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Isomorphisms on interpolation spaces generated by the method of means

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Abstract. We investigate the stability of isomorphisms acting between interpolation spaces generated by the method of means. We focus on the methods which are determined by balanced sequences for non-degenerate quasi-concave functions. The key point for our investigation is that these methods have orbital description by a single element generated by a special ideal of operators between Banach couples. We prove that if an operator is invertible in one orbit it is also invertible by nearby orbits provided that the corresponding indices of quasi-concave functions generated these orbits are close to each other. In particular, these results apply to the real method of interpolation.

Keskiarvomenetelmän tuottamien interpolaatioavaruuksien isomorfismit

Tiivistelmä. Tarkastelemme keskiarvomenetelmän tuottamien interpolaatioavaruuksien välisten isomorfismien häiriöherkkyyttä. Keskitymme ei-degeneroituneiden kvasikonkaavien funktioiden suhteen tasapainotettujen jonojen määräämiin menetelmiin. Tutkimuksemme avainhavainto on, että nämä menetelmät voidaan kuvailla radoittain Banachin parien välillä toimivan erityisen operaattori-ihanteen virittämän yksittäisen alkion avulla. Osoitamme, että jos operaattori on kääntyvä yhdellä radalla, niin se on kääntyvä myös läheisillä radoilla, mikäli näiden ratojen virittämien kvasikonkaavien funktioiden vastaavat indeksit ovat lähellä toisiaan. Erityisesti tulokset soveltuvat reaaliseen interpolointimenetelmään.

1. Introduction

One of the major problems in interpolation theory is the study of stability properties of operators acting between interpolation Banach spaces. Studying these types of properties spins off applications in various areas of analysis. Development of the theory of Fredholm operators and a general recognition of the importance of the subject in applications to the solvability of partial differential equations motivates the study of stability of the Fredholm properties under interpolation. The first result on the stability of Fredholm property is due to Shneiberg [20]; it states that, if $T\colon (X_0, X_1) \to (Y_0, Y_1)$ is an operator between compatible couples of complex Banach spaces, then the set of all $\theta \in (0,1)$, for which the operator $T\colon [X_0, X_1]_\theta \to [Y_0, Y_1]_\theta$ is Fredholm between Calderón interpolation spaces is open. This result was overlooked at first, but after a while it became crucial for further research (see [1, 3, 5, 10, 13, 21]) and references given there. It is perhaps appropriate to remark that the stability of Fredholm properties of operators between interpolation scales of Banach spaces is deeply connected with the stability of interpolated isomorphisms. This phenomenon was discussed in [4] for interpolation scales constructed by using vector-valued analytic functions introduced in [11], which recover, up to equivalence of norms, the real and the complex methods of interpolation. These results were used in [5] to study

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the stability of the inverses of isomorphisms acting on interpolation scales of Banach spaces.

The main purpose of the present paper is to investigate the stability of invertibility property for operators between interpolation spaces generated by the method of means. It is a natural continuation of previous work in the literature related to the problem mentioned above (see, e.g., [1, 4, 5, 15]). We focus on methods of means determined by quasi concave-functions which proved to play a key role in the theory of interpolation of linear operators. As we will see, the key to our study is that these methods have an orbital description by a single element generated by a special ideal of operators. We prove that if an operator is invertible in one orbit it is also invertible in nearby orbits provided that the corresponding indices of quasi-concave functions generated these orbits are close to each other. In particular, these results apply to the real method of interpolation.

Throughout the paper we will use standard notation. As usual, for a given Banach space X we denote by $L(X)$ the Banach space of all bounded linear operators on X equipped with the uniform norm. If X and Y are Banach spaces such that $X \subset Y$ and the inclusion map $\mathrm{id}\colon X \to Y$ is bounded, then we write $X \hookrightarrow Y$. We write $X \cong Y$ whenever $X = Y$, with *equality* of norms.

2. Notation and main results

First, we introduce some essential definitions and notation. For basic notation of interpolation theory, we refer to [7] and [8]. We recall that a mapping $F\colon \vec{\mathcal{B}} \to \mathcal{B}$, from the category $\vec{\mathcal{B}}$ of all couples of Banach spaces into the category $\mathcal{B}$ of all Banach spaces is said to be an interpolation *functor* (or *method*) if, for any couple $\vec{X} := (X_0, X_1)$, the Banach space $F(X_0, X_1)$ is intermediate with respect to $\vec{X}$ (i.e., $X_0 \cap X_1 \subset F(\vec{X}) \subset X_0 + X_1$), and $T\colon F(X_0, X_1) \to F(Y_0, Y_1)$ for all $T\colon (X_0, X_1) \to (Y_0, Y_1)$; here, as usual, the notation $T\colon (X_0, X_1) \to (Y_0, Y_1)$ means that $T\colon X_0 + X_1 \to Y_0 + Y_1$ is a linear operator, such that the restrictions of T to the space X_j is a bounded operator from X_j to Y_j, for $j = 0$ and $j = 1$. Notice that by the closed graph theorem, for any Banach couples $\vec{X}$ and $\vec{Y}$ one has

$$\|T\|_{F(\vec{X}) \to F(\vec{Y})} \le C\, \|T\|_{\vec{X} \to \vec{Y}} := C \max_{j=0,1} \|T\|_{X_j \to Y_j}.$$

If C may be chosen independently of $\vec{X}$ and $\vec{Y}$, then F is called a bounded interpolation functor and it is called exact if $C = 1$.

An operator $T\colon (X_0, X_1) \to (Y_0, Y_1)$ between Banach couples is said to be invertible whenever the restriction $T|_{X_j}\colon X_j \to Y_j$ is invertible (i.e., T is an isomorphism of X_j onto Y_j) for $j = 0$ and $j = 1$. In what follows we will often omit the domain of the restricted operator. We will use Peetre's K-functional. Let $\vec{X} = (X_0, X_1)$ be a Banach couple. For $t > 0$

$$K(t, x; \vec{X}) = K(t, x; X_0, X_1) := \inf\big\{\|x_0\|_{X_0} + t\|x_1\|_{X_1};\, x_0 + x_1 = x\big\}, \quad x \in X_0 + X_1.$$

Following [15] we will consider a special class of operators between Banach couples. For a fixed Banach couple $\vec{A}$, we define a left operator ideal $\mathcal{I}$ (from $\vec{A}$) as a subclass of $L(\vec{A}, \cdot)$ such that for any Banach couple $\vec{X}$ its components

$$\mathcal{I}(\vec{A}, \vec{X}) := L(\vec{A}, \vec{X}) \cap \mathcal{I}$$

are linear spaces which satisfy the following properties for all Banach couples $\vec{X}$ and $\vec{Y}$:

(i) $\mathcal{I}(\vec{A}, \vec{X})$ equipped with a norm $\|\cdot\|_{\mathcal{I}}$ is a Banach space such that

$$\gamma(\vec{A}) := \sup_{\vec{X}\in\vec{\mathcal{B}}} \|\mathrm{id}\colon \mathcal{I}(\vec{A}, \vec{X}) \to L(\vec{A}, \vec{X})\| < \infty.$$

(ii) $\mathcal{I}(\vec{A}, \vec{X})$ contains all one rank operators from $\vec{A}$ to $\vec{X}$.

(iii) (The left ideal property) If $T \in \mathcal{I}(\vec{A}, \vec{X})$, $S \in L(\vec{X}, \vec{Y})$, then $ST \in \mathcal{I}(\vec{A}, \vec{Y})$ and

$$\|ST\|_{\mathcal{I}} \le \|S\|_{\vec{X}\to\vec{Y}} \|T\|_{\mathcal{I}}.$$

Let $\vec{A}$ be a Banach couple and let $\mathcal{I}(\vec{A}, \cdot)$ be a left operator ideal. For an arbitrary element $a \neq 0$ in $A_0 + A_1$, we define the orbit $\mathrm{Orb}_a^{\mathcal{I}}(\vec{X})$ to be the space of all elements in the form Ta, where $T \in \mathcal{I}(\vec{A}, \vec{X})$. This space is equipped with the norm

$$\|x\| = \inf\{\|T\|_{\mathcal{I}};\, x = Ta\}.$$

Since

$$\|Ta\|_{X_0+X_1} \le \|T\|_{\vec{A}\to\vec{X}} \|a\|_{A_0+A_1} \le \gamma(\vec{A})\, \|T\|_{\mathcal{I}} \|a\|_{A_0+A_1},$$

the map $\delta_a\colon \mathcal{I}(\vec{A}, \vec{X}) \to X_0 + X_1$ given by

$$\delta_a(T) := Ta, \quad T \in \mathcal{I}(\vec{A}, \vec{X}),$$

is continuous. This implies that $\mathrm{Orb}_a^{\mathcal{I}}(\vec{X})$ is isometrically isomorphic to the quotient space $\mathcal{I}(\vec{A}, \vec{X})/\mathrm{Ker}_a^{\mathcal{I}}(\vec{X})$, where

$$\mathrm{Ker}_a^{\mathcal{I}}(\vec{X}) := \ker(\delta_a) = \big\{T \in \mathcal{I}(\vec{A}, \vec{X});\, Ta = 0\big\}$$

and so $\mathrm{Orb}_a^{\mathcal{I}}(\vec{X})$ is a Banach space. Clearly, the left ideal property yields that $\vec{\mathcal{B}} \ni \vec{X} \mapsto \mathrm{Orb}_a^{\mathcal{I}}(\vec{X})$ is an exact interpolation functor.

We will use the theorem on stability of invertible operators acting between interpolation orbits generated by the left ideal operator. We need first some notation and collect some results which will be used extensively in the remainder.

Given a Banach space U and any closed subspaces U_0, U_1 of U we let

$$\rho(U_0, U_1) := \sup_{\|u\|_U=1} |\mathrm{dist}(u, U_0) - \mathrm{dist}(u, U_1)|,$$

where $\mathrm{dist}(u, U_j)$ is the distance from $u \in U$ to U_j, that is,

$$\mathrm{dist}(u, U_j) := \inf_{u_j\in U_j} \|u - u_j\|_U, \quad j \in \{0, 1\}.$$

Let U, V be Banach spaces and U_0, U_1 and V_0, V_1 be closed subspaces of U and V, respectively. Suppose that H is a linear bounded operator from U to V which maps U_0 to V_0 and U_1 to V_1. Since for $j = 0$ and $j = 1$ one has $H(u+u_j) = H(u)+H(u_j) \in H(u) + V_j$ for all $u_j \in U_j$, we can define the quotient operators $H_j\colon U/U_j \to V/V_j$ by

$$H_j(u + U_j) := H(u) + V_j, \quad u \in U.$$

The key result we need is the following theorem from [3, Theorem 3.4] which is a slightly modified variant Theorem 9 from [15].

Theorem 2.1. *Suppose that $H\colon U \to V$ maps U_j to V_j for each $j \in \{0, 1\}$, and the quotient operator $H_0\colon U/U_0 \to V/V_0$ is invertible. If*

$$\max\{\mathrm{dist}(U_0, U_1), \mathrm{dist}(V_0, V_1)\} < \frac{1}{2\big(1 + \|H\|_{U\to V} \|H_0^{-1}\|_{V/V_0\to U/U_0}\big)},$$

then the quotient operator $H_1\colon U/U_1 \to V/V_1$ is invertible. Moreover, an upper estimate for the norm of H_1 is given by

$$\|H_1^{-1}\|_{V/V_1\to U/U_1} \le 2\|H_0^{-1}\|_{V/V_0\to U/U_0}.$$

Let $\mathcal{I}(\vec{A},\cdot)$ be the left ideal of operators. Following [15] in the case of an ideal L of all operators, let ρ be a distance defined on closed subspaces of the spaces of $\mathcal{I}(\vec{A},\vec{X})$, and let $a_0, a_1 \in A_0 + A_1$. Then we let

$$\rho\big(\operatorname{Orb}_{a_0}^{\mathcal{I}}(\vec{X}), \operatorname{Orb}_{a_1}^{\mathcal{I}}(\vec{X})\big) := \rho(\ker\delta_{a_0}, \ker\delta_{a_1}).$$

We define a distance between orbits $\operatorname{Orb}_{a_0}^{\mathcal{I}}$ and $\operatorname{Orb}_{a_1}^{\mathcal{I}}$ by the following pseudo-metric on $A_0 + A_1$:

$$\rho_{\mathcal{I}}(a_0, a_1) := \rho\big(\operatorname{Orb}_{a_0}^{\mathcal{I}}, \operatorname{Orb}_{a_1}^{\mathcal{I}}\big) := \sup_{\vec{X}\in\vec{\mathcal{B}}} \rho\big(\operatorname{Orb}_{a_0}^{\mathcal{I}}(\vec{X}), \operatorname{Orb}_{a_1}^{\mathcal{I}}(\vec{X})\big).$$

We start with an obvious observation which follows from the the definition.

Lemma 2.2. *For every $a_0, a_1 \in A_0 + A_1$ and any left ideal $\mathcal{I}$ one has*

$$\rho_{\mathcal{I}}(a_0, a_1) = \sup\big|\|Ta_0\|_{\operatorname{Orb}_{a_0}^{\mathcal{I}}(\vec{X})} - \|Ta_1\|_{\operatorname{Orb}_{a_1}^{\mathcal{I}}(\vec{X})}\big|,$$

where the supremum is taken over all $T\colon \vec{A} \to \vec{X}$ such that $\|T\|_{\mathcal{I}(\vec{A},\vec{X})} \le 1$.

Applying Lemma 2.2 we get estimates of the metric $\rho_{\mathcal{I}}$ using operators which allow a suitable cancellations. We omit the proof since it is an inessential modification of the proof of Proposition 3 in [15].

Proposition 2.3. *Let $\vec{A}$ be a Banach couple, and let $\mathcal{I}(\vec{A},\cdot)$ be a left ideal of operators. Assume that $\varepsilon > 0$ and a_0, $a_1 \in A_0 + A_1$ are such that the following conditions are satisfied*:

(i) *For any Banach couple $\vec{X}$ and any operator $T \in \mathcal{I}(\vec{A},\vec{X})$ with $Ta_0 = 0$, it follows that*

$$\|Ta_0\|_{\operatorname{Orb}_{a_0}^{\mathcal{I}}(\vec{X})} \le \varepsilon\|T\|_{\mathcal{I}(\vec{A},\vec{X})}.$$

(ii) *For any Banach couple $\vec{X}$ and any operator $T \in \mathcal{I}(\vec{A},\vec{X})$ with $Ta_1 = 0$, it follows that*

$$\|Ta_0\|_{\operatorname{Orb}_{a_1}^{\mathcal{I}}(\vec{X})} \le \varepsilon\|T\|_{\mathcal{I}(\vec{A},\vec{X})}.$$

Then one has

$$\rho_{\mathcal{I}}(a_0, a_1) \le 2\varepsilon.$$

The theorem which follows now will be useful in proving results on stability of invertible operators acting between spaces generated by method of means.

Theorem 2.4. *Let $\vec{A}$, $\vec{X}$, $\vec{Y}$ be Banach couples, and let $\mathcal{I}(\vec{A},\cdot)$ be a left ideal of operators. Assume that for some $a_0 \in A_0 + A_1$ an operator $T\colon \vec{X} \to \vec{Y}$ is such that $T\colon \operatorname{Orb}_{a_0}^{\mathcal{I}}(\vec{X}) \to \operatorname{Orb}_{a_0}^{\mathcal{I}}(\vec{Y})$ is invertible. Then $T\colon \operatorname{Orb}_{a}^{\mathcal{I}}(\vec{X}) \to \operatorname{Orb}_{a}^{\mathcal{I}}(\vec{Y})$ is also invertible for all $a \in A_0 + A_1$ such that*

$$\rho_{\mathcal{I}}(a_0, a) \le \big[2\big(1 + \|T\|_{\vec{X}\to\vec{Y}}\,\|T^{-1}\|_{\operatorname{Orb}_{a_0}^{\mathcal{I}}(\vec{Y})\to\operatorname{Orb}_{a_0}^{\mathcal{I}}(\vec{X})}\big)\big]^{-1}.$$

Proof. The argument is similar to the proof in [15, Theorem 10(c)] for $\mathcal{I} = L$. We include the proof for completeness. We apply Theorem 2.1 for Banach spaces

$U := \mathcal{I}(\vec{A}, \vec{X})$, $V := \mathcal{I}(\vec{A}, \vec{Y})$, and subspaces $U_0 := \mathrm{Ker}^{\mathcal{I}}_{a_0}(\vec{X})$, $V_0 := \mathrm{Ker}^{\mathcal{I}}_{a_0}(\vec{Y})$ and $U_1 := \mathrm{Ker}^{\mathcal{I}}_{a_1}(\vec{X})$, $V_1 := \mathrm{Ker}^{\mathcal{I}}_{a_1}(\vec{Y})$ and $H\colon U \to V$ given by

$$H(S) := T \circ S, \quad S \in L(\vec{A}, \vec{X}),$$

Observe that for all $a \in A_0 + A_1$, we have $H\colon \mathrm{Ker}^{\mathcal{I}}_{a}(\vec{X}) \to \mathrm{Ker}^{\mathcal{I}}_{a}(\vec{X})$. Thus using H, we can define the quotient operators $H_j\colon \mathcal{I}(\vec{A}, \vec{X})/U_j \to \mathcal{I}(\vec{A}, \vec{Y})/V_j$ by

$$H_j(S + U_j) := H(S) + V_j, \quad S \in \mathcal{I}(\vec{A}, \vec{X}), \quad j \in \{0, 1\}.$$

Clearly, for all $a \in A_0 + A_1$, we have $H\colon \mathrm{Ker}^{\mathcal{I}}_{a}(\vec{X}) \to \mathrm{Ker}^{\mathcal{I}}_{a}(\vec{X})$. This implies that $U\colon U_j \to V_j$ for $j = 0$ and $j = 1$. Since for any $B \in \vec{\mathcal{B}}$ and for any operator $S \in \mathcal{I}(\vec{A}, \vec{B})$ one has

$$\|Sa\|_{\mathrm{Orb}^{\mathcal{I}}_{a}(\vec{B})} = \|[S]\|_{\mathcal{I}(\vec{A},\vec{B})/\mathrm{Ker}^{\mathcal{I}}_{a}(\vec{B})}, \quad [S] := S + \mathrm{Ker}^{\mathcal{I}}_{a}(\vec{B}),$$

it follows that

$$\|H_j\|_{U/U_j \to V/V_j} = \|T\|_{\mathrm{Orb}_{a_j,\mathcal{I}}(\vec{X}) \to \mathrm{Orb}_{a_j,\mathcal{I}}(\vec{Y})}, \quad j \in \{0, 1\}.$$

Finally, applying this and an obvious estimate $\|H\|_{U \to V} \leq \|T\|_{\vec{X} \to \vec{Y}}$, we obtain the desired statement from Theorem 2.1. □

We consider Banach sequence lattices on $\mathbb{Z}$. For any such lattice E and any positive sequence $w = \{w_n\} := \{w_n\}_{n \in \mathbb{Z}}$, we define the weighted lattice $E(\{w_n\})$ on $\mathbb{Z}$ to be the Banach space of all scalar sequences $\xi = \{\xi_n\}$ such that $\xi w := \{\xi_n w_n\} \in E$. The norm on $E(\{w_n\})$ is defined in the usual way, i.e., $\|x\|_{E(\{w_n\})} = \|xw\|_E$. In the following, we briefly write $E(w_n)$ instead of $E(\{w_n\})$.

Let E be a Banach sequence lattice on $\mathbb{Z}$ and let X be a Banach space. The vector sequence $x = \{x_n\}_{n \in \mathbb{Z}}$ in X is called strongly E-summable if the scalar sequence $\{\|x_n\|_X\}$ is in E. We denote by $E(X)$ the set of all such sequences in X. This forms a Banach space under pointwise operations, and a natural norm in $E(X)$ is given by $\|x\|_{E(X)} := \|\{\|x_n\|_X\}\|_E$.

A pair $\bar{\Phi} = (\Phi_0, \Phi_1)$ of Banach sequence lattices on $\mathbb{Z}$ is called a parameter of the method of means if $\Phi_0 \cap \Phi_1 \subset \ell_1$. The space $J_{\vec{\Phi}}(\vec{X}) = J_{\Phi_0,\Phi_1}(\vec{X})$ built by the method of means consists of all $x \in X_0 + X_1$ which may be represented in the form

$$x = \sum_{n \in \mathbb{Z}} u_n \quad \text{(convergence in } X_0 + X_1\text{)}$$

with $\{u_n\} \in \Phi_0(X_0) \cap \Phi_1(X_1)$. It is well known that $J_{\vec{\Phi}}$ is an exact interpolation functor when the Banach space $J_{\vec{\Phi}}(\vec{X})$ is equipped with the norm

$$\|x\|_{J_{\vec{\Phi}}(\overline{X})} = \inf \max\big\{\|\{u_n\}\|_{\Phi_0(X_0)},\, \|\{u_n\}\|_{\Phi_1(X_1)}\big\},$$

where the infimum is taken over all the above representations of x (see, e.g., [8, 14]).

As usual for a given quasi-concave function φ, we define φ^* by $\varphi^*(t) := t/\varphi(t)$ for all $t > 0$ and $\varphi^*(0) := 0$. We are interested in the special method of means generated by weighted Banach sequence spaces determined by *nondegenerate* quasi-concave functions, i.e., such quasi-concave functions φ that the images of the semi-infinite interval $(0, \infty)$ under φ and φ^* are $(0, \infty)$. In what follows we consider some important sequences in $(0, \infty)$ connected with quasi-concave functions. The importance of these sequences in the study of abstract real interpolation spaces was discovered independently by Brudnyi and Kruglyak, and Janson (see [8, 12]).

Following [9, Definitions 3.1, 3.2], a positive sequence $\{t_k\}_{k\in I}$ is said to be a partition of $\mathbb{R}_+ := (0,\infty)$ if

(i) I is nonempty interval of integers, that is, $\emptyset \neq I = \mathbb{Z} \cap [\inf I, \sup I]$,
(ii) $t_{k-1} < t_k$ for each k such that $k-1, k \in I$,
(iii) $\mathbb{R}_+ \subset \bigcup_{k-1,k\in I} [t_{k-1}, t_k]$.

Let φ be a quasi-concave function. A partition $\{t_k\}_{k\in I}$ is called a *balanced sequence* for φ is there exists a constant $\gamma \geq 1$ and positive integer N such that

(i) For each $k \in I$ such that $k-1 \in I$, at least one of the inequalities
$$\varphi(t_k) \leq \gamma\varphi(t_{k-1}), \quad \varphi^*(t_k) \leq \gamma\varphi^*(t_{k-1})$$
holds.
(ii) for every $a > 0$,
$$\operatorname{card}\{k \in I;\, a < \varphi(t_k) \leq 2a\} \leq N, \quad \operatorname{card}\{k \in I;\, a < \varphi^*(t_k) \leq 2a\} \leq N.$$

It is well known that every quasi-concave function φ has a balanced sequence. Note that such sequences can be constructed by induction (see [9, Proposition 3.4]).

From the point of view of applications such functions φ are interesting only when they have a balanced sequence $\{t_i\}_{i\in I}$ in which I is infinite. For this reason, we will focus below only on this case. The proofs presented in this section apply to corresponding Banach sequence lattices E and quasi-concave functions having balanced sequences modelled on $I \in \{\mathbb{Z}_-, \mathbb{Z}_+, \mathbb{Z}\}$, where $I = \mathbb{Z}_-$ (resp., $I = \mathbb{Z}_+$) in the setting of the subclass of ordered Banach couples (X_0, X_1) with $X_1 \hookrightarrow X_0$ (resp., $X_0 \hookrightarrow X_1$) and $I = \mathbb{Z}$ in the general case of all Banach couples. For simplicity of presentation, we consider only the case when $I = \mathbb{Z}$ is the whole set of integers, which corresponds to nondegenerate quasi-concave functions φ. Notice that it follows from [9, Proposition 3.5] that if $\{t_k\}_{k\in\mathbb{Z}}$ is a balanced sequence for a nondegenerate quasi-concave function φ, then there exists a positive constant C such that
$$C^{-1} \sum_{k\in\mathbb{Z}} \varphi(t_k) \min\Big\{1, \frac{t}{t_k}\Big\} \leq \varphi(t) \leq C \sup_{k\in\mathbb{Z}} \varphi(t_k) \min\Big\{1, \frac{t}{t_k}\Big\}, \quad t > 0.$$
In terms of the K-functional of Peetre the above inequalities are equivalent to
$$C^{-1} K(t, \{\varphi(t_k)\};\, \ell_1, \ell_1(1/t_k)) \leq \varphi(t) \leq C\, K(t, \{\varphi(t_k)\};\, \ell_\infty, \ell_\infty(1/t_k)).$$
We point out that a balanced sequence $\{t_k\}_{k\in\mathbb{Z}}$ can be constructed by induction so that the above inequalities hold with $C = 4$ (see [9, Remark 3.7]). It is straightforward to check that $\{2^n\}$ is the balanced sequence for φ_θ given by $\varphi_\theta(t) = t^\theta$ for all $t \geq 0$.

Let $\{t_n\}_{n\in\mathbb{Z}}$ be a balanced sequence for a nondegenerate quasi-concave function φ. From the above inequalities, it follows that $\{\varphi(t_n)\} \in \ell_1 + \ell_1(1/t_n)$. This implies that for any couple (E_0, E_1) of Banach sequence lattices on $\mathbb{Z}$ such that $E_j \hookrightarrow \ell_\infty$ for $j = 0, 1$ one has
$$\ell_\infty \hookrightarrow E_0'(\varphi(t_n)) + E_1'(\varphi(t_n)/t_n),$$
where E' denotes the Köthe dual space of a Banach sequence lattice E on $\mathbb{Z}$. Thus, by Köthe duality, it follows that $(\Phi_0, \Phi_1) := (E_0(1/\varphi(t_n)), E_1(t_n/\varphi(t_n))$ is a parameter of the method of means. In this case the space $J_{\vec{\Phi}}(\vec{X})$ is denoted by $\vec{X}_{\varphi,E_0,E_1}$. If $1 \leq p_0, p_1 \leq \infty$, $\varphi(t) = t^\theta$ for all $t \geq 0$ with $0 < \theta < 1$ and $\{t_n\} = \{2^n\}$, then $\vec{X}_{\varphi,\ell_{p_0},\ell_{p_1}}$ is the classical Lions–Peetre method of means denoted by $J_{\theta,p_0,p_1}(\vec{X})$ (see [16]). If $E_0 = E_1 = E$, then $J_{\Phi_0,\Phi_1}(\vec{X})$ is the classical abstract J-space, which is denoted by $J_E(\vec{X})$ (see [8, 14]).

We will give an orbital description of the spaces $(X_0, X_1)_{\varphi,E_0,E_1}$. In order to do this we need to introduce some additional notation. Suppose we are given a balanced sequence $\{t_n\}_{n\in\mathbb{Z}}$ for a nondegenerate quasi-concave function φ, and a couple $\vec{E} = (E_0, E_1)$ of Banach sequence lattices on $\mathbb{Z}$ such that $E_j \hookrightarrow \ell_\infty$. For any Banach couple $\vec{X} = (X_0, X_1)$, we define $\mathcal{I}_{\varphi,\vec{E}}((\ell_1, \ell_1(1/t_n)), \vec{X})$ to be the space of all operators $T\colon (\ell_1, \ell_1(1/t_n)) \to (X_0, X_1)$ such that $\{T(e_n)\} \in E_0(X_0)$ and $\{T(t_n e_n)\} \in E_1(X_1)$, equipped with the norm given by

$$\|T\|_{\mathcal{I}_{\varphi,\vec{E}}(\vec{X})} = \max\big\{\|\{T(e_n)\}\|_{E_0(X_0)}, \|\{T(t_n e_n)\}\|_{E_1(X_1)}\big\}.$$

Frequently, for simplicity, we write $\mathcal{I}_{\varphi,\vec{E}}(\vec{X})$ instead of $\mathcal{I}_{\varphi,\vec{E}}((\ell_1, \ell_1(1/t_n)), \vec{X})$.

It can be easily verified that $\mathcal{I}_{\varphi,\vec{E}}$ is a left ideal of the Banach couple $(\ell_1, \ell_1(1/t_n))$. If $1 \le q \le \infty$, $0 < \theta < 1$ and $\varphi(t) = t^\theta$ for all $t \ge 0$ and $E_0 = E_1 = \ell_q$, then we recover the left ideal $\mathcal{I}_q := \mathcal{I}_{\varphi,\ell_q}$ introduced in [15].

Note that under the above assumptions, we have $a_\varphi := \{\varphi(t_n)\}_{n\in\mathbb{Z}} \in \ell_1 + \ell_1(1/t_n)$ and so the orbit $\mathrm{Orb}^{\mathcal{I}}_{a_\varphi}(\vec{X})$ from the couple $(\ell_1, \ell_1(1/t_n))$ to $\vec{X}$ generated by an ideal $\mathcal{I} := \mathcal{I}_{\varphi,\vec{E}}$ is well defined. It is denoted by $\mathrm{Orb}_{a_\varphi,\vec{E}}(\vec{X})$. In the case $E_0 = E_1 = E$, we write $\mathcal{I}_{\varphi,E}$ and $\mathrm{Orb}_{a_\varphi,E}(\vec{X})$ for short instead of $\mathcal{I}_{\varphi,\vec{E}}$ and $\mathrm{Orb}_{a_\varphi,\vec{E}}$. Moreover if $\mathcal{I} := \mathcal{I}_{\varphi,E}$ we write ρ_E instead of $\rho_{\mathcal{I}}$.

The starting point for our investigation is the following result. We omit the proof which is an inessential modification of the proof Theorem 12 in the work by Kruglyak-Milman [15] in the case of the left ideal $\mathcal{I}_q = \mathcal{I}_{\varphi,\ell_q}$ with $\varphi(t) = t^\theta$ for all $t \ge 0$ and for some $\theta \in (0,1)$.

Theorem 2.5. *For every Banach couple $\vec{X}$ the following isometrical formula holds*

$$\mathrm{Orb}_{a_\varphi,\vec{E}}(\vec{X}) \cong J_{\varphi,\vec{E}}(\vec{X}).$$

The following variant of Lemma 2 from [15] is the essential ingredient in the proofs of the main results.

Lemma 2.6. *Let $\vec{X} = (X_0, X_1)$ be a Banach couple and let $x \in X_0 + X_1$. Assume that $w = \{w_k\}_{k\in\mathbb{Z}}$ is a positive sequence and $\{x_k^0\}_{k\in\mathbb{Z}}$ and $\{x_k^1\}_{k\in\mathbb{Z}}$ are sequences in X_0 and X_1, respectively such that $x_k^0 + x_k^1 = x$ for each $k \in \mathbb{Z}$ and the series $\sum_{k\in\mathbb{Z}} w_k(x_k^0 - x_{k-1}^0)$ converges in $X_0 + X_1$. If for a given $n \in \mathbb{Z}$ the series $\sum_{k<n} |w_{k+1} - w_k|\, \|x_k^0\|_{X_0}$ and $\sum_{k\ge n} |w_{k+1} - w_k|\, \|x_k^1\|_{X_1}$ converge absolutely in X_0 and X_1, respectively, then for every $t > 0$ one has*

$$K(t, x_w - w_n x) \le \sum_{k<n} |w_{k+1} - w_k|\, \|x_k^0\|_{X_0} + t \sum_{k\ge n} |w_{k+1} - w_k|\, \|x_k^1\|_{X_1},$$

where $x_w := \sum_{k\in\mathbb{Z}} w_k\big(x_k^0 - x_{k-1}^0\big)$.

Proof. Clearly, our hypotheses imply that the following series converge in $X_0 + X_1$ and it is follows readily that

$$\begin{aligned} x_w - w_n x &= \sum_{k\le n} w_k\big(x_k^0 - x_{k-1}^0\big) - w_n x_n^0 + \sum_{k>n} w_k\big(x_{k-1}^1 - x_k^1\big) - w_n x_n^1 \\ &= \sum_{k<n} (w_k - w_{k+1}) x_k^0 + \sum_{k\ge n} (w_{k+1} - w_k) x_k^1. \end{aligned}$$

Since $\sum_{k<n}(w_k - w_{k+1}) x_k^0 \in X_0$ and $\sum_{k\ge n}(w_{k+1} - w_k) x_k^1 \in X_1$, the desired estimate follows. □

We need to introduce the following notion. Fix pair (φ_0, φ_1) of nondegenerate quasi-concave functions. A positive sequence $\{t_k\}_{k\in I}$ is said to be *admissible* of φ_0 and φ_1 if it is a common balanced sequence for φ_0 and φ_1 such that

$$\sup_{k,k+1\in I} \frac{w_{k+1}}{w_k} < \infty,$$

where $w_k := \frac{\varphi_1(t_k)}{\varphi_0(t_k)}$ for each $k \in I$. A pair (φ_0, φ_1) is said to be admissible whenever there exists an *admissible* sequence $\{t_k\}_{k\in I}$ of φ_0 and φ_1 such that

$$\inf_{k,k+1\in I} \frac{w_{k+1}}{w_k} > 0.$$

For any balanced sequence $t := \{t_n\}_{n\in\mathbb{Z}}$ for a nondegenerate quasi-concave function φ, the Calderón operator S^t defined on $\ell_1 + \ell_1(1/t_n)$ is given by

$$S^t(\xi) := \Big\{ \sum_{k\le n} \xi_k + t_n \sum_{k>n} \frac{\xi_k}{t_k} \Big\}_{n\in\mathbb{Z}}, \quad \xi = \{\xi_n\} \in \ell_1 + \ell_1(1/t_n).$$

Now we are ready to state the following result.

Theorem 2.7. *Let $t = \{t_n\}_{n\in\mathbb{Z}}$ be a common balanced sequence for nondegenerate quasi-concave functions φ_0 and φ_1. Assume that the Calderón operator $S := S^t$ is bounded in spaces F_0 and F_1, where $F_j := E(1/\varphi_j(t_n))$ for $j \in \{0,1\}$ and E is a Banach sequence lattice on $\mathbb{Z}$ with $E \hookrightarrow \ell_\infty$. Then for any Banach couple $\vec{X} = (X_0, X_1)$ and any operator $T\colon (\ell_1, \ell_1(1/t_n)) \to (X_0, X_1))$ with $T \in \mathcal{I}_{\varphi_0,E}(\vec{X})$, one has*

$$\begin{aligned}&\Big\| \Big\{ K\Big(t_n, T\Big(\frac{a_{\varphi_1}}{\varphi_1(t_n)} - \frac{a_{\varphi_0}}{\varphi_0(t_n)}\Big); \vec{X}\Big) \Big\} \Big\|_E \\ &\le 2 \sup_{n\in\mathbb{Z}} \Big| \frac{w_{n+1}}{w_n} - 1 \Big| \|S\|_{F_0\to F_0} \|S\|_{F_1\to F_1} \|T\|_{\mathcal{I}_{\varphi_0,E}(\vec{X})},\end{aligned}$$

where $w_n = \frac{\varphi_1(t_n)}{\varphi_0(t_n)}$ for each $n \in \mathbb{Z}$.

Proof. Fix an operator $T\colon (\ell_1, \ell_1(1/t_n)) \to (X_0, X_1)$. Since for both $j = 0$ and $j = 1$ one has $a_{\varphi_j} = \sum_{k\in\mathbb{Z}} \varphi_j(t_k) e_k$ (convergence in $\ell_1 + \ell_1(1/t_n)$),

$$T(a_{\varphi_j}) = \sum_{k\in\mathbb{Z}} T(\varphi_j(t_k) e_k) \quad \text{(convergence in } X_0 + X_1).$$

Let $x := T(a_{\varphi_0})$. Since for each $k \in \mathbb{Z}$ the series $\sum_{i<k} \varphi_0(t_i) e_i$ and $\sum_{i\ge k} \varphi_0(t_i) e_i$ converge absolutely in ℓ_1 and $\ell_1(1/t_n)$ respectively, it follows that $x = x_k^0 + x_k^1$, where

$$x_k^0 := \sum_{i<k} T(\varphi_0(t_i) e_i) \in X_0, \quad x_k^1 := \sum_{i\ge k} T(\varphi_0(t_i) e_i) \in X_1.$$

The above formulas yield

$$T(a_{\varphi_1}) = \sum_{k\in\mathbb{Z}} T(\varphi_1(t_k) e_k) = \sum_{k\in\mathbb{Z}} \frac{\varphi_1(t_k)}{\varphi_0(t_k)} T(\varphi_0(t_k) e_k)) = \sum_{k\in\mathbb{Z}} w_k \big(x_k^0 - x_k^1\big).$$

We set $\gamma(w) := \sup_{n\in\mathbb{Z}} \big| \frac{w_{n+1}}{w_n} - 1 \big|$ and $\widetilde{K}(t_k, x) := \|x_k^0\|_{X_0} + t_k \|x_k^1\|_{X_1}$ for each $k \in \mathbb{Z}$. Applying Lemma 2.6 with $x = T(a_{\varphi_0})$ and $x_w = T(a_{\varphi_1})$, we conclude that for each

$n \in \mathbb{Z}$ one has

$$K(t_n, T(a_{\varphi_1}) - w_n T(a_{\varphi_0})) \le \sum_{k<n} |w_{k+1} - w_k| \|x_k^0\|_{X_0} + t_n \sum_{k\ge n} |w_{k+1} - w_k| \|x_k^1\|_{X_1}$$
$$\le \Big(\sup_{k\in\mathbb{Z}} \Big|\frac{w_{k+1}}{w_k} - 1\Big|\Big) \Big(\sum_{k<n} w_k \|x_k^0\|_{X_0} + t_n \sum_{k\ge n} w_k \|x_k^1\|_{X_1}\Big)$$
$$\le \gamma(w) \Big(\sum_{k<n} w_k\, \widetilde{K}(t_k, x) + t_n \sum_{k\ge n} \frac{w_k}{t_k} \widetilde{K}(t_k, x)\Big) = \gamma(w)\, S(\{w_k \widetilde{K}(t_k, x)\}).$$

Consequently

$$K\Big(t_n, T\Big(\frac{a_{\varphi_1}}{\varphi_1(t_n)} - \frac{a_{\varphi_0}}{\varphi_0(t_n)}\Big) \le \gamma(w)\, \frac{S(\{w_k \widetilde{K}(t_k, x)\})}{\varphi_1(t_n)}, \quad n \in \mathbb{Z},$$

whence

$$(*) \qquad \Big\|\Big\{K\Big(t_n, T\Big(\frac{a_{\varphi_1}}{\varphi_1(t_n)} - \frac{a_{\varphi_0}}{\varphi_0(t_n)}\Big); \vec{X}\Big)\Big\}\Big\|_E \le \gamma(w) \|S\|_{F_1\to F_1} \big\|\{\widetilde{K}(t_n, x)\}\big\|_{F_0}.$$

The hypothesis $T \in \mathcal{I} := \mathcal{I}_{\varphi_0,E}(\vec{X})$ yields

$$\big\|\{\varphi_0(t_k)\|T(e_k)\|_{X_0}\}\big\|_{F_0} = \|\{T(e_k)\}\|_{E(X_0)} \le \|T\|_{\mathcal{I}},$$

and

$$\big\|\{\varphi_0(t_k)\|T(t_k e_k)\|_{X_1}\}\big\|_{F_0} = \|\{T(t_k e_k)\}\|_{E(X_1)} \le \|T\|_{\mathcal{I}}.$$

Now observe that

$$\widetilde{K}(t_n, x) = \|x_n^0\|_{X_0} + t_n \|x_n^1\|_{X_1} \le \sum_{k<n} \varphi_0(t_k) \|T(e_k)\|_{X_0} + t_n \sum_{k\ge n} \frac{\varphi_0(t_k)}{t_k} \|T(t_k e_k)\|_{X_1}$$
$$\le S(\{\varphi_0(t_k)\|T(e_k)\|_{X_0}\}) + S(\{\varphi_0(t_k)\|T(t_k e_k)\|_{X_1}\}).$$

Combining this with the estimates above gives

$$\|\{\widetilde{K}(t_n, x)\}\|_{F_0} \le 2\|S\|_{F_0\to F_0} \|T\|_{\mathcal{I}},$$

so the required estimate follows from the inequality $(*)$. □

To present applications of Theorem 2.7 we need a connection between the special method of means and the K-method. Suppose we are given a positive sequence $s = \{s_n\} \in c_0(\mathbb{Z})$ (that is $\lim_{|n|\to\infty} s_n = 0$) and a Banach sequence lattice on $\mathbb{Z}$ such that

$$\ell_\infty \cap \ell_\infty(1/s_n) \subset \Phi \subset \ell_1 + \ell_1(1/s_n).$$

Note that $\Phi \subset \ell_1 + \ell_1(1/s_n)$ implies that $\Phi \cap \Phi(s) \subset \ell_1$, so the method of means $J_{\Phi,\Phi(s)}$ is well defined. For any Banach couple $\vec{X} = (X_0, X_1)$, we define $K_\Phi^s(\vec{X})$ to be a Banach space of all $x \in X_0 + X_1$ equipped with the norm

$$\|x\|_{K_\Phi^s(\vec{X})} := \big\|\{K(s_n, x; \vec{X})\}\big\|_\Phi.$$

The proof of the following lemma is similar to the equivalence theorem for the classical Lions-Peetre $J_{\theta,q}$- and $K_{\theta,q}$-methods. For the sake of completeness we include a proof.

Lemma 2.8. *Let $\{s_n\} \in c_0(\mathbb{Z})$ be a positive sequence. Then under the above conditions on Φ the following hold for any Banach couple $\vec{X} = (X_0, X_1)$:*

(i) *$K_\Phi^s(\vec{X}) \hookrightarrow J_{\Phi,\Phi(s)}(\vec{X})$ with the norm less than or equal to 4.*

(ii) *If the Calderón operator $S = S^s$ is bounded in Φ, then $J_{\Phi,\Phi(s)}(\vec{X}) \hookrightarrow K_\Phi^s(\vec{X})$ with norm less than or equal to $2\|S\|_{\Phi\to\Phi}$.*

Proof. (i). For any $x \in K^s_\Phi(\vec{X})$ one has $\{K(s_k, x; \vec{X})\} \in \ell_1 + \ell_1(1/s_k)$. Hence

$$\min\big\{1, 1/s_k\big\} K(s_k, x; \vec{X}) \to 0 \quad \text{as} \ \ |k| \to \infty.$$

Then it follows from the proof of a fundamental lemma for interpolation (see [7, Lemma 3.3.2]) that there exists a representation of x, such that

$$x = \sum_{k\in\mathbb{Z}} u_k \quad (\text{convergence in } \ X_0 + X_1)$$

and

$$\max\{\|u_k\|_{X_0}, s_k\|u_k\|_{X_1}\} \leq 4K(s_k, x; \vec{X}), \quad k \in \mathbb{Z}.$$

This shows that

$$\big\|\mathrm{id}\colon K^s_\Phi(\vec{X}) \to J_{\Phi,\Phi(s)}(\vec{X})\big\| \leq 4.$$

(ii). For given $\varepsilon > 0$ and $x \in X := J_{\Phi,\Phi(v)}(\vec{X})$ there exists a representation of x, such that

$$x = \sum_{k\in\mathbb{Z}} u_k \quad (\text{convergence in } \ X_0 + X_1)$$

and

$$\max\{\|\{u_k\}\|_{\Phi(X_0)},\, \|\{s_k u_k\}\|_{\Phi(X_1)}\} \leq \|x\|_X + \varepsilon.$$

Hence, letting $\xi = \{\xi_k\}$ with $\xi_k := \max\{\|u_k\|_{X_0},\, s_k\|u_k\|_{X_1}\}$ for each $k \in \mathbb{Z}$, we get

$$\begin{aligned} K(s_n, x; \vec{X}) &\leq \sum_{k\in\mathbb{Z}} K(s_n, u_k; \vec{X}) \leq \sum_{k\in\mathbb{Z}} \min\{\|u_k\|_{X_0}, s_n\|u_k\|_{X_1}\} \\ &\leq \sum_{k\in\mathbb{Z}} \min\Big\{1, \frac{s_n}{s_k}\Big\} \max\{\|u_k\|_{X_0}, s_k\|u_k\|_{X_1}\} = (S\xi)_n. \end{aligned}$$

The above estimate combined with

$$\|\xi\|_\Phi \leq 2\, \max\{\|\{u_k\}\|_{\Phi(X_0)},\, \|\{s_k u_k\}\|_{\Phi(X_1)}\} \leq 2\, \|x\|_X + \varepsilon$$

yields

$$\|x\|_{K^s_\Phi(\vec{X})} \leq 2\|S\|_{\Phi\to\Phi}(\|x\|_X + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, the required statement follows. □

To state the next result we note that in what follows for a given balanced sequence $s = \{s_n\}$ for a nondegenerate quasi-concave function φ and a Banach sequence lattice $E \hookrightarrow \ell_\infty$, we let

$$K_{\varphi,E}(\vec{X}) := K^s_\Phi(\vec{X}),$$

for any Banach couple $\vec{X}$ where $\Phi := E(1/\varphi(s_n))$.

Theorem 2.9. *Assume that φ_0, φ_1 and E satisfy the conditions of Theorem 2.7, where E is a Banach sequence lattice on $\mathbb{Z}$ intermediate between ℓ_1 and ℓ_∞. Then there exists a constant $C > 0$ such that for $S := S^t$ with $t = \{t_n\}_{n\in\mathbb{Z}}$ one has*

$$\rho_E(a_{\varphi_0}, a_{\varphi_1}) \leq C \sup_{k\in\mathbb{Z}} \left|\frac{w_{k+1}}{w_k} - 1\right| \|S\|_{F_0\to F_0} \|S\|_{F_1\to F_1},$$

where $w_k = \frac{\varphi_1(t_k)}{\varphi_0(t_k)}$ for each $k \in \mathbb{Z}$ and $F_j := E(1/\varphi_j(t_n))$ for $j \in \{0, 1\}$.

Proof. From Lemma 2.8 it follows that $K_{\varphi,E}(\vec{X}) \hookrightarrow J_{\varphi,E}(\vec{X})$ with the norm of the inclusion map less than or equal to 4.

By Theorem 2.7, we conclude that there exists a universal constant $C > 0$ such that

$$\big\|\mathrm{id}\colon K_{\varphi,E}(\vec{X}) \to \mathrm{Orb}_{a_\varphi,E}(\vec{X})\big\| < C/4.$$

Now observe that if $T\colon (\ell_1, \ell_1(1/t_n)) \to (X_0, X_1)$ is any operator in $\mathcal{I} := \mathcal{I}_{\varphi,E}(\vec{X})$ such that $\|T\|_{\mathcal{I}} \le 1$ and $T(a_{\varphi_j}) = 0$ for $j = 0$. The above continuous inclusion combined with Theorem 2.7 yields,

$$\|Ta_{\varphi_j}\|_{\mathrm{Orb}_{a_{\varphi_j},E}(\vec{X})} \le \frac{C}{2} \sup_{k\in\mathbb{Z}} \Big|\frac{w_{k+1}}{w_k} - 1\Big| \|S\|_{F_0\to F_0} \|S\|_{F_1\to F_1}, \quad j = 0.$$

Similar statement holds for $j = 1$. This completes the proof by Proposition 2.3. □

We make applications to a special class of pairs of quasi-concave functions. Following Brudnyi and Shteinberg [9], quasi-concave functions φ_0 and φ_1 are said to be in the same *scale* if there exist quasi-concave functions ψ_0, ψ_1 and θ_0, $\theta_1 \in (0,1)$ such that

$$\varphi_0 = \psi_0^{1-\theta_0}\psi_1^{\theta_0}, \quad \varphi_1 = \psi_0^{1-\theta_1}\psi_1^{\theta_1}.$$

We will use the following lemma.

Lemma 2.10. *Let φ_0 and φ_1 be quasi-concave functions in the same scale. Then the pair (φ_0, φ_1) is admissible.*

Proof. By [9, Proposition 3.9], it follows that if quasi-concave functions φ_0 and φ_1 are in the same scale, then $\{t_k\}_{k\in I}$ is a balanced sequence for φ_0 if and only if it is a balanced sequence for φ_1 and, moreover for any common balanced sequence $\{t_k\}_{k\in I}$ for φ_0 and φ_1 there exists a constant γ such that for each k such that $k-1$, $k \in I$ at least one of the following conditions holds:

$$\varphi_j(t_k) \le \gamma\varphi_j(t_{k-1}), \quad j \in \{0,1\},$$

or

$$\varphi_j^*(t_k) \le \gamma\varphi_j^*(t_{k-1}), \quad j \in \{0,1\}.$$

Now we assume that $\{t_k\}_{k\in I}$ is a common balanced sequence for φ_0 and φ_1. We claim that this sequence satisfies the required conditions, which ensure that the pair (φ_0, φ_1) is admissible. To see this let $\{w_k\}_{k\in I}$ denote the sequence given by $w_k := \varphi_1(t_k)/\varphi_0(t_k)$ for each $k \in I$. Recall that $t_k < t_{k+1}$ for each pair $(k, k+1) \in I \times I$. Since φ_j and φ_j^* are non-decreasing functions for $j = 0$ and $j = 1$, the first (resp., the second) estimate above yields for each k such that k, $k+1 \in I$,

$$\frac{1}{\gamma} \le \frac{\varphi_0(t_k)}{\varphi_0(t_{k+1})} \le \frac{w_{k+1}}{w_k} = \frac{\varphi_1(t_{k+1})}{\varphi_1(t_k)}\,\frac{\varphi_0(t_k)}{\varphi_0(t_{k+1})} \le \frac{\varphi_1(t_{k+1})}{\varphi_1(t_k)} \le \gamma$$

$\Big($resp.,

$$\frac{1}{\gamma} \le \frac{\varphi_1^*(t_k)}{\varphi_1^*(t_{k+1})} \le \frac{w_{k+1}}{w_k} = \frac{\varphi_1^*(t_k)}{\varphi_1^*(t_{k+1})}\,\frac{\varphi_0^*(t_{k+1})}{\varphi_0^*(t_k)} \le \frac{\varphi_0^*(t_{k+1})}{\varphi_0^*(t_k)} \le \gamma\Big).$$

This completes the proof. □

Recall that a Banach sequence lattice E on $\mathbb{Z}$ is called *rearrangement invariant* (r.i. for short) if $\xi = \{\xi_k\}_{k\in\mathbb{Z}} \in E$ and π is an arbitrary permutation of $\mathbb{Z}$, then the sequence $\xi_\pi = \{\xi_{\pi(k)}\}_{k\in\mathbb{Z}}$ belongs to E and $\|\xi_\pi\|_E = \|\xi\|_E$.

We are ready to state the following result.

Theorem 2.11. *Let nondegenerate quasi-concave functions φ_0 and φ_1 be given by $\varphi_0 = \psi_0^{1-\theta_0}\psi_1^{\theta_0}$ and $\varphi_1 = \psi_0^{1-\theta_1}\psi_1^{\theta_1}$, where ψ_0, ψ_1 are quasi-concave functions and $\theta_0, \theta_1 \in (0,1)$ with $\theta_0 \neq \theta_1$. Suppose that there exist $C > 1$ and a balanced sequence $\{t_n\}_{n\in\mathbb{Z}}$ for φ_0 such that*

$$\frac{1}{C} \leq \frac{\psi_1(t_{k+1})}{\psi_1(t_k)}\,\frac{\psi_0(t_k)}{\psi_0(t_{k+1})} \leq C, \quad k \in \mathbb{Z}.$$

Then for any r.i. Banach sequence lattice E on $\mathbb{Z}$ one has

$$\rho_E(a_{\varphi_0}, a_{\varphi_1}) \leq \left(C^{|\theta_0-\theta_1|} - 1\right) \|S\|_{F_0\to F_0}\|S\|_{F_1\to F_1},$$

where $F_j := E(1/\varphi_j(t_n))$ for $j \in \{0,1\}$.

Proof. Without loss of generality we may assume that $\theta_0 < \theta_1$. The result by Brudnyi and Shteinberg mentioned in the proof of Lemma 2.10 shows that $\{t_n\}$ is also a balanced sequence for φ_1. Since E is a r.i. Banach sequence lattice, the analysis of the proof of Lemma 6 in [12] shows that the Calderón operator S generated by a sequence $\{t_n\}$ is bounded in $E(1/\varphi_j(t_n))$ for $j = 0, 1$.

Let $\{w_n\}$ be given by $w_n = \frac{\varphi_1(t_n)}{\varphi_0(t_n)}$ for each $n \in \mathbb{Z}$. Then we have

$$\frac{1}{\gamma} \geq \frac{\psi_0(t_k)}{\psi_0(t_{k+1})} \geq \frac{w_{k+1}}{w_k} = \left(\frac{\psi_1(t_{k+1})}{\psi_1(t_k)}\,\frac{\psi_0(t_k)}{\psi_0(t_{k+1})}\right)^{\theta_1-\theta_0}, \quad k \in \mathbb{Z}.$$

The hypotheses on φ_0 and φ_1 combined with

$$\sup_{k\in\mathbb{Z}}\left|\frac{w_{k+1}}{w_k} - 1\right| \leq \max\left\{ -1 + \sup_{k\in\mathbb{Z}}\frac{w_{k+1}}{w_k},\, 1 - \inf_{k\in\mathbb{Z}}\frac{w_{k+1}}{w_k}\right\}$$

gives

$$\sup_{k\in\mathbb{Z}}\left|\frac{w_{k+1}}{w_k} - 1\right| \leq \max\left\{C^{\theta_1-\theta_0} - 1,\, 1 - C^{\theta_0-\theta_1}\right\} \leq C^{\theta_1-\theta_0} - 1.$$

To finish we apply Theorem 2.9. □

We give applications to quasi-power functions. We first recall that the *lower* and *upper dilation indices* of a quasi-concave function $\phi\colon [0,a) \to [0,\infty)$ $(0 < a \leq \infty)$ are given by

$$\gamma_\phi := \lim_{t\to 0+}\frac{\ln s_\varphi(t)}{\ln t}, \quad \delta_\phi := \lim_{t\to\infty}\frac{\ln s_\phi(t)}{\ln t},$$

respectively, where $s_\phi(t) := \sup_{0<s<a, 0<st<a}\frac{\phi(st)}{\phi(t)}$. We have $0 \leq \gamma_\phi \leq \delta_\phi \leq 1$

A quasi-concave function $\varphi(0,\infty) \to (0,\infty)$ is said to be quasi-power if $0 < \gamma_\phi \leq \delta_\phi < 1$. It is well-known that $\{2^n\}_{n\in\mathbb{Z}}$ is a balanced sequence for any quasi-power concave function φ. In what follows for any quasi-power function φ, we let $a_\varphi := \{\varphi(2^n)\}_{n\in\mathbb{Z}}$ and $S := S^t$ with $t := \{2^n\}$.

Corollary 2.12. *Let quasi-power functions φ_0 and φ_1 be given by $\varphi_0 = \psi_0^{1-\theta_0}\psi_1^{\theta_0}$ and $\varphi_1 = \psi_0^{1-\theta_1}\psi_1^{\theta_1}$, where ψ_0, ψ_1 are quasi-concave functions and $\theta_0, \theta_1 \in (0,1)$ with $\theta_0 \neq \theta_1$. Then for any r.i. Banach sequence lattice E on $\mathbb{Z}$ one has*

$$\rho_E(a_{\varphi_0}, a_{\varphi_1}) \leq |\theta_0 - \theta_1|\, \|S\|_{F_0\to F_0}\|S\|_{F_1\to F_1},$$

where $F_j := E(1/\varphi_j(2^n))$ for $j \in \{0,1\}$.

Proof. Note that any quasi-power function ρ is nondegenerate. By quasi-concavity of ψ_0 and ψ_1 we get

$$\frac{1}{2} \leq \frac{\psi_1(2^{k+1})}{\psi_1(2^k)}\,\frac{\psi_0(2^k)}{\psi_0(2^{k+1})} \leq 2, \quad k \in \mathbb{Z}.$$

We now apply Theorem 2.11 for $\{t_k\}_{k\in\mathbb{Z}} := \{2^k\}$ and $C = 2$ to get by the obvious inequality $2^\alpha - 1 \le \alpha$ for all $\alpha \in [0,1]$, the required statement. □

Theorem 2.13. *Assume that quasi-power functions φ_0 and φ_1 are such that for some $\theta_0, \theta_1 \in (0,1)$ one of the following conditions is satisfied*:

(i) $\{\varphi_1(2^n)/\varphi_0(2^n)\}_{n\in\mathbb{Z}}$ *is a nondecreasing sequence, $t \mapsto \varphi_0(t)/t^{\theta_0}$ is a nondecreasing function on $\mathbb{R}_+$ and $t \mapsto \varphi_1(t)/t^{\theta_1}$ is a nonincreasing on $\mathbb{R}_+$.*

(ii) $\{\varphi_1(2^n)/\varphi_0(2^n)\}_{n\in\mathbb{Z}}$ *is a nonincreasing sequence, $t \mapsto \varphi_0(t)/t^{\theta_0}$ is a nonincreasing function on $\mathbb{R}_+$ and $t \mapsto \varphi_1(t)/t^{\theta_1}$ is nondecreasing on $\mathbb{R}_+$.*

Then for any translation-invariant Banach sequence lattice on $\mathbb{Z}$ one has

$$\rho_E(a_{\varphi_0}, a_{\varphi_1}) \le C|\theta_0 - \theta_1| \, \|S\|_{F_0\to F_0} \|S\|_{F_1\to F_1},$$

where $F_j := E(1/\varphi_j(2^n))$ for $j \in \{0,1\}$.

Proof. Since E is an r.i. Banach sequence lattice, it is easy to verify that the Calderón operator S generated by a sequence $\{2^n\}$ is bounded in $E(1/\varphi(2^n))$ for any quasi-power function φ.

Assume that the condition (i) (resp., (ii)) holds. Let $w_n = \frac{\varphi_1(2^n)}{\varphi_0(2^n)}$ for each $n \in \mathbb{Z}$. Then our hypotheses on φ_0 and φ_1 imply that for each $k \in \mathbb{Z}$ one has

$$1 \le \frac{w_{k+1}}{w_k} = \frac{\varphi_1(2^{k+1})}{\varphi_1(2^k)} \cdot \frac{\varphi_0(2^k)}{\varphi_0(2^{k+1})} \le 2^{\theta_1} \cdot 2^{-\theta_0} = 2^{\theta_1 - \theta_0}$$

(resp., $2^{\theta_1-\theta_0} \le \frac{w_{k+1}}{w_k} \le 1$); in particular, $\theta_0 \le \theta_1$ (resp., $\theta_1 \le \theta_0$). These estimates combined with the above-mentioned inequality $2^\alpha - 1 \le \alpha$ for all $\alpha \in [0,1]$ yield

$$\sup_{k\in\mathbb{Z}} \left| \frac{w_{k+1}}{w_k} - 1 \right| \le 2^{|\theta_0-\theta_1|} - 1 \le |\theta_0 - \theta_1|.$$

Thus the required statement follows from Theorem 2.9. □

We give examples of non-power functions satisfying the conditions of Theorem 2.13. Fix a quasi-concave function φ and any $\alpha_0, \alpha_1, \beta_0, \beta_1, \theta_0 \in (0,1)$ such that $0 < \theta_0 < \alpha_1 < 1/2$ and $0 < \beta_0 < \beta_1 < \alpha_1$. For $j \in \{0,1\}$ we define

$$\varphi_j(t) := t^{\alpha_j}\varphi(t)^{\beta_j}, \quad t \ge 0,$$

and note that

$$\gamma_{\varphi_j} = \alpha_j + \beta_j\gamma_\varphi \le \delta_{\varphi_j} = \alpha_j + \beta_j\delta_\varphi.$$

Since φ is quasi-concave, $0 \le \gamma_\varphi \le \delta_\varphi \le 1$. Thus our hypotheses imply that φ_0 and φ_1 are quasi-power functions. If we put $\theta_0 := \alpha_0$ and $\theta_1 := 2\alpha_1$, then the functions $t \mapsto \varphi_1(t)/\varphi_0(t)$, $t \mapsto \varphi_0(t)/t^{\theta_0}$ are nondecreasing and $t \mapsto \varphi_1(t)/t^{\theta_1}$ is nonincreasing on $(0,\infty)$. Thus the quasi-power functions φ_0, φ_1 satisfy the condition (i) of Theorem 2.13. In a similar way, we can construct an example of functions satisfying the second condition (ii).

Before we state the next result we introduce the following definition. A family $\{\varphi_\theta\}_{\theta\in(0,1)}$ of nondegenerate quasi-concave functions is said to be *stable* with respect to a Banach sequence lattice E on $\mathbb{Z}$ with $E \hookrightarrow \ell_\infty$ if there exists a common balanced sequence $\{t_n\}_{n\in\mathbb{Z}}$ for all functions of this family having the property: if $\theta_0 \in (0,1)$ is fixed, then for any $\delta > 0$ there exists $\varepsilon > 0$ such that if $|\theta - \theta_0| < \varepsilon$, then

$$\rho_E(a_{\varphi_\theta}, a_{\varphi_{\theta_0}}) < \delta.$$

We are ready to formulate a variant of Shneiberg's result on stability of bounded invertible operators acting on a corresponding scale of interpolation spaces generated by the method of means.

Theorem 2.14. *Let $\{\varphi_\theta\}_{\theta\in(0,1)}$ a family of nondegenerate quasi-concave functions which is stable with respect to a Banach sequence lattice $E \hookrightarrow \ell_\infty$. If $T\colon \vec{X} \to \vec{Y}$ is an operator between Banach couples such that $T\colon J_{\varphi_{\theta_0},E}(\vec{X}) \to J_{\varphi_{\theta_0},E}(\vec{Y})$ is invertible for some $\theta_0 \in (0,1)$, then there exists $\varepsilon > 0$ such that $T\colon J_{\varphi_\theta,E}(\vec{X}) \to J_{\varphi_\theta,E}(\vec{Y})$ is invertible for all $\theta \in (0,1)$ with $|\theta - \theta_0| < \varepsilon$.*

Proof. Applying Theorem 2.5 we infer that for every $\theta \in (0,1)$ one has

$$\mathrm{Orb}_{a_{\varphi_\theta},E}(\cdot) \cong J_{\varphi_\theta,\vec{E}}(\cdot).$$

Thus if we put $X := \mathrm{Orb}_{a_{\varphi_{\theta_0}},E}(\vec{X})$ and $Y := \mathrm{Orb}_{a_{\varphi_{\theta_0}},E}(\vec{Y})$, then $T\colon X \to Y$ is invertible. Now choose $\delta > 0$ such that

$$\delta < \big[2(1 + \|T\|_{\vec{X}\to\vec{Y}} \|T^{-1}\|_{Y\to X})\big]^{-1}.$$

By our hypothesis, it follows that there exists $\varepsilon > 0$ such that if $|\theta - \theta_0| < \varepsilon$, then

$$\rho_E(a_{\varphi_\theta}, a_{\varphi_{\theta_0}}) < \delta.$$

Consequently, the desired result follows from the Theorem 2.4. □

The following result provides examples of families which have the *stability property.*

Lemma 2.15. *Let ψ_0 and ψ_1 be nondegenerate quasi-concave functions with the same balanced sequence $\{t_n\}_{n\in\mathbb{Z}}$ generated by induction. Then the family $\{\varphi_\theta\}_{\theta\in(0,1)}$ where $\varphi_\theta = \psi_0^{1-\theta}\psi_1^\theta$ for all $\theta \in (0,1)$ is stable with respect to any r.i. Banach sequence lattice E on $\mathbb{Z}$.*

Proof. Let $\{t_n\}_{n\in\mathbb{Z}}$ be a balanced sequence for both ψ_0 and ψ_1 generated by induction (via a fixed $q > 1$), that is, for $j = 0$ and $j = 1$,

$$\min\left\{\frac{\psi_j(t_{k+1})}{\psi_j(t_k)}, \frac{t_{k+1}\psi_j(t_k)}{t_k\psi_j(t_{k+1})}\right\} = q, \quad k \in \mathbb{Z}.$$

Then for every $\theta \in (0,1)$ and each $k \in \mathbb{Z}$ one has

$$\min\left\{\frac{\varphi_\theta(t_{k+1})}{\varphi_\theta(t_k)}, \frac{t_{k+1}\varphi_\theta(t_k)}{t_k\varphi_\theta(t_{k+1})}\right\} = q$$

and so $\{t_n\}$ is a balanced sequence of φ_θ for all $\theta \in (0,1)$.

Fix an r.i. Banach sequence lattice E on $\mathbb{Z}$. Then the Calderón operator S generated by $\{t_n\}$ is bounded in $E_j := E(1/\psi_j(t_n))$ for $j \in \{0,1\}$. Since S is positive operator, it follows from an interpolation theorem for positive operators between the Calderón product spaces ($X(\theta) := X_0^{1-\theta}X_1^\theta$, generated by Banach function lattices X_0 and X_1 on any σ-finite measure space) that S is bounded on $E_0^{1-\theta}E_1^\theta$ with (see, e.g., [14, p. 246])

$$\|S\|_{E(\theta)\to E(\theta)} \le (\|S\|_{E_0\to E_0})^{1-\theta}(\|S\|_{E_1\to E_1})^\theta \le \max_{j=0,1}\|S\|_{E_j\to E_j}.$$

We shall use the fact (which is easily seen) that

$$E(\theta) = E(1/\psi_0(t_n))^{1-\theta}E(1/\psi_1(t_n))^\theta = E(1/\varphi_\theta(t_n))$$

with $2^{-1}\|\cdot\|_{E(1/\varphi_\theta(t_n))} \le \|\cdot\|_{E(\theta)} \le \|\cdot\|_{E(1/\varphi_\theta(t_n))}$. Hence

$$\|S\|_{E(1/\varphi_\theta(t_n))} \le K := 2\max_{j=0,1}\|S\|_{E(1/\psi_j(t_n))}, \quad \theta\in(0,1),$$

where the constant K is independent of θ. Now apply Theorem 2.11 to find

$$\begin{aligned}\rho_E(a_{\varphi_{\theta_0}}, a_{\varphi_{\theta_1}}) &\le \big(C^{|\theta_0-\theta_1|}-1\big)\,\|S\|_{E(1/\varphi_{\theta_0}(t_n))}\|S\|_{E(1/\varphi_{\theta_1}(t_n))}\\ &\le \big(C^{|\theta_0-\theta_1|}-1\big)K^2.\end{aligned}$$

This completes the proof. □

We conclude this section with the following result.

Corollary 2.16. *Let $\{\varphi_\theta\}_{\theta\in(0,1)}$ a family of nondegenerate quasi-concave functions which is stable with respect to a translation-invariant Banach sequence lattice E on $\mathbb{Z}$. If $T\colon \vec{X}\to\vec{Y}$ is an operator between Banach couples such that $T\colon K_{\varphi_{\theta_0},E}(\vec{X})\to K_{\varphi_{\theta_0},E}(\vec{Y})$ is invertible for some $\theta_0\in(0,1)$, then there exists $\varepsilon>0$ such that $T\colon K_{\varphi_\theta,E}(\vec{X})\to K_{\varphi_\theta,E}(\vec{Y})$ is invertible for all $\theta\in(0,1)$ with $|\theta-\theta_0|<\varepsilon$.*

Proof. The assumption that the Calderón operator S generated by any balanced sequence $\{t_n\}$ of nondegenerate quasi-concave function φ is bounded in $E(1/\varphi(t_n))$ yields the continuous inclusion for any Banach couple $\vec{X}$,

$$J_{\varphi,E}(\vec{X}) \hookrightarrow K_{\varphi,E}(\vec{X}).$$

Since the opposite inclusion holds (see proof of Theorem 2.9), $K_{\varphi,E}(\vec{X}) = J_{\varphi,E}(\vec{X})$. This completes the proof by Theorem 2.14. □

3. Applications to rearrangement invariant function spaces

Throughout this section $\mathbb{I}=[0,1]$ or $\mathbb{I}=[0,\infty)$ is equipped with the Lebesgue measure m and $L^0(\mathbb{I},m)$ denotes the space of equivalence classes of all real Lebesgue measurable functions on $\mathbb{I}$. Given $f\in L^0(\mathbb{I},m)$, its distribution function is defined by $m_f(\lambda)=m(\{t\in\mathbb{I};\,|f(t)|>\lambda\})$, and its decreasing rearrangement by $f^*(t)=\inf\{\lambda\ge 0;\, m_f(\lambda)\le t\}$ for $t>0$. A Banach lattice $(X,\|\cdot\|_X)$ is called a rearrangement invariant (r.i. for short) function space provided $m_f=m_g$, $f\in X$ implies $g\in X$, $\|f\|_X=\|g\|_X$.

If X is an r.i. function space on $(\mathbb{I},m)$ (for short on $\mathbb{I}$) and χ_A denotes the characteristic function of a measurable set A, clearly $\|\chi_A\|_X$ depends only on $m(A)$. The function $\varphi_X(t)=\|\chi_A\|_X$, where $m(A)=t$, $0\le t\le 1$, is called the *fundamental function* of X.

Important examples of r.i. function spaces include the L_p-spaces with $1\le p\le\infty$, Marcinkiewicz and Lorentz spaces. Let $\varphi\colon\mathbb{I}\to[0,\infty)$ be a quasi-concave function. The Marcinkiewicz space M_φ is the space of all $f\in L^0(\mathbb{I},m)$ equipped with the norm

$$\|f\|_{M_\varphi} := \sup_{0<t\in\mathbb{I}}\frac{\varphi(t)}{t}\int_0^t f^*(s)\,ds<\infty.$$

In the case when $\varphi\colon\mathbb{I}\to[0,\infty)$ is a concave function with $\varphi(0)=0$, the Lorentz space Λ_φ consists of all $f\in L^0(\mathbb{I},m)$ equipped with the norm

$$\|f\|_{\Lambda_\varphi} := \int_{\mathbb{I}} f^*(t)\,d\varphi(t)<\infty.$$

Note that the fundamental functions of these spaces are $\varphi_{\Lambda_\varphi}=\varphi_{M_\varphi}=\varphi$.

Recall that if $\mathbb{I} = [0, 1]$ (resp., $\mathbb{I} = [0, \infty)$), then L_1 and L_∞ (resp., $L_1 \cap L_\infty$ and $L_1 + L_\infty$) are, respectively, the largest and the smallest r.i. function spaces on $\mathbb{I}$. Moreover, if X is an r.i. function space on $\mathbb{I}$ with fundamental function φ, then φ is quasi-concave and the following continuous embeddings hold (see [14, Theorems II.5.5 and II.5.7] or [9, Theorem II.5.13]):

$$\Lambda(\widetilde{\varphi}) \hookrightarrow X \hookrightarrow M(\varphi),$$

where $\widetilde{\varphi}$ is a least concave majorant of φ.

For a given $t > 0$ the *dilation operator* σ_t is defined for all $f \in L^0(\mathbb{I}, m)$ by $\sigma_t f(s) = f(s/t)\chi_{\mathbb{I}}(s/t)$, $s \in \mathbb{I}$. Then σ_t is bounded in every r.i. function space X and so the *lower* and the *upper Boyd indices* are well defined by

$$\alpha_X = \lim_{t\to 0+} \frac{\ln \|\sigma_t\|}{\ln t}, \quad \beta_X = \lim_{t\to\infty} \frac{\ln \|\sigma_t\|}{\ln t},$$

respectively. In general, $0 \le \alpha_X \le \beta_X \le 1$. Since $s_{\varphi_X}(t) \le \|\sigma_t\|_X$ for every $t > 0$, it follows that $\alpha_X \le p_X \le q_X \le \beta_X$, where $p_X := \gamma_{\varphi_X}$ and $q_X := \delta_{\varphi_X}$. We refer the reader to [9, 14] for more details about r.i. function spaces.

An r.i. function space X on $\mathbb{I}$ is said to be *ultrasymmetric* if X is an interpolation space between the Lorentz space Λ_φ and the Marcinkiewicz space M_φ with $\varphi := \varphi_X$. Ultrasymmetric spaces were studied by Pustylnik [17]. Applications of these spaces are given in [18, 19].

A description of a certain class of ultrasymmetric spaces was given by Pustylnik in terms of real interpolation spaces. Using our notation, his main result Theorem 2.1 in [17] states that an r.i. function space X on $[0, 1]$ with $\gamma_X > 0$ is ultrasymmetric if and only if one has

$$X = K_{E(1/\varphi(t_n))}(L_1, L_\infty),$$

where $\varphi = \varphi_X$, $\{t_n\} = \{2^{-n}\}_{n\ge 0}$ and E is an interpolation space between ℓ_1 and ℓ_∞ defined on $\mathbb{Z}_+$. Combining this result with our variants of the results for estimates of orbits of elements $a_\varphi = \{\varphi(t_n)\}_{n\in\mathbb{I}}$ for $\mathbb{I} = \mathbb{Z}_+$ allows us to obtain Shnieberg's type results on stability of invertible operators $T\colon (L_1, L_\infty) \to (L_1, L_\infty)$ acting on ultrasymmetric spaces. We leave details to the interested reader.

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