use the fact (which is easily seen) that
\[
E(\theta) = E(1/\psi_0(t_n))^{1-\theta} E(1/\psi_1(t_n))^\theta = E(1/\varphi_\theta(t_n))
This completes the proof.

\[ \square \]

yields the continuous inclusion for any Banach couple sequence \( \{ \psi_j \} \).

Corollary 2.16. Let \( \{ \varphi_{\theta} \}_{\theta \in (0,1)} \) a family of nondegenerate quasi-concave functions which is stable with respect to a translation-invariant Banach sequence lattice \( E \) on \( \mathbb{Z} \). If \( T: \tilde{X} \rightarrow \tilde{Y} \) is an operator between Banach couples such that \( T: K_{\varphi_{\theta_0},E}(\tilde{X}) \rightarrow K_{\varphi_{\theta_0},E}(\tilde{Y}) \) is invertible for some \( \theta_0 \in (0,1) \), then there exists \( \varepsilon > 0 \) such that \( T: K_{\varphi,E}(\tilde{X}) \rightarrow K_{\varphi,E}(\tilde{Y}) \) is invertible for all \( \theta \in (0,1) \) with \( |\theta - \theta_0| < \varepsilon \).

Proof. The assumption that the Calderón operator \( S \) generated by any balanced sequence \( \{ t_n \} \) of nondegenerate quasi-concave function \( \varphi \) is bounded in \( E(1/\varphi(t_n)) \) yields the continuous inclusion for any Banach couple \( \tilde{X} \),

\[ J_{\varphi,E}(\tilde{X}) \hookrightarrow K_{\varphi,E}(\tilde{X}). \]

This completes the proof. \( \square \)

3. Applications to rearrangement invariant function spaces

Throughout this section \( \mathbb{I} = [0,1] \) or \( \mathbb{I} = [0,\infty) \) is equipped with the Lebesgue measure \( m \) and \( L^0(\mathbb{I}, m) \) denotes the space of equivalence classes of all real Lebesgue measurable functions on \( \mathbb{I} \). Given \( f \in L^0(\mathbb{I}, m) \), its distribution function is defined by \( m_f(\lambda) = m\{ t \in \mathbb{I}; |f(t)| > \lambda \} \), and its decreasing rearrangement by \( f^*(t) = \text{inf}\{ \lambda \geq 0; m_f(\lambda) \leq t \} \) for \( t > 0 \). A Banach lattice \( (X, \| \cdot \|_X) \) is called a rearrangement invariant (r.i. for short) function space provided \( m_f = m_g, f \in X \) implies \( g \in X, \|f\|_X = \|g\|_X \).

If \( X \) is an r.i. function space on \( (\mathbb{I}, m) \) (for short on \( \mathbb{I} \)) and \( \chi_A \) denotes the characteristic function of a measurable set \( A \), clearly \( \|\chi_A\|_X \) depends only on \( m(A) \). The function \( \varphi_X(t) = \|\chi_A\|_X \), where \( m(A) = t, 0 \leq t \leq 1 \), is called the fundamental function of \( X \).

Important examples of r.i. function spaces include the \( L_p \)-spaces with \( 1 \leq p \leq \infty \), Marcinkiewicz and Lorentz spaces. Let \( \varphi: \mathbb{I} \rightarrow [0,\infty) \) be a quasi-concave function. The Marcinkiewicz space \( M_{\varphi} \) is the space of all \( f \in L^0(\mathbb{I}, m) \) equipped with the norm

\[ \|f\|_{M_{\varphi}} := \sup_{0 < t < 1} \frac{\varphi(t)}{t} \int_0^t f^*(s) \, ds < \infty. \]

In the case when \( \varphi: \mathbb{I} \rightarrow [0,\infty) \) is a concave function with \( \varphi(0) = 0 \), the Lorentz space \( \Lambda_{\varphi} \) consists of all \( f \in L^0(\mathbb{I}, m) \) equipped with the norm

\[ \|f\|_{\Lambda_{\varphi}} := \int_1^\infty f^*(t) \, d\varphi(t) < \infty. \]

Note that the fundamental functions of these spaces are \( \varphi_{\Lambda_{\varphi}} = \varphi_{M_{\varphi}} = \varphi \).
Recall that if \( \mathbb{I} = [0, 1] \) (resp., \( \mathbb{I} = [0, \infty) \)), then \( L_1 \) and \( L_\infty \) (resp., \( L_1 \cap L_\infty \) and \( L_1 + L_\infty \)) are, respectively, the largest and the smallest r.i. function spaces on \( \mathbb{I} \). Moreover, if \( X \) is an r.i. function space on \( \mathbb{I} \) with fundamental function \( \varphi \), then \( \varphi \) is quasi-concave and the following continuous embeddings hold (see [14, Theorems II.5.5 and II.5.7] or [9, Theorem II.5.13]):

\[
\Lambda(\tilde{\varphi}) \hookrightarrow X \hookrightarrow M(\varphi),
\]

where \( \tilde{\varphi} \) is a least concave majorant of \( \varphi \).

For a given \( t > 0 \) the \textit{dilation operator} \( \sigma_t \) is defined for all \( f \in L^0(\mathbb{I}, m) \) by \( \sigma_tf(s) = f(s/t)\chi_1(s/t), s \in \mathbb{I} \). Then \( \sigma_t \) is bounded in every r.i. function space \( X \) and so the \textit{lower} and the \textit{upper Boyd indices} are well defined by

\[
\alpha_X = \lim_{t \to 0^+} \frac{\ln \|\sigma_t\|}{\ln t}, \quad \beta_X = \lim_{t \to \infty} \frac{\ln \|\sigma_t\|}{\ln t},
\]

respectively. In general, \( 0 \leq \alpha_X \leq \beta_X \leq 1 \). Since \( s_{\varphi_X}(t) \leq \|\sigma_t\|_X \) for every \( t > 0 \), it follows that \( \alpha_X \leq p_X \leq q_X \leq \beta_X \), where \( p_X := \gamma_{\varphi_X} \) and \( q_X := \delta_{\varphi_X} \). We refer the reader to [9, 14] for more details about r.i. function spaces.

An r.i. function space \( X \) on \( \mathbb{I} \) is said to be \textit{ultrasymmetric} if \( X \) is an interpolation space between the Lorentz space \( \Lambda_\varphi \) and the Marcinkiewicz space \( M_\varphi \), with \( \varphi := \varphi_X \). Ultrasymmetric spaces were studied by Pustylnik [17]. Applications of these spaces are given in [18, 19].

A description of a certain class of ultrasymmetric spaces was given by Pustylnik in terms of real interpolation spaces. Using our notation, his main result Theorem 2.1 in [17] states that an r.i. function space \( X \) on \([0,1]\) with \( \gamma_X > 0 \) is ultrasymmetric if and only if one has

\[
X = K_{E(1/\varphi(t_n))}(L_1, L_\infty),
\]

where \( \varphi = \varphi_X \), \( \{t_n\} = \{2^{-n}\}_{n \geq 0} \) and \( E \) is an interpolation space between \( \ell_1 \) and \( \ell_\infty \) defined on \( \mathbb{Z}_+ \). Combining this result with our variants of the results for estimates of orbits of elements \( a_\varphi = \{\varphi(t_n)\}_{n \in \mathbb{Z}_+} \) for \( \mathbb{I} = \mathbb{Z}_+ \) allows us to obtain Shnieberg’s type results on stability of invertible operators \( T: (L_1, L_\infty) \to (L_1, L_\infty) \) acting on ultrasymmetric spaces. We leave details to the interested reader.

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Mieczysław Mastyło
Adam Mickiewicz University, Poznań
Faculty of Mathematics and Computer Science
Uniwersytetu Poznańskiego 4, 61-614 Poznań, Poland
mieczyslaw.mastylo@amu.edu.pl