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Space of chord-arc curves and BMO/VMO Teichmüller space

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Abstract. This paper focuses on the structure of the subspace T_c of the BMO Teichmüller space T_b corresponding to chord-arc curves, which contains the VMO Teichmüller space T_v. We prove that T_c is not a subgroup with respect to the group structure of T_b, but it is preserved under the inverse operation and the left and the right translations by any element of T_v. Moreover, we show that T_b has a fiber structure induced by T_v, and the complex structure of T_b can be projected down to the quotient space $T_v \backslash T_b$. Then, we see that T_c consists of fibers of this projection, and its quotient space also has the induced complex structure.

Jännekaarikäyrien avaruus ja Teichmüllerin BMO/VMO-avaruus

Tiivistelmä. Tämä työ keskittyy Teichmüllerin BMO-avaruuden T_b jännekaarikäyriin liittyvän aliavaruuden T_c rakenteeseen. Tämä T_c sisältää Teichmüllerin VMO-avaruuden T_v. Osoitamme, että T_c ei ole aliryhmä avaruuden T_b ryhmärakenteen suhteen, mutta T_c on kuitenkin suljettu sekä käänteisoperaation että aliavaruuden T_v suuntaisten vasemman- ja oikeanpuoleisten siirtojen suhteen. Lisäksi osoitamme, että T_v antaa avaruudelle T_b säierakenteen, ja että avaruuden T_b kompleksinen rakenne voidaan projisoida tekijäavaruuteen $T_v \backslash T_b$. Sen jälkeen näemme, että T_c koostuu tämän projektion säikeistä, ja että myös sen tekijäavaruudella on johdettu kompleksinen rakenne.

1. Introduction

Chord-arc curves have been studied with great interest in the fields of harmonic analysis and complex analysis. This concept is related to BMO and VMO functions. Astala and Zinsmeister [3] brought chord-arc curves and BMO functions into the theory of Teichmüller spaces and introduced the BMO Teichmüller space T_b as an analogous space with the universal Teichmüller space T. A brief summary concerning the background on this classical Teichmüller space of quasicircles and quasiconformal maps is given in Section 2. In this paper, we consider the structure of the space of chord-arc curves. We realize this space T_c in the BMO Teichmüller space T_b so that it contains the VMO Teichmüller space T_v. The precise definitions of these spaces are given in Section 3. The main theme of this paper is the investigation of the fiber structure of T_c with respect to the projection of T_b onto T_v.

In the universal Teichmüller space T, the little subspace T_0 is defined by a certain vanishing property of its elements, and the quotient $T_0 \backslash T$ was considered by Gardiner and Sullivan [13]. By taking the projection of the Bers embedding providing

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a complex Banach manifold structure for T and its closed submanifold structure for T_0, they showed that the asymptotic Teichmüller space $AT = T_0 \backslash T$ possesses the quotient Banach manifold structure. A formulation of this result in terms of the foliation of T by T_0 is given in Theorem A and Corollary B. Our first result stated in Theorem 1 is an analogue of this formulation in the relation between T_b and T_v, which is based on Theorem C about the Bers embedding and the complex Banach manifold structures of T_b and T_v proved in [26].

The main results in this paper are Theorem 4 and its corollaries. Having the projection from T_b onto T_v compatible with their complex structures, we will prove that the subspace T_c of chord-arc curves is composed as a union of the fibers of this projection. This can be shown by investigating the set T_c in view of the group structures of T_v and T_b. Then, we see that T_c and its quotient have desirable structures in this fiber space. The precise statements are put in Section 4, where a motivation of our study of T_c and related problems are also mentioned.

We organize this paper by dividing it into sections. Some background on the original theory and preliminaries for some necessary material are summarized in Sections 2 and 3, respectively. The new results are all gathered in Section 4. Instead of giving an introduction of these results in the first section, we ask the reader to refer to the content developed in Sections 2 and 3. The rest of the paper after Section 4 is all devoted to the proofs of the main theorems.

2. Background on the universal Teichmüller space

In this section, we review the theory of the universal Teichmüller space (see [1, 16, 20] for the details). In particular, we explain the construction and the structure of the quotient space by the little universal Teichmüller space, called the asymptotic Teichmüller space (see [10, 12] for the details). The purpose of this paper is to give an analogue of the asymptotic Teichmüller space for the BMO and VMO Teichmüller spaces defined in the next section.

2.1. The universal Teichmüller space. A sense-preserving homeomorphism h of the unit circle $\mathbb{S} = \{z \in \mathbb{C} \mid |z| = 1\}$ is said to be *quasisymmetric* if there exists a least positive constant $C(h)$, called the quasisymmetry constant of h, such that the quasisymmetry quotient

$$m_h(x,t) = \frac{|\ h(e^{i(x+t)}) - h(e^{ix})\ |}{|\ h(e^{ix}) - h(e^{i(x-t)})\ |}$$

takes values in the interval $[1/C(h), C(h)]$ for all $x \in [0, 2\pi)$ and $t \in (0, \pi)$. Let QS denote the group of all quasisymmetric homeomorphisms of $\mathbb{S}$. Beurling and Ahlfors [4] proved that a sense-preserving homeomorphism h of $\mathbb{S}$ is quasisymmetric if and only if there exists some quasiconformal homeomorphism of the unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ onto itself that has boundary value h. Later, Douady and Earle [9] gave a quasiconformal extension of a quasisymmetric homeomorphism of $\mathbb{S}$, called the Douady–Earle extension, in a conformally natural way.

The *universal Teichmüller space* T can be defined as the group QS modulo the left action of the group Möb($\mathbb{S}$) of all Möbius transformations of $\mathbb{S}$, i.e., $T = \text{Möb}(\mathbb{S})\backslash \text{QS}$. Alternatively, T can be identified with the set of representatives $h \in \text{QS}$ fixing three points, 1, -1 and i, which can be realized as a subgroup of QS. The topology on T is induced from that on QS defined by the quasisymmetry constant. This is the real model of the universal Teichmüller space T.

The universal Teichmüller space T is also represented by quasiconformal maps. Let

$$M(\mathbb{D}) = \{\mu \in L^\infty(\mathbb{D}) \mid \|\mu\|_\infty < 1\}$$

denote the open unit ball of the Banach space $L^\infty(\mathbb{D})$ of all essentially bounded measurable functions on $\mathbb{D}$. An element in $M(\mathbb{D})$ is called a *Beltrami coefficient*. By the measurable Riemann mapping theorem, for any $\mu \in M(\mathbb{D})$, there is a unique normalized quasiconformal map f^μ of $\mathbb{D}$ onto itself whose complex dilatation is μ, where the normalization is given by fixing 1, -1 and i. Two elements μ and ν in $M(\mathbb{D})$ are equivalent, denoted by $\mu \sim \nu$, if $f^\mu|_{\mathbb{S}} = f^\nu|_{\mathbb{S}}$. Then, the set $M(\mathbb{D})/\!\sim$ of all equivalence classes $[\mu]$ is the Beltrami coefficient model of T. Let π be the *Teichmüller projection* from $M(\mathbb{D})$ onto T defined by $\pi(\mu) = [\mu]$. This is continuous and open, which induces a homeomorphism between $M(\mathbb{D})/\!\sim$ and $\text{Möb}(\mathbb{S})\backslash\text{QS}$.

There is also a unique normalized quasiconformal map f_μ of the Riemann sphere $\widehat{\mathbb{C}}$ with complex dilatation μ in $\mathbb{D}$ and 0 in $\mathbb{D}^* = \widehat{\mathbb{C}} - \overline{\mathbb{D}}$. The normalization of a holomorphic map $f\colon \mathbb{D}^* \to \widehat{\mathbb{C}}$ is given by

$$f(z) = z + \frac{b_1}{z} + \cdots \quad (z \to \infty). \tag{1}$$

We consider the Bers map $\Phi\colon M(\mathbb{D}) \to B(\mathbb{D}^*)$ sending μ to the Schwarzian derivative $\mathcal{S}(f_\mu|_{\mathbb{D}^*})$ of the conformal map $f_\mu|_{\mathbb{D}^*}$, where

$$B(\mathbb{D}^*) = \{\varphi \mid \|\varphi\|_B = \sup_{z\in\mathbb{D}^*}(|z|^2-1)^2|\varphi(z)| < \infty\}$$

is the Banach space of all holomorphic mappings φ on $\mathbb{D}^* = \widehat{\mathbb{C}} - \overline{\mathbb{D}}$ with the norm $\|\varphi\|_B$. The *Bers embedding* $\beta\colon T \to B(\mathbb{D}^*)$ is given by the factorization of the map Φ by the Teichmüller projection π, i.e., $\beta \circ \pi = \Phi$. This is a well-defined injection due to the fact that $f_\mu|_{\mathbb{D}^*} = f_\nu|_{\mathbb{D}^*}$ is equivalent to $f^\mu|_{\mathbb{S}} = f^\nu|_{\mathbb{S}}$. It can be proved that Φ is a holomorphic split submersion (see [16, Theorem V.5.3], [20, Sections 3.4, 3.5]). In particular, there is a local holomorphic right inverse of Φ at every point of $\Phi(M(\mathbb{D}))$. This implies that the Bers embedding β is a homeomorphism onto its image, and thus it induces a complex structure of T as a domain in the Banach space $B(\mathbb{D}^*)$. This is the unique complex structure on T such that the projection π is holomorphic.

It is well known that a quasiconformal homeomorphism of $\mathbb{D}$ onto itself induces a biholomorphic automorphism of the universal Teichmüller space T. Precisely, the normalized quasiconformal homeomorphism f^μ for $\mu \in M(\mathbb{D})$ induces a biholomorphic automorphism $r_\mu\colon M(\mathbb{D}) \to M(\mathbb{D})$ which sends ν to

$$\nu * \mu^{-1} = \left(\frac{\nu-\mu}{1-\bar{\mu}\nu}\frac{\partial f^\mu}{\overline{\partial f^\mu}}\right) \circ (f^\mu)^{-1}.$$

This is the complex dilatation of the composition $f^\nu \circ (f^\mu)^{-1}$. We denote by $\nu * \mu$ the complex dilatation of the composition $f^\nu \circ f^\mu$, and by μ^{-1} the complex dilatation of the inverse $(f^\mu)^{-1}$. The map r_μ descends down to a biholomorphic automorphism $R_{[\mu]}$ of T defined by $R_{[\mu]} \circ \pi = \pi \circ r_\mu$ (see [16, Section V.5.4], [20, Section 3.6.2]).

The group operation $*$ on $M(\mathbb{D})$ also descends down to the operation $*$ on T by $[\mu] * [\nu] = [\mu * \nu]$. Combined with the inverse operation $[\mu]^{-1} = [\mu^{-1}]$, this turns out to be the group structure $(T, *)$ of the universal Teichmüller space, which is the same as the group structure of T regarded as a subgroup of QS. Then, $R_{[\mu]}$ is the right translation of T by $[\mu]$ which sends $[\nu]$ to $[\nu] * [\mu]^{-1}$.

2.2. The asymptotic Teichmüller space. A quasisymmetric homeomorphism $h \in \mathrm{QS}$ of $\mathbb{S}$ is called *symmetric* if, in addition, the quasisymmetry quotient satisfies $m_h(x,t) \to 1$ as $t \to 0$ uniformly. We denote the subgroup of QS consisting of all symmetric homeomorphisms of $\mathbb{S}$ by Sym. It was proved by Gardiner and Sullivan [13] that Sym is the characteristic topological subgroup of QS. This in particular implies that, for any $h \in \mathrm{Sym}$, f_n converges to f in QS if and only if $h \circ f_n$ converges to $h \circ f$. The *little universal Teichmüller space* is defined by $T_0 = \mathrm{M\ddot{o}b}(\mathbb{S})\backslash\mathrm{Sym}$. This can be regarded as a closed topological subgroup of $(T, *)$.

It it known that h is symmetric if and only if h can be extended to an asymptotically conformal homeomorphism f of the unit disk $\mathbb{D}$ onto itself. In fact, the Douady–Earle extension of h is asymptotically conformal when h is symmetric. Here, by an *asymptotically conformal homeomorphism* f of the unit disk $\mathbb{D}$, we mean that its complex dilatation μ vanishes at the boundary, that is, $\mathrm{ess.sup}_{r \leqslant |z| < 1}|\mu(z)| \to 0$ as $r \to 1$. The closed subspace of $M(\mathbb{D})$ consisting of all Beltrami coefficients vanishing at the boundary is denoted by $M_0(\mathbb{D})$. Then, $\pi(M_0(\mathbb{D}))$ is a closed subspace of T, which coincides with T_0. The image of $M_0(\mathbb{D})$ under the Bers map $\Phi\colon M(\mathbb{D}) \to B(\mathbb{D})$ is contained in the Banach subspace $B_0(\mathbb{D}^*)$ of $B(\mathbb{D}^*)$ consisting of those holomorphic functions φ such that $(|z|^2-1)^2|\varphi(z)| \to 0$ as $|z| \to 1$ uniformly. Moreover, the local holomorphic inverse of Φ at every point of $\Phi(M_0(\mathbb{D}))$ maps its neighborhood in $B_0(\mathbb{D}^*)$ to $M_0(\mathbb{D})$. Thus, T_0 has the structure of the complex Banach submanifold of T modeled on $B_0(\mathbb{D}^*)$. In particular, we have that $\beta(T_0) = \Phi(M_0(\mathbb{D})) = \beta(T) \cap B_0(\mathbb{D}^*)$.

We consider the quotient space $AT = T_0\backslash T$, which is called the *asymptotic Teichmüller space* (see [10, 13]). If we regard $T = \mathrm{M\ddot{o}b}(\mathbb{S})\backslash\mathrm{QS}$ and $T_0 = \mathrm{M\ddot{o}b}(\mathbb{S})\backslash\mathrm{Sym}$, then AT can be realized as the set of all cosets $\mathrm{Sym}\backslash\mathrm{QS}$ in the group QS. Alternatively, the equivalence class in T under the quotient of T_0 containing $\tau \in T$ is given by $R_\tau^{-1}(T_0)$. In order to introduce a complex structure to AT modeled on the quotient Banach space $B_0(\mathbb{D}^*)\backslash B(\mathbb{D}^*)$, we have to verify the compatibility of the quotients $T_0\backslash T$ and $B_0(\mathbb{D}^*)\backslash B(\mathbb{D}^*)$ under the Bers embedding $\beta\colon T \to B(\mathbb{D}^*)$.

It was proved in [13] that the quotient Bers embedding

$$\widehat{\beta}\colon T_0\backslash T \to B_0(\mathbb{D}^*)\backslash B(\mathbb{D}^*)$$

is well-defined and locally injective. Then, $AT = T_0\backslash T$ becomes a complex manifold having local coordinates in the quotient Banach space $B_0(\mathbb{D}^*)\backslash B(\mathbb{D}^*)$. Later, Kahn (see [12, Section 16.8]) proved further that $\widehat{\beta}$ is globally injective. Earle, Markovic and Saric [11, Theorem 4] gave a different proof for this by using the Douady–Earle extension and generalized the result in a compatible way with Fuchsian group action. As in the following theorem, we can formulate these claims so that the decomposition of T into the submanifolds $R_\tau^{-1}(T_0)$ corresponds bijectively to the decomposition of $\beta(T) \subset B(\mathbb{D}^*)$ into the intersections of the affine subspaces isometric to $B_0(\mathbb{D}^*)$. We call this decomposition the *affine foliated structure* of T induced by T_0. See [18] for a comprehensive exposition.

Theorem A. *$\beta \circ R_\tau^{-1}(T_0) = \beta(T) \cap \{B_0(\mathbb{D}^*) + \beta(\tau)\}$ for every $\tau \in T$.*

The inclusion $\subset$ in the above equality implies that the quotient Bers embedding $\widehat{\beta}$ is well-defined. The converse inclusion $\supset$ implies that $\widehat{\beta}$ is injective. Combining the well-definedness and the injectivity of $\widehat{\beta}$ with the homeomorphy of β from T onto its image in $B(\mathbb{D}^*)$, we have the following result naturally.

Corollary B. *The quotient Bers embedding $\widehat{\beta}\colon T_0\backslash T \to B_0(\mathbb{D}^*)\backslash B(\mathbb{D}^*)$ is a homeomorphism onto its image. Consequently, $AT = T_0\backslash T$ possesses a complex* *structure such that $\hat{\beta}$ is a biholomorphic homeomorphism from $T_0\backslash T$ onto its image in $B_0(\mathbb{D}^*)\backslash B(\mathbb{D}^*)$.*

In this paper, we will obtain the analogous results to these two claims in the setting of the BMO theory of the universal Teichmüller space.

3. Preliminaries on the BMO and VMO Teichmüller spaces

In this section, we give basic definitions and fundamental results on the BMO theory of the universal Teichmüller space (see [3, 25, 26] for the details). We explain these in an analogous way to those for the universal Teichmüller space given in the previous section.

3.1. The BMO Teichmüller space. A sense-preserving homeomorphism h of the unit circle $\mathbb{S}$ is called *strongly quasisymmetric* if for any $\varepsilon > 0$ there is some $\delta > 0$ such that for any arc $I \subset \mathbb{S}$ and any Borel set $E \subset I$,

$$|E| \leqslant \delta|I| \quad \Rightarrow \quad |h(E)| \leqslant \varepsilon|h(I)|.$$

By this definition, a strongly quasisymmetric homeomorphism is quasisymmetric, and the composition of strongly quasisymmetric homeomorphisms is strongly quasisymmetric. Moreover, it is known that the inverse of a strongly quasisymmetric homeomorphism is strongly quasisymmetric (see [7]). We denote by SQS the subgroup of QS consisting of all strongly quasisymmetric homeomorphisms of $\mathbb{S}$.

Any strongly quasisymmetric homeomorphism h is absolutely continuous with $\log h' \in \mathrm{BMO}(\mathbb{S})$ (see [14, Chap. 6]), but the converse is not true. Here, an integrable function ϕ on $\mathbb{S}$ belongs to $\mathrm{BMO}(\mathbb{S})$ if

$$\|\phi\|_{\mathrm{BMO}} = \sup_{I \subset \mathbb{S}} \frac{1}{|I|} \int_I |\phi(e^{i\theta}) - \phi_I| \frac{d\theta}{2\pi} < \infty,$$

where the supremum is taken over all arcs I on $\mathbb{S}$, $|I| = \int_I d\theta/2\pi$ is the normalized length of I, and ϕ_I denotes the average of ϕ over I. The *BMO Teichmüller space* is defined by $T_b = \mathrm{M\ddot{o}b}(\mathbb{S})\backslash \mathrm{SQS}$. This can be regarded as a subgroup of $(T, *)$. The topology on T_b is induced from that on SQS given by the BMO norm, that is, by the distance $d(h_1, h_2) = \| \log h_1' - \log h_2' \|_{\mathrm{BMO}}$.

As in the case of the universal Teichmüller space, the BMO Teichmüller space T_b has the corresponding space for Beltrami coefficients. A measure $\lambda = \lambda(z)\, dx\, dy$ on $\mathbb{D}$ is called a *Carleson measure* if

$$\|\lambda\|_c = \sup \frac{\lambda(S_{h,\theta_0})}{h} < \infty$$

where the supremum is taken over all sectors

$$S_{h,\theta_0} = \{re^{i\theta} \in \mathbb{D} \mid 1 - h \leqslant r < 1,\ |\theta - \theta_0| \leqslant \pi h\}$$

for $h \in (0, 1]$ and $\theta_0 \in [0, 2\pi)$. We denote by $\mathrm{CM}(\mathbb{D})$ the set of all Carleson measures on $\mathbb{D}$. For $\mathbb{D}^* = \widehat{\mathbb{C}} - \overline{\mathbb{D}}$, the set $\mathrm{CM}(\mathbb{D}^*)$ of the Carleson measures on $\mathbb{D}^*$ can be defined similarly. For $\mu \in L^\infty(\mathbb{D})$ and for the Poincaré density $\rho_{\mathbb{D}}(z) = (1 - |z|^2)^{-1}$ (with curvature constant equal to -4) on $\mathbb{D}$, we set

$$\lambda_\mu = |\mu(z)|^2 \rho_{\mathbb{D}}(z)\, dx\, dy.$$

Then the linear subspace $\mathcal{L}(\mathbb{D}) \subset L^\infty(\mathbb{D})$ consisting of all μ with $\lambda_\mu \in \mathrm{CM}(\mathbb{D})$ is a Banach space with a norm $\|\mu\|_* = \|\mu\|_\infty + \|\lambda_\mu\|_c^{1/2}$. Moreover, we consider the corresponding space of Beltrami coefficients as $\mathcal{M}(\mathbb{D}) = M(\mathbb{D}) \cap \mathcal{L}(\mathbb{D})$. Then, T_b is the image of $\mathcal{M}(\mathbb{D})$ under the Teichmüller projection $\pi\colon M(\mathbb{D}) \to T$. The quotient topology on T_b induced from $\mathcal{M}(\mathbb{D})$ by π coincides with the topology on T_b induced from the BMO norm.

There is also a subspace of holomorphic mappings corresponding to T_b. For $\varphi \in B(\mathbb{D}^*)$, another norm is given by $\|\varphi\|_{\mathcal{B}} = \|\tilde{\lambda}_\varphi\|_c^{1/2}$, where

$$\tilde{\lambda}_\varphi = |\varphi(z)|^2 \rho_{\mathbb{D}^*}^{-3}(z)\, dx\, dy$$

is a Carleson measure on $\mathbb{D}^*$ for the Poincaré density $\rho_{\mathbb{D}^*}(z) = (|z|^2 - 1)^{-1}$. We consider the Banach space $\mathcal{B}(\mathbb{D}^*) \subset B(\mathbb{D}^*)$ consisting of all such elements φ that $\tilde{\lambda}_\varphi \in \mathrm{CM}(\mathbb{D}^*)$ equipped with the norm $\|\varphi\|_{\mathcal{B}}$. The following result was proved in [26, Theorem 5.1].

Theorem C. *The Bers map Φ restricted to $\mathcal{M}(\mathbb{D})$ is a holomorphic map into $\mathcal{B}(\mathbb{D}^*)$ with a local holomorphic right inverse at every point of the image $\Phi(\mathcal{M}(\mathbb{D}))$. The Bers embedding β of T_b is a homeomorphism onto the domain $\beta(T_b) = \Phi(\mathcal{M}(\mathbb{D}))$ in $\mathcal{B}(\mathbb{D}^*)$. In particular, T_b has a complex structure modeled on $\mathcal{B}(\mathbb{D}^*)$.*

3.2. The VMO Teichmüller space. We say that a strongly quasisymmetric homeomorphism $h \in \mathrm{SQS}$ is *strongly symmetric* if $\log h' \in \mathrm{VMO}(\mathbb{S})$, where a function $\phi \in \mathrm{BMO}(\mathbb{S})$ belongs to $\mathrm{VMO}(\mathbb{S})$ if

$$\lim_{|I|\to 0} \frac{1}{|I|} \int_I |\phi(e^{i\theta}) - \phi_I| \frac{d\theta}{2\pi} = 0$$

uniformly. In fact, $\mathrm{VMO}(\mathbb{S})$ is a closed subspace of $\mathrm{BMO}(\mathbb{S})$, and it is precisely the closure of the space of all continuous functions on $\mathbb{S}$ under the BMO topology (see [23]). We denote by SS the subgroup of SQS consisting of all strongly symmetric homeomorphisms. It is easy to see that the inclusion relation $\mathrm{SS} \subset \mathrm{Sym}$ holds. It was proved in [28] that SS is the characteristic topological subgroup of SQS. This in particular implies that, for any $h \in \mathrm{SS}$, f_n converges to f in SQS if and only if $h \circ f_n$ converges to $h \circ f$. The *VMO Teichmüller space* T_v is defined to be $T_v = \mathrm{Möb}(\mathbb{S})\backslash\mathrm{SS}$, which can be regarded as a closed topological subgroup of $(T_b, *)$.

A Carleson measure $\lambda \in \mathrm{CM}(\mathbb{D})$ is called a *vanishing Carleson measure* if

$$\lim_{h\to 0} \frac{\lambda(S_{h,\theta_0})}{h} = 0$$

uniformly for $\theta_0 \in [0, 2\pi)$. We denote the set of all vanishing Carleson measures on $\mathbb{D}$ by $\mathrm{CM}_0(\mathbb{D})$. The set $\mathrm{CM}_0(\mathbb{D}^*)$ of the vanishing Carleson measures on $\mathbb{D}^*$ can be defined similarly. Let $\mathcal{M}_0(\mathbb{D})$ be the closed subspace of $\mathcal{M}(\mathbb{D})$ consisting of all Beltrami coefficients μ such that $\lambda_\mu \in \mathrm{CM}_0(\mathbb{D})$. Then, $\pi(\mathcal{M}_0(\mathbb{D}))$ is a closed subspace of T_b, which coincides with T_v. We denote by $\mathcal{B}_0(\mathbb{D}^*)$ the Banach subspace of $\mathcal{B}(\mathbb{D}^*)$ consisting of all elements φ such that $\tilde{\lambda}_\varphi = |\varphi(z)|^2 \rho_{\mathbb{D}^*}^{-3}(z)\, dx\, dy \in \mathrm{CM}_0(\mathbb{D}^*)$. Then $\mathcal{B}_0(\mathbb{D}^*) \subset B_0(\mathbb{D}^*)$ by [26, Lemma 4.1]. It was proved in [26, Theorems 4.1, 5.2] that Φ maps $\mathcal{M}_0(\mathbb{D})$ into $\mathcal{B}_0(\mathbb{D}^*)$ and the Bers embedding β of T_v is a homeomorphism onto a domain $\beta(T_v) = \Phi(\mathcal{M}_0(\mathbb{D}))$ in $\mathcal{B}_0(\mathbb{D}^*)$.

3.3. The chord-arc curve space. A rectifiable closed Jordan curve Γ in the complex plane $\mathbb{C}$ is called a *chord-arc curve* if

$$l_\Gamma(z_1, z_2)/|z_1 - z_2| \leqslant K \qquad (2)$$

for any $z_1, z_2 \in \Gamma$, where $l_\Gamma(z_1, z_2)$ denotes the Euclidean length of the shorter arc of Γ between z_1 and z_2. The smallest such constant $K > 1$ is called the chord-arc constant for Γ. A chord-arc curve is in particular a quasicircle. Further, if the ratio in (2) tends to 1 uniformly as $|z_1 - z_2| \to 0$, then the curve Γ is called *asymptotically smooth* in the sense of Pommerenke [21].

The following fact is well-known as the characterization of a chord-arc curve (see [15, Proposition 1.13]).

Proposition D. *A chord-arc curve is the image of $\mathbb{S}$ under a bi-Lipschitz homeomorphism f of $\mathbb{C}$ onto itself with respect to the Euclidean distance. That is, there exists a homeomorphism $f\colon \mathbb{C} \to \mathbb{C}$ with a constant $C \geqslant 1$ such that $f(\mathbb{S}) = \Gamma$ and $C^{-1}|z - w| \leqslant |f(z) - f(w)| \leqslant C|z - w|$ for all $z, w \in \mathbb{C}$.*

If Γ is a simple curve passing through ∞ satisfying the chord-arc condition (2), we can transfer it to a bounded chord-arc curve by a Möbius transformation because the chord-arc condition is Möbius invariant, or equivalently, the distance in the chord-arc condition can be replaced with the spherical distance (see [17, p. 877]). This also implies that Proposition D can be stated equivalently for an unbounded chord-arc curve by the same bi-Lipschitz condition.

Let Γ be a bounded Jordan curve in the Riemann sphere $\widehat{\mathbb{C}}$, let Ω and Ω^* denote its inner and outer domains in $\widehat{\mathbb{C}}$, respectively, and let g and f be conformal maps of $\mathbb{D}$ and $\mathbb{D}^*$ onto Ω and Ω^*, respectively. We always assume that $f\colon \mathbb{D}^* \to \Omega^*$ is normalized so that it satisfies (1). These two maps extend homeomorphically to the boundary, and hence $h = (g|_{\mathbb{S}})^{-1} \circ (f|_{\mathbb{S}})$ determines a sense-preserving homeomorphism of $\mathbb{S}$ onto itself, which is called the *conformal welding homeomorphism* with respect to Γ.

It is well known that h is strongly quasisymmetric if and only if the curve Γ is a quasicircle satisfying the so-called Bishop–Jones condition (see [6]): for any $z \in \Omega$ there exists a domain $\Omega_z (\subset \Omega)$ containing z bounded by a chord-arc curve with constant K such that the diameter of Ω_z is uniformly comparable to $\operatorname{dist}(z, \Gamma)$ and the Hausdorff linear measure of $\Gamma \cap \partial\Omega_z$ is bounded from below by $C\operatorname{dist}(z, \Gamma)$, where $K > 1$ and $C > 0$ depend only on Γ. The Bishop–Jones condition is invariant under a bi-Lipschitz homeomorphism of $\mathbb{C}$ onto itself in the Euclidean metric, and hence, any chord-arc curve satisfies this condition. It was proved in [21] that h is strongly symmetric if and only if the curve Γ is asymptotically smooth. However, although chord-arc curves are in a very special class of quasicircles, no characterization has been found in terms of their conformal welding homeomorphisms of $\mathbb{S}$.

We denote the set of all these conformal welding homeomorphisms for chord-arc curves by CQS. Then, we have the following proper inclusion relations: $\mathrm{SS} \subsetneqq \mathrm{CQS} \subsetneqq \mathrm{SQS}$. Here, the strictness of the second inclusion is seen by a fact that a quasicircle satisfying the Bishop–Jones condition is not necessarily rectifiable (see [5, 24]). Similarly to $T_b = \mathrm{M\ddot{o}b}(\mathbb{S})\backslash\mathrm{SQS}$ and $T_v = \mathrm{M\ddot{o}b}(\mathbb{S})\backslash\mathrm{SS}$, we define $T_c = \mathrm{M\ddot{o}b}(\mathbb{S})\backslash\mathrm{CQS}$. The following fact was essentially shown by Zinsmeister [31] (see also [3]).

Proposition E. *T_c is an open subset of T_b containing T_v.*

4. Statement of the results

We state our results in this paper, which fall into two parts. The first part is concerning the affine foliated structure of the BMO Teichmüller space T_b by the VMO Teichmüller space T_v. This is an analogous result with Theorem A, which gives a foundation to investigate the structure of the quotient Teichmüller space.

Theorem 1. *$\beta \circ R_\tau^{-1}(T_v) = \beta(T_b) \cap \{\mathcal{B}_0(\mathbb{D}^*) + \beta(\tau)\}$ for every $\tau \in T_b$.*

We note that $R_\tau^{-1}(T_v)$ for each $\tau \in T_b$ is an equivalence class of the quotient space $T_v \backslash T_b \cong \mathrm{SS}\backslash\mathrm{SQS}$ containing τ, which is also a closed subspace of T_b biholomorphically equivalent to T_v. By this theorem, we have the decomposition of the Bers embedding as

$$\beta(T_b) = \bigsqcup_{[\tau]\in T_v\backslash T_b} \beta \circ R_\tau^{-1}(T_v) = \bigsqcup_{[\psi]\in \mathcal{B}_0(\mathbb{D}^*)\backslash \mathcal{B}(\mathbb{D}^*)} \beta(T_b) \cap \{\mathcal{B}_0(\mathbb{D}^*) + \psi\}.$$

Based on Theorems 1 and C, the quotient space $T_v\backslash T_b$ is provided with a complex structure modeled on the quotient Banach space $\mathcal{B}_0(\mathbb{D}^*)\backslash\mathcal{B}(\mathbb{D}^*)$. The argument for this is the same as that for Corollary B in the case of the asymptotic Teichmüller space.

Corollary 2. *The quotient Bers embedding*

$$\widehat{\beta}\colon T_v\backslash T_b \to \mathcal{B}_0(\mathbb{D}^*)\backslash\mathcal{B}(\mathbb{D}^*)$$

is well-defined and injective to be a homeomorphism of $T_v\backslash T_b$ onto its image. Consequently, $T_v\backslash T_b$ possesses a complex structure such that $\widehat{\beta}$ is biholomorphic.

Moreover, we have the following result on biholomorphic automorphisms of $T_v\backslash T_b$ with respect to its complex structure. This is well-known in the theory of asymptotic Teichmüller space (see [19, Proposition 4.1]).

Corollary 3. *Let $p\colon T_b \to T_v\backslash T_b$ be the quotient projection from T_b onto $T_v\backslash T_b$. For every $\tau \in T_b$, the biholomorphic automorphism R_τ of T_b induces a biholomorphic automorphism $\widehat{R}_\tau$ of $T_v\backslash T_b$ satisfying $p \circ R_\tau = \widehat{R}_\tau \circ p$.*

The second part of our results is concerning the structure of the space of chord-arc curves. We will show that CQS does not carry a group structure under the composition. This follows from the claim that every element of SQS can be represented as a finite composition of elements in CQS. However, CQS is preserved under the inverse operation and under the left and right actions of SS. In particular, CQS is preserved under the conjugation by SS. We state these claims in the framework of Teichmüller spaces; the chord-arc curve space is identified with a subspace $T_c = \mathrm{M\ddot{o}b}(\mathbb{S})\backslash\mathrm{CQS}$ of the BMO Teichmüller space $(T_b, *)$. For each $\sigma \in T_b$, the left translation $L_\sigma\colon T_b \to T_b$ is defined by $L_\sigma(\tau) = \sigma * \tau$ for every $\tau \in T_b$.

Theorem 4. *The following statements hold.*

(a) *Each element of T_b can be represented as a finite composition of elements in T_c. Hence, T_c is not a subgroup of T_b.*
(b) *The inverse element τ^{-1} belongs to T_c for every $\tau \in T_c$.*
(c) *$L_\sigma(T_c) = T_c$ and $R_\sigma(T_c) = T_c$ for every $\sigma \in T_v$.*

Remark. Statement (a) is a general fact not only for $U = T_c$ but also for any open neighborhood $U \subsetneqq T_b$ of the origin $o = [0]$ of the Teichmüller space. This is seen from its proof. Statement (b) does not assert that the inverse operation $\tau \mapsto \tau^{-1}$ is continuous. In fact, this is not necessarily continuous on T_c but continuous on T_v. For statement (c), we have only to show the inclusion $L_\sigma(T_c) \subset T_c$. The inverse inclusion follows from the facts that $L_\sigma^{-1} = L_{\sigma^{-1}}$ and $\sigma^{-1} \in T_v$. The equality for the right translation is obtained by taking the inverse of the equality for the left translation and applying statement (b).

The above condition $L_\sigma(T_c) \subset T_c$ for every $\sigma \in T_v$ is equivalent to that $R_\tau^{-1}(T_v) \subset T_c$ for every $\tau \in T_c$. We verify this property in the proof. Concerning the fiber structure of T_c with respect to the projection $p\colon T_b \to T_v\backslash T_b$, this condition implies that T_c consists of all fibers of p that intersect T_c. Moreover, if we consider this as the property of the right translation R_σ by $\sigma \in T_v$ preserving T_c, the projection p restricted to T_c is also obtained as the quotient by the group action of T_v.

Corollary 5. *For each $\tau \in T_c$, the fiber $R_\tau^{-1}(T_v)$ of the projection $p\colon T_b \to T_v \backslash T_b$ is entirely contained in T_c. Hence, $T_c = \bigsqcup_{[\tau]\in p(T_c)} R_\tau^{-1}(T_v)$. For each $\sigma \in T_v$, the right translation R_σ acts on T_c as a biholomorphic automorphism, and $p(T_c)$ is given as the quotient $T_v \backslash T_c$ of this group action.*

Problem. We have seen that T_v acts on T_c as a group of biholomorphic automorphisms. Then, we may ask about whether this property characterizes T_v, namely, if the stabilizer subgroup

$$\{\tau \in T_b \mid R_\tau(T_c) = T_c = L_\tau(T_c)\}$$

containing T_v coincides with T_v or not. As a related question, for any $\tau \in T_c - T_v$, we ask about the existence of an integer n such that $\tau^n = \tau * \cdots * \tau \notin T_c$.

Corollary 5 implies the following equation in the Bers embedding. Combined with Theorem 1, this yields the affine foliated structure of T_c by T_v.

Corollary 6. $\beta(T_b)\cap\{\mathcal{B}_0(\mathbb{D}^*)+\beta(\tau)\} = \beta(T_c)\cap\{\mathcal{B}_0(\mathbb{D}^*)+\beta(\tau)\}$ *for every* $\tau \in T_c$.

The quotient Bers embedding from $T_v \backslash T_c$ into $\mathcal{B}_0(\mathbb{D}^*)\backslash\mathcal{B}(\mathbb{D}^*)$, considered in [29, Theorem 2.2], is well-defined and injective. This can be extended to the embedding of $T_v \backslash T_b$ as in Corollary 2. Then, we provide the quotient space $T_v \backslash T_c$ with a complex structure modeled on the quotient Banach space $\mathcal{B}_0(\mathbb{D}^*)\backslash\mathcal{B}(\mathbb{D}^*)$.

Corollary 7. *The quotient Bers embedding $\widehat{\beta}$ maps $T_v \backslash T_c$ homeomorphically onto its image in $\mathcal{B}_0(\mathbb{D}^*)\backslash\mathcal{B}(\mathbb{D}^*)$, and $T_v \backslash T_c$ possesses a complex structure as a domain in $T_v \backslash T_b$.*

Connectivity of T_c is an open problem (see [2], [3, p. 614]). Since T_v is connected, Corollary 5 also implies the following reduction on this problem.

Corollary 8. *T_c is connected if and only if $T_v \backslash T_c$ is connected.*

Corollaries 5, 6, 7, and 8 follow directly from the preceding results. The remainder of this paper is devoted to the proofs of Theorem 1 with Corollaries 2 and 3 and Theorem 4.

5. Proof of Theorem 1

For every $\tau \in T_b$, let $f^\nu\colon \mathbb{D} \to \mathbb{D}$ be a normalized quasiconformal extension of τ with complex dilatation $\nu \in \mathcal{M}(\mathbb{D})$ (i.e. $\pi(\nu) = \tau$) that is bi-Lipschitz under the Poincaré metric on $\mathbb{D}$ (for instance, the Douady–Earle extension of τ satisfies this condition; see [8]). Let $\psi = \Phi(\nu) \in \mathcal{B}(\mathbb{D}^*)$.

5.1. Proof of the inclusion $\subset$. We divide the arguments into two steps. We first deal with the special case that $\mu \in \mathcal{M}_0(\mathbb{D})$ has a compact support. Then, we extend this to the general case by means of an approximation process.

We take such a Beltrami coefficient $\mu \in \mathcal{M}_0(\mathbb{D})$ with compact support. We will show that $\Phi(\mu * \nu) - \Phi(\nu) \in \mathcal{B}_0(\mathbb{D}^*)$. Then, the inclusion $\subset$ follows from

$$\Phi(\mu * \nu) - \Phi(\nu) = \beta \circ \pi(\mu * \nu) - \beta \circ \pi(\nu) = \beta \circ R_\tau^{-1}(\pi(\mu)) - \beta(\tau).$$

Let $f_{\mu*\nu}\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the quasiconformal homeomorphism with complex dilatation $\mu * \nu$ on $\mathbb{D}$ that is conformal on $\mathbb{D}^*$. Set $\widehat{f} = f_{\mu*\nu} \circ f_\nu^{-1}$. Then, $\widehat{f}$ is a quasiconformal homeomorphism with complex dilatation $\widehat{\mu}$ on $\Omega = f_\nu(\mathbb{D})$ whose support is contained in a Jordan domain Ω_0 with $\overline{\Omega_0} \subset \Omega$, and is conformal on $\Omega^* = f_\nu(\mathbb{D}^*)$ with

$$|\mathcal{S}(\widehat{f})|^2 \rho_{\Omega^*}^{-3} = \left(|\mathcal{S}(f_{\mu*\nu}) - \mathcal{S}(f_\nu)|^2 \rho_{\mathbb{D}^*}^{-3}\right) \circ f_\nu^{-1} |(f_\nu^{-1})'|. \tag{3}$$

In fact, $\widehat{f}$ is conformal on the larger domain $\Omega_0^* = \widehat{\mathbb{C}} - \overline{\Omega_0}$.

It is known that $|\mathcal{S}(\widehat{f})(z)| \rho_{\Omega_0^*}^{-2}(z) \leqslant 12$ for $z \in \Omega_0^*$ (see [16, p. 67]). Since the Poincaré density is monotone with respect to the domain, we have $\rho_{\Omega_0^*}(z) \leqslant \rho_{\Omega^*}(z)$. Then, there exists a constant C such that

$$|\mathcal{S}(\widehat{f})(z)|^2 \rho_{\Omega^*}^{-3}(z) \leqslant 144 \rho_{\Omega_0^*}^4(z) \rho_{\Omega^*}^{-3}(z) \leqslant 144 \rho_{\Omega_0^*}(z) \leqslant C$$

for $z \in \Omega^*$. From this, we deduce that $|\mathcal{S}(\widehat{f})|^2 \rho_{\Omega^*}^{-3} \in \mathrm{CM}_0(\Omega^*)$. By (3) and well-definedness of the pull-back operator from $\mathrm{CM}_0(\Omega^*)$ into $\mathrm{CM}_0(\mathbb{D}^*)$ (see [29, Theorem 3.1]), we have that $|\mathcal{S}(f_{\mu*\nu}) - \mathcal{S}(f_\nu)|^2 \rho_{\mathbb{D}^*}^{-3} \in \mathrm{CM}_0(\mathbb{D}^*)$, and thus $\Phi(\mu * \nu) - \Phi(\nu) \in \mathcal{B}_0(\mathbb{D}^*)$.

Next, we consider the general case. For any $\sigma \in T_v$, the complex dilatation of the Douady–Earle extension of σ is denoted by μ. Then, $\mu \in \mathcal{M}_0(\mathbb{D})$ by [27, Theorem 3.7] (see also [30]). We take an increasing sequence of positive numbers $r_n < 1$ $(n = 1, 2, \ldots)$ tending to 1. Let Δ_n be an open disk of radius r_n centered at the origin, and set $A_n = \mathbb{D} - \overline{\Delta_n}$. We define

$$\mu_n = \begin{cases} \mu & \text{on } \overline{\Delta_n}, \\ 0 & \text{on } A_n. \end{cases}$$

Then, $\{\mu_n\}$ is a sequence of complex dilatations with compact support such that

$$\|\mu - \mu_n\|_* = \|\mu - \mu_n\|_\infty + \|\lambda_{\mu-\mu_n}\|_c^{1/2} = \|\mu|_{A_n}\|_\infty + \|\lambda_{\mu|_{A_n}}\|_c^{1/2} \to 0 \tag{4}$$

as $n \to \infty$. Indeed, it was proved in [11] that the complex dilatation of the Douady–Earle extension of a symmetric homeomorphism is in $M_0(\mathbb{D})$. Combining this with the inclusion relation $\mathrm{SS} \subset \mathrm{Sym}$, we see that μ belongs to $M_0(\mathbb{D})$, which implies that the first term of the second line of (4) tends to 0. By the definition of $\mathcal{M}_0(\mathbb{D})$, the second term also tends to 0.

Since f^ν is bi-Lipschitz under the Poincaré metric as we mentioned at the beginning of this section, ν induces a biholomorphic automorphism $r_\nu^{-1} \colon \mathcal{M}(\mathbb{D}) \to \mathcal{M}(\mathbb{D})$ (see [26, Remark 5.1]). Then, we have

$$\|r_\nu^{-1}(\mu) - r_\nu^{-1}(\mu_n)\|_* = \|\mu * \nu - \mu_n * \nu\|_* \to 0$$

as $n \to \infty$. The continuity of Φ yields that

$$\|(\Phi(\mu * \nu) - \Phi(\nu)) - (\Phi(\mu_n * \nu) - \Phi(\nu))\|_{\mathcal{B}} = \|\Phi(\mu * \nu) - \Phi(\mu_n * \nu)\|_{\mathcal{B}} \to 0$$

as $n \to \infty$. We have proved that $\Phi(\mu_n * \nu) - \Phi(\nu) \in \mathcal{B}_0(\mathbb{D}^*)$ in the first step. Then, it follows that $\Phi(\mu * \nu) - \Phi(\nu) \in \mathcal{B}_0(\mathbb{D}^*)$ from the fact that $\mathcal{B}_0(\mathbb{D}^*)$ is closed in $\mathcal{B}(\mathbb{D}^*)$. This proves the inclusion $\subset$. □

5.2. Proof of the inclusion $\supset$. This can be proved by using the following claim, which is shown in [18, Proposition 3.3].

Claim. Let $f_\nu \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a quasiconformal homeomorphism with complex dilatation $\nu \in M(\mathbb{D})$ that is bi-Lipschitz between $\mathbb{D}$ and $\Omega = f_\nu(\mathbb{D})$ under their Poincaré metrics, and is conformal on $\mathbb{D}^*$ with $\mathcal{S}(f_\nu|_{\mathbb{D}^*}) = \psi$. Then, for every $\varphi \in B_0(\mathbb{D}^*)$, there exists a quasiconformal homeomorphism $\widehat{f} \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with complex dilatation $\widehat{\mu}$ on Ω

vanishing at the boundary that is conformal on $\Omega^* = f_\nu(\mathbb{D}^*)$ with $\mathcal{S}(\widehat{f} \circ f_\nu|_{\mathbb{D}^*}) = \varphi + \psi$ such that the following statements are valid: $\widehat{f}$ is decomposed into two quasiconformal homeomorphisms $\widehat{f}_0$ and $\widehat{f}_1$ of $\widehat{\mathbb{C}}$ with $\widehat{f} = \widehat{f}_0 \circ \widehat{f}_1$, where $\widehat{f}_1$ is conformal on Ω^* with $\mathcal{S}(\widehat{f}_1 \circ f_\nu|_{\mathbb{D}^*}) = \varphi_1 + \psi$, satisfying the following properties:

(i) the complex dilatation $\widehat{\mu}_1$ of $\widehat{f}_1$ on Ω satisfies

$$|\widehat{\mu}_1 \circ f_\nu(z)| \leqslant \frac{1}{\varepsilon}\rho_{\mathbb{D}^*}^{-2}(z^*)|\varphi_1(z^*)| \quad (z^* = \bar{z}^{-1})$$

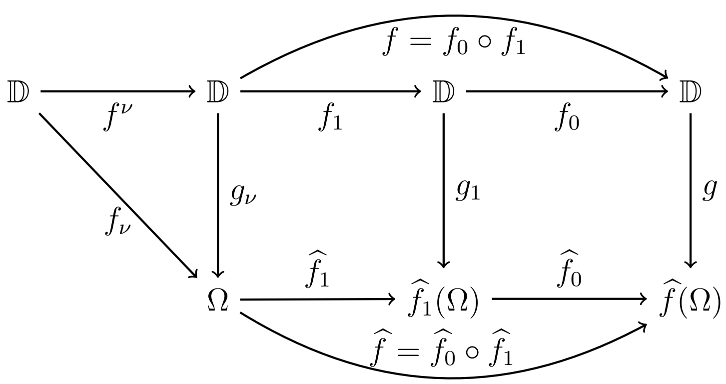
for some $\varepsilon > 0$ and for every $z \in \mathbb{D}$;

(ii) the support of the complex dilatation μ_0 of the normalized quasiconformal homeomorphism $f_0 \colon \mathbb{D} \to \mathbb{D}$, which is conformally conjugate to $\widehat{f}_0 \colon \widehat{f}_1(\Omega) \to \widehat{f}(\Omega)$, is contained in a compact subset of $\mathbb{D}$;

(iii) for the complex dilatation μ_1 of the normalized quasiconformal homeomorphism $f_1 \colon \mathbb{D} \to \mathbb{D}$, which is conformally conjugate to $\widehat{f}_1 \colon \Omega \to \widehat{f}_1(\Omega)$, we have

$$\varphi - \varphi_1 = \Phi(\mu_0 * \mu_1 * \nu) - \Phi(\mu_1 * \nu).$$

Combining all those maps in the claim above, we have the following commutative diagram, where g_ν, g_1, and g are the conjugating conformal maps:



We take $\varphi \in \mathcal{B}_0(\mathbb{D}^*)$ such that $\varphi + \psi \in \beta(T_b)$. Since $\mathcal{B}_0(\mathbb{D}^*) \subset B_0(\mathbb{D}^*)$, there is a quasiconformal homeomorphism $\widehat{f} \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ conformal on Ω^* and asymptotically conformal on Ω such that $\mathcal{S}(\widehat{f} \circ f_\nu|_{\mathbb{D}^*}) = \varphi + \psi$. According to the claim above, we consider the decomposition $\widehat{f} = \widehat{f}_0 \circ \widehat{f}_1$ together with other maps that appear in it, and apply the properties shown there.

Since $\varphi \in \mathcal{B}_0(\mathbb{D}^*)$, if $\varphi - \varphi_1 \in \mathcal{B}_0(\mathbb{D}^*)$, then $\varphi_1 \in \mathcal{B}_0(\mathbb{D}^*)$. By property (ii), μ_0 in particular belongs to $\mathcal{M}_0(\mathbb{D})$, and by property (iii), $\varphi - \varphi_1 = \Phi(\mu_0 * \mu_1 * \nu) - \Phi(\mu_1 * \nu)$. By the previous arguments showing the inclusion $\subset$, we see that $\varphi - \varphi_1 \in \mathcal{B}_0(\mathbb{D}^*)$. Hence, $\varphi_1 \in \mathcal{B}_0(\mathbb{D}^*)$.

By property (i), $\varphi_1 \in \mathcal{B}_0(\mathbb{D}^*)$ implies that $\widehat{\mu}_1 \circ f_\nu \in \mathcal{M}_0(\mathbb{D})$. Since $|\widehat{\mu}_1 \circ f_\nu| = |\mu_1 \circ f^\nu|$, we have $\mu_1 \circ f^\nu \in \mathcal{M}_0(\mathbb{D})$. It follows from the bi-Lipschitz continuity of f^ν and [27, Proposition 3.5] that $\mu_1 \in \mathcal{M}_0(\mathbb{D})$. By property (ii), the support of the complex dilatation μ_0 of f_0 is contained in a compact subset of $\mathbb{D}$. Hence, we see that the complex dilatation $\mu_f = \mu_0 * \mu_1$ of $f = f_0 \circ f_1$ belongs to $\mathcal{M}_0(\mathbb{D})$. Since the complex dilatation of the quasiconformal homeomorphism $\widehat{f} \circ f_\nu$ on $\mathbb{D}$ is $r_\nu^{-1}(\mu_f)$, we have that

$$\varphi + \psi = \Phi(\mu_0 * \mu_1 * \nu) = \Phi \circ r_\nu^{-1}(\mu_f) \in \Phi \circ r_\nu^{-1}(\mathcal{M}_0(\mathbb{D})) = \beta \circ R_\tau^{-1}(T_v),$$

which proves the inclusion $\supset$. $\square$

6. Proofs of Corollaries 2 and 3

6.1. Proof of Corollary 2. For the quotient maps $p\colon T_b \to T_v \backslash T_b$ and $P\colon \mathcal{B}(\mathbb{D}^*) \to \mathcal{B}_0(\mathbb{D}^*)\backslash\mathcal{B}(\mathbb{D}^*)$, the following commutative diagram holds:

$$\begin{array}{ccc} T_b & \xrightarrow{\quad\beta\quad} & \mathcal{B}(\mathbb{D}^*) \\ \Big\downarrow p & & \Big\downarrow P \\ T_v\backslash T_b & \xrightarrow{\quad\widehat{\beta}\quad} & \mathcal{B}_0(\mathbb{D}^*)\backslash\mathcal{B}(\mathbb{D}^*) \end{array}$$

The well-definedness and the injectivity of $\widehat{\beta}$ are direct consequences from Theorem 1. Since P is the projection onto the quotient Banach space, the image $P(\beta(T_b)) = \widehat{\beta}(T_v\backslash T_b)$ is an open subset of $\mathcal{B}_0(\mathbb{D}^*)\backslash\mathcal{B}(\mathbb{D}^*)$. Moreover, $\widehat{\beta}\colon T_v\backslash T_b \to \widehat{\beta}(T_v\backslash T_b)$ is open and continuous because so is $\beta\colon T_b \to \beta(T_b)$. Combined with the injectivity of $\widehat{\beta}$, this implies that $\widehat{\beta}$ is a homeomorphism of $T_v\backslash T_b$ onto its image. $\square$

6.2. Proof of Corollary 3. For each $\sigma \in T_b$, we have that $R_\tau(T_v * \sigma) = T_v * (\sigma * \tau^{-1})$. This shows that the correspondence $[\sigma] \mapsto [\sigma * \tau^{-1}]$ yields a well-defined map $\widehat{R}_\tau\colon p(T_b) \to p(T_b)$ that satisfies $p \circ R_\tau = \widehat{R}_\tau \circ p$. By considering the inverse mapping $R_\tau^{-1} = R_{\tau^{-1}}$, we see that $\widehat{R}_\tau$ is bijective. In the same way as the proof of Corollary 2, $\widehat{R}_\tau$ is shown to be a homeomorphism. Thus, it suffices to prove that $\widehat{R}_\tau$ is holomorphic.

We may identify T_b with the domain $\beta(T_b)$ in $\mathcal{B}(\mathbb{D}^*)$. The conjugate $\widetilde{R}_\varphi = \beta \circ R_\tau \circ \beta^{-1}$ for $\varphi = \beta(\tau)$ is a biholomorphic automorphism of $\beta(T_b) \subset \mathcal{B}(\mathbb{D}^*)$. We use its projection $\widehat{R}_\varphi$ to $P(\beta(T_b)) = \widehat{\beta}(p(T_b))$ as a replacement of $\widehat{R}_\tau$, which satisfies $P \circ \widetilde{R}_\varphi = \widehat{R}_\varphi \circ P$. Let $\phi_1, \phi_2 \in \beta(T_b)$ with $\phi_1 - \phi_2 \in \mathcal{B}_0(\mathbb{D}^*)$ and let $\psi_1, \psi_2 \in \mathcal{B}(\mathbb{D}^*)$ with $\psi_1 - \psi_2 \in \mathcal{B}_0(\mathbb{D}^*)$. The derivative of $\widetilde{R}_\varphi$ satisfies

$$d_{\phi_1}\widetilde{R}_\varphi(\psi_1) = \lim_{t\to 0}\frac{1}{t}(\widetilde{R}_\varphi(\phi_1 + t\psi_1) - \widetilde{R}_\varphi(\phi_1));$$
$$d_{\phi_2}\widetilde{R}_\varphi(\psi_2) = \lim_{t\to 0}\frac{1}{t}(\widetilde{R}_\varphi(\phi_2 + t\psi_2) - \widetilde{R}_\varphi(\phi_2)),$$

where the limits refer to the convergence under the norm $\|\cdot\|_{\mathcal{B}}$. From this, we see that $d_{\phi_1}\widetilde{R}_\varphi(\psi_1) - d_{\phi_2}\widetilde{R}_\varphi(\psi_2)$ belongs to $\mathcal{B}_0(\mathbb{D}^*)$ because $\mathcal{B}_0(\mathbb{D}^*)$ is closed and

$$\begin{aligned} &\{\widetilde{R}_\varphi(\phi_1 + t\psi_1) - \widetilde{R}_\varphi(\phi_1)\} - \{\widetilde{R}_\varphi(\phi_2 + t\psi_2) - \widetilde{R}_\varphi(\phi_2)\} \\ &= \{\widetilde{R}_\varphi(\phi_1 + t\psi_1) - \widetilde{R}_\varphi(\phi_2 + t\psi_2)\} - \{\widetilde{R}_\varphi(\phi_1) - \widetilde{R}_\varphi(\phi_2)\} \end{aligned}$$

belongs to $\mathcal{B}_0(\mathbb{D}^*)$. Thus, for every $[\phi] \in P(\beta(T_b))$, a linear map $A^\varphi_{[\phi]}\colon \mathcal{B}_0(\mathbb{D}^*)\backslash\mathcal{B}(\mathbb{D}^*) \to \mathcal{B}_0(\mathbb{D}^*)\backslash\mathcal{B}(\mathbb{D}^*)$ is well-defined by $A^\varphi_{[\phi]}([\psi]) = [d_\phi\widetilde{R}_\varphi(\psi)]$. This satisfies $A^\varphi_{[\phi]} \circ P = P \circ d_\phi\widetilde{R}_\varphi$.

The linear operator $A^\varphi_{[\phi]}$ is bounded and the operator norm satisfies $\|A^\varphi_{[\phi]}\| \leqslant \|d_\phi\widetilde{R}_\varphi\|$. Indeed, for every $[\psi] \in \mathcal{B}_0(\mathbb{D}^*)\backslash\mathcal{B}(\mathbb{D}^*)$ and every $\varepsilon > 0$, we may choose

$\psi \in \mathcal{B}(\mathbb{D}^*)$ such that $P(\psi) = [\psi]$ and $\|\psi\| \leqslant \|[\psi]\| + \varepsilon$. Then,

$$\|A^{\varphi}_{[\phi]}([\psi])\| = \|P \circ d_\phi \widetilde{R}_\varphi(\psi)\| \leqslant \|d_\phi \widetilde{R}_\varphi(\psi)\| \leqslant \|d_\phi \widetilde{R}_\varphi\| \cdot \|\psi\| \leqslant \|d_\phi \widetilde{R}_\varphi\|(\|[\psi]\| + \varepsilon).$$

Making $\varepsilon > 0$ arbitrarily small, we obtain the claim.

Moreover, since we may assume that $\|\psi\| \leqslant 2\|[\psi]\|$ in the above choice of ψ, we have that

$$\begin{aligned} &\|\widehat{R}_\varphi([\phi] + [\psi]) - \widehat{R}_\varphi([\phi]) - A^{\varphi}_{[\phi]}([\psi])\| \\ &= \|P \circ \widetilde{R}_\varphi(\phi + \psi) - P \circ \widetilde{R}_\varphi(\phi) - P \circ d_\phi \widetilde{R}_\varphi(\psi)\| \\ &\leqslant \|\widetilde{R}_\varphi(\phi + \psi) - \widetilde{R}_\varphi(\phi) - d_\phi \widetilde{R}_\varphi(\psi)\| = o(\|[\psi]\|). \end{aligned}$$

This implies that $\widehat{R}_\varphi$ is differentiable at every $[\phi] \in P(\beta(T_b))$ in every direction $[\psi] \in \mathcal{B}_0(\mathbb{D}^*) \backslash \mathcal{B}(\mathbb{D}^*)$ with the derivative $d_{[\phi]} \widehat{R}_\varphi([\psi]) = A^{\varphi}_{[\phi]}([\psi])$. □

7. Proof of Theorem 4

7.1. Proof of statement (a). Let V denote a subset of T_b consisting of all τ for which there exists an open neighborhood W such that each $\tau' \in W$ can be represented as a finite composition of elements in T_c. Since T_b is connected, in order to prove that V coincides with T_b, it suffices to show that V is non-empty, open, and closed. By Proposition E, T_c is an open subset of T_b containing the origin $o = [0]$. We see that $o \in V$, and hence V is non-empty. By the definition of V, this is open.

Now we prove that V is closed. Let $\{\tau_n\} \subset V$ be a sequence such that $\tau_n \to \tau$ as $n \to \infty$. We will show that $\tau \in V$. Let U be an open neighborhood of o in T_c. Then, $R_\tau(\tau_n) \in U$ for all sufficiently large n, that is, $\sigma_n = \tau_n * \tau^{-1} \in U$. Since $\sigma_n \to o$ as $n \to \infty$, we have $\sigma_n^{-1} \to o$. Thus, we may assume that $\sigma_n^{-1} \in U$. Let $W = R_\tau^{-1}(U)$, which is a neighborhood of τ. For each $\tau' \in W$, there exists an element $\sigma' \in U$ such that $\tau' * \tau^{-1} = \sigma'$. It follows that $\tau' = \sigma' * \tau = \sigma' * \sigma_n^{-1} * \tau_n$. Therefore, $\tau' \in W$ can be represented as a finite composition of elements in T_c. This shows that $\tau \in V$. □

7.2. Proof of statement (b). If $h = g^{-1} \circ f$ is the conformal welding homeomorphism corresponding to a chord-arc curve Γ, then $h^{-1} = f^{-1} \circ g = (j \circ f \circ j)^{-1} \circ (j \circ g \circ j)$ is the conformal welding homeomorphism corresponding to $j(\Gamma)$, where $j(z) = z^* = \bar{z}^{-1}$ is the standard reflection with respect to $\mathbb{S}$. Since j is an isometry with respect to the spherical metric of $\widehat{\mathbb{C}}$, $j(\Gamma)$ is a chord-arc curve. This proves that if $\tau = [h] \in T_c$ then $\tau^{-1} \in T_c$. □

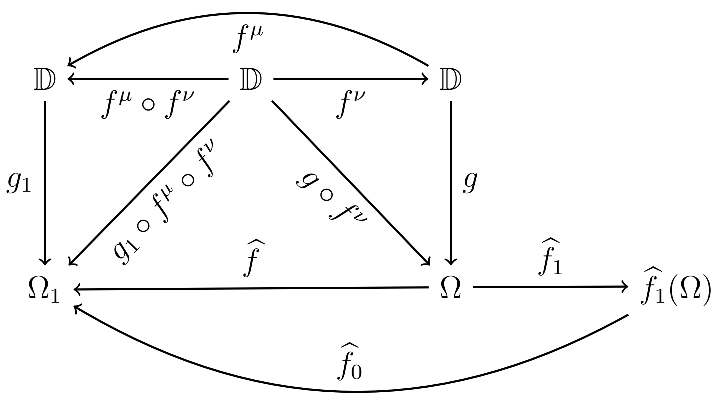
7.3. Proof of statement (c). For any $\tau \in T_c$ and $\sigma \in T_v$, we will show that $R_\tau^{-1}(\sigma)$ belongs to T_c. Set $\widehat{\sigma} = R_\tau^{-1}(\sigma) = \sigma * \tau$. Let $g^{-1} \circ f$ and $g_1^{-1} \circ f_1$ be the conformal welding homeomorphisms such that $[g^{-1} \circ f] = \tau$ and $[g_1^{-1} \circ f_1] = \widehat{\sigma}$, respectively. Here, g and g_1 are conformal maps on $\mathbb{D}$, and f and f_1 are conformal maps on $\mathbb{D}^*$ with the normalization (1). We set $\Omega = g(\mathbb{D})$, $\Omega^* = f(\mathbb{D}^*)$, and $\Gamma = \partial\Omega = \partial\Omega^*$ which is a chord-arc curve. Similarly, we set $\Omega_1 = g_1(\mathbb{D})$, $\Omega_1^* = f_1(\mathbb{D}^*)$, and $\Gamma_1 = \partial\Omega_1 = \partial\Omega_1^*$ which is a quasicircle at this moment. Let f^ν and f^μ be the normalized quasiconformal self-homeomorphisms of $\mathbb{D}$ corresponding to τ and σ, respectively. As $\sigma \in T_v$, we can assume that the complex dilatation μ of f^μ induces a vanishing Carleson measure $\lambda_\mu \in \mathrm{CM}_0(\mathbb{D})$. Then, $g \circ f^\nu$ and $g_1 \circ f^\mu \circ f^\nu$ are quasiconformal extensions of f and f_1 to $\mathbb{D}$, respectively.

We define

$$\widehat{f} = \begin{cases} f_1 \circ f^{-1} & \text{on } \Omega^*, \\ (g_1 \circ f^\mu \circ f^\nu) \circ (g \circ f^\nu)^{-1} = g_1 \circ f^\mu \circ g^{-1} & \text{on } \Omega. \end{cases}$$

Then, $\widehat{f}\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is conformal on Ω^* and asymptotically conformal on Ω whose complex dilatation $\widehat{\mu}$ satisfies $|\widehat{\mu}|^2\rho_\Omega = \lambda_\mu \circ g^{-1}|(g^{-1})'|$ for the Poincaré density ρ_Ω on Ω. As $\lambda_\mu \in \mathrm{CM}_0(\mathbb{D})$, we have that $|\widehat{\mu}|^2\rho_\Omega \in \mathrm{CM}_0(\Omega)$ by [29, Theorem 3.2].

We decompose $\widehat{f}$ into $\widehat{f}_0 \circ \widehat{f}_1$ as follows. The quasiconformal homeomorphism $\widehat{f}_1\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is chosen so that its complex dilatation $\widehat{\mu}_1$ coincides with $\widehat{\mu}$ on $\Omega - \Omega_0$ for some compact subset Ω_0 of Ω homeomorphic to a closed disk, and zero elsewhere. We may assume that $\widehat{f}_1$ satisfies the normalization (1). Then, $\widehat{f}_0$ is defined to be $\widehat{f} \circ \widehat{f}_1^{-1}$. We have the following commutative diagram:



Here, the compact subset $\Omega_0 \subset \Omega$ is chosen so that $|\widehat{\mu}_1|^2\rho_\Omega \in \mathrm{CM}_0(\Omega)$ has a sufficiently small norm as a Carleson measure. It follows from [29, Lemma 4.1] that $|\mathcal{S}(\widehat{f}_1)|^2\rho_{\Omega^*}^{-3} \in \mathrm{CM}_0(\Omega^*)$ with a small norm. By [29, Theorem 3.1], we have that

$$|\mathcal{S}(\widehat{f}_1 \circ f) - \mathcal{S}(f)|^2\rho_{\mathbb{D}^*}^{-3} = (|\mathcal{S}(\widehat{f}_1)|^2\rho_{\Omega^*}^{-3}) \circ f|f'| \in \mathrm{CM}_0(\mathbb{D}^*),$$

and moreover, we see that it can be of a small norm according to that of $|\mathcal{S}(\widehat{f}_1)|^2\rho_{\Omega^*}^{-3}$. Combined with the facts that Γ is a chord-arc curve and that the subspace T_c is open in T_b by Proposition E, this implies that $\partial\widehat{f}_1(\Omega)$ is also a chord-arc curve.

Since the complex dilatation $\widehat{\mu}_0$ of $\widehat{f}_0$ has the compact support $\widehat{f}_1(\Omega_0) \subset \widehat{f}_1(\Omega)$, we conclude that Γ_1 is the image of $\partial\widehat{f}_1(\Omega)$ under $\widehat{f}_0$, which is conformal when restricted to $\mathbb{C} - \widehat{f}_1(\Omega_0)$. In particular, $\widehat{f}_0$ is bi-Lipschitz on the chord-arc curve $\partial\widehat{f}_1(\Omega)$. Then, this extends to a bi-Lipschitz homeomorphism of $\mathbb{C}$ (see [22, Theorem 7.10]), and thus Γ_1 is again a chord-arc curve by Proposition D. This implies that $\widehat{\sigma} \in T_c$. $\square$

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